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# CHARACTERISATION OF BRANCHED SURFACES FULLY CARRYING A LAMINATION

Skander ZANNAD

## Abstract

We give a necessary and sufficient condition for a branched surface in a closed orientable 3-manifold to fully carry a lamination. This answers D. Gabai's problem **3.4** of [GO] (which is also problem **2.1** of [Ga]).

The last part is devoted to contact structures carried by branched surfaces. An application of the result of the previous part is given.

## 1 Introduction

Branched surfaces are combinatoric objects which have been proven useful to study laminations, in particular essential laminations. Maybe one of the most striking topological result is theorem **6.1** of [GO] :

**Theorem 1.1** ([GO]) *If a compact orientable 3-manifold contains an essential lamination (or equivalently, according to [Li], a laminar branched surface), then its universal cover is homeomorphic to  $\mathbb{R}^3$ .*

Our goal is to link the theory of branched surfaces with the theory of contact structures, and to use this link to prove topological results. This leads to the notion of contact structure *carried* by a branched surface. Such a way has already been taken by U. Oertel and J. Świątkowski in [OS1] and [OS2], where they have obtained numerous correspondences between the properties of contact structures and the properties of branched surfaces. However, we use in this paper a more general definition of contact structure carried by a branched surface (see section **8**), which can be linked to the definition of a pair formed by a contact structure and a Reeb vector field *adjusted* to a sutured hierarchy (see [HKM] and [CH] for instance). The starting point of this paper was the following question : if a branched surface carries a positive contact structure and a negative one, does it fully carry a lamination ? By studying which properties must have such a branched surface, theorem **4.6** emerged :

**Theorem 4.6** *Let  $\mathcal{B}$  be a branched surface in  $M$  having no twisted disk of contact. Then  $\mathcal{B}$  fully carries lamination.*

which implies :

**Theorem 4.3** *Let  $\mathcal{B}$  be a branched surface in  $M$ . Then the two following assertions are equivalent :*

- (A)  $\mathcal{B}$  fully carries a lamination.
- (B) All the twisted disks of contact of  $\mathcal{B}$  are carrying.

This theorem answers D. Gabai's problem **3.4** of [GO] (which is also problem **2.1** of [Ga]) : *When does a branched surface in a 3-manifold fully carry a lamination ?*

It also gives a reassuring precision to L. Mosher's theorem :

**Theorem 1.2 (L. Mosher)** *The problem of whether or not a general branched surface abstractly carries a lamination is algorithmically unsolvable.*

Let us give brief explanations on the terms “general branched surface” and “abstractly carries” : the definition of a branched surface we will use in this text is the definition of a branched surface embedded in a 3-manifold. However, there is a more general definition of a branched surface, which does not imply that the branched surface is embedded or even immersed in a 3-manifold. In [Ch], J. Christy gives necessary and sufficient conditions for a general branched surface to be immersed or embedded in a 3-manifold, and some examples. “Abstractly carrying” a lamination is the generalisation for general branched surfaces of “fully carrying” a lamination. Precise definitions can be found in [MO].

According to theorem **4.3**, the problem of whether or not a branched surface embedded in a 3-manifold fully carries a lamination is algorithmically solvable, and theorem **1.2** reduces to branched surfaces which can not be embedded in a 3-manifold.

A proof of theorem **1.2** is given in [Ga].

Let us also recall theorem **2** of T. Li in [Li] :

**Theorem 1.3 ([Li])** *A 3-manifold contains an essential lamination if and only if it contains a laminar branched surface.*

A *laminar* branched surface is a branched surface which satisfies conditions (ii) to (vi) in definition **8.2.1** and has no sink disk. Actually, the last condition “has no sink disk” can be replaced by the weaker condition “has no twisted disk of contact and no disk of contact”. Indeed, the first condition implies the second one, and a branched surface which satisfies conditions (i) to (vi) in definition **8.2.1** and has no twisted disk of contact is essential by theorem **4.3**.

We conclude with proposition **8.2.3**.

An algorithm to decide whether a 3-manifold is laminar has been given by I. Agol and T. Li in [AL].

Theorem **4.3** can be compared to D. Calegari's following result, which is theorem **2.16** of [Ca] :

**Theorem 1.4** ([Ca]) *There exists a branched surface  $\mathcal{B}$  in a 3-manifold  $M$  which does not carry a lamination (of any kind) but which virtually carries a lamination (and even an essential lamination), in the sense that there is a finite cover  $\widehat{M}$  of  $M$  so that the pullback of  $\mathcal{B}$  to  $\widehat{M}$  carries an essential lamination. In fact, for any  $n$  there is a branched surface  $\mathcal{B}_n$  in a manifold  $M_n$  and an  $n$ -fold cover  $\widehat{M}_n$  of  $M_n$  such that the pullback  $\widehat{\mathcal{B}}_n$  fully carries an essential lamination, but there is no  $m$ -fold cover of  $M_n$  with  $m < n$  for which the pullback  $\mathcal{B}_n$  carries anything.*

Theorem **1.4** is an answer to D. Gabai's question : *Are there useful branched surfaces which carry nothing ?*

Corollary **4.7** partially answers the initial question :

**Corollary 4.7** *Let  $\mathcal{B}$  be a branched surface in  $M$  which carries both a positive and a negative contact form. Then  $\mathcal{B}$  fully carries a lamination.*

According to results of V. Colin in [Co] , we can use theorem **1.1** and get corollary **4.8** :

**Corollary 4.8** *If a closed orientable 3-manifold contains a branched surface which carries a positive tight form of contact and a negative tight one, then its universal cover is homeomorphic to  $\mathbb{R}^3$ .*

Future investigation must answer questions such as : which branched surfaces carry a (tight, hypertight) contact structure (form) ? If a branched surface carries a positive tight contact structure and a negative one, does this imply topological results on the manifold ? Are there other contact conditions on a manifold so that its universal cover is  $\mathbb{R}^3$  ?

Sections **2** and **3** give the basic definitions about branched surfaces and surfaces of contact. Section **4** contains the statement of the results. One of the main tools, the splitting, is defined in section **5**. Theorem **4.6** is then proved in section **6**. The proof of theorem **4.3** comes to an end in section **7**. The last section is devoted to the contact structures carried by branched surfaces.

Throughout the paper,  $M$  is an oriented closed 3-manifold.

## 2 Branched surfaces

**Definition 2.1** A *branched surface*  $\mathcal{B}$  in  $M$  is a union of smooth surfaces locally modeled on one of the three models of figure 2.2. The *branch locus*  $\mathcal{L}$  of  $\mathcal{B}$  is the set of points, called *branch points*, none of whose neighbourhoods is a disk. Its *regular part* is  $\mathcal{B} \setminus \mathcal{L}$ . The closure of a connected component of the regular part is called a *sector* of  $\mathcal{B}$ .

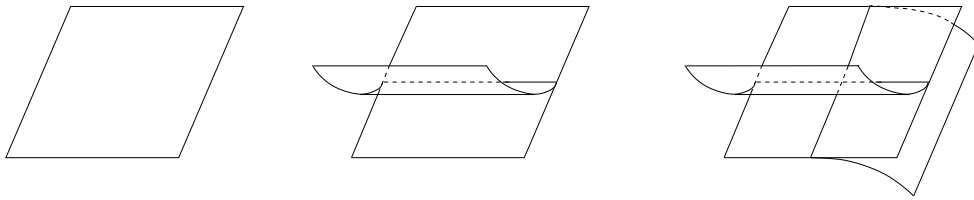


FIG. 2.2 – Local models of a branched surface

The branch locus may have double points, as it is the case in the third model of figure 2.2. At each regular point of  $\mathcal{L}$ , we can define a *branch direction*, as on figure 2.3.

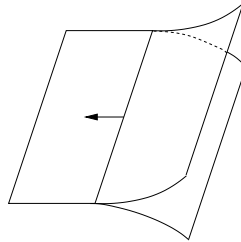


FIG. 2.3 – branch direction

**Definition 2.4** A *fibred neighbourhood*  $N(\mathcal{B})$  of  $\mathcal{B}$  is an interval bundle over  $\mathcal{B}$ , as seen on figure 2.5. The leaves of this foliation are the *fibres* of  $N(\mathcal{B})$ . The boundary of  $N(\mathcal{B})$  can be decomposed into an *horizontal boundary*  $\partial_h N(\mathcal{B})$  transverse to the fibres and a *vertical boundary*  $\partial_v N(\mathcal{B})$ , tangent to the fibres (see figure 2.5, a)).

We define the projection map  $\pi : N(\mathcal{B}) \rightarrow \mathcal{B}$  which sends a fibre of  $N(\mathcal{B})$  onto its base point. In particular,  $\pi(\partial_v N(\mathcal{B})) = \mathcal{L}$ . We can see  $N(\mathcal{B})$  as a part of  $M$ , but in this case  $\mathcal{B}$  is not included in  $N(\mathcal{B})$ . However,  $N(\mathcal{B})$  contains a branched surface  $\mathcal{B}_1$  which is isomorphic to  $\mathcal{B}$  (see figure 2.5, b)). The branched surface  $\mathcal{B}_1$  is a *splitting* of  $\mathcal{B}$ . Splittings will be defined in section 5.

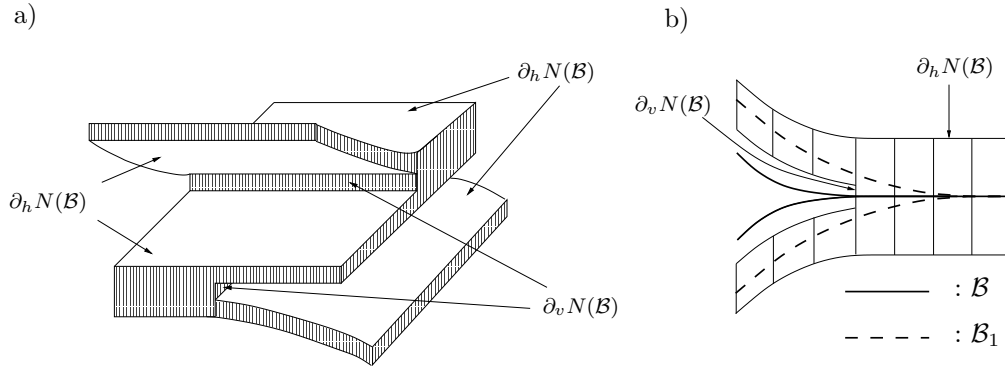


FIG. 2.5 – Fibred neighbourhood

Let's see how we can put a sign on each double point of  $\mathcal{L}$ . Locally, two smooth parts of  $\mathcal{L}$  run through  $p$ . They are cooriented by their branch direction, and we call them  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Set an orientation of the fibre of  $N(\mathcal{B})$  passing through  $p$ . Hence, it makes sense to say that one of the branching  $\mathcal{L}_1$  or  $\mathcal{L}_2$  is over the other at  $p$ . Say for example that  $\mathcal{L}_1$  is under  $\mathcal{L}_2$ . Let  $v_1$  be a vector of  $T_p M$  defining the branch direction of  $\mathcal{L}_1$  at  $p$ , and  $v_2$  be a vector of  $T_p M$  defining the branch direction of  $\mathcal{L}_2$  at  $p$ .

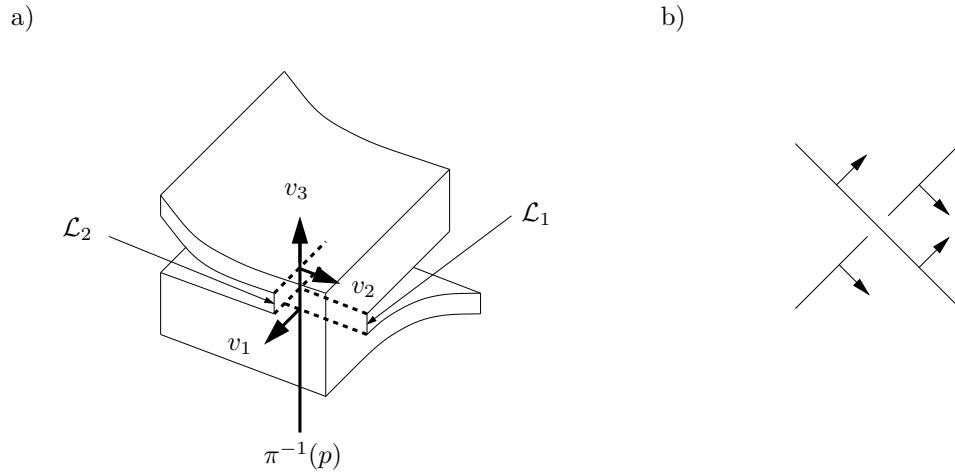


FIG. 2.6 – a) :  $\{v_1, v_2, v_3\}$ ; b) positive double point

At last, let  $v_3$  be a vector giving the chosen orientation of the fibre of  $N(\mathcal{B})$  passing through  $p$ , as seen on figure 2.6, a). We then call  $p$  a *positive double point* (resp. *negative double point*) if the base  $\{v_1, v_2, v_3\}$  of  $T_p M$  is direct (resp. indirect) in respect with the orientation of  $M$ . With this convention, the positive double points will be drawn in the plane as on the diagramm b) of figure 2.6.

**Remark 2.7** The sign of a double point depends on the orientation of  $M$  : if this one is reversed, the signs of the double points are reversed as well. Though, this sign is independant of the chosen orientation of the fibre passing through the double point in the preceding definition.

**Definition 2.8** A branched surface  $\mathcal{B}$  carries a surface  $\mathcal{S}$  immersed in  $V$  if  $\mathcal{S}$  is contained in the interior of a fibred neighbourhood of  $\mathcal{B}$ , and if  $\mathcal{S}$  is transverse to the fibres of this fibred neighbourhood. We say that  $\mathcal{S}$  is *fully carried* by  $\mathcal{B}$  if, moreover,  $\mathcal{S}$  meets every fibres of the fibred neighbourhood.

**Definition 2.9** A codimension 1 *lamination* in a dimension 3 (resp. 2) manifold  $V$  is the decomposition of a closed subset  $\lambda$  of  $M$  into injectively immersed surfaces (resp. curves) called *leaves*, such that  $\lambda$  is covered by charts of the form  $I^2 \times I$  (resp.  $I \times I$ ) in which the leaves have the form  $I^2 \times \{point\}$  (resp.  $I \times \{point\}$ ).

**Definition 2.10** A branched surface  $\mathcal{B}$  carries a lamination  $\lambda$  of codimension 1 if  $\lambda$  is contained in a fibred neighbourhood of  $\mathcal{B}$  and if its leaves are transverse to the fibres. We say that  $\lambda$  is *fully carried* if moreover it meets all the fibres.

### 3 Surfaces of contact

Let  $\mathcal{B}$  be a branched surface.

**Definition 3.1** A *surface of contact* is the immersion of an orientable surface  $\mathcal{S}$  in  $\mathcal{B}$ , whose boundary is sent onto smooth circles of the branch locus of  $\mathcal{B}$ , such that the branch directions along these boundary components point into  $\mathcal{S}$ .

If we consider a lift of  $\mathcal{S}$  into  $N(\mathcal{B})$ , we see that the existence of such a surface is equivalent to the existence of an immersion  $f : \mathcal{S} \rightarrow N(\mathcal{B})$  satisfying :

- (i)  $f(Int(\mathcal{S})) \subset Int(N(\mathcal{B}))$  and is transverse to the fibres ;
- (ii)  $f(\partial\mathcal{S}) \subset Int(\partial_v N(\mathcal{B}))$  and is transverse to the fibres.

Hence, the expression *surface of contact* will be used for both definitions.

An example is given in figure 3.4, a).

**Remark 3.2** In general, a surface of contact is not a sector, but a union of sectors : the branch locus of the branched surface may meet the interior of the surface of contact. The same is true for the sink surfaces, the twisted surfaces of contact and the source surfaces defined further.

**Definition 3.3** A *sink surface* is the immersion of an orientable surface  $\mathcal{S}$  in  $\mathcal{B}$ , whose boundary is sent onto piecewise smooth circles of the branch locus of  $\mathcal{B}$ , at least one of whose is not smooth, such that the branch directions along these boundary components point into  $\mathcal{S}$ . A double point in the boundary of  $\mathcal{S}$  which is the intersection of two smooth components of the boundary of  $\mathcal{S}$  is called a *corner* of  $\mathcal{S}$ . A sink surface has thus at least one corner.

Equivalently, if we consider a non smooth lift of  $\mathcal{S}$  into  $N(\mathcal{B})$ , we can say that a sink surface is an immersion  $f : \mathcal{S} \rightarrow N(\mathcal{B})$  satisfying :

- (i)  $f(Int(\mathcal{S})) \subset Int(N(\mathcal{B}))$  and is transverse to the fibres;
- (ii)  $f(\partial\mathcal{S})$  is included in  $Int(\partial_v N(\mathcal{B}))$  except in a finite and non empty number of closed disjoint intervals  $C_1 \dots C_k$ . Outside these  $C_i$ ,  $f(\partial\mathcal{S})$  is transverse to the fibres of  $\partial_v N(\mathcal{B})$ . Each  $C_i$  is included in a fibre of  $N(\mathcal{B})$  corresponding to a double point of  $\mathcal{L}$ , and must intersect  $Int(N(\mathcal{B}))$ . Thus,  $\pi(f(\partial\mathcal{S}))$  is not smooth. The  $C_i$  s are called the *corners* of  $\mathcal{S}$ .

An example is given in figure 3.4, b).

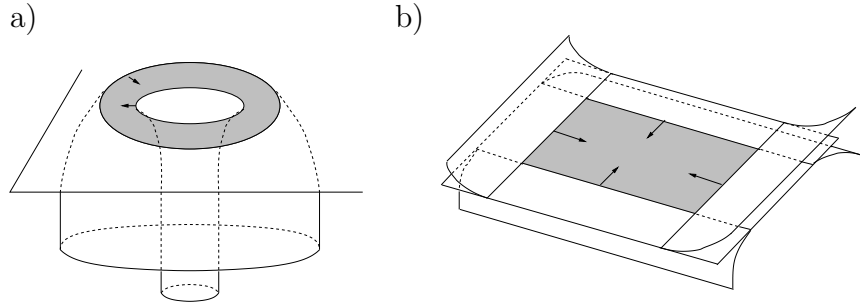


FIG. 3.4 – Annulus of contact and sink disk

**Definition 3.5** A *twisted surface of contact* is a sink surface whose every corners, which are double points, have the same sign, and which satisfies : when two components of  $\mathcal{L}$ ,  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , form a corner of  $\mathcal{S}$ , then, in a neighbourhood of this corner, neither  $\mathbf{a}_1$  nor  $\mathbf{a}_2$  run inside  $\pi(\mathcal{S})$ . Then, for some Riemannian metric for which  $\mathcal{L}$  in the neighbourhood of a double point cuts  $\mathcal{B}$  into four sectors of angle  $\pi/2$ , the corners of a twisted surface of contact are all of angle  $\pi/2$ .

A twisted surface of contact is *positive* (resp. *negative*) if all its corners are positive (resp. negative).

An example is given in figure 3.6.

In the same spirit, we give a definition which will often be used in section 7 :

**Definition 3.7** A *source surface* is the immersion of a surface into  $\mathcal{B}$ , whose



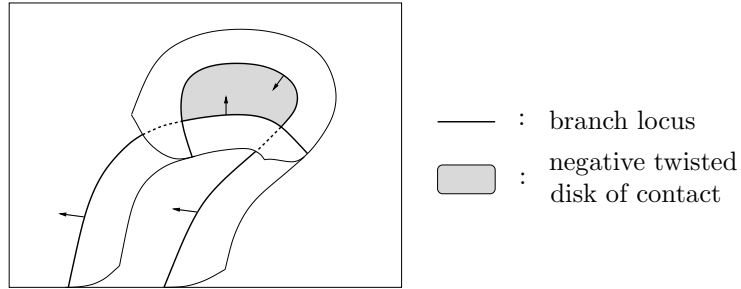


FIG. 3.6 – Negative twisted disk of contact

boundary is sent onto circles of  $\mathcal{L}$ , with the branch directions along these boundary components pointing outwards.

## 4 Results and course of the proof

Before giving the main result, we need a last definition :

**Definition 4.1** Let  $\mathcal{D}$  be a twisted disk of contact in a branched surface  $\mathcal{B}$ . We say that  $\mathcal{D}$  is a *carrying* disk if it satisfies the three following conditions :

- (i)  $\mathcal{D}$  contains a source disk  $d$  embedded in the source of  $\text{Int}(\mathcal{D})$ ; the boundary of  $d$ , denoted  $\mathcal{C}$ , may have corners ;
- (ii) all the sectors which branch on  $d$  along  $\mathcal{C}$  lie, in a neighbourhood of  $\mathcal{C}$ , on the same side of  $d$ , for a given coorientation of  $d$  (this implies that the angles of  $\mathcal{C}$  are  $3\pi/2$ , as seen later in section 7.2) ;
- (iii) if we consider all the connected, compact or not, surfaces immersed in  $\mathcal{B}$ , bounded by  $\mathcal{C}$ , and which lie, in a neighbourhood of  $\mathcal{C}$ , behind  $\mathcal{C}$  for its branching orientation, then all these surfaces are not disks.

Figure 4.2 shows some examples : in **a**), we see a carrying disk. In **b**), a torus contains a disk sector bounded by  $\mathcal{C}$  and a circle  $\delta$  of the branch locus, with one corner. This circle is the boundary of infinitely many carrying disks, because these disks are immersed, and can thus run around the torus as many times as we want. They are all carrying because behind the branching of  $\mathcal{C}$ , there is an immersed non compact cylinder with one boundary component,  $\mathcal{C}$ . These cylinders are sent onto the torus and run around it infinitely many times. In **c**), the twisted disk of contact satisfies points **(i)** and **(ii)** but not point **(iii)**. In **d**), the twisted disk of contact does not satisfy point **(ii)**.

The word *carrying* has been chosen because, as said further, such a carrying disk  $\mathcal{D}$  is not an obstruction to the existence of a lamination fully carried by  $\mathcal{B}$ , and thus fully carried by  $\mathcal{D}$ .

The purpose is then to prove the following result :

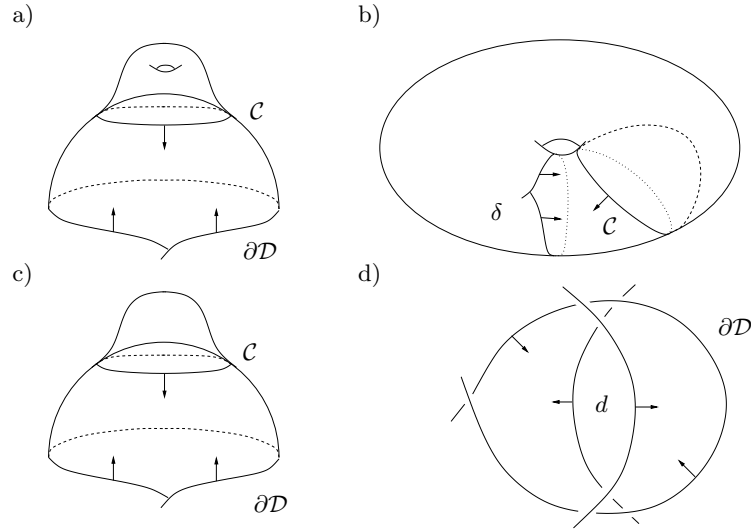


FIG. 4.2 – Twisted disks of contact, carrying or not

**Theorem 4.3** *Let  $\mathcal{B}$  be a branched surface in  $M$ . Then the two following assertions are equivalent :*

- (A)  $\mathcal{B}$  fully carries a lamination.
- (B) All the twisted disks of contact of  $\mathcal{B}$  are carrying.

**Remark 4.4** In the statement of theorem 4.3, the assumption of non existence of an immersed twisted disk of contact can not be weakened into an assumption of non existence of an embedded twisted disk of contact.

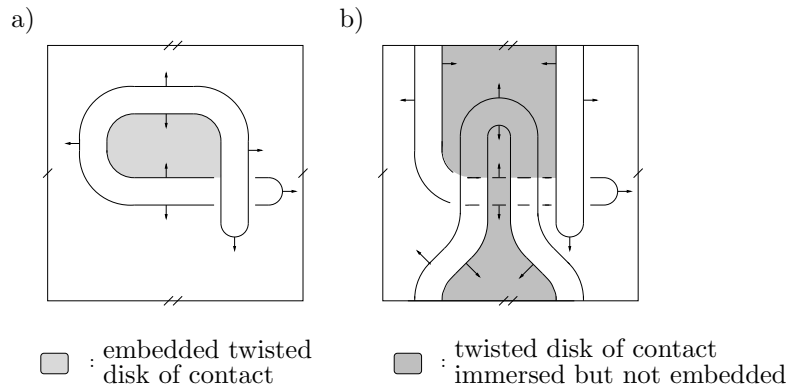


FIG. 4.5 – Branched surface having an immersed but not embedded twisted disk of contact

Indeed, there exist branched surfaces having an immersed and not embedded twisted disk of contact, and which do not fully carry a lamination. An example is shown in figure 4.5. In this example, we start from a torus, represented by a square whose opposite sides are identified, and on which we glue

a disk, in **a**). We then deform the branch locus in order to get an immersed but not embedded twisted disk of contact, in **b**). This branched surface in **b**) does not have any embedded twisted surface of contact, and, as the branched surface in **a**), it does not fully carry a lamination (this last fact is well-known, and a proof is given in lemma 6.4.8).

The main difficulty of the proof of theorem 4.3 lies in the proof of the implication **(B)**  $\Rightarrow$  **(A)**, and in particular in the proof of the following theorem :

**Theorem 4.6** *Let  $\mathcal{B}$  be a branched surface in  $M$  having no twisted disk of contact. Then  $\mathcal{B}$  fully carries lamination.*

Theorem 4.3 is useful to prove the following result, which will be explained and proved in section 8 :

**Corollary 4.7** *Let  $\mathcal{B}$  be a branched surface in  $M$  which carries both a positive and a negative contact form. Then  $\mathcal{B}$  fully carries a lamination.*

which in turn implies, using results of [GO]and [Co] :

**Corollary 4.8** *If a closed orientable 3-manifold contains a branched surface which carries a positive tight form of contact and a negative tight one, then its universal cover is homeomorphic to  $\mathbb{R}^3$ .*

## 5 Splittings

### 5.1 Definition and splitting along an arc

**Definition 5.1.1** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be two branched surfaces in  $M$ . We say that  $\mathcal{B}'$  is a *splitting* of  $\mathcal{B}$  if there exists a fibred neighbourhood  $N(\mathcal{B})$  of  $\mathcal{B}$  and an  $I$ -bundle  $J$  in  $N(\mathcal{B})$ , over a subsurface of  $\mathcal{B}$ , such that :

- (i)  $N(\mathcal{B}) = N(\mathcal{B}') \cup J$ ;
- (ii)  $J \cap N(\mathcal{B}') \subset \partial J$ ;
- (iii)  $\partial_h J \subset \partial_h N(\mathcal{B}')$ ;
- (iv)  $\partial_v J \cap N(\mathcal{B}')$  is included in  $\partial_v N(\mathcal{B}')$ , has finitely many components, and their fibres are fibres of  $\partial_v N(\mathcal{B}')$ .

**Remark 5.1.2** When  $\mathcal{B}'$  is a splitting of  $\mathcal{B}$ , the following notation will be used :  $\mathcal{B}' \xrightarrow{p} \mathcal{B}$ . Actually,  $\mathcal{B}'$  is included in a fibred neighbourhood  $N(\mathcal{B})$  of  $\mathcal{B}$ , endowed with a projection  $\pi$  on  $\mathcal{B}$ , and the restriction  $p$  of  $\pi$  to  $\mathcal{B}'$  is the projection we wanted.

There is a first obvious result which be useful in section 7 :

**Lemma 5.1.3** *Let  $\mathcal{B}'$  be a splitting of a branched surface  $\mathcal{B}$ . If  $\mathcal{B}'$  fully carries a lamination  $\lambda$ , then  $\lambda$  is fully carried by  $\mathcal{B}$  as well.*

**Proof** Notice that every lamination carried by  $\mathcal{B}'$  is obviously carried by  $\mathcal{B}$ . Let  $J$  be the  $I$ -bundle such that  $N(\mathcal{B}) = J \cup N(\mathcal{B}')$ . Let  $f$  be a fibre of  $N(\mathcal{B})$ . Point (iii) of definition 5.1.1 implies that the ends of  $f$  can not lie in  $J$ . Thus,  $f \setminus J$  is not empty. Since  $\lambda$  is fully carried by  $N(\mathcal{B}')$ , it intersects  $f \setminus J$  and hence intersects  $f$ . As a matter of fact, it is fully carried by  $\mathcal{B}$ .  $\square$

**Definition 5.1.4** Let  $\mathcal{B}$  be a branched surface. Let  $\Sigma$  be a sector of  $\mathcal{B}$  whose boundary contains a smooth part  $\alpha$  of  $\mathcal{L}$  and whose branching direction points into  $\Sigma$ . Let  $\gamma : I \rightarrow \Sigma$  be an embedded arc in  $\Sigma$  such that  $\gamma(0) \in \alpha$  and  $\gamma(t) \in \text{Int}(\Sigma)$  for  $t \neq 0$ . A *splitting along  $\gamma$*  is a branched surface  $\mathcal{B}'$  defined as in definition 5.1.1, where  $J$  is an  $I$ -bundle over a tubular neighbourhood of  $\gamma$  in  $\Sigma$  (see figure 5.1.5).

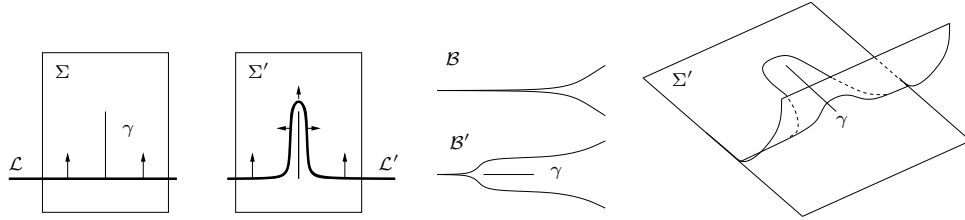


FIG. 5.1.5 – Splitting along  $\gamma$

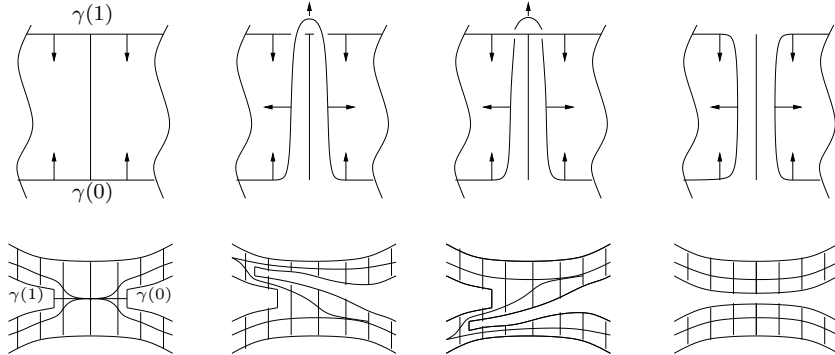


FIG. 5.1.6 – Over, under and neutral splittings

**Definition 5.1.7** We keep the notations of definition 5.1.4. Suppose now that  $\gamma(1)$  is in  $\mathcal{L}$ , in a point where the branching direction points into  $\Sigma$  as well.

We then say that  $\gamma$  is in *face-to-face* position. If an orientation of the fibres of  $N(\mathcal{B})$  along  $\gamma$  is chosen, there are three possible splittings along  $\gamma$  : the *over splitting*, the *under splitting* and the *neutral splitting*, drawn in figure 5.1.6.

**Remark 5.1.8** If  $\Sigma$  is a non compact sector, a splitting can be performed along a non compact arc  $\gamma : [0, 1[ \rightarrow \Sigma$ , verifying the same conditions as in definition 5.1.4. This splitting can be seen as a neutral splitting “at infinity”.

**Remark 5.1.9** It is possible to perform a splitting along an arc  $\gamma$  which comes from a sector to another one through the branch locus in the branch direction. In this case, there is only one possible splitting, called a *backward splitting* (see figure 5.1.10).

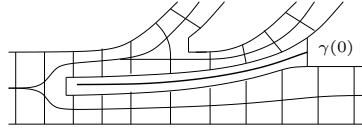


FIG. 5.1.10 – Backward splitting

## 5.2 Inverse limit of a sequence of splittings

**Definition 5.2.1** Let  $\mathcal{B}$  be a branched surface. A *sequence of splittings* is a sequence  $\dots \mathcal{B}_{k+1} \xrightarrow{p_{k+1}} \mathcal{B}_k \xrightarrow{p_k} \dots \xrightarrow{p_2} \mathcal{B}_1 \xrightarrow{p_1} \mathcal{B} = \mathcal{B}_0$  of branched surfaces  $(\mathcal{B}_i)_{i \in \mathbb{N}}$  such that :

- (i) for all  $i$ ,  $\mathcal{B}_{i+1}$  is a splitting of  $\mathcal{B}_i$  ;
- (ii) for all  $i$ ,  $\mathcal{B}_i$  is endowed with a fibred neighbourhood  $N(\mathcal{B}_i)$ , and those fibred neighbourhoods are such that  $N(\mathcal{B}_{i+1})$  is contained in  $N(\mathcal{B}_i)$ .

Thus, the fibres of each  $N(\mathcal{B}_i)$  are tangent to the fibres of  $N(\mathcal{B})$ .

For such a sequence, we denote, for all  $k \geq 1$  :

$$P_k = p_1 \circ p_2 \circ \dots \circ p_k = \pi|_{\mathcal{B}_k} : \mathcal{B}_k \rightarrow \mathcal{B}$$

the projection from  $\mathcal{B}_k$  onto  $\mathcal{B}$ .

We will also denote by  $\pi_n : N(\mathcal{B}_n) \rightarrow \mathcal{B}_n$  the projection along the fibres from  $N(\mathcal{B}_n)$  to  $\mathcal{B}_n$ .

The following definition is inspired by [MO] :

**Definition 5.2.2** A sequence of splittings  $\dots \mathcal{B}_{k+1} \xrightarrow{p_{k+1}} \dots \xrightarrow{p_1} \mathcal{B} = \mathcal{B}_0$  is said *resolving* if it satisfies :

- (i) there exist points of  $\mathcal{B}$  denoted  $(x_i)_{i \in \mathbb{N}}$ , a real number  $\rho > 0$  and disks embedded in  $\mathcal{B}$  denoted  $(d_i)_{i \in \mathbb{N}}$ , centred at  $x_i$  and of radius  $\rho$  for some metric on  $\mathcal{B}$ , such that the  $d_i$  's cover  $\mathcal{B}$  ;
- (ii) for all  $i \in \mathbb{N}$ , there exists a subsequence  $(\mathcal{B}_{\varphi_i(n)})_{n \in \mathbb{N}}$  such that the branch loci of the branched surfaces of this subsequence do not intersect

$\pi^{-1}(d_i)$ . That is, for all  $k$ ,  $\mathcal{B}_{\varphi_i(k)}$  does not have any branching over  $d_i$ : the branch points over  $d_i$  have been *resolved*, and  $P_{\varphi_i(k)}^{-1}(d_i)$  is thus a union of disjoint disks.

When such a sequence exists, we say that  $\mathcal{B}$  *admits a resolving sequence of splittings*.

**Remark 5.2.3** In particular, a branched surface admitting a resolving sequence of splittings is *fully splittable* in the sense of [GO].

**Lemma 5.2.4** ([GO],[MO]) *Let  $\mathcal{B}$  be a branched surface admitting a resolving sequence of splittings. Then  $\mathcal{B}$  fully carries a lamination.*

**Proof** : It can be found in p. 84-85 of [MO].

Let  $\dots \mathcal{B}_{k+1} \xrightarrow{p_{k+1}} \dots \xrightarrow{p_1} \mathcal{B} = \mathcal{B}_0$  be a resolving sequence.

Let us define  $\lambda = \bigcap_{n \in \mathbb{N}} N(\mathcal{B}_n)$ . As an intersection of closed subsets,  $\lambda$  is closed. We will now find an adapted atlas, whose charts will be the  $\pi^{-1}(d_i)$  's, where the  $d_i$  's are the disks from point (i) of definition 5.2.2.

Let  $i \in \mathbb{N}$ , and  $y \in d_i$ . Then  $\lambda \cap \pi^{-1}(y)$  is some closed subset  $T$  in  $[0, 1]$ . The sequence of splittings being resolving, let us consider the subsequence  $(\mathcal{B}_{\varphi_i(n)})_{n \in \mathbb{N}}$  from point (ii) of definition 5.2.2. Since the  $N(\mathcal{B}_n)$  form a decreasing sequence of closed subsets, we get :  $\lambda = \bigcap_{n \in \mathbb{N}} N(\mathcal{B}_{\varphi_i(n)})$ . But, for all  $y$  in  $d_i$  and for all integer  $n$ ,  $P_{\varphi_i(n)}^{-1}(x) = P_{\varphi_i(n)}^{-1}(y)$ , according to point (ii) of definition 5.2.2. Hence, for all  $i$ ,  $\lambda \cap \pi^{-1}(d_i)$  is topologically a product  $d_i \times T$ . If the transversal  $T$  contains an interval  $I_T$  whose interior is non empty, we remove  $Int(I_T)$  from  $T$ . We then reduce  $T$  to a transversal  $T' = T \setminus Int(T) = \partial T$ , whose interior is empty, and which is totally discontinuous. Hence  $\pi^{-1}(d_i)$  is a laminated chart, the leaves being the  $\{t\} \times d_i$  s, for  $t \in T'$ . The set  $\lambda' = \bigcup_{i \in \mathbb{N}} (d_i \times T')$  is a lamination. Moreover,  $\lambda'$  meets all the fibres of  $N(\mathcal{B})$  transversally.  $\square$

**Definition 5.2.5** Let  $\dots \mathcal{B}_{k+1} \xrightarrow{p_{k+1}} \dots \xrightarrow{p_1} \mathcal{B} = \mathcal{B}_0$  be a resolving sequence of splittings. The fully carried lamination  $\lambda' = \bigcup_{i \in \mathbb{N}} (d_i \times T')$  defined in the previous proof is called the *inverse limit* of this sequence of splittings.

## 6 Proof of theorem 4.6

Let  $\mathcal{B}$  be a branched surface satisfying the hypotheses of theorem 4.6.

### 6.1 Principle of the proof

The principle is the same as the one of the construction of a lamination whose holonomy is strictly negative, in section 4 of [OS2]. We will build a resolving sequence of splittings, whose inverse limit induces a null holonomy lamination on the fibred neighbourhood of the neighbourhood of the 1-skeleton of some cell decomposition into disks and half-planes of  $\mathcal{B}$ .

## 6.2 Cell decomposition of $\mathcal{B}$

The branch locus  $\mathcal{L}$  cuts  $\mathcal{B}$  into sectors. This is a first cell decomposition  $X$  of  $\mathcal{B}$ . The 2-cells are the sectors, and are not disks or half-planes in general. The edges are the smooth parts of  $\mathcal{L}$  having no double points in their interior and such that : if an edge is compact, both ends are double points (they may be the same double point) ; if an edge is diffeomorphic to  $[0, 1[$ , then its end is a double point ; if an edge is diffeomorphic to  $\mathbb{R}$ , it does not meet any double point. But this first decomposition is not fine enough. For reasons which should become clear after the statement of lemma 6.4.4, this decomposition must be refined into a decomposition whose compact cells are disks. This is made by adding as many vertices and edges (compact or not) as necessary. We also add vertices and edges so that the non compact cells are half-planes, and vertices so that no edge is a loop (i.e. its two ends coincide). We denote  $Y$  the obtained decomposition.

**Remark 6.2.1** The “boundary of a 2-cell” is not the topological boundary, but the combinatorial one. An edge can be found twice, with different orientations, in the boundary of the same 2-cell.

## 6.3 First splitting

The first step is to perform a first splitting of  $\mathcal{B}$ , denoted  $\mathcal{B}_1$ , which is fully carried by  $N(\mathcal{B})$ , as in definition 2.4. Let us describe it more precisely.

Let  $\varepsilon$  be a non negative real number, such that, for some metric on  $\mathcal{B}$ , the edges of  $Y$  are all strictly longer than  $5\varepsilon$  (we shall see why later). Let us look at the intersection of  $\mathcal{B}$  with an  $\varepsilon$ -tubular neighbourhood of  $\mathcal{L}$  in  $M$ . We chose  $\varepsilon$  small enough for this tubular neighborhood to be regular. This intersection is the union of  $\mathcal{L}$  and of two other parts, which meet together at the double points : one part lies behind  $\mathcal{L}$ , for the coorientation of  $\mathcal{L}$  given by the branch directions, and the other part, denoted  $T_{\mathcal{L}}$ , lies in front of  $\mathcal{L}$ . The boundary of  $T_{\mathcal{L}}$  is included in the union of  $\mathcal{L}$  with a parallel copy of  $\mathcal{L}$ , called  $\mathcal{L}_1$ . It is just “included in” and not “equal to” this union, because of what happens at the double points. The first splitting is a splitting over  $T_{\mathcal{L}}$ , which means that we remove from  $N(\mathcal{B})$  an  $I$ -bundle over  $T_{\mathcal{L}}$ . The branched surface  $\mathcal{B}_1$  we get is isomorphic to  $\mathcal{B}$ , and its branch locus is  $\mathcal{L}_1$  (see figure 6.3.1).

The trace of  $\pi^{-1}(\mathcal{L})$  on  $\mathcal{B}_1$  is made of two copies of  $\mathcal{L}$ , drawn on two different sectors (at least locally), as seen on figure 6.3.1. The trace of  $\pi^{-1}(Y)$  on  $\mathcal{B}_1$ , denoted  $Y_1$ , is then more complicated than a cell decomposition into disks and half-planes, since some of the cells and some of the edges are branched. But all these branchings lie in a closed  $\varepsilon$ -neighbourhood of  $\mathcal{L}_1$ , and  $\mathcal{B}_1$  minus a closed  $\varepsilon$ -neighbourhood of the 1-skeleton of  $Y_1$  is the same union of disks and half-planes as  $\mathcal{B}$  minus  $\mathcal{L}$ .

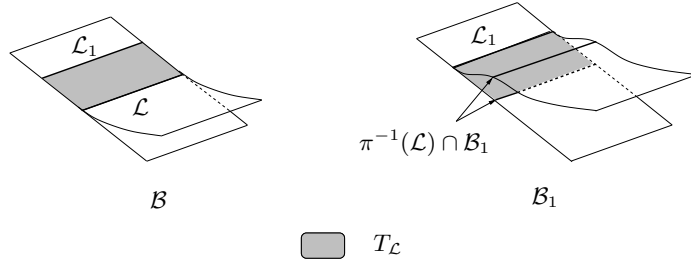


FIG. 6.3.1 – First splitting

## 6.4 Train tracks

Each 2-cell  $\Sigma$  of  $Y$  inherits from  $N(\mathcal{B})$  an interval bundle  $N(\Sigma)$  built in the following way : we denote by  $N(Int(\Sigma))$  the set of all the fibres of  $N(\mathcal{B})$  whose base point lies in  $Int(\Sigma)$  and we set  $N(\Sigma) = \overline{N(Int(\Sigma))}$ .

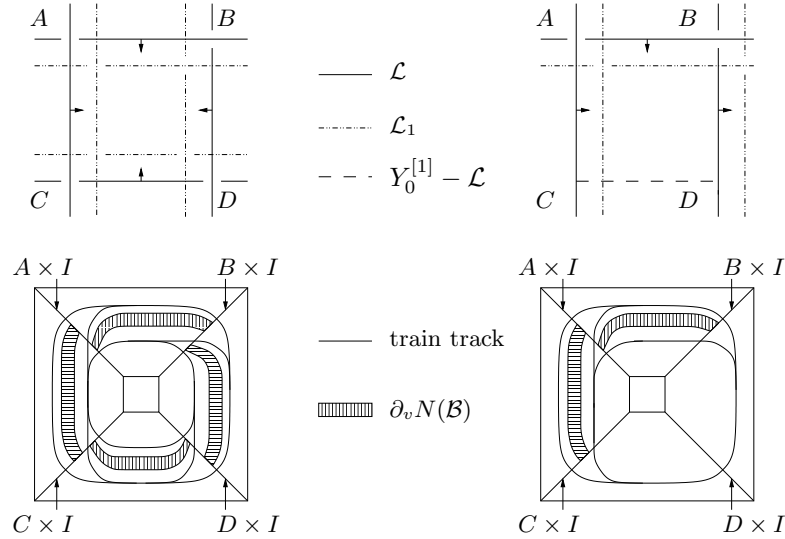


FIG. 6.4.1 – Train tracks

This  $N(\Sigma)$  can be decomposed into an horizontal boundary (included in  $\partial_h N(\mathcal{B})$ ) and a vertical boundary (not included in  $\partial_v N(\mathcal{B})$ ). Since all the compact 2-cells of  $Y$  are disks and are orientable, the vertical boundary of  $N(\Sigma)$ , denoted  $\partial_v N(\Sigma)$ , is in fact of the form  $S^1 \times I$ . For the non compact 2-cells, the vertical boundary is of the form  $\mathbb{R} \times [0, 1]$ . For each 2-cell  $\Sigma$ , let us look at the trace of  $\mathcal{B}_1$  on  $\partial_v N(\Sigma)$ , which is also the boundary of  $\Sigma_1 = p_1^{-1}(\Sigma)$ . It is a *train track*, i.e. a branched curve fully carried by  $\partial_v N(\Sigma)$ . This train track does not have a boundary and avoids the trace of  $\partial_v N(\mathcal{B})$  on  $\partial_v N(\Sigma)$ . It is compact if and only if  $\Sigma$  is compact. Figure 6.4.1 shows two examples of compact train tracks.



An orientation of a 2-cell gives an orientation of its boundary. The corresponding train track is then oriented as well. For each 2-cell  $\Sigma$ , we set an orientation of the fibres of  $\partial_v N(\Sigma)$ . We introduce the following definitions :

**Definition 6.4.2** A branching of a train track is said *direct* when a track followed in the direct way divides itself into two tracks at this branching, and it is said *backward* when two tracks followed in the direct way meet at this branching.

We can go a bit further in the classification of the branchings of a train track :

**Definition 6.4.3** Let  $\mathcal{V}$  be an oriented compact train track without boundary, fully carried by a trivial bundle  $\mathbb{S}^1 \times [0, 1]$ . We set an orientation of the fibres. Let  $\mathcal{C}$  be a smooth closed curve of  $\mathcal{V}$ . It cuts  $\mathbb{S}^1 \times [0, 1]$  into two parts :  $(\mathbb{S}^1 \times [0, 1])^+$ , containing the points which lie over  $\mathcal{C}$  for the orientation of the fibres, and  $(\mathbb{S}^1 \times [0, 1])^-$  containing the points which lie under. A branching along a smooth closed curve of  $\mathcal{V}$  is called an *over branching* (resp. *under branching*) if the branch which leaves or meets  $\mathcal{C}$  there lies in  $(\mathbb{S}^1 \times [0, 1])^+$  (resp.  $(\mathbb{S}^1 \times [0, 1])^-$ ).

We can thus state the following lemma :

**Lemma 6.4.4** Let  $\dots \mathcal{B}_{k+1} \xrightarrow{p_{k+1}} \dots \xrightarrow{p_1} \mathcal{B} = \mathcal{B}_0$  be a sequence of splittings of  $\mathcal{B}$ . Let  $\Sigma$  be a compact 2-cell of  $Y$  ( $\Sigma$  is a disk), and  $\Sigma_n$  be its trace  $P_n^{-1}(\Sigma)$  on  $\mathcal{B}_n$ . Let  $\mathcal{V}$  be the boundary of  $\Sigma_n$  for some  $n$ . It is an oriented compact train track of  $\partial_v N(\Sigma) = \mathbb{S}^1 \times [0, 1]$ , without boundary and fully carried. We set an orientation of the fibres. The three following assertions are equivalent :

- (i) when we follow a smooth closed curve of  $\mathcal{V}$ , either no under branching is met or at least one direct under branching and one backward under branching are met;
- (ii)  $\mathcal{V}$  can be split into a union of smooth circles;
- (iii)  $P_n^{-1}(\Sigma)$  contains a twisted disk of contact.

**Proof**

★ (i)  $\Rightarrow$  (ii) : if this is true for each connected component of  $\mathcal{V}$ , then it is true for  $\mathcal{V}$ . So, we suppose that  $\mathcal{V}$  is connected, and different from a smooth curve. For each  $\theta \in \mathbb{S}^1$  we define  $max(\theta) = \max\{t \in [0, 1] \mid (\theta, t) \in \mathcal{V}\}$ , which is in  $[0, 1]$ , and then we define  $max(\mathcal{V}) = \{(\theta, max(\theta)), \theta \in \mathbb{S}^1\}$ . This  $max(\mathcal{V})$  is a smooth circle of  $\mathcal{V}$ , along which we meet at least one direct under branching and one backward under branching, and no over branching. In particular, there exists an oriented arc  $\mathcal{A}$  of  $max(\mathcal{V})$ , going (for the orientation of  $\mathcal{V}$ ), from a direct branching to a backward branching, with no branching between the two previous ones.  $\mathcal{V} \setminus \mathcal{A}$  is then an oriented compact train track without boundary denoted  $\mathcal{V}_1$ , fully carried

by  $\mathbb{S}^1 \times [0, 1]$ , and  $\mathcal{V} = \mathcal{V}_1 \cup \text{max}(\mathcal{V})$ . Each smooth closed curve of  $\mathcal{V}_1$  is a smooth closed curve of  $\mathcal{V}$  as well, and its under branchings remain unchanged by the previous splitting. Hence,  $\mathcal{V}_1$  satisfies point (ii) of the lemma. If  $\mathcal{V}_1$  is not a circle, we perform the same operation again using  $\text{max}(\mathcal{V}_1)$ , and after a finite number of steps, we have decomposed  $\mathcal{V}$  into a union of smooth circles. An example is shown in figure 6.4.5.

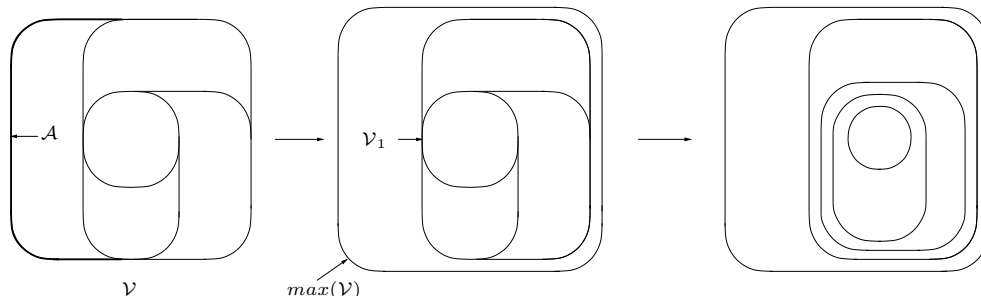


FIG. 6.4.5 – Splitting of a train track into a union of smooth circles

- ★  $\neg(i) \Rightarrow \neg(ii)$  : Let  $\mathcal{C}$  be a smooth closed curve of  $\mathcal{V}$  having, for example, only direct under branchings. If we follow  $\mathcal{C}$  in the direct way, and if we take a direct under branching, then, whatever the smooth path we follow on  $\mathcal{V}$ , we will never be able to go on  $\mathcal{C}$  again, for it would imply the existence of a backward under branching along  $\mathcal{C}$ . Thus, no branch leaving  $\mathcal{C}$  by a direct under branching is included in a union of smooth circles.
- ★  $\neg(iii) \Rightarrow \neg(i)$  : A part of the trace of  $\partial_v N(\mathcal{B})$  on  $\mathcal{B}_n$  when  $P_n^{-1}(\Sigma)$  contains a twisted disk of contact is always as on figure 6.4.6, i.e. it is the union of two smooth circles and of segments joining them at branch points.

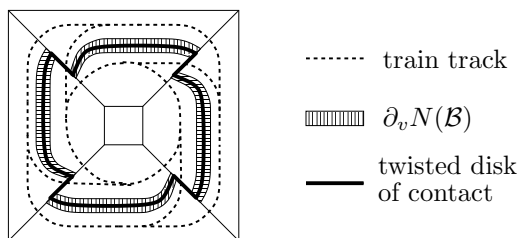


FIG. 6.4.6 –  $\neg(i) \Leftrightarrow \neg(ii)$

The top circle has only under branchings, and it has at least one branching because a twisted disk of contact has at least one corner. Moreover, these branchings are of one type because the corners of a twisted disk of contact all have the same sign.

- ★  $\neg(i) \Rightarrow \neg(iii)$  : Suppose that there exists a closed smooth curve of  $\mathcal{V}$  whose branchings are all under branchings of a single type, for example

direct, and which has at least one under branching. Look at the trace of  $\partial_v N(\mathcal{B})$  on  $\partial_v N(\Sigma)$ . Each of its connected component has a vertical boundary with two connected components and a horizontal boundary, also with two connected components. Each component of the vertical boundary is included in a fibre of  $\partial_v N(\Sigma)$  whose base point is a branch point of  $\mathcal{B}$ .

Stand at a point  $p$  on  $C$ , and follow  $C$  in the direct way. When we meet the first branch point  $p_1$ ,  $C$  divides into two branches : the top branch passes over a component  $b_1$  of  $\partial_v N(\mathcal{B})$ , and the bottom branch passes under  $b_1$ . We go on until we meet the fibre where  $b_1$  ends, whose base point is some branch point  $p_2$ . If  $p_2$  is not a double point, then the branch of  $\mathcal{V}$  which is over  $b_1$  joins the branch which is under  $b_1$ . But these two branches are the two previous branches, and that would imply that there is a backward branching on  $C$ . Hence  $p_2$  is a double point. At  $p_2$ , there are thus two branchings, one is direct and the other is backwards. One of them is on  $C$ , so this is the direct one.  $C$  divides again into two branches which surrounds another component  $b_2$  of the trace of  $\partial_v N(\mathcal{B})$ . Since this branching is direct,  $b_2$  lies over  $b_1$  at  $p_2$ . We carry on following  $C$  until we return at  $p$ . We have then met  $k$  components  $b_1 \dots b_k$  of  $\partial_v N(\mathcal{B})$  and  $k$  double points  $p_1 \dots p_k$ . Each  $b_i$  goes from  $p_i$  to  $p_{i+1}$  for  $i = 1 \dots k$  modulo  $k$ . At  $p_i$ ,  $b_{i-1}$  lies under  $b_i$ , for all  $i$ . As a result, all the double points have the same sign, and  $P_n^{-1}(\Sigma)$  contains a twisted disk of contact with  $k$  corners. □

**Remark 6.4.7** In the proof of point (i)  $\Rightarrow$  (ii), we could also define  $\min(\mathcal{V})$  in the same way as  $\max(\mathcal{V})$ , and show that points (ii) and (iii) are equivalent to a point (i') : when we follow a smooth closed curve of  $\mathcal{V}$ , either we meet no over branching, or we meet at least one direct over branching and at least one backward over branching. Points (i) and (i') are thus equivalent.

With the same ideas, we can also prove the following well known lemma, whose result has already been mentioned in remark 4.4 :

**Lemma 6.4.8** *Let  $\mathcal{B}$  be a branched surface having a disk sector  $\mathcal{D}$  which is twisted disk of contact as well. Then  $\mathcal{B}$  can not fully carry a lamination.*

**Proof** Suppose that  $\mathcal{B}$  fully carries a lamination  $\lambda$ . We consider  $N(\mathcal{D})$ , the fibred neighbourhood over  $\mathcal{D}$  in  $N(\mathcal{B})$ . Let  $\mathcal{V}$  be the train track which is the trace of  $\mathcal{B}_1$  in  $\partial_v(N(\mathcal{D}))$ . The intersection of the leaves of  $\lambda$  passing through  $\mathcal{D}$  with  $N(\mathcal{D})$  are disks. Their boundaries are circles which form a 1-dimension lamination in  $\partial_v(N(\mathcal{D}))$ . Since  $\lambda$  is fully carried by  $\mathcal{B}$ , this 1-dimension lamination by circles is fully carried by a fibred neighbourhood of  $\mathcal{V}$ . However, as seen in the example on figure 6.4.9, b),  $\mathcal{V}$  is the union of two smooth circles,

and of segments which join these two circles at branch points. And since  $\mathcal{V}$  can not be decomposed into an union of circle, there is no circle carried by a fibred neighbourhood of  $\mathcal{V}$  which passes over one of these segments (c)), which is a contradiction.

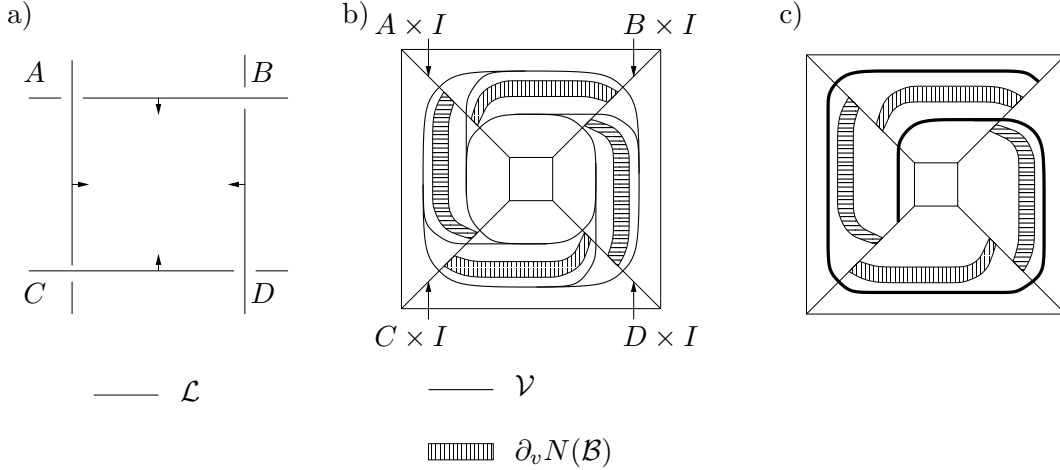


FIG. 6.4.9 – Twisted disk of contact

□

That is why the existence of a twisted disk of contact prevents the proof to work. That is also why we have refined the first cell decomposition of  $\mathcal{B}$  in subsection 6.2.

**Remark 6.4.10** For the non compact cells, it is much simpler, since a train track fully carried by a fibred neighbourhood  $\mathbb{R} \times [0, 1]$  can always be decomposed into a union of smooth lines.

By mean of train tracks, the following lemma, which will later be useful, can also be proven :

**Lemma 6.4.11** *Let  $\mathcal{B}$  a branched surface. There is no surface  $\mathcal{S}$  in  $\mathcal{B}$  with a boundary component containing an arc of  $\mathcal{L}$  of the form of the one shown in figure 6.4.12 ,a), i.e. an arc formed by  $n$  consecutive smooth parts of  $\mathcal{L}$ , denoted  $l_i$ ,  $i = 1 \dots n$ , and satisfying :*

- ★ *the branch directions along  $l_1$  and  $l_n$  point into  $\mathcal{S}$  ;*
- ★ *the branch directions along the  $l_i$  's for  $i$  different from 1 and  $n$ , point outside of  $\mathcal{S}$  ;*
- ★ *the corners of  $\mathcal{S}$  defined by  $l_1$  and  $l_2$  and by  $l_{n-1}$  and  $l_n$  have angle  $\pi/2$ , and the other ones have angle  $3\pi/2$ , for some metric on  $\mathcal{B}$  ;*

★ all the corners along this arc have the same sign : each  $l_i$  passes over  $l_{i-1}$  and under  $l_{i+1}$ , or vice-versa.

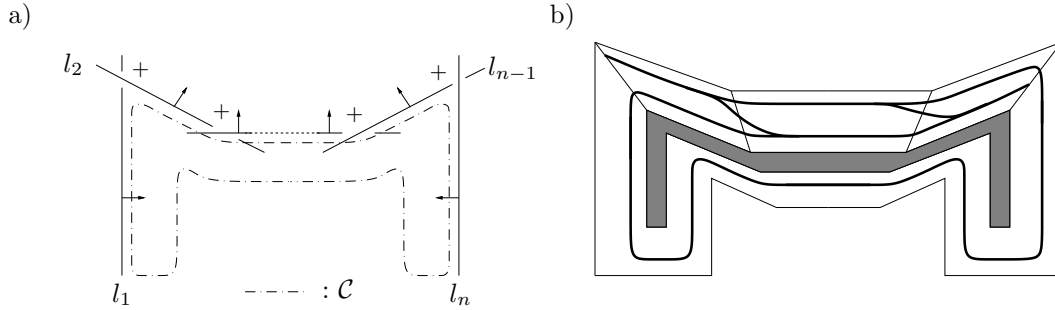


FIG. 6.4.12 – a) arc in the boundary of  $\mathcal{S}$  and  $\mathcal{C}$  ; b) train track on  $\mathcal{C} \times [0, 1]$

**Proof** There exists a branched surface  $\mathcal{B}_{-1}$  isomorphic to  $\mathcal{B}$ , a fibred neighbourhood of whose fully carries  $\mathcal{B}$ . The relation between  $\mathcal{B}_{-1}$  and  $\mathcal{B}$  is the same as the one between  $\mathcal{B}$  and  $\mathcal{B}_1$ . In the same way as  $\varepsilon$  is the distance between the branch loci of  $\mathcal{B}$  and  $\mathcal{B}_1$ , we can define a  $\rho$  which is the distance between the branch loci of  $\mathcal{B}_{-1}$  and  $\mathcal{B}$ . Moreover, there is a projection  $p_{-1}$  from this fibred neighbourhood onto  $\mathcal{B}$ .

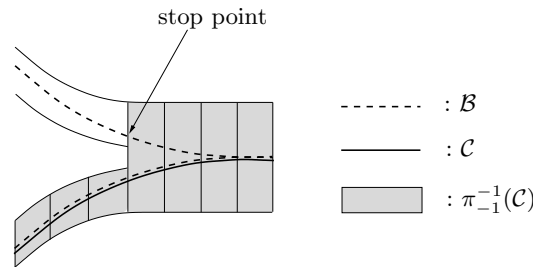


FIG. 6.4.13 –

If  $\mathcal{S}$  exists, then there exists a circle  $\mathcal{C}$  on  $\mathcal{S}$ , contractible in  $\mathcal{S}$ , and a part of whose runs along the arc of the lemma at a distance smaller than  $\rho$ , and whose other part returns through the inside of  $\mathcal{S}$ , at a distance greater than  $\rho$ , as in **a)** of figure 6.4.12. Let us look at the trace of  $\mathcal{B}$  on  $p_{-1}^{-1}(\mathcal{C})$ . This is a train track which has two end points. Indeed,  $p_{-1}^{-1}(\mathcal{C})$  is not a regular  $I$ -bundle over  $\mathcal{C}$  : when we cross a branch point in the branch direction, there is a gap in the size of the fibres, and  $\mathcal{B}$  comes out of the  $I$ -bundle through this gap. Figure 6.4.13 shows this phenomenon. The trace of  $\mathcal{B}$  on  $p_{-1}^{-1}(\mathcal{C})$  is then drawn in **b)** of figure 6.4.12. But this train track does not contain any smooth circle, and hence does not contain  $\mathcal{C}$ , which is absurd.  $\square$

**Example 6.4.14** If, in the previous lemma, we take  $n = 3$ , we are in the situation of figure 6.4.15, and the result is easier to visualize.

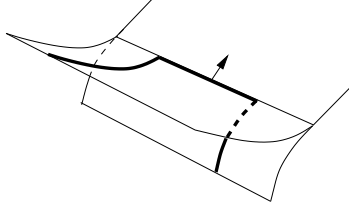


FIG. 6.4.15 –

## 6.5 Resolving sequence of splittings

We keep the real number  $\varepsilon > 0$  defined in subsection 6.3 for the splitting from  $\mathcal{B}$  to  $\mathcal{B}_1$ . Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a strictly decreasing sequence of real numbers such that for all  $n$ ,  $\frac{\varepsilon}{2} < \varepsilon_n < \varepsilon$ .

Let  $Y_\varepsilon$  be a  $\frac{\varepsilon}{2}$ -neighbourhood of  $Y^{[1]}$  in  $\mathcal{B}$ . Then the trace of  $\mathcal{B}_1$  on  $\pi^{-1}(Y_\varepsilon)$  is a branched surface with boundary denoted  $\mathcal{B}'_1$ .

The purpose of this subsection is to explain how to build a sequence of splittings of  $\mathcal{B}$ , whose sequence of splittings it induces on  $\mathcal{B}'_1$  is resolving.

We denote  $(y_i)_{i \in J}$  the set of vertices of  $Y$ , where  $J$  is a subset of  $\mathbb{N}$ . To each vertex  $y_i$  correspond several vertices of  $Y_1$ , at least 2 and at most 3, whether  $y_i$  is a double point or a regular point of the branch locus. We denote these vertices  $y_i(j)$  for  $j = 1, 2$  or  $3$ . We then call  $d_i(j)$  the projection by  $P_1$  of the disk of  $\mathcal{B}_1$  centred at  $y_i(j)$  and of radius  $2\varepsilon$ , such that  $d_i = \cup_{j=1 \dots 3} d_i(j)$  is a branched disk, neighbourhood of  $y_i$  in  $\mathcal{B}$  (see figure 6.5.1).

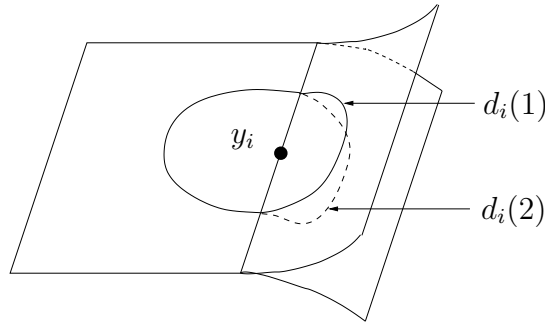


FIG. 6.5.1 –

The branch locus of  $\mathcal{B}'_1$  is included in the union of the  $P_1^{-1}(d_i(j))$ 's, for all the  $i$ 's and  $j$ 's. Notice that the branch locus of  $\mathcal{B}'_1$  has no double point.

Moreover, since we supposed that the edges of  $Y$  are strictly longer than  $5\varepsilon$ , if  $y_{i_1}$  is different from  $y_{i_2}$ , then  $d_{i_1}$  and  $d_{i_2}$  are disjoint.

At last, we define a sequence of vertices of  $Y$ ,  $(y_{\psi(n)})_{n \in \mathbb{N}}$ , for  $\psi$  a map from  $\mathbb{N}$  to  $J$ , such that each vertex appears infinitely many times. This is possible since there is only countably many vertices in  $Y$ .

Define now which are the splittings from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ . We take all the edges of  $Y$ , a vertex of whose is  $y_{\psi(1)}$ . We orient them from  $y_{\psi(1)}$  to their second vertex. Let  $\mathbf{a}$  be one of these edges. Its second vertex is  $y_k$ , different from  $y_{\psi(1)}$ . We call  $\mathcal{V}_{\mathbf{a}}$  the trace of  $\mathcal{B}_1$  on  $\pi^{-1}(\mathbf{a})$ . Since  $\mathbf{a}$  is oriented, it makes sense to talk of direct and backward branchings along  $\mathcal{V}_{\mathbf{a}}$ . According to the definition of  $\mathcal{B}_1$ , the backward branchings all lie in  $\pi^{-1}(d_{\psi(1)})$ , and the direct branchings all lie in  $\pi^{-1}(d_k)$ . Moreover, each branching lies at a distance  $\varepsilon$  from the ends of  $\mathcal{V}_{\mathbf{a}}$ . Actually, at this step of the sequence of splittings, there is at most one direct branching and one backward branching along  $\mathcal{V}_{\mathbf{a}}$ . If there is no backward branching, no splitting will be made along  $\mathcal{V}_{\mathbf{a}}$ . Else, we will perform a splitting along a path inscribed on  $\mathcal{V}_{\mathbf{a}}$ , going from the backward branching to the direct one if it exists, or to the end of  $\mathcal{V}_{\mathbf{a}}$ , in an  $\varepsilon_1$ -neighbourhood of this path. If a direct branching is met, this splitting can be an over, under or neutral splitting. The following subsection **6.6** will tell which one must be chosen. If it is the neutral splitting, the splitting stops at this branching point. Else, we can split on along a path in  $\mathcal{V}_{\mathbf{a}}$  which goes to the end of  $\mathcal{V}_{\mathbf{a}}$ . Since  $\varepsilon_1 < \varepsilon$ , along this path, no other backward branching is met, and hence, there is no backward splitting. The same process is applied to the other edges having  $y_{\psi(1)}$  as a vertex. The second splitting takes place in an  $\varepsilon_2$ -neighbourhood of the corresponding path, the third splitting takes place in an  $\varepsilon_3$ -neighborhood of the corresponding path, and so on. This allows to avoid backward splittings. The order of the edges does not matter.

After these splittings, we get a branched surface  $\mathcal{B}_2$ . We take all the edges of  $Y$ , a vertex of whose is  $y_{\psi(2)}$ . We orient them from  $y_{\psi(2)}$  to their second vertex. Let  $\mathbf{a}$  be one of these edges. Its second vertex is  $y_k$ , different from  $y_{\psi(2)}$ . We call  $\mathcal{V}_{\mathbf{a}}$  the trace of  $\mathcal{B}_2$  on  $\pi^{-1}(\mathbf{a})$ . The situation is as previously, except for one detail : there can now be more than one direct branching and one backward branching along  $\mathcal{V}_{\mathbf{a}}$ . However, all the backward branchings lie in  $\pi^{-1}(d_{\psi(2)})$ , and all the direct branchings lie in  $\pi^{-1}(d_k)$ . All these branchings lie at a distance at least  $\varepsilon_i$  from the ends of  $\mathcal{V}_{\mathbf{a}}$ , where  $i$  is the number of splittings performed on  $\mathcal{B}_1$ .

Look at the backward branchings of  $\mathcal{V}_{\mathbf{a}}$  : there are  $j$  such branchings. Since the successive splittings have been performed in smaller and smaller neighbourhoods, we can order these branchings from the furthest from  $\pi^{-1}(y_{\psi(2)})$  to the nearest. We note them  $b_1 \dots b_j$ ,  $b_i$  being strictly further than  $b_{i+1}$ . We

will make splittings along paths going from the  $b_i$ 's, in smaller and smaller neighbourhoods, whose size is set by the  $(\varepsilon_n)$  sequence. To avoid any backward splitting, we begin by the splitting along a path starting from  $b_1$ . The second splitting will start from  $b_2$ , and so on until the last splitting, which will start from  $b_j$ . When a direct branching is met, one of the over, under and neutral splittings must be chosen : this is done in subsection 6.6. As previously, if the neutral splitting is chosen, the splitting stops here. Else, we can split on until another direct branching is met, or until the end of  $\mathcal{V}_a$ . Again, thanks to the choice of the  $\varepsilon_n$ , backward splittings are avoided. The same process is applied to all the edges having  $y_{\psi(2)}$  as a vertex. The order of the edges does not matter.

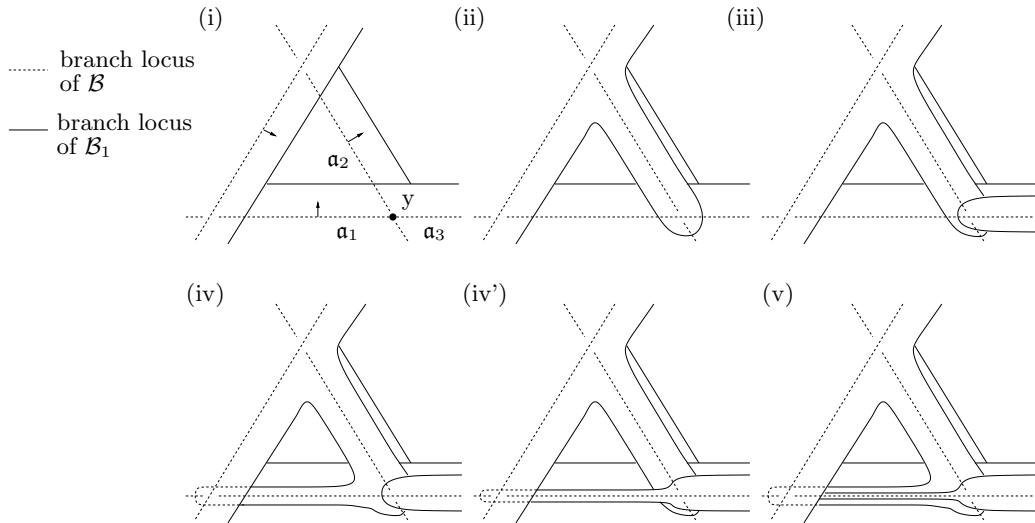


FIG. 6.5.2 – Example of a sequence of splittings

We iterate these operations at each step : the splittings from  $\mathcal{B}_n$  to  $\mathcal{B}_{n+1}$  are performed along arcs whose image by  $\pi$  is included in an edge having  $y_{\psi(n)}$  as a vertex. The backward branchings are always over  $d_{\psi(n)}$  : they are more and more numerous, but they are always strictly ordered, from the furthest to the nearest. Moreover, the branch locus of  $\mathcal{B}'_{n+1}$  does not intersect  $P_{n+1}^{-1}(d_{\psi(n)})$  any more : the singularities over  $d_{\psi(n)}$  have then been *resolved*. Since the vertex  $y_{\psi(n)}$  will reappear infinitely many times in the sequence  $(y_{\psi(n)})_{n \in \mathbb{N}}$ , the sequence of splittings is resolving.

Figure 6.5.2 shows an example of such a sequence of splittings. On this figure, the branch loci are seen “from above”, and only the top parts are drawn. The three first points show a sequence of splittings at the end of which there are several direct branchings along some edges having  $y$  as a vertex. The first splitting to be performed along  $\mathbf{a}_1$  is the one drawn in (iv), but not the one drawn in (iv') , where a backward splitting occurs. The second splitting is the



one drawn in (v) . It then remains to split along  $\mathfrak{a}_2$  and  $\mathfrak{a}_3$ .

## 6.6 Adapted splittings

We will now see how it is possible to perform the splittings along the edges previously defined, in such a way that none of the  $\mathcal{B}_n$  has a twisted disk of contact.

If an arc of splitting from  $\mathcal{B}_n$  to  $\mathcal{B}_{n+1}$  is not in a face-to-face position, then the branch locus of  $\mathcal{B}_{n+1}$  remains the same as the branch locus of  $\mathcal{B}_n$  : it is deformed, but there is no new double point.

When the arc of splitting is in a face-to-face position, then we have the following fact : an over splitting introduces two new double points in the branch locus, a positive and a negative one, and an under splitting introduces two double points at the same place but of opposite signs. Figure 5.1.6 shows this.

Being given an arc to split along, we now have to find a splitting which will not create a twisted disk of contact. Such a splitting will be said *adapted*.

The following proposition is fundamental (we keep the previous notations) :

**Proposition 6.6.1** *Let  $\mathcal{B}_n$  be a branched surface obtained from  $\mathcal{B}_1$  by a sequence of splittings, and which does not have any twisted disk of contact. We denote  $\mathcal{L}_n$  its branch locus. Then, for every arc of splitting in a face-to-face position, at least one of the three splittings, over, under or neutral, is adapted.*

**Proof** Let  $\mathfrak{a}$  be the arc of splitting, and  $p_1$  and  $p_2$  be its ends. We suppose that  $\mathfrak{a}$  is in a face-to-face position, and then the double points  $p_1$  and  $p_2$  are of opposite signs. Suppose that both the over and the under splittings along  $\mathfrak{a}$  create a twisted disk of contact. We denote  $\mathcal{D}_+$  the twisted disk of contact created by the over splitting, and  $\mathcal{D}_-$  the one created by the under splitting. We can suppose that  $\mathcal{D}_+$  is positive. Here are all the possible cases :

- (i) There is a smooth part of the branch locus of  $\mathcal{B}_n$  whose ends are  $p_1$  and  $p_2$ , denoted  $\mathfrak{a}'$ , such that, at  $p_1$  and  $p_2$ , the branch orientation of  $\mathfrak{a}'$  points to the interior of the edge  $[p_1, p_2]$  (see figure 6.6.2), and such that  $\mathfrak{a} \cup \mathfrak{a}'$  bounds a disk  $\mathcal{D}$  immersed in  $\mathcal{B}_n$ .

In this case, the over and under splittings turn  $\mathcal{D}$  into a twisted disk of contact with one single corner. Hence, only the neutral splitting can be adapted.

Suppose that the neutral splitting creates a twisted disk of contact as well, as in figure 6.6.3.

Then there exists a union of smooth parts of the branch locus of  $\mathcal{B}_n$ , denoted  $L_1$ , whose corners are all of the same signs, for example positive, such that  $\mathfrak{a} \cup L_1$

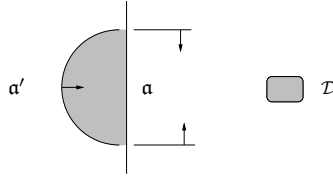


FIG. 6.6.2 – (i)

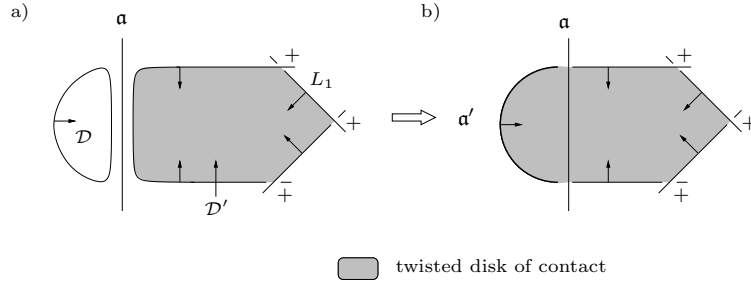


FIG. 6.6.3 – a) the neutral splitting creates a twisted disk of contact ;  
 b) there was a twisted disk of contact before the splitting

bounds a disk  $\mathcal{D}_+$ . The neutral splitting transforms  $\mathcal{D}_+$  into a positive twisted disk of contact. But then  $\mathbf{a}' \cup L_1$  bounds  $\mathcal{D} \cup \mathcal{D}_+$ , which is a sink disk whose corners all have the same sign. It is thus a twisted disk of contact, which is absurd. The neutral splitting is adapted.

We will prove that in the other cases we come to a contradiction :

(ii) in a neighbourhood of  $\mathbf{a}$ ,  $\mathcal{D}_+$  lies on one side of  $\mathbf{a}$ , and  $\mathcal{D}_-$  lies on the other side (see figure 6.6.4) :

In this case, both disks have the same sign, since one of them is created on one side of  $\mathbf{a}$  by an over splitting, and the other is created on the other side of  $\mathbf{a}$  by an under splitting.

The union of  $\mathcal{D}_+$  and  $\mathcal{D}_-$  is then a twisted disk of contact of  $\mathcal{B}_n$ , which is a contradiction.

(iii) in a neighbourhood of  $\mathbf{a}$ ,  $\mathcal{D}_+$  and  $\mathcal{D}_-$  lie on the same side of  $\mathbf{a}$  :

This case can not happen, but this is by far the most difficult case. We denote  $\mathbf{a}_1$  and  $\mathbf{a}_2$  the edges different from  $\mathbf{a}$  and passing respectively through  $p_1$  and  $p_2$ , and lying in the boundary of both  $\mathcal{D}_+$  and  $\mathcal{D}_-$ . Following these two edges from  $p_1$  or  $p_2$ , we stay on the boundary of  $\mathcal{D}_+$  and  $\mathcal{D}_-$ , until we arrive to a double point which is a corner of  $\mathcal{D}_+$  or  $\mathcal{D}_-$ , and at these double points, the boundary of  $\mathcal{D}_+$  and the boundary of  $\mathcal{D}_-$  part. The boundaries of  $\mathcal{D}_+$  and  $\mathcal{D}_-$  being in  $\mathcal{L}_n$ , and after the local models of a branched surface (figure 2.2),

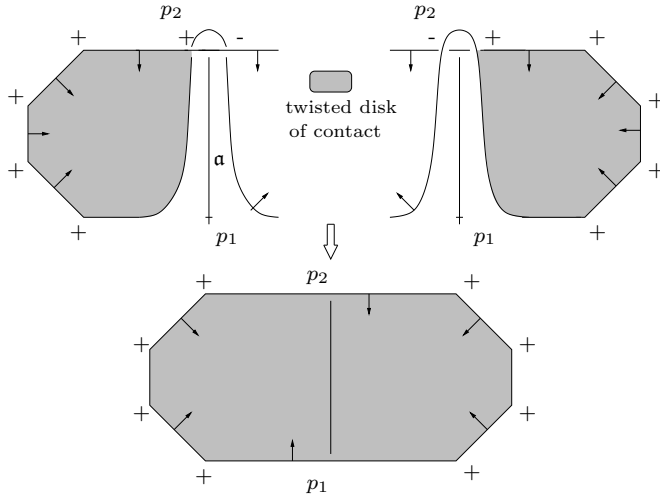


FIG. 6.6.4 – (ii)

these boundaries can part only at a double point, which is a corner for one of the disks of contact, but not for the other one.

Let us look at  $\mathcal{D}_+ \cap \mathcal{D}_-$ . According to the local models of a branched surface,  $\mathcal{D}_+$  and  $\mathcal{D}_-$  can not intersect transversally in  $M$ . If an isolated point or an isolated line is in their intersection, they can only lie in the boundary of  $\mathcal{D}_+$  or  $\mathcal{D}_-$ . But in this case, for branching reasons,  $\mathcal{D}_+$  and  $\mathcal{D}_-$  must lie on the same side of this intersection, and so they intersect in their interior as well. Hence,  $\mathcal{D}_+ \cap \mathcal{D}_-$  is a surface. Its boundary, which is not smooth, can be decomposed into maximal segments of four types :

- type 1 : the segment lies in  $\partial\mathcal{D}_+ \cap \partial\mathcal{D}_-$  ;
- type 2 : the segment lies in the boundary of  $\mathcal{D}_+$ , but not in the boundary of  $\mathcal{D}_-$  ;
- type 3 : the segment lies in the boundary of  $\mathcal{D}_-$ , but not in the boundary of  $\mathcal{D}_+$  ;
- type 4 : the segment is in the *branching border*, denoted  $\mathcal{F}$ , which is the place in the branch locus where  $\mathcal{D}_+$  and  $\mathcal{D}_-$  part. The easiest example and a more complicated one are drawn in figure **6.6.5**.

This branching border is a union of smooth parts of  $\mathcal{L}_n$ , whose branch orientations point into  $\mathcal{D}_+ \cap \mathcal{D}_-$ . There may be double points of both signs along  $\mathcal{F}$ , as in **b)** of figure **6.6.5**.

$\mathcal{F}$  is neither in the boundary of  $\mathcal{D}_+$ , nor in the boundary of  $\mathcal{D}_-$ , but the interior of  $\mathcal{F}$  lies in the interior of both  $\mathcal{D}_+$  and  $\mathcal{D}_-$ . The interiors of the segments of type 2 lie in the interior of  $\mathcal{D}_-$ , and the interiors of the segments of type 3 lie in the interior of  $\mathcal{D}_+$ . At last, when  $\mathcal{F}$  meets the boundary of  $\mathcal{D}_+$ , for example, it is at a double point, where two smooth parts of  $\mathcal{L}_n$  intersect. One of them,  $\alpha$ , lies in the boundary of  $\mathcal{D}_+$ , and the other one,  $\beta$ , lies in the

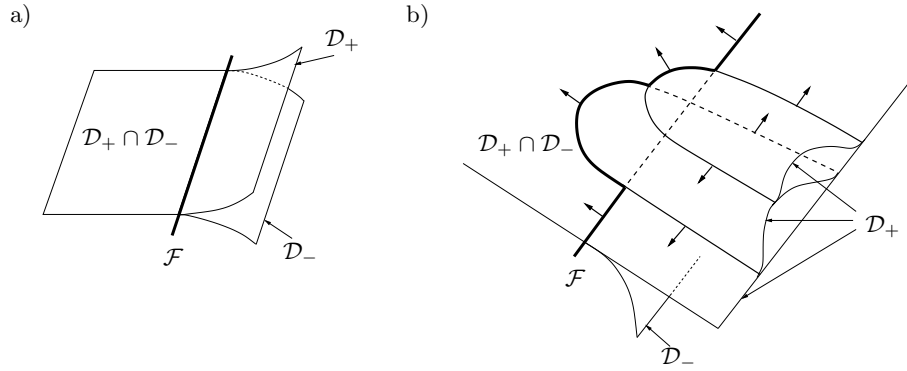


FIG. 6.6.5 – Branching border

inside of  $\mathcal{D}_-$ , as seen on figure 6.6.6.

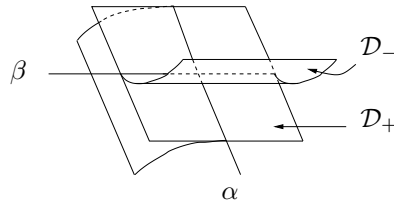


FIG. 6.6.6 –

Anyway, the boundary of  $\mathcal{D}_+ \cap \mathcal{D}_-$  is included in  $\mathcal{L}_n$ . Hence, when we follow the boundary of  $\mathcal{D}_+ \cap \mathcal{D}_-$  and we pass from a segment of some type, to a segment of another type, we cross a double point. These double points at which we pass from a type to another one, are called the *bifurcations* of the boundary of  $\mathcal{D}_+ \cap \mathcal{D}_-$ . The sign of a bifurcation is its sign as a double point. Along the boundary of  $\mathcal{D}_+ \cap \mathcal{D}_-$ , the branch direction always points into the interior of  $\mathcal{D}_+ \cap \mathcal{D}_-$ . Notice that a segment of type 4 can not be adjacent to a segment of type 1, because the boundaries of  $\mathcal{D}_+$  and  $\mathcal{D}_-$  being in  $\mathcal{L}_n$ , we would be in the situation of figure 6.6.7. But this configuration does not fit the local models of a branched surface.

Since  $\alpha \cup \alpha_1 \cup \alpha_2$  lies in a boundary component of  $\mathcal{D}_+ \cap \mathcal{D}_-$ , and since this whole component can not be of type 1, then this component must contain a sequence of the form : a segment of type 1 ; a segment of type 2 or 3 ; a finite number of segments of different types, but not of type 1 ; a segment of type 2 or 3 ; a segment of type 1. The union of the segments of this sequence, minus the first and the last ones, form a path  $\gamma$  in  $\mathcal{L}_n$ , whose ends will be denoted  $q_1$  and  $q_2$ . We orient  $\gamma$  from  $q_1$  to  $q_2$ . Each segment of  $\gamma$  inherits an orientation from the orientation of  $\gamma$ , and has then a first and a last end. The double points  $q_1$  and  $q_2$  are corners of  $\mathcal{D}_+$  or  $\mathcal{D}_-$ , so their signs are known.

We must now determine the signs of the bifurcations of  $\gamma$ . Knowing the

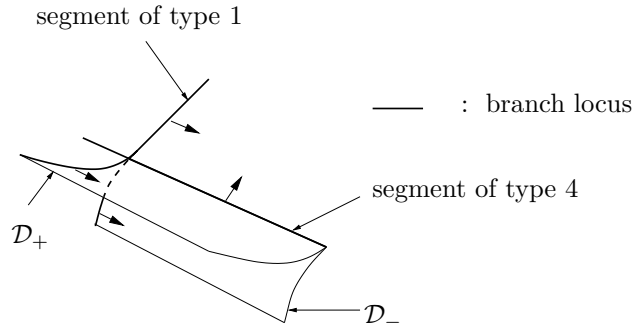


FIG. 6.6.7 –

sign of a bifurcation, we will determine the sign of the following one, from  $q_1$  until  $q_2$ . We will find of what sign must be  $q_2$ , but this sign will not be the sign of  $q_2$  already known. Hence a contradiction.

Let  $[b_1, b_2]$  be a segment of  $\gamma$  : it is not of type 1. The double points  $b_1$  and  $b_2$  are bifurcations. Let us see how we can deduce the sign of  $b_2$  from the sign of  $b_1$ , according to the type of  $[b_1, b_2]$  :

★  $[b_1, b_2]$  is of type 2 or 3 :

Suppose that  $[b_1, b_2]$  is of type 2. The case where it is of type 3 is dealt in the same way. The segment  $[b_1, b_2]$  is in the boundary of  $\mathcal{D}_+$ , and its interior lies in the interior of  $\mathcal{D}_-$ . The double point  $b_1$  is the intersection of two smooth parts of  $\mathcal{L}_n$  : one of them is the continuation of  $[b_1, b_2]$  and the other one,  $l_1$ , is the continuation of a segment of type 1, 3 or 4, and is thus in  $\mathcal{D}_-$ , the branch direction pointing into  $\mathcal{D}_-$ . We can decompose  $l_1$  into two parts,  $l_1^+$  and  $l_1^-$ , defined as follows :  $l_1^+ = l_1 \cap \mathcal{D}_+$  and  $l_1^- = l_1 \setminus l_1^+$ . The situation is the same at  $b_2$  :  $[b_1, b_2]$  meets there a smooth part  $l_2$  of  $\mathcal{L}_n$  which lies in  $\mathcal{D}_-$ , and can be decomposed into  $l_2^+$  and  $l_2^-$  (see figure 6.6.8).

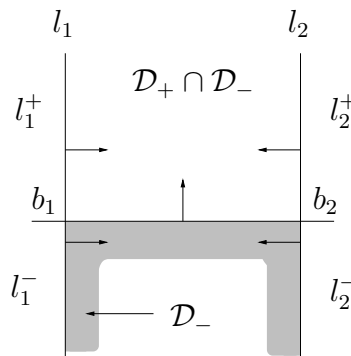


FIG. 6.6.8 –

Thus,  $l_1^- \cup [b_1, b_2] \cup l_2^-$  is in the boundary of some subsurface of  $\mathcal{D}_-$ . Lemma 6.4.11 implies that  $b_1$  and  $b_2$  are of opposite signs. Hence, the ends of a segment

of type 2 or 3 are of opposite signs.

★★  $[b_1, b_2]$  is of type 4 :

Then,  $[b_1, b_2]$  lies in the branching border of  $\mathcal{D}_+ \cap \mathcal{D}_-$ . At  $b_1$ , we pass from a segment of type 2 or 3 to a segment of type 4, i.e. we leave the boundary of  $\mathcal{D}_+$  or  $\mathcal{D}_-$ . At  $b_2$ , this is the reversed situation : we meet the boundary of  $\mathcal{D}_+$  or  $\mathcal{D}_-$ . We can set an orientation of the fibres of  $\mathcal{N}(\mathcal{B}_n)$  along this segment, for which  $\mathcal{D}_+$  is over  $\mathcal{D}_-$  at  $b_1$ . Then,  $\mathcal{D}_+$  is over  $\mathcal{D}_-$  all along the segment.

Suppose that  $b_1$  is on the boundary of  $\mathcal{D}_+$ . The double point  $b_1$  is the intersection of a smooth part  $f$  of the branching boundary, lying on  $\mathcal{D}_+ \cap \mathcal{D}_-$  and thus on  $\mathcal{D}_-$ , with a smooth part  $l$  of the boundary of  $\mathcal{D}_+$ . Hence,  $f$  lies under  $l$ . The double point  $b_2$  is also the intersection of  $f$  with a smooth part  $l'$ . If  $b_2$  lies on the boundary of  $\mathcal{D}_+$ ,  $l'$  lies on  $\mathcal{D}_+$ , and  $f$  is under  $l'$ . Hence,  $b_1$  and  $b_2$  have opposite signs, as in **a)** of figure 6.6.9. However, if  $b_2$  lies on the boundary of  $\mathcal{D}_-$ ,  $l'$  is on  $\mathcal{D}_-$  and  $f$  is over  $l'$ . In this case,  $b_2$  and  $b_1$  have the same sign, as in **b)** of figure 6.6.9.

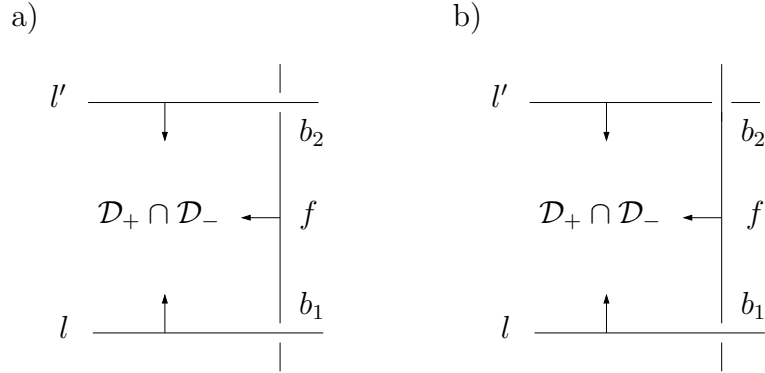


FIG. 6.6.9 –

The case where  $b_1$  lies on the boundary of  $\mathcal{D}_-$  is dealt in the same way, and we come to the result : if  $b_1$  and  $b_2$  are the ends of some segment of type 4, if they both lie on the boundary of  $\mathcal{D}_+$  or both lie on the boundary of  $\mathcal{D}_-$ , they have opposite signs ; if one of them lies on the boundary of  $\mathcal{D}_+$  and the other one lies on the boundary of  $\mathcal{D}_-$ , then they have the same sign.

There are two ways along  $\gamma$  to pass from the boundary of  $\mathcal{D}_+$  to the boundary of  $\mathcal{D}_-$ . The first one is to cross a bifurcation from a segment  $I_1$  of type 2 to a segment  $I_2$  of type 3. The first end of  $I_2$  and the first end of  $I_1$  are then of opposite signs.

The second way is to pass from a segment  $I_1 = [b_1, b_2]$  of type 2 to a segment  $I_2 = [b_2, b_3]$  of type 4, then to a segment  $I_3 = [b_3, b_4]$  of type 3. According to what has previously been said,  $b_1$  and  $b_2$  have opposite signs and  $b_2$  and  $b_3$  have the same sign, so  $b_1$  and  $b_3$  are of opposite signs.

At last, if we have a sequence : segment  $I_1 = [b_1, b_2]$  of type 2 (resp. 3) ;

segment  $I_2 = [b_2, b_3]$  of type 4; segment  $I_3 = [b_3, b_4]$  de type 2 (resp. 3), then  $b_1$  and  $b_3$  have the same sign.

All of this proves that the first ends of all the segments of type 2 are of one sign, and that the first ends of all the segments of type 3 are of the other sign.

It remains to pass from  $q_1$  to  $q_2$ , distinguishing two cases :

★  $q_1$  and  $q_2$  have the same sign :

For example,  $q_1$  and  $q_2$  are positive. They are thus both corners of  $\mathcal{D}_+$ , and  $q_1$  is the first end of a segment of type 2, and  $q_2$  is the last end of a segment of type 2. But  $q_1$  and  $q_2$  have the same sign, which is a contradiction.

★★  $q_1$  and  $q_2$  have opposite signs :

For example,  $q_1$  is positive and is thus a corner of  $\mathcal{D}_+$ , and  $q_2$  is negative, and is a corner of  $\mathcal{D}_-$ . Then  $q_1$  is the first end of a segment of type 2, and  $q_2$  is the last end of a segment of type 3, hence  $q_1$  and  $q_2$  must have the same sign, which is again a contradiction.  $\square$

## 6.7 Conclusion

After the two previous subsections, we have built a sequence of splittings of  $\mathcal{B}$ , none of whose having a twisted disk of contact. This sequence induces a resolving sequence of splittings of  $\mathcal{B}'_1$ , whose inverse limit is a lamination  $\lambda$  fully carried by  $\mathcal{B}'_1$ . We aim at proving that  $\lambda$  has null holonomy.

Let  $\Sigma$  be a 2-cell of  $Y$ , and  $\partial\Sigma \times [0, 1]$  be the subbundle of  $N(\mathcal{B})$  over  $\partial\Sigma$ . Then  $\lambda \cap (\partial\Sigma \times I)$  is an oriented dimension 1 lamination denoted  $l_\Sigma$ , fully carried by  $\partial\Sigma \times I$ , and obtained as the inverse limit (in the sense of definition 5.2.5) of the oriented train tracks  $v_n = \mathcal{B}_n \cap (\partial\Sigma \times I)$ .

If  $\Sigma$  is not compact, there are no holonomy problems since there is no first-return map on a fibre.

Then, suppose that  $\Sigma$  is compact.

**Definition 6.7.1** Let  $\lambda$  be an oriented lamination carried by a trivial bundle  $\mathbb{S}^1 \times [0, 1]$ . An *increasing leaf* (resp. a *decreasing leaf*) of  $\lambda$  is a leaf which goes, in the direct way, from a point  $p_1 = (\theta, t_1)$  to a point  $p_2 = (\theta, t_2)$ , with  $t_1 < t_2$  (resp.  $t_1 > t_2$ ).

**Lemma 6.7.2** *The lamination  $l_\Sigma$  is a lamination by circles.*

**Proof** We denote  $N(v_n) = N(\mathcal{B}_n) \cap (\partial\Sigma \times I)$ , which is a fibred neighbourhood of  $v_n$ . We call  $L_\Sigma = \bigcap_{n \in \mathbb{N}} N(v_n)$ , and we then have  $l_\Sigma = \partial L_\Sigma$ , according to definition 5.2.5.

Let  $L$  be an increasing leaf of  $l_\Sigma$ . This leaf is a spiral with two limit circles :  $C^+$ , limit when  $L$  is followed in the direct way, and  $C^-$ , limit when  $L$  is followed in the indirect way. We call  $A$  the annulus between  $C^+$  and  $C^-$ . Look at  $L_\Sigma \cap A$ . If this intersection is equal to  $A$ ,  $l_\Sigma \cap \text{Int}(A) = \emptyset$  and  $L$  is a leaf in the boundary

of  $A$ , and is thus a circle. Else, this means that  $A \setminus L_\Sigma$  contains some subset of the form  $\gamma \times [0, 1]$ , where  $\gamma$  is a compact oriented path fully carried by  $\partial\Sigma \times I$ , and which is increasing (see. figure **6.7.3**).

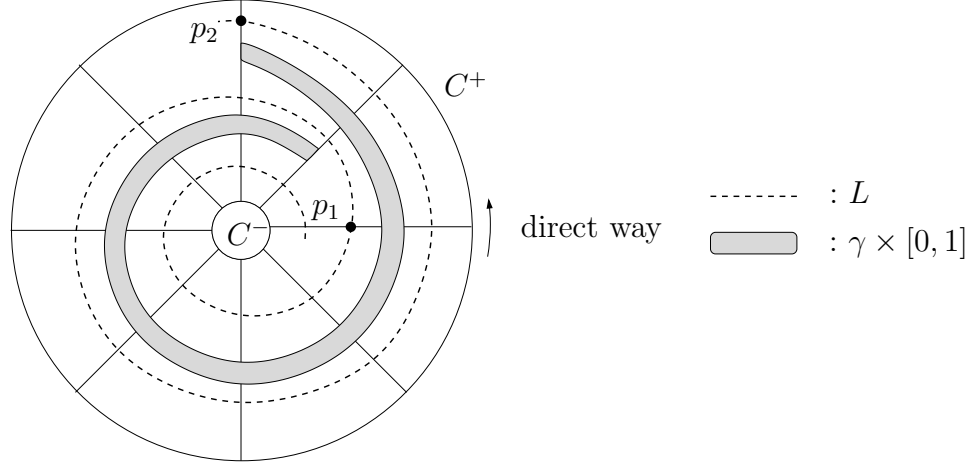


FIG. 6.7.3 –

Hence, there exists an integer  $N$  such that for all integer  $n$  greater than  $N$ , we have  $(\gamma \times [0, 1]) \cap N(\mathcal{B}_n) = \emptyset$ . If not, it would imply the existence of a sequence of points  $(q_n)$  such that  $q_n \in (\gamma \times [0, 1]) \cap N(\mathcal{B}_n)$ . Since  $(\gamma \times [0, 1]) \cap N(\mathcal{B}_n)$  is compact, there would be a subsequence of  $(q_n)$  converging towards a point  $q$  contained in  $\bigcap_{n \in \mathbb{N}} ((\gamma \times [0, 1]) \cap N(\mathcal{B}_n))$ . But this last set is equal to  $\gamma \cap L_\Sigma$ , which is empty.

It is then impossible to find a path in  $v_N$  going in the direct way from  $p_2$  to  $p_1$ , where  $p_1$  and  $p_2$  are two points of  $L$  placed as in figure **6.7.3**.

However, because  $\Sigma$  is not a twisted disk of contact and according to lemma **6.4.4**, the existence of a path of  $v_N$  going in the direct way from  $p_1$  to  $p_2$  implies the existence of a path of  $v_N$  going in the indirect way from  $p_1$  to  $p_2$ . This is a contradiction, and  $L$  must be a circle. In the same way,  $l_\Sigma$  does not have any decreasing leaf.  $\square$

Hence,  $\lambda$  has null holonomy.

To get a lamination fully carried by  $\mathcal{B}$ , it only remains to “fill the holes” of leaves of  $\lambda$ , these holes being in fact diffeomorphic to the 2-cells of  $Y$ , which are disks and half planes. This is possible because  $\lambda$  is null holonomic. This ends the proof of theorem **4.6**.



## 7 Proof of theorem 4.3

### 7.1 (A) $\Rightarrow$ (B)

Let  $\mathcal{B}$  be a branched surface fully carrying a lamination  $\lambda$ , and  $\mathcal{D}$  be a twisted disk of contact in  $\mathcal{B}$ .

Look at the train track corresponding to the boundary of  $\mathcal{D}$ . Since  $\mathcal{D}$  is twisted, this train track must contain non circular leaves (of dimension 1). But the corresponding leaves of  $\lambda$  (of dimension 2) can not be fully carried by  $\mathcal{D}$ , so they must be carried by some subsurface of  $\mathcal{D}$ , and then they leave  $\mathcal{D}$  by mean of branchings. This part of the branch locus of  $\mathcal{B}$  which allows these leaves to escape is a circle  $\mathcal{C}$  embedded inside  $\mathcal{D}$ , which may have corners. The sectors which branch on  $\mathcal{D}$  along  $\mathcal{C}$  must lie on the same side of  $\mathcal{D}$ , at least locally. And  $\mathcal{C}$  bounds a disk  $d$  embedded in  $\mathcal{D}$ , such that  $\mathcal{D} \setminus d$  is a sink annulus. The conditions (i) and (ii) of definition 4.1 are thus satisfied by  $\mathcal{D}$ .

At last, since the previous leaves leaving  $\mathcal{D}$  are carried by  $\mathcal{B}$  behind the branchings of  $\mathcal{C}$ , the following fact must be satisfied : all the immersed surfaces with boundary in  $\mathcal{B}$ , whose boundary is sent onto  $\mathcal{C}$ , and which lie behind the branchings of  $\mathcal{C}$ , in a neighbourhood of this boundary, are not disks. This is exactly condition (iii) of definition 4.1. Thus,  $\mathcal{D}$  is a carrying disk.

### 7.2 (B) $\Rightarrow$ (A)

Let  $\mathcal{B}$  be a branched surface satisfying condition (B). Since  $M$  is compact,  $\mathcal{B}$  has countably many twisted disks of contact. We denote them  $(\mathcal{D}_j)_{j \in J}$ , where  $J$  is a subset of  $\mathbb{N}$ .

Let  $\mathcal{D}_j$  be a twisted disk of contact of  $\mathcal{B}$ . Let  $d(j)$  be a disk embedded in  $\mathcal{D}_j$ , with boundary  $\mathcal{C}(j)$  and satisfying the three points of definition 4.1 of a carrying disk.

**Lemma 7.2.1** *Keeping the previous notations,  $d(j)$  contains a smooth circle  $\mathcal{C}_m(j)$  of  $\mathcal{L}$  bounding a source disk  $d_m(j)$  embedded in  $d(j)$ .*

**Proof** Let us look at the intersection of  $\mathcal{L}$  and  $d(j)$ . It is a finite set whose elements are either smooth compact segments of  $\mathcal{L}$ , which are embedded in  $d(j)$  and whose ends are double points (see figure 7.2.2), or smooth circles of  $\mathcal{L}$ , embedded in  $d(j)$  as well. Since  $d(j)$  is orientable, it has a top and a bottom. We can thus divide  $\mathcal{L} \cap d(j)$  into two parts : on one hand the smooth parts of  $\mathcal{L}$  for which the sector branching on  $d(j)$  lies, at least locally, over  $d(j)$ ; on the other hand the smooth parts of  $\mathcal{L}$  for which the sector branching on  $d(j)$  lies, at least locally, under  $d(j)$ . We will respectively denote them  $\mathcal{L}_+$  and  $\mathcal{L}_-$ .

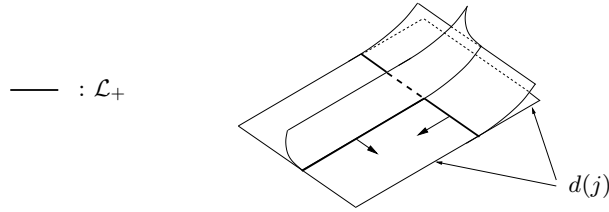


FIG. 7.2.2 –

Notice that two elements of  $\mathcal{L}_+$  can not meet in their interior, because both are inscribed on  $d(j)$ . If two elements of  $\mathcal{L}_+$  meet, the intersection point lies in the interior of one of them, and is an end of the second element, which, as a result, is a segment. Moreover, this segment lies in front of the branching defined by the first element (see figure 7.2.2). The same is true for the elements of  $\mathcal{L}_-$ . Two elements of  $\mathcal{L} \cap d$  intersect in their interior only if one is in  $\mathcal{L}_+$  and the other one is in  $\mathcal{L}_-$ .

At last, remember that  $\mathcal{C}$  must be entirely in  $\mathcal{L}_+$  or  $\mathcal{L}_-$ .

Among the finite set of all circles having the same properties as  $\mathcal{C}(j)$ , we consider a minimal element, i.e. the disk it bounds does not contain any circle of this set. We denote it  $\mathcal{C}_m(j)$ . We denote  $d_m(j)$  the disk it bounds in  $\mathcal{D}_j$ . The minimality of  $\mathcal{C}_m(j)$  implies that there can not be any twisted disk of contact inside  $d_m(j)$ . If it were so, this twisted disk of contact, being carrying, would also contain a disk of the previous set, which would contradict the minimality of  $\mathcal{C}_m(j)$ .

Suppose for example that  $\mathcal{C}_m(j)$  is a union of elements of  $\mathcal{L}_+$ , and prove that  $\mathcal{C}_m(j)$  is smooth. Suppose that it is not, and thus has at least one corner. The two possible configurations are drawn in figure 7.2.3.

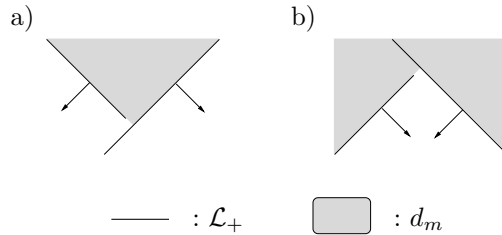


FIG. 7.2.3 –

But in **a)** we see an element of  $\mathcal{L}_+$  coming from behind another element of  $\mathcal{L}_+$ , which is impossible. We are thus in situation **b)**, and this implies that the branch locus of  $\mathcal{B}$  runs in the interior of  $d_m(j)$ .

Let us stand at a corner  $p_0$  of  $\mathcal{C}_m(j)$ . This corner is the intersection of two elements  $s_0$  and  $s_1$  of  $\mathcal{L}_+$ . It is in the interior of one of these elements, for

example  $s_1$ . Since  $s_0$  lies, at  $p_0$ , in front of the branching of  $s_1$ ,  $s_1$  can not be a circle bounding a sink disk. Actually, a part of  $\mathcal{C}_m(j)$  would lie outside of this disk of contact, and there would be an element of  $\mathcal{L}_+$  which would arrive to another element of  $\mathcal{L}_+$  from behind, which is impossible. By minimality of  $\mathcal{C}_m(j)$ , it can neither be a circle bounding a source disk. As a result,  $s_1$  is a segment.

Now, follow  $s_1$  from  $p_0$ , going into  $d_m(j)$ . Doing this, we never meet  $\mathcal{C}_m(j)$ , else an element  $\mathcal{L}_+$  would arrive on another element of  $\mathcal{L}_+$  from behind. When we arrive at the end of  $s_1$ , we stand in the interior of another element  $s_2$  of  $\mathcal{L}_+$ , at a double point  $p_1$ . For the same reasons which prevented  $s_1$  from being a circle,  $s_2$  is a segment.

Suppose that  $s_1 = s_2$ . In this case, there exists a disk embedded in  $d_m(j)$ , bounded by a subsegment of  $s_1$ . If this disk is a source disk, this contradicts the minimality of  $\mathcal{C}_m(j)$ . If this disk is a sink disk, then this disk is twisted, which again contradicts the minimality of  $\mathcal{C}_m(j)$ . Hence  $s_1$  and  $s_2$  are different.

Follow  $s_2$  this time, in the way of the branch direction of  $s_1$  at  $p_1$ . Once we arrive at the end of  $s_2$ , we stand in the interior of a third element  $s_3$  of  $\mathcal{L}_+$ , at a double point  $p_2$ . As before,  $s_3$  is a segment. However, since  $p_1$  and  $p_2$  are both the intersection of two segments of  $\mathcal{L}_+$ , they are of the same sign.

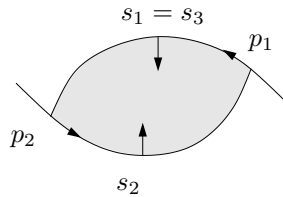


FIG. 7.2.4 –

Again,  $s_3$  is different from  $s_2$  and  $s_1$ . If not,  $d_m(j)$  would contain either a source disk or a twisted disk of contact (see figure 7.2.4). Both of these cases would contradict the minimality of  $\mathcal{C}_m(j)$ .

We can iterate this process infinitely many times, whereas  $\mathcal{L}_+$  is a finite set. This leads to a contradiction, and  $\mathcal{C}_m(j)$  is smooth.  $\square$

**Remark 7.2.5** Let  $p$  be a double point in  $d_m(j)$ , intersection of two elements of  $\mathcal{L}_\pm$ . There exists a path, an end of whose is  $p$ , whose interior is contained in the interior of  $d_m(j)$ , and which is built in the same way as the path starting from  $p_0$  built in the previous proof. If  $p$  lies on the boundary of  $d_m(j)$ , this path is uniquely determined. If  $p$  lies in the interior of  $d_m(j)$ , for each initial direction chosen at  $p$ , there exists a unique such path. Since such a path will be extensively used later, we call it a *good path from  $p$  inside  $d_m(j)$* . We will call *corner* of a good path a double point where we pass from an element of  $\mathcal{L}_\pm$  to another one.

Using the same arguments as in the previous proof, if we follow a good path, we must arrive on the boundary of  $d_m(j)$  or on a smooth circle of  $\mathcal{L}_\pm$  bounding a disk of contact.

In the same way as we have defined  $\mathcal{L}_+$  and  $\mathcal{L}_-$ , we can share the elements of  $\mathcal{L} \cap d_m(j)$  into two parts,  $\mathcal{L}_{m+}$  and  $\mathcal{L}_{m-}$ .

We still suppose that  $\mathcal{C}_m(j)$  is in  $\mathcal{L}_{m+}$ . Thus, only the elements of  $\mathcal{L}_{m-}$  can meet  $\mathcal{C}_m(j)$ .

**Lemma 7.2.6** *With the previous notations,  $\mathcal{L}_{m+} \cap \text{Int}(d_m(j))$  is contained in a union of disjoint disks of contact, embedded in  $d_m(j)$ , and bounded by smooth circles of  $\mathcal{L}_{m+}$ .*

**Proof** If  $\mathcal{L}_{m+}$  contains only smooth circles, we get the result.

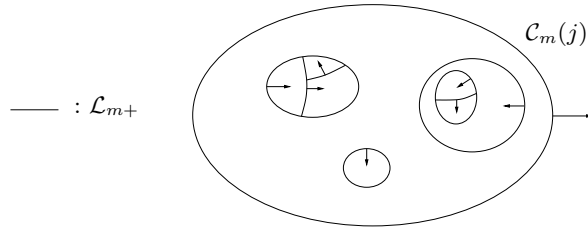


FIG. 7.2.7 –  $\mathcal{L}_{m+} \cap \text{Int}(d_m(j))$  is contained in a finite union of disks of contact

Else,  $\mathcal{L}_{m+}$  contains two segments which meet at a point  $p$ . Following a good path from  $p$ , since we can not arrive on  $\mathcal{C}_m(j)$ , we arrive on a smooth circle of  $\mathcal{L}_{m+}$  which bounds a disk of contact. Every segment of  $\mathcal{L}_{m+}$  is then contained in a disk of contact bounded by a circle of  $\mathcal{L}_{m+}$ . Since  $d_m(j)$  is compact, there are only finitely many such disks of contact. Figure 7.2.7 shows an example.  $\square$

We now set an orientation of  $\mathcal{C}_m(j)$ .

**Definition 7.2.8** Let  $p$  the intersection of  $\mathcal{C}_m(j)$  with an element  $l$  of  $\mathcal{L}_{m-}$ . We say that  $p$  is *direct* (resp. *indirect*) if the branch orientation on  $l$  at  $p$  is (resp. is reversed to) the orientation of  $\mathcal{C}_m(j)$  at  $p$ .

**Remark 7.2.9** The sign of  $p$  as a double point tells if  $p$  is direct or indirect. Nevertheless, it would be much easier to use definition 7.2.8 rather than to take care of the signs of the double points.

**Lemma 7.2.10** *The intersection of  $\mathcal{C}_m(j)$  with  $\mathcal{L}_{m-}$  satisfies the two following points :*

- (i) *If it is not empty, then it contains at least one direct point and at least one indirect point ;*

(ii) If moreover this intersection contains only two points, these points lie on a single segment of  $\mathcal{L}_{m-}$ .

**Proof**

(i) Let  $p_1$  be in  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$ . Following the good path from  $p_1$  inside  $d_m(j)$ , we must arrive to a point  $p_2$ , which lies in  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  as well. This good path is globally cooriented by the branch orientations along this path. So, if  $p_1$  is direct (resp. indirect),  $p_2$  is indirect (resp. direct).

(ii) Suppose now that  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  only contains two points, but that these two points are not on the same segment of  $\mathcal{L}_{m-}$ . Then, following the good path  $\gamma_1$  from  $p_1$  inside  $d_m(j)$ , we must arrive at  $p_2$ , but this good path has at least one corner. Let  $c$  be the last corner met before  $p_2$ . Follow now the good path  $\gamma_2$  from  $p_2$  inside  $d_m(j)$ . Both paths  $\gamma_1$  and  $\gamma_2$  are equal from  $p_2$  to  $c$ . However,  $c$  is a corner for  $\gamma_1$  but not for  $\gamma_2$ . The two good paths diverge at  $c$  :  $\gamma_2$  goes on behind the branchings of the segments of  $\gamma_1$ . Hence,  $\gamma_1$  can not meet  $\gamma_2$  anymore. The second end of  $\gamma_2$  is thus a point  $p_3$  in  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$ , which is different from  $p_1$ , which is a contradiction.  $\square$

**Lemma 7.2.11** *There exists a finite sequence of splittings of  $\mathcal{B}$  with support in  $d(j)$ , leading to a branched surface  $\mathcal{B}(j)$  satisfying :*

- (i)  $\mathcal{B}(j)$  satisfies point **(B)** ;
- (ii) if  $\mathcal{L}(j)$  is the branch locus of  $\mathcal{B}(j)$ , then  $\mathcal{L}(j) \cap \text{Int}(d_m(j))$  is an union of disjoint smooth circles bounding disks of contact.

**Proof** Look at  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$ . All its points can not be of the same type (direct or indirect), according to point **(i)** of lemma 7.2.10.

If we follow  $\mathcal{C}_m(j)$  in the direct way, there must be a direct point  $p_1$  such that the following point of  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  is an indirect point  $p_2$ . Then, the portion  $[p_1, p_2]$  of  $\mathcal{C}_m(j)$  which lies between  $p_1$  and  $p_2$  is a path in a face-to-face situation between the element  $l_1$  of  $\mathcal{L}_{m-}$  which runs through  $p_1$  and the element  $l_2$  of  $\mathcal{L}_{m-}$  which runs through  $p_2$ . We perform a splitting along this path. If possible, we perform any splitting (over, under or neutral) if this does not create a twisted disk of contact in  $\mathcal{B}$ . If this is not possible, we show, as in the proof of proposition 6.6.1, that there exists in  $\mathcal{B}$  a twisted disk of contact  $\mathcal{D}_k$  whose boundary contains  $p_1$  and  $p_2$ , and which follows  $l_1$  and  $l_2$  in neighbourhoods of  $p_1$  and  $p_2$ . This disk is cut into two disks  $d_1$  and  $d_2$  by  $[p_1, p_2]$ . But  $\mathcal{D}_k$  is carrying, and thus contains a circle  $\mathcal{C}(k)$  bounding a disk  $d(k)$ , satisfying the three points of definition 4.2. Since  $]p_1, p_2[$  does not contain any double point,  $\mathcal{C}(k)$  does not meet  $[p_1, p_2]$ , and  $d(k)$  is entirely included in  $d_1$  or  $d_2$ . For example, it is contained in  $d_1$ .

If all the corners of  $\mathcal{D}_k$  are on the boundary of  $d_1$ , we can perform a neutral move :  $d_2$  becomes a disk of contact, and  $d_1$  becomes a carrying twisted disk of contact.

If some corner of  $\mathcal{D}_k$  is on the boundary of  $d_2$ , among the over and the

under moves, we chose the one which adds to  $d_1$  a corner of the same sign as  $\mathcal{D}_k$ , and adds to  $d_2$  a corner of the opposite sign. Then, after this splitting,  $d_2$  is not a twisted disk of contact because it has a corner of each sign, and  $d_1$  is a carrying twisted disk of contact.

In the new branched surface we get,  $\mathcal{D}_k$  has vanished, to the profit of a new carrying twisted disk of contact,  $d_1$ . No other twisted disk of contact has been created.

In the particular case where  $p_1$  and  $p_2$  are the only points in  $\mathcal{C}_m \cap \mathcal{L}_{m-}$ , according to point (ii) of lemma 7.2.10, they lie on the same segment  $s$  of  $\mathcal{L}_{m-}$ . Hence, with the previous notations,  $s$  is in the boundary of  $\mathcal{D}_k$ . The disk in  $d_m$  bounded by  $[p_1, p_2] \cup s$  is included in  $\mathcal{D}_k$ , and it is equal to  $d_1$  or  $d_2$ . By minimality of  $\mathcal{C}_m$ , it can not contain  $\mathcal{C}(k)$ , so this disk is  $d_2$ . In this case there are no corners of  $\mathcal{D}_k$  on the boundary of  $d_2$ , and a neutral move can be performed.

It remains to verify that the modification of the branch locus induced by this splitting does not destroy the carrying character of a twisted disk of contact of  $\mathcal{B}$ . This could happen only if  $l_1$  and  $l_2$  lie both (at least in neighbourhoods of  $p_1$  and  $p_2$ ) in a circle (possibly with corners) bounding a source disk satisfying the three points of definition 4.2.

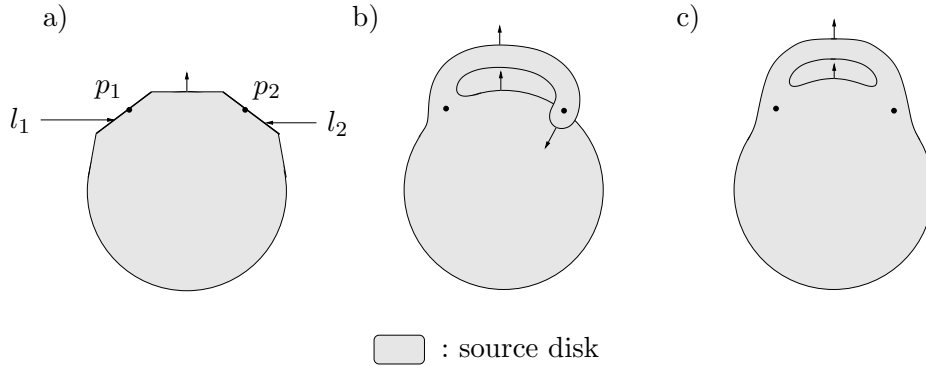


FIG. 7.2.12 – source disk : a) before splitting ; b) after an under or over splitting ; c) after a neutral splitting

In this case, as seen on figure 7.2.12, there still exists after splitting a circle bounding a source disk, and this circle satisfies the three points, because it bounds the same surfaces (topologically) than the circle before splitting. The new branched surface thus still satisfies condition (B).

After this splitting, the number of points in  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  has strictly decreased. Since  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  can not consist of a single point, by repeating this operation a finite number of times,  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  contains only two points. As seen previously, a neutral move can be performed, and  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  becomes empty.

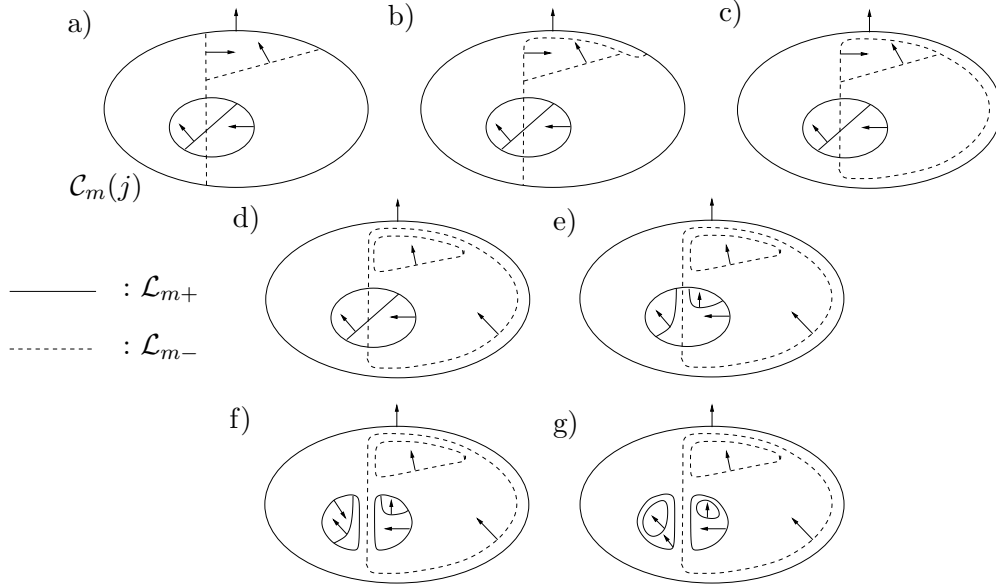


FIG. 7.2.13 – example of a sequence of splittings

The intersection  $\mathcal{C}_m(j) \cap \mathcal{L}_{m-}$  being empty, according to lemma 7.2.6,  $\mathcal{L}_{m+}$  is entirely contained in a finite union of disjoint disks of contact embedded in  $d_m(j)$ . The same is true with  $\mathcal{L}_{m-}$ . For each circle bounding a disk of contact in  $d_m(j)$ , we can apply the previous method. These circles being in finite number, after finitely many splittings, the trace of the branch locus on  $d_m(j)$  is a finite union of smooth circles bounding disks of contact.

Figure 7.2.13 shows an example of such a sequence of splittings.  $\square$

**Lemma 7.2.14** *There exists a finite sequence of splittings of  $\mathcal{B}(j)$  with support in  $d_m(j)$  leading to a branched surface  $\mathcal{B}'(j)$  satisfying :*

- (i)  $\mathcal{B}'(j)$  satisfies **(B)** ;
- (ii) if  $\mathcal{L}'(j)$  is the branch locus of  $\mathcal{B}'(j)$ , then  $\mathcal{L}'(j) \cap \text{Int}(d_m(j))$  is empty.

**Proof** The trace of the branch locus of  $\mathcal{B}(j)$  in  $d_m(j)$  is a finite union of smooth circles bounding disks of contact.

Let  $C$  be a minimal element in this set of smooth circles, i.e. the disk  $d_C$  bounded by  $C$  does not contain any other smooth circle of the branch locus. We then remove from  $N(\mathcal{B}(j))$  an  $I$ -bundle over  $d_C$ , as on figure 7.2.15. Since this operation does not create nor eliminate any corner, no twisted disk of contact is created.

Let us verify a last point. It could be possible that  $C$ , on top of bounding a disk of contact, also bounds a source disk. An example of such a circle is shown in figure 7.2.16, where we can see a branched surface of revolution. But even in this case, since the interior of  $d_C$  does not meet the branch locus,  $C$  can not

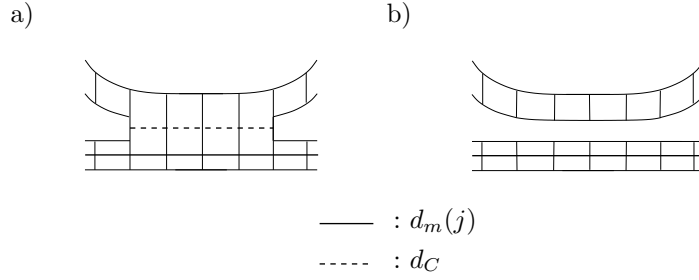


FIG. 7.2.15 –

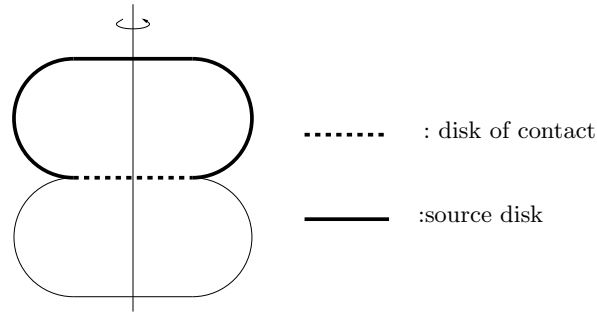


FIG. 7.2.16 –

be a circle turning a twisted disk of contact into a carrying one. Finally, the previous operation can not destroy the carrying character of a twisted disk of contact. After this operation,  $C$  has vanished from the branch locus.

Doing this from minimal circle to minimal circle, we get after a finite number of steps, the branched surface  $\mathcal{B}'(j)$  we wanted.  $\square$

**Lemma 7.2.17** *There exists a finite sequence of splittings of  $\mathcal{B}(j)$  with support in  $d_m(j)$  leading to a branched surface  $\mathcal{B}'$  satisfying :*

- (i)  $\mathcal{B}'$  satisfies point **(B)** ;
- (ii) *there exists a finite set  $\mathcal{E}$  whose elements are source disks of  $\mathcal{B}'$  with smooth boundary, and which are sectors (they are thus disjoint), and such that for every carrying twisted disk of contact  $\mathcal{D}$  of  $\mathcal{B}'$  there exists a disk of  $\mathcal{E}$  embedded in  $\mathcal{D}$ .*

**Proof** Since  $M$  is compact, there is a finite number of smooth circles bounding source disks. Hence, the set of the  $d_m(j)$  's is finite, even if the set of indices  $J$  is not. We apply lemmas 7.2.11 and 7.2.14 to one of those disks, then to another one, until we have done that for all those disks. After a finite number of splittings, we get the branched surface  $\mathcal{B}'$ .  $\square$

**Remark 7.2.18** Lemma 7.2.17 is not in contradiction with the fact that  $\mathcal{B}$  can have infinitely many carrying twisted disks of contact. In the example **b)**



of figure 4.2 , two source disks which are sectors are included in infinitely many carrying twisted disks of contact.

**Lemma 7.2.19** *We set  $\mathcal{B}'' = \mathcal{B}' \setminus (\cup_{d \in \mathcal{E}} \text{Int}(d))$ . Then  $\mathcal{B}''$  is a branched surface having no twisted disk of contact.*

**Proof** According to [Li], a branched surface minus the interior of a source sector is always a branched surface without boundary, except if two smooth components of the boundary of this sector are identified in  $\mathcal{L}$ . This is not the case here, since the boundary of a circle in  $\mathcal{E}$  is smooth, and since  $\mathcal{L}$  does not meet the interior of these disks. Since  $\mathcal{E}$  is finite and since its elements are disjoint,  $\mathcal{B}''$  is branched surface without boundary.

Moreover, by removing those disks, no twisted disk of contact is created. At last, a disk is removed from the interior of each twisted disk of contact of  $\mathcal{B}'$  : there is no twisted disk of contact anymore in  $\mathcal{B}''$ .  $\square$

**Lemma 7.2.20**  *$\mathcal{B}'$  fully carries a lamination.*

**Proof** The branched surface  $\mathcal{B}''$  satisfies the hypotheses of theorem 4.6 : it then fully carries a lamination  $\lambda'$ . We now have to add leaves to  $\lambda'$ , to turn it into a lamination fully carried by  $\mathcal{B}'$ .

Let  $d_m(j)$  be one of the disks taken away from  $\mathcal{B}'$ . Look at the train track corresponding to the boundary of  $d_m(j)$ . This train track is fully carried, and all its leaves (of dimension 1) correspond to leaves of  $\lambda'$  (of dimension 2) which take the branching of  $\mathcal{C}_m$ , and which go over  $d_m(j)$  since  $\mathcal{C}_m(j)$  is in  $\mathcal{L}_+$ . Among all these leaves, the bottom leaf (for the chosen coorientation of  $d_m(j)$ ) must be a circle, either because it is the lowest leaf of a set of circular leaves, or because it is the limit leaf of a set of spiral leaves. We denote  $l$  this leaf, and  $L$  the leaf of  $\lambda'$  which contains  $l$ .

We must now look at two possibilities : either  $L \setminus l$  has two connected components, or it is still connected.

The first case is the easiest one. We denote  $L \setminus l = L_1 \cup L_2$ , where  $L_1$  is the part of  $L$  which lies behind the branching of  $d_m(j)$ , and  $L_2$  is the one which lies in front of the branching of  $d_m(j)$ . We add to  $\lambda'$  an isolated copy of  $L_1$ , just under  $L$ . We “fill” this copy by adding along  $l$  a disk fully carried by  $d_m(j)$ , lying under all the leaves of  $\lambda'$  taking branchings of  $\mathcal{L}_+$  in  $d_m(j)$ . By adding this isolated leaf to  $\lambda'$ , we get a lamination  $\lambda$  fully carried by  $\mathcal{B}$ .

In the second case, we make an isolated copy of  $L$  and  $l$  in  $\lambda'$ , just under  $L$ , denoted  $L'$  and  $l'$ , and we realize the product  $L' \times [0, 1]$ , foliated by leaves  $L'(t) = L' \times \{t\}$ . This new lamination is still fully carried by  $\mathcal{B}'$ . We then

consider a diffeomorphism  $\varphi$  strictly increasing from  $[0, 1]$  to  $[\alpha, 1]$ , for some  $\alpha \in ]0, 1]$ , and satisfying : for each  $t \in [0, 1[$ ,  $\varphi(t) > t$ . We next realize the suspension of  $L' \times [0, 1]$  by  $\varphi$ , which is the following construction : we consider the foliation  $(L' \times [0, 1]) \setminus (l' \times [0, 1])$ , and we call  $L''(t)$  the leaf  $L'(t) \setminus (l' \times \{t\})$ . Locally, in a neighbourhood of  $(l' \times [0, 1])$ ,  $L''(t)$  is the union of two parts of  $L''(t)$ , denoted  $L''_1(t)$  and  $L''_2(t)$ , where  $L''_1(t)$  lies in front of the branching of  $d_m$ , and  $L''_2(t)$  lies behind. We then perform a shift between these leaves on each side of  $(l' \times [0, 1])$  : we glue  $L''_2(t)$  with  $L''_1(\varphi(t))$  along a copy of  $l'$ . After this operation,  $L' \times [0, 1]$  is a new set of leaves. Among these leaves, only the leaf  $L'(1)$  remains unchanged. The others are leaves with one boundary component of the form  $l' \times \{t\}$ , for some  $t \in [0, \varphi(0)[$ , spiraling towards  $L'(1)$ , their limit leaf. From this set of leaves, we only retain  $L'(1)$  and one of the other leaves, denoted  $L'_0$ , and whose boundary is  $l'(t_0)$ , for  $t_0 \in [0, \varphi(0)[$ . We “fill”  $L'_0$  by adding to it, along  $l'(t_0)$ , a disk fully carried by  $d_m$ , as in the first case. The set  $\lambda = \lambda' \cup L'(1) \cup L'_0$  is a lamination since  $L'(1)$  is the limit leaf of  $L'_0$ , and it is fully carried by  $d_m(j)$ .

Repeating this operation for all the disks of  $\mathcal{E}$ , we get a lamination  $\Lambda$  fully carried by  $\mathcal{B}'$ .  $\square$

The proof comes to an end by using lemma 5.1.3 :  $\mathcal{B}'$  is a splitting of  $\mathcal{B}$ , and  $\Lambda$  is fully carried by  $\mathcal{B}$  as well.

## 8 Contact structures carried by a branched surface - an application

### 8.1 Definitions

The following definitions are taken from [Co].

**Definition 8.1.1** Let  $\xi$  be a positive (resp. negative) contact structure defined in a fibred neighbourhood  $N(\mathcal{B})$  of  $\mathcal{B}$ . We say that  $\xi$  is a contact structure *carried* by  $\mathcal{B}$  if  $\xi$  satisfies :

- (i)  $\xi$  is transverse to the fibres of  $N(\mathcal{B})$  ;
- (ii) each component of  $\partial_h N(\mathcal{B})$  is transverse to a germ of Reeb vector field  $R$  and  $\partial(\partial_h N(\mathcal{B}))$  is transversally ascending (resp. descending) to  $\xi$ , for the orientation of  $\partial(\partial_h N(\mathcal{B}))$  induced by the orientation of  $\partial_h N(\mathcal{B})$  (cooriented by  $R$ ) and the local orientation of  $\xi$  given by  $R$ .

**Remark 8.1.2** If  $\mathcal{B}$  carries a contact structure, then  $\partial_h N(\mathcal{B})$  does not contain any closed surface. Indeed, in a neighbourhood of the horizontal boundary,  $\xi$  is defined by a 1-form  $\alpha$  whose Reeb vector field is transversal to the horizontal boundary. Because  $d\alpha$  is non degenerate on the horizontal boundary, and

because of Stokes theorem, the horizontal boundary can not contain a closed surface.

**Definition 8.1.3** Let  $\xi$  be a contact structure carried by  $\mathcal{B}$ . If moreover all the fibres of  $N(\mathcal{B})$  are tangent to a Reeb vector field for  $\xi$  defined on all  $N(\mathcal{B})$ , then it is said that  $\mathcal{B}$  carries a contact form.

**Remark 8.1.4** If  $\mathcal{B}$  carries a contact form, then, of course, it also carries a contact structure, and, for the same reasons as in remark 8.1.2, it does not carry any closed surface.

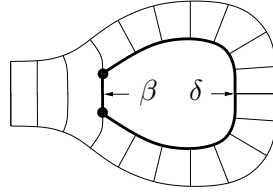


FIG. 8.1.5 – monogon

**Remark 8.1.6** If  $\mathcal{B}$  carries a contact form, the fibres of  $N(\mathcal{B})$  are globally orientable. This prevents the existence of a *monogon* : a monogon is a disk  $D \subset V \setminus \text{Int}(N(\mathcal{B}))$  with  $\partial D = D \cap N(\mathcal{B}) = \beta \cup \delta$ , where  $\beta \subset \partial_v N(\mathcal{B})$  is in a fibre of  $\partial_v N(\mathcal{B})$  and  $\delta \subset \partial_h N(\mathcal{B})$  (see figure 8.1.5).

**Definition 8.1.7** A branched surface *strongly carries* a positive (resp. negative) contact structure  $\xi$  if  $\mathcal{B}$  is everywhere transverse to a Reeb vector field  $R$  of  $\xi$  and if its branch locus is transversally ascending (resp. descending) to  $R$ , for the orientation induced by the branch direction.

**Remark 8.1.8** By a thickening along  $R$ , a branched surface which strongly carries a contact structure  $\xi$  also carries a contact form corresponding to  $\xi$ .

**Remark 8.1.9** Definitions 8.1.1, 8.1.3 and 8.1.7 establish a hierarchy in the notion of “contact structure carried by a branched surface”.

**Definition 8.1.10** Let  $\xi$  be a contact structure carried by a branched surface  $\mathcal{B}$ . An *overtwisting disk*  $\mathcal{D}$  for  $\xi$  is a disk whose boundary is tangent to  $\partial \mathcal{D}$  and which is transverse to  $\xi$  along  $\partial \mathcal{D}$  (see [El] for instance). We say that  $\xi$  is a *tight contact structure carried by*  $\mathcal{B}$  if there is no overtwisting disk in  $M$  whose boundary is in  $N(\mathcal{B})$ . The same definition can be applied to contact forms (resp. structures) carried (resp. strongly carried) by branched surfaces.

## 8.2 Some known results

We first need some last definitions :

**Definition 8.2.1** If  $\mathcal{B}$  is a branched surface embedded in  $M$ , then  $\mathcal{B}$  is *essential* if it satisfies the following conditions :

- (i)  $\mathcal{B}$  has no disk of contact ;
- (ii)  $\partial_h N(\mathcal{B})$  is incompressible in  $M \setminus \text{Int}(N(\mathcal{B}))$  ;
- (iii) no component of  $\partial_h N(\mathcal{B})$  is a sphere ;
- (iv) there is no monogon in  $M \setminus \text{Int}(N(\mathcal{B}))$  ;
- (v)  $M \setminus \text{Int}(N(\mathcal{B}))$  is irreducible ;
- (vi)  $\mathcal{B}$  does not carry a torus that bounds a solid torus in  $M$  ;
- (vii)  $\mathcal{B}$  fully carries a lamination.

**Definition 8.2.2** If  $\mathcal{B}$  is a branched surface embedded in  $M$ , then  $\mathcal{B}$  is *laminar* if it satisfies conditions (ii) to (vi) of definition 8.2.1 and has no sink disk.

As showed by T. Li in [Li], a laminar branched surface is an essential branched surface. The converse is not true. Essential and laminar branched surfaces are important because of the two following results :

**Theorem 8.2.3** ([GO]) *A lamination is essential if and only if it is fully carried by an essential branched surface.*

**Theorem 1.3** ([Li]) *A 3-manifold contains an essential lamination if and only if it contains a laminar branched surface.*

We then recall results from [Co] :

**Proposition 8.2.4** ([Co]) *Let  $\mathcal{B}$  be a branched surface which carries a tight contact form. Then  $\mathcal{B}$  satisfies points (i) to (vi) of definition 8.2.1.*

which is used to prove

**Proposition 8.2.5** ([Co]) *If  $\xi$  is a tight contact structure strongly carried by  $\mathcal{B}$ , then  $\mathcal{B}$  is laminar, and ([Li]) the universal cover of  $M$  is homeomorphic to  $\mathbb{R}^3$ .*

If we want to go down one step into the hierarchy of contact structures carried by branched surfaces, i.e. we consider contact forms carried by  $\mathcal{B}$  instead of contact structures strongly carried by  $\mathcal{B}$ , proposition 8.2.5 does not hold anymore, but corollary 4.8 gives a condition which is stronger but in the same spirit, to get the same conclusion.

### 8.3 Proof of corollary 4.7

We may refer to [Gi1] and [Gi2] for sign conventions.

Suppose that  $\mathcal{B}$  has a positive twisted surface of contact  $\mathcal{S}$ . Let  $\alpha_1$  be the positive contact form carried by  $\mathcal{B}$ . The existence of  $\alpha_1$  implies that there is a global orientation of the fibres of  $N(\mathcal{B})$ , defined by the Reeb vector field of  $\alpha_1$ , and that, maybe after having reversed the orientation of  $\mathcal{S}$ , the orientation of  $N(\mathcal{S})$  is the same as the orientation of the fibres of  $N(\mathcal{B})$ . It is impossible that at two different points of  $f(\mathcal{S})$ , the fibres of  $N(\mathcal{S})$  at these points are included in the same fibre of  $N(\mathcal{B})$  with opposite orientations. This is possible in general, for example along a monogon.

The boundary of  $\mathcal{S}$  is piecewise smooth, and each smooth portion of the boundary is included in the inside of the vertical boundary of  $N(\mathcal{B})$ . The coorientation of the connected components of  $\partial_h N(\mathcal{B})$  by the Reeb vector field of  $\alpha_1$  gives an orientation of the boundary of these components, which itself induces an orientation of each smooth component of the boundary of  $\mathcal{S}$ . For this orientation, each smooth component is transversal ascending to  $\alpha_1$ . However, this orientation is reversed to the orientation of the boundary of  $\mathcal{S}$ , when  $\mathcal{S}$  is also cooriented by the Reeb vector field of  $\alpha_1$ . Finally, the boundary of  $\mathcal{S}$  for this coorientation is transversal descending to  $\alpha_1$ . As a result, the characteristic foliation of  $\mathcal{S}$  points inwards along its boundary.

Moreover, for the contact form  $\alpha_1$ ,  $\mathcal{S}$  is convex (in a contact sense, see [Gi1] for definitions) and its characteristic foliation has only positive singularities, and an empty dividing curve. This is in contradiction with the fact that the boundary of  $\mathcal{S}$  is transversally descending for the Reeb vector field of  $\alpha_1$ .

In the same way,  $\mathcal{B}$  can not have any negative twisted surface of contact or else there would be a convex surface whose characteristic foliation points outwards along its boundary, but having only negative singularities and an empty dividing curve.

It remains to apply theorem 4.3 (actually, theorem 4.6 is here sufficient).

### 8.4 Proof of corollary 4.8

If  $\mathcal{B}$  carries a positive tight contact form and a negative one, it fully carries a lamination according to corollary 4.7, and it is essential, according to proposition 8.2.4. This lamination is thus essential, thanks to proposition 8.2.3. Theorem 1.1 then implies that the universal cover of  $M$  is homeomorphic to  $\mathbb{R}^3$ .

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