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## Weak KAM Pairs and Monge-Kantorovich Duality

Patrick Bernard and Boris Buffoni

### Abstract.

The dynamics of globally minimizing orbits of Lagrangian systems can be studied using the Barrier function, as Mather first did, or using the pairs of weak KAM solutions introduced by Fathi. The central observation of the present paper is that Fathi weak KAM pairs are precisely the admissible pairs for the Kantorovich problem dual to the Monge transportation problem with the Barrier function as cost. We exploit this observation to recover several relations between the Barrier functions and the set of weak KAM pairs in an axiomatic and elementary way.

### §1. Introduction

Let  $M$  be a compact connected manifold and consider a  $C^2$  Lagrangian function

$$L : TM \times \mathbb{R} \rightarrow \mathbb{R}$$

that satisfies the standard hypotheses of the calculus of variations,

$$(L1) \quad L(x, v, t + 1) = L(x, v, t) \quad \text{on } TM \times \mathbb{R},$$

$$(L2) \quad \partial_{vv}^2 L(x, v, t) > 0 \quad \text{on } TM \times \mathbb{R},$$

$$(L3) \quad \lim_{\|v\| \rightarrow \infty} L(x, v, t) / \|v\| = \infty \quad \text{on } M \times \mathbb{R}.$$

It is standard that, under these assumptions, there exists a well-defined time-periodic continuous vectorfield  $E(x, v, t)$  on  $TM$  such that the integral curves of  $E$  satisfy the Euler-Lagrange equations associate to  $L$ . We assume in addition that this vectorfield generates a complete flow, and denote by  $\varphi$  the time-one flow, which is a diffeomorphism of  $TM$ .

In this paper we show that the theory developed by Mather [10], Mañé [13] and Fathi [9] amounts for a large part to the analysis of the function  $A : M \times M \rightarrow \mathbb{R}$  defined by the expression

$$A(x, y) = \min_{\gamma} \int_0^1 L(\gamma(t), \dot{\gamma}(t), t) dt,$$

where the minimum is taken on the set of  $C^2$  curves  $\gamma : [0, 1] \rightarrow M$  which satisfy  $\gamma(0) = x$  and  $\gamma(1) = y$ .

To emphasize this point of view, we develop an abstract theory based solely on an arbitrary continuous function  $A : M \times M \rightarrow \mathbb{R}$ , where  $M$  is a connected compact metric space. We then define  $A_1 = A$  and

$$A_n(x, y) = \min_{z_1, \dots, z_{n-1} \in M} A(x, z_1) + A(z_1, z_2) + \dots + A(z_{n-1}, y)$$

for all integers  $n \geq 2$ . It turns out that the family  $(A_n)$  is equicontinuous and our only hypothesis on  $A$  is that the family  $(A_n)$  is uniformly bounded (this can be achieved by adding some constant to  $A$ ). It then follows that the expression

$$c(x, y) = \liminf_{n \rightarrow \infty} A_n(x, y)$$

defines a continuous function  $c : M \times M \rightarrow \mathbb{R}$ .

We call  $(\phi_0, \phi_1)$  an *admissible Kantorovich pair* for  $c$  if

$$\forall y \in M \quad \phi_1(y) = \min_{x \in M} \phi_0(x) + c(x, y)$$

and

$$\forall x \in M \quad \phi_0(x) = \max_{y \in M} \phi_1(y) - c(x, y).$$

The first main result (Theorem 12) states that  $(\phi_0, \phi_1)$  is an admissible Kantorovich pair for  $c$  if and only if

- $\phi_0(x) = \max_{y \in M} \phi_0(y) - A(x, y)$  for all  $x \in M$ ,
- $\phi_1(x) = \min_{y \in M} \phi_1(y) + A(y, x)$  for all  $x \in M$ ,
- and  $\phi_0(x) = \phi_1(x)$  whenever  $c(x, x) = 0$ .

