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# Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds

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Let  $M$  be a compact manifold without boundary. A function  $H(x, p) : T^*M \rightarrow \mathbb{R}$  is called a Tonelli Hamiltonian if it is  $C^2$  and if, for each  $x \in M$ , the function  $p \mapsto H(x, p)$  is convex with positive definite Hessian and superlinear on the fibre  $T_x^*M$ . We consider the Hamilton-Jacobi equation

$$H(x, du_x) = c, \tag{HJ}$$

with a special emphasis on sub-solutions. A function  $u : M \rightarrow \mathbb{R}$  is called a sub-solution of (HJ) if it is Lipschitz and satisfies the inequality  $H(x, du_x) \leq c$  at all its points of differentiability. Note that this definition is equivalent to the notion of viscosity sub-solutions, see [3]. The goal of the present paper is to present a short and self-contained proof of:

**THEOREM** *Let  $H$  be a Tonelli Hamiltonian. If the Hamilton-Jacobi equation (HJ) has a sub-solution, then it has a  $C^{1,1}$  sub-solution. Moreover, the set of  $C^{1,1}$  sub-solutions is dense for the uniform topology in the set of sub-solutions.*

Fathi and Siconolfi recently proved the existence of a  $C^1$  sub-solution. Our result is optimal in the sense that examples are known where a  $C^{1,1}$  sub-solution exist, but no  $C^2$  solution. There exists a real number  $\alpha(H)$ , called the Mañé critical value in the literature, such that that the equation (HJ) has sub-solutions if and only if  $c \geq \alpha(H)$ . It is not hard to prove the existence of smooth sub-solutions for  $c > \alpha(H)$  by regularization, see [2]. As a consequence, our Theorem is relevant for the sub-solutions of the critical equation  $H(x, du_x) = \alpha(H)$ , which are called the critical sub-solutions of (HJ). The study of the critical Hamilton-Jacobi equation  $H(x, du_x) = \alpha(H)$  is the core of Fathi's weak KAM theory.

A sub-solution  $u$  is called strict on the open set  $U \subset M$  if there exists a smooth non-negative function  $V : M \rightarrow \mathbb{R}$  which is positive on  $U$  and such that  $u$  is also a sub-solution of the equation  $H(x, du_x) + V(x) = c$ . By applying the Theorem to the Hamiltonian  $H + V$ , we obtain:

**ADDENDUM** *If there exists a sub-solution of (HJ) which is strict on the open set  $U$ , then there exists a  $C^{1,1}$  sub-solution which is strict on  $U$ .*

We then obtain the important:

COROLLARY *There exists a critical  $C^{1,1}$  sub-solution which is strict outside of the Aubry set.*

We refer to the literature, for example [3] and [4], for the definition of the Aubry set. The Corollary holds if there exist sub-solutions which are strict outside of the Aubry set. This fact is proved in [4]. In the paper of Fathi and Siconolfi, finding a strict sub-solution is a key technical step in order to build  $C^1$  sub-solutions. In our approach, the problems of finding strict and regular solutions are independent. We do not improve on [4] for the existence of strict sub-solutions, which is an important problem in itself.

Let us now proceed to the proof. It is necessary to start with more definitions. We define the Lagrangian  $L : TM \rightarrow \mathbb{R}$  associated to  $H$  by the relation

$$L(x, v) = \max_{p \in T_x^*M} p(v) - H(x, p).$$

Then we define, for each  $t \geq 0$ , the action functionals  $A_t : M^2 \rightarrow \mathbb{R}$  by

$$A_t(x, y) := \min_{\gamma} \int_0^t c + L(\gamma(s), \dot{\gamma}(s)) ds$$

where the minimum is taken on the set of curves  $\gamma \in C^2([0, t], M)$  which satisfy  $\gamma(0) = x$  and  $\gamma(t) = y$ . Following Fathi, we define the Lax-Oleinik semi-groups  $T_t$  and  $\check{T}_t$  on  $C^0(M, \mathbb{R})$  by

$$T_t u(x) = \min_{y \in M} u(y) + A_t(y, x) \quad \text{and} \quad \check{T}_t u(x) = \max_{y \in M} u(y) - A_t(x, y).$$

Fathi proves in his book that a function  $u$  is a sub-solution of (HJ) if and only if the functions  $t \mapsto T_t u(x)$  and  $t \mapsto -\check{T}_t u(x)$  are non-decreasing. As a consequence, both the operators  $T_t$  and  $\check{T}_t$ ,  $t \geq 0$ , preserve the set of sub-solutions. Another important property of these semigroups is that, for all  $t > 0$  and all continuous function  $u$ , the function  $T_t u$  is semi-concave and the function  $\check{T}_t u$  semi-convex.

We say that a function  $u$  is semi-concave if it can be written

$$u(x) = \min_{f \in F} f(x)$$

with a bounded family  $F \subset C^2(M, \mathbb{R})$  of functions. Similarly, a function is called semi-convex if it can be written  $u(x) = \max_{f \in F} f(x)$ . See for example [1] for more on semi-concave functions. It is known that a function is  $C^{1,1}$  if and only if it is both semi-concave and semi-convex, see [3] or [1].

If  $u$  is a sub-solution of (HJ), then for each  $s \geq 0$  and  $t \geq 0$ , the function  $T_s \check{T}_t u$  is a sub-solution. Since  $\check{T}_t u$  is semi-convex, the Theorem follows from the following result, which may have other applications.

PROPOSITION *Let  $H$  be a Tonelli Hamiltonian. For each semi-convex function  $v$ , the function  $T_s v$  is  $C^{1,1}$  for each sufficiently small  $s > 0$ .*

In order to prove this proposition, it is enough to prove that the function  $T_s v$  is semi-convex for small  $s$ , since we already know that it is semi-concave for  $s > 0$ . Let us write  $v(x) = \max_{f \in F} f(x)$ , where  $F \subset C^2(M, \mathbb{R})$  is bounded. It is plain that

$$T_s v = \max_{f \in F} T_s f.$$

Now for each  $f \in F$ , let us call  $\Gamma_f$  the graph of the differential of  $f$  in  $T^*M$ , which is a  $C^1$  Lagrangian manifold transversal to the verticals. It is known that, for each  $f$ , there exists a time  $s_0 > 0$  such that, for  $s \in [0, s_0]$ , the image  $\psi_s(\Gamma_f)$  of  $\Gamma_f$  under the Hamiltonian flow  $\psi_s$  is

the graph of a  $C^2$  function  $f_s$ . In addition, we have  $f_s = T_s f$ . Since the Graphs  $\Gamma_f$ ,  $f \in F$  are contained in a common compact set and are uniformly transversal to the verticals, the time  $s_0$  can be taken uniformly for all these graphs, and in addition the family  $F_s$  of the  $C^2$  functions  $T_s f$ ,  $f \in F$  is bounded. It follows that the function

$$v_s(x) = \max_{f_s \in F_s} f_s(x) = \max_{f \in F} T_s f(x) = T_s v(x)$$

is semi-convex for all  $s \leq s_0$ . This ends the proof of the Proposition, and of the Theorem.

LAST REMARK *If  $u$  is a viscosity sub-solution of the critical Hamilton-Jacobi equation, then there exists a  $C^{1,1}$  critical sub-solution  $v$  whose restriction to the Aubry set is equal to  $u$ .*

I wish to thank Cedric Villani, not only because he is sick at the moment I am writing these lines, but also because the questions he asked me on the geometry of optimal transportation led me to the Proposition which is the key of the proof.

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