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Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds

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Let M be a compact manifold without boundary. A function $H(x, p) : T^*M \rightarrow \mathbb{R}$ is called a Tonelli Hamiltonian if it is C^2 and if, for each $x \in M$, the function $p \mapsto H(x, p)$ is convex with positive definite Hessian and superlinear on the fibre T_x^*M . Each Tonelli Hamiltonian generates a complete C^1 flow ψ_t . We consider the Hamilton-Jacobi equation

$$H(x, du_x) = c, \tag{HJ}$$

with a special emphasis on sub-solutions. A function $u : M \rightarrow \mathbb{R}$ is called a sub-solution of (HJ) if it is Lipschitz and satisfies the inequality $H(x, du_x) \leq c$ at all its points of differentiability. Note that this definition is equivalent to the notion of viscosity sub-solutions, see [5]. We denote by $C^{1,1}(M, \mathbb{R})$ the set of differentiable functions with Lipschitz differential. The goal of the present paper is to present a short and selfcontained proof of:

THEOREM *Let H be a Tonelli Hamiltonian. If the Hamilton-Jacobi equation (HJ) has a sub-solution, then it has a $C^{1,1}$ sub-solution. Moreover, the set of $C^{1,1}$ sub-solutions is dense for the uniform topology in the set of sub-solutions.*

Fathi and Siconolfi recently proved the existence of a C^1 sub-solution in [6]. Our result is optimal in the sense that examples are known where a $C^{1,1}$ sub-solution exist, but no C^2 solution. There exists a real number $\alpha(H)$, called the Mañé critical value in the literature, such that that the equation (HJ) has sub-solutions if and only if $c \geq \alpha(H)$. It is not hard to prove the existence of smooth sub-solutions for $c > \alpha(H)$ by regularization, see [4]. As a consequence, our Theorem is relevant for the sub-solutions of the critical equation $H(x, du_x) = \alpha(H)$, which are called the critical sub-solutions of (HJ). The study of the critical Hamilton-Jacobi equation $H(x, du_x) = \alpha(H)$ is the core of Fathi's weak KAM theory.

A sub-solution u is called strict on the open set $U \subset M$ if there exists a smooth non-negative function $V : M \rightarrow \mathbb{R}$ which is positive on U and such that u is also a sub-solution of the equation $H(x, du_x) + V(x) = c$. By applying the Theorem to the Hamiltonian $H + V$, we obtain:

ADDENDUM *If there exists a sub-solution of (HJ) which is strict on the open set U , then there exists a $C^{1,1}$ sub-solution which is strict on U .*

We then obtain the important:

COROLLARY *There exists a critical $C^{1,1}$ sub-solution which is strict outside of the Aubry set.*

We refer to the literature, for example [5] and [6], for the definition of the Aubry set. The Corollary holds if there exist sub-solutions which are strict outside of the Aubry set. This fact is proved in [6]. In the paper of Fathi and Siconolfi, finding a strict sub-solution is a key technical step in order to build C^1 sub-solutions. In our approach, the problems of finding strict and regular sub-solutions are independent. We do not improve on [6] for the existence of strict sub-solutions, which is an important problem in itself. See however [7, 2] for new methods of construction of strict sub-solutions.

Let us now proceed to the proof. It is necessary to start with more definitions. We define the Lagrangian $L : TM \rightarrow \mathbb{R}$ associated to H by the relation

$$L(x, v) = \max_{p \in T_x^*M} p(v) - H(x, p).$$

Then we define, for each $t \geq 0$, the function $A_t : M^2 \rightarrow \mathbb{R}$ by

$$A_t(x, y) := \min_{\gamma} \int_0^t c + L(\gamma(s), \dot{\gamma}(s)) ds$$

where the minimum is taken on the set of curves $\gamma \in C^2([0, t], M)$ which satisfy $\gamma(0) = x$ and $\gamma(t) = y$. Following Fathi, we define the Lax-Oleinik semi-groups T_t and \check{T}_t on $C^0(M, \mathbb{R})$ by

$$T_t u(x) = \min_{y \in M} u(y) + A_t(y, x) \quad \text{and} \quad \check{T}_t u(x) = \max_{y \in M} u(y) - A_t(x, y).$$

Fathi proves in his book that a Lipschitz function u is a sub-solution of (HJ) if and only if the function $t \mapsto T_t u(x)$ is non-decreasing. As a consequence, the operators T_t preserve the set of sub-solutions. Similarly, the function u is a sub-solution if and only if $t \mapsto \check{T}_t u(x)$ is non-increasing, hence the operators \check{T}_t preserve the set of sub-solutions. Another important property of these semigroups is that, for all $t > 0$ and all continuous function u , the function $T_t u$ is semi-concave and the function $\check{T}_t u$ semi-convex.

We say that a function u is semi-concave if it can be written

$$u(x) = \min_{f \in F} f(x)$$

with a bounded family $F \subset C^2(M, \mathbb{R})$ of functions. Similarly, a function is called semi-convex if it can be written $u(x) = \max_{f \in F} f(x)$. See for example [3, 5, 1] for more on semi-concave functions. It is known that a function is $C^{1,1}$ if and only if it is both semi-concave and semi-convex.

If u is a sub-solution of (HJ), then for each $s \geq 0$ and $t \geq 0$, the function $T_s \check{T}_t u$ is a sub-solution. Since $\check{T}_t u$ is semi-convex, the Theorem follows from the following result, which may have other applications.

PROPOSITION *Let H be a Tonelli Hamiltonian. For each semi-convex function v , the function $T_s v$ is $C^{1,1}$ for each sufficiently small $s > 0$.*

PROOF. In order to prove this proposition, it is enough to prove that the function $T_s v$ is semi-convex for small s , since we already know that it is semi-concave for all $s > 0$. This follows from two Lemmas:

LEMMA *For each bounded subset $F \subset C^2(M, \mathbb{R})$ there exists a time $s_0 > 0$ such that, for each $s \in [0, s_0]$, the image $T_s(F)$ is a bounded subset of $C^2(M, \mathbb{R})$ and the following relation*

holds for all functions $f \in F$ and all $x \in M$

$$T_s f(x(s)) = f(x) + \int_0^s c + L(x(t), \dot{x}(t)) dt, \quad (1)$$

where $x(t)$ is the curve $\pi \circ \psi_t(x, df(x))$ ($\pi : T^*M \rightarrow M$ is the projection and ψ_t is the Hamiltonian flow).

PROOF. Let us consider a C^2 function f and the graph $\Gamma_f \subset T^*M$ of its differential. This graph is a C^1 Lagrangian manifold transversal to the fibers. It is known that, for $s \geq 0$ small enough, the Lagrangian manifold $\psi_s(\Gamma_f)$ is the graph of a C^2 function, and that this C^2 function is $T_s f$. Then, we have (1). The maximum time s_0 such that these properties hold is uniform for families of functions which are bounded in C^2 norm (for then the associated graphs are contained in a given compact set, and are uniformly transversal to the verticals). In addition, one can choose s_0 in such a way that the set $\{T_s f, s \in [0, s_0], f \in F\}$ is bounded in the C^2 topology, (which amounts to say that the manifolds $\psi_s(\Gamma_f)$ are uniformly transversal to the fibers). \square

LEMMA *Let v be a semi-convex function. Then there exists a bounded subset $F \subset C^2(M, \mathbb{R})$ and a time $s_0 > 0$ such that*

$$T_s v = \max_{f \in F} T_s f$$

for all $s \in [0, s_0]$, hence $T_s v$ is a semi-convex function for $s > 0$ small enough.

PROOF. Let us define, for each $x \in M$, the set $\partial^- v(x)$ of proximal sub-differentials of v at point x . This is the set of linear forms $df(x)$, where f is a C^2 function such that $u - f$ has a minimum at x . If v is semi-convex, then there exists a bounded subset $F \subset C^2(M, \mathbb{R})$ such that $v = \max_F f$ and such that for each x and each $p \in \partial^- v(x)$, there exists a function $f \in F$ such that $(f(x), df(x)) = (v(x), p)$ (see [1]). Let us fix from now on such a family F of functions, and consider the time s_0 associated to this family by the first Lemma.

Notice that

$$T_s v \geq \sup_{f \in F} T_s f$$

for all s , because for each $f \in F$ we have $f \leq v$ hence $T_s f \leq T_s v$. In order to prove that the equality holds for $s \in [0, s_0]$, let us fix a point $x \in M$. There exists a point y such that

$$T_s v(x) = u(y) + A_s(y, x).$$

Now let $(x(t), p(t)) : [0, s] \rightarrow T^*M$ be a Hamiltonian trajectory which is optimal for $A_s(y, x)$. We mean that $x(0) = y$, $x(s) = x$, and

$$A_s(y, x) = \int_0^s c + L(x(t), \dot{x}(t)) dt.$$

It is known (see [5, 1]) that $-p(0)$ is then a proximal super-differential of the function $z \mapsto A_s(z, x)$ at point y . Since the function $z \mapsto u(z) + A_s(z, x)$ is minimal at y , the linear form $p(0)$ is a sub-differential of the function u at point y . Let us consider a function $f \in F$ such that $(f(y), df(y)) = (u(y), p(0))$. Then we have $(x(t), p(t)) = \psi_t(y, df(y))$ and, by the first Lemma,

$$T_s f(x) = T_s f(x(s)) = f(y) + \int_0^s c + L(x(t), \dot{x}(t)) dt = u(y) + A_s(y, x) = T_s u(x).$$

We have proved that, for each point $x \in M$, there exists a function $f \in F$ such that $T_s f(x) = T_s u(x)$. This ends the proof. \square

This proof also gives:

LAST REMARK *If u is a critical sub-solution of the Hamilton-Jacobi equation, then there exists a $C^{1,1}$ critical sub-solution v whose restriction to the Aubry set is equal to u .*

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