The second main result (Theorem 13) concerns the minimization problem

$$\min_{\eta} \int_{M \times M} A(x, y) d\eta(x, y),$$

where the minimum is taken on the set of Borel probability measures  $\eta$  on  $M \times M$  with equal marginal measures, that is,  $\pi_{0\#}(\eta) = \pi_{1\#}(\eta)$  with  $\pi_0$  and  $\pi_1$  denoting the canonical projections on  $M$ . Among all

admissible measures, the minimizing ones are shown to be exactly those supported on the set

$$D = \{(x, y) \in M \times M \mid A(x, y) + c(y, x) = 0\}.$$

This is also restated in the following way in Theorem 15. Let  $X = M^{\mathbb{Z}}$  be endowed with the product topology and denote by  $\mathcal{M}_T(X)$  the set of Borelian probability measures on  $X$  which are invariant by translation. Consider the minimization problem

$$\min_{\nu \in \mathcal{M}_T(X)} \int_X A(x_0, x_1) d\nu(x),$$

where a generic  $x \in X$  is written  $x = (\dots, x_{-1}, x_0, x_1, \dots)$ . Then we show with the help of the Ergodic Decomposition Theorem that  $\nu$  in  $\mathcal{M}_T(X)$  is minimizing exactly when the push-forward of  $\nu$  by the projection  $x \rightarrow (x_0, x_1)$  is concentrated on  $D$ .

The paper ends with the interpretation of these abstract theorems in the setting of the Aubry-Mather theory, recovering in this way some key results of [13, 10, 9].

## §2. Monge-Kantorovich theory

We present some standard facts of Monge-Kantorovich theory, first in the general case, and then when the cost satisfies some given assumptions.

### 2.1. Generalities

We recall the basics of Monge-Kantorovich duality. The proofs are available in many texts on the subjects, for example [1, 14, 15]. We assume that  $M$  and  $N$  are compact metric spaces, and that  $c(x, y)$  is a continuous cost function on  $M \times N$ . Given Borel probability measures  $\mu_0$  on  $M$  and  $\mu_1$  on  $N$ , a transport plan between  $\mu_0$  and  $\mu_1$  is a measure on  $M \times N$  which satisfies

$$\pi_{0\#}(\eta) = \mu_0 \text{ and } \pi_{1\#}(\eta) = \mu_1,$$

where  $\pi_0 : M \times N \rightarrow M$  is the projection on the first factor, and  $\pi_1 : M \times N \rightarrow N$  is the projection on the second factor. We denote by  $\mathcal{K}(\mu_0, \mu_1)$ , after Kantorovich, the set of transport plans. Kantorovich proved the existence of a minimum in the expression

$$(1) \quad C(\mu_0, \mu_1) = \min_{\eta \in \mathcal{K}(\mu_0, \mu_1)} \int_{M \times N} c d\eta$$

for each pair  $(\mu_0, \mu_1)$  of probability measures. The plans which realize this minimum are called optimal transfer plans. Let  $\phi_0$  be a real function on  $M$  and  $\phi_1$  a real function on  $N$ . The pair  $(\phi_0, \phi_1)$  is called an admissible Kantorovich pair if it satisfies the relations

$$\phi_1(y) = \min_{x \in M} \phi_0(x) + c(x, y) \text{ and } \phi_0(x) = \max_{y \in N} \phi_1(y) - c(x, y)$$

for all point  $x \in M$  and  $y \in N$ . Another discovery of Kantorovich is that

$$(2) \quad C(\mu_0, \mu_1) = \max_{\phi_0, \phi_1} \left( \int_N \phi_1 d\mu_1 - \int_M \phi_0 d\mu_0 \right)$$

where the maximum is taken on the non-empty set of admissible Kantorovich pairs  $(\phi_0, \phi_1)$ . This maximization problem is called the dual Kantorovich problem, the admissible pairs which reach this maximum are called optimal Kantorovich pairs. The direct problem (1) and dual problem (2) are related as follows.

**Proposition 1.** *If  $\eta$  is an optimal transfer plan, and if  $(\phi_0, \phi_1)$  is a Kantorovich optimal pair, then the support of  $\eta$  is contained in the set*

$$\{(x, y) \in M \times N \text{ such that } \phi_1(y) - \phi_0(x) = c(x, y)\} \subset M \times N.$$

Let us remark that the knowledge of the set of Kantorovich admissible pairs is equivalent to the knowledge of the cost function  $c$ .

**Lemma 2.** *We have*

$$c(x, y) = \max_{(\phi_0, \phi_1)} \phi_1(y) - \phi_0(x)$$

where the maximum is taken on the set of Kantorovich admissible pairs.

*Proof.* This Lemma is elementary and can be proved by easy manipulation of inequalities, see [4]. However, we present a short proof based on the non-elementary Monge-Kantorovich duality. Let us fix points  $x \in M$  and  $y \in N$ , and let  $\mu_0$  be the Dirac measure at  $x$  and  $\mu_1$  be the Dirac measure at  $y$ . There exists one and only one transport plan between  $\mu_0$  and  $\mu_1$ , it is the Dirac measure at  $(x, y)$ . As a consequence, we have  $c(x, y) = C(\mu_0, \mu_1)$ . Hence the equality above is precisely the conclusion of Kantorovich duality for the transportation problem between  $\mu_0$  and  $\mu_1$ .  $\square$

**Proposition 3.** *Let  $(\phi_0, \phi_1)$  be an admissible pair, and let  $\mu_0$  be a probability measure on  $M$ . Then there exists a probability measure  $\mu_1$  on*

$N$  such that the pair  $(\phi_0, \phi_1)$  is optimal for the transportation problem of the measure  $\mu_0$  onto the measure  $\mu_1$

*Proof.* If  $\mu_0$  is the Dirac at  $x$ , then take a point  $y$  such that  $\phi_1(y) = \phi_0(x) + c(x, y)$ , and observe that the conclusion obviously holds if  $\mu_1$  is the Dirac at  $y$ . The set of measures  $\mu_0$  for which the conclusion holds (given  $\phi_0, \phi_1$ ) is clearly convex and closed (with respect to the weak topology), it contains the Dirac measures, hence it is the whole set of probability measures.  $\square$

## 2.2. Distance-like costs

Kantorovich stated his duality theorem first in the case where  $M = N$  and the cost is a distance. Then, the dual problem takes a simpler form that we now describe. In fact, it is not necessary to assume that the cost is a distance. It is sufficient to assume that, for all  $x, y$  and  $z$  in  $M$ , we have

$$(C1) \quad c(x, z) \leq c(x, y) + c(y, z),$$

$$(C2) \quad c(x, x) = 0.$$

A function  $\phi : M \rightarrow \mathbb{R}$  is called  $c$ -Lipschitz if it satisfies the inequality

$$\phi(y) - \phi(x) \leq c(x, y)$$

for all  $x$  and  $y$  in  $M$ . Note that, in the above and in what follows, we assume that  $M = N$  is a compact and connected metric space, and that  $c : M \times M \rightarrow \mathbb{R}$  is a continuous cost function.

**Theorem 4.** *Assume that the cost  $c \in C(M^2, \mathbb{R})$  satisfies the assumptions (C1 – 2). Then for each pair  $\mu_0, \mu_1$  of probability measures on  $M$ , we have*

$$C(\mu_0, \mu_1) = \max_{\phi} \int_M \phi d(\mu_1 - \mu_0)$$

where the maximum is taken on the set of  $c$ -Lipschitz functions  $\phi$ .

This is a well-known direct rewriting of Kantorovich duality in view of the following description of admissible pairs.

**Lemma 5.** *If the cost satisfies (C1 – 2), then the Kantorovich admissible pairs are precisely the pairs of the form  $(\phi, \phi)$ , with  $\phi$   $c$ -Lipschitz.*

*Proof.* If  $\phi$  is a  $c$ -Lipschitz function, then  $(\phi, \phi)$  is an admissible pair. Indeed, let us prove for example that  $\phi(x) = \min_y \phi(y) + c(y, x)$ . On the one hand, we have  $\phi(x) \leq \phi(y) + c(y, x)$  because  $\phi$  is  $c$ -Lipschitz, hence  $\phi(x) \leq \min_y \phi(y) + c(y, x)$ . On the other hand,  $\phi(x) = \phi(x) + c(x, x) \geq \min_y \phi(y) + c(y, x)$ . One can prove similarly that  $\phi(x) = \max_y \phi(y) - c(x, y)$ . It follows that  $(\phi, \phi)$  is an admissible pair. Conversely, if  $(\phi_0, \phi_1)$  is an admissible pair, then  $\phi_0 = \phi_1$  is a  $c$ -Lipschitz function. This is a special case of Lemma 6 below.  $\square$

Let us now study costs which satisfy (C1) but not necessarily (C2). It is then useful to define the set

$$\mathcal{A} := \{x \in M, c(x, x) = 0\} \subset M.$$

Note that the restriction of the cost  $c$  to  $\mathcal{A} \times \mathcal{A}$  obviously satisfies (C1–2). In this more general case, we have:

**Lemma 6.** *Let  $c \in C(M^2, \mathbb{R})$  satisfy (C1). Let  $(\phi_0, \phi_1)$  be an admissible pair. Then the functions  $\phi_0$  and  $\phi_1$  are  $c$ -Lipschitz. In addition, we have  $\phi_0 \leq \phi_1$  with equality on  $\mathcal{A}$ .*

*Proof.* Let us first prove that the function  $\phi_1$  is  $c$ -Lipschitz. Given  $x \in M$ , there exists  $y$  such that  $\phi_1(x) = \phi_0(y) + c(y, x)$ , and then, for each  $z$ ,

$$\phi_1(x) = \phi_0(y) + c(y, x) \geq \phi_1(z) - c(y, z) + c(y, x) \geq \phi_1(z) - c(x, z).$$

One can prove similarly that  $\phi_0$  is  $c$ -Lipschitz.

We then have

$$\phi_0(x) = \max_y \phi_1(y) - c(x, y) \leq \max_y \phi_1(x) = \phi_1(x).$$

because  $\phi_1$  is  $c$ -Lipschitz. If  $x \in \mathcal{A}$ , we have, in addition,

$$\phi_0(x) = \max_y \phi_1(y) - c(x, y) \geq \phi_1(x) - c(x, x) = \phi_1(x).$$

$\square$

We now introduce another hypothesis which is certainly less natural than (C1) and (C2), but is useful for the applications we have in mind. We assume that

$$(C3) \quad \mathcal{A} \neq \emptyset \quad \text{and} \quad c(x, y) = \min_{a \in \mathcal{A}} c(x, a) + c(a, y)$$

for each  $x$  and  $y$  in  $M$ . Note that, under (C1), (C3) is implied by (C2). The hypothesis (C3) implies that each optimal transport can be factored through the set  $\mathcal{A}$ .

**Lemma 7.** *If the cost satisfies (C1) and (C3), then for each pair  $(\mu_0, \mu_1)$  of probability measures, there exists a probability measure  $\mu$  supported on  $\mathcal{A}$  and such that*

$$C(\mu_0, \mu_1) = C(\mu_0, \mu) + C(\mu, \mu_1)$$

*Proof.* First note that  $C(\mu_0, \mu_1) \leq C(\mu_0, \mu) + C(\mu, \mu_1)$  is true for all Borelian probability measures  $\mu$  on  $M$ . This can be seen as follows. Let  $\eta_0$  and  $\eta_1$  be optimal transport plans for  $(\mu_0, \mu)$  and  $(\mu, \mu_1)$  respectively. Disintegrate  $\eta_0$  with respect to  $\pi_1$  and  $\eta_1$  with respect to  $\pi_0$ :  $\eta_0 = \int_M \eta_{0z} d\mu(z)$  and  $\eta_1 = \int_M \eta_{1z} d\mu(z)$  (see e.g. Theorem 5.3.1 in [2] for the disintegration theorem; here  $\eta_{0z}$  and  $\eta_{1z}$  are seen as probability measures on  $M$ ). Following Section 5.3 in [2], define the probability measure  $\eta$  on  $M^2$  by

$$\eta(A \times B) = \int_M \eta_{0z}(A) \eta_{1z}(B) d\mu(z)$$

for all Borelian subsets  $A, B \subset M$ . Then  $\eta \in \mathcal{K}(\mu_0, \mu_1)$  and

$$\begin{aligned} \int_{M^2} c d\eta &= \int_{M^3} c(x, y) d\eta_{0z}(x) d\eta_{1z}(y) d\eta(z) \\ &\leq \int_{M^3} \{c(x, z) + c(z, y)\} d\eta_{0z}(x) d\eta_{1z}(y) d\eta(z) = \int_{M^2} c d\eta_0 + \int_{M^2} c d\eta_1. \end{aligned}$$

Let us now prove the reverse inequality when  $\mu_0$  and  $\mu_1$  are Dirac measures supported in  $x$  and  $y$ . In this case, one can take for  $\mu$  the Dirac measure supported at  $a$ , where  $a$  is any point such that  $c(x, y) = c(x, a) + c(a, y)$ . The general case is then deduced once again using the fact that, on  $M^2$ , the set of probability measures is the closed convex envelop of the set of Dirac measures, so that we can approximate any optimal transfer plan in  $\mathcal{K}(\mu_0, \mu_1)$  by Dirac measures.  $\square$

**Proposition 8.** *If the cost  $c \in C(M^2, \mathbb{R})$  satisfies (C1) and (C3), then for each admissible pair  $(\phi_0, \phi_1)$ , there exists a function  $\phi : \mathcal{A} \rightarrow \mathbb{R}$ , which is  $c$ -Lipschitz, and such that*

$$\phi_1(a) = \phi_0(a) = \phi(a)$$

for all  $a \in \mathcal{A}$ ,

$$(3) \quad \phi_1(x) = \min_{a \in \mathcal{A}} \phi(a) + c(a, x)$$

for all  $x \in M$  and

$$(4) \quad \phi_0(x) = \max_{a \in \mathcal{A}} \phi(a) - c(x, a).$$



Conversely, given any  $c$ -Lipschitz function  $\phi$  on  $\mathcal{A}$ , the functions  $\phi_0$  and  $\phi_1$  defined by (4) and (3) form an admissible pair. In other words, there is a bijection between the set of admissible pairs and the set of  $c$ -Lipschitz functions on  $\mathcal{A}$ .

*Proof.* The fact that  $\phi_0$  and  $\phi_1$  are  $c$ -Lipschitz and that, on  $\mathcal{A}$ ,  $\phi_0 = \phi_1 := \phi$  results from Lemma 6. Let us prove (4), the proof of (3) being similar:

$$\begin{aligned}\phi_0(x) &= \max_y \phi_1(y) - c(x, y) \stackrel{(C3)}{=} \max_{y \in M, a \in \mathcal{A}} \phi_1(y) - c(x, a) - c(a, y) \\ &= \max_{a \in \mathcal{A}} \phi_0(a) - c(x, a) = \max_{a \in \mathcal{A}} \phi(a) - c(x, a).\end{aligned}$$

Conversely, let  $\phi$  be a  $c$ -Lipschitz function on  $\mathcal{A}$ , and let  $\phi_0$  and  $\phi_1$  be defined by (4) and (3). The reader will easily check that  $\phi_0$  and  $\phi_1$  are  $c$ -Lipschitz, and that  $\phi_1 \leq \phi \leq \phi_0$  on  $\mathcal{A}$ . We now prove that  $\phi_0 \leq \phi_1$  (and then that there is equality on  $\mathcal{A}$ ):

$$\begin{aligned}\phi_0(x) - \phi_1(x) &= \max_{a, b \in \mathcal{A}} \phi(a) - c(x, a) - \phi(b) - c(b, x) \\ &\leq \max_{a, b \in \mathcal{A}} \phi(a) - \phi(b) - c(b, a) \leq 0\end{aligned}$$

because  $\phi$  is  $c$ -Lipschitz on  $\mathcal{A}$ . In order to check that the pair  $(\phi_0, \phi_1)$  is an admissible pair, we shall prove that

$$\phi_0(x) = \max_y \phi_1(y) - c(x, y)$$

and live the other half to the reader. For each  $x$  in  $M$ , we have

$$\phi_0(x) = \max_{a \in \mathcal{A}} \phi(a) - c(x, a) = \max_{a \in \mathcal{A}} \phi_1(a) - c(x, a) \leq \max_{y \in M} \phi_1(y) - c(x, y).$$

In order to obtain the other inequality, let us prove that

$$\phi_1(y) - \phi_0(x) \leq c(x, y)$$

for all  $x$  and  $y$  in  $M$ . Indeed, we have

$$\begin{aligned}\phi_1(y) - \phi_0(x) &= \min_{a, b \in \mathcal{A}} \phi(a) + c(a, y) - \phi(b) + c(x, b) \\ &\leq \min_{a \in \mathcal{A}} c(a, y) + c(x, a) = c(x, y)\end{aligned}$$

by (C3). □

Since  $\phi_1$  and  $\phi_2$  are  $c$ -Lipschitz (Lemma 6), equations (3) and (4) imply

$$\phi_1(x) = \min_{y \in M} \phi_1(y) + c(y, x) \quad \text{and} \quad \phi_0(x) = \max_{y \in M} \phi_0(y) - c(x, y)$$

for all  $x \in M$ .

### §3. Abstract Mather-Fathi Theory

In this section, we consider a continuous function  $A(x, y) : M \times M \rightarrow \mathbb{R}$ . Recall that  $M$  is a compact connected metric space. We shall build several functions out of  $A$ . First, we define the sequence of functions  $A_n(x, y)$  by setting  $A_1 = A$  and

$$\begin{aligned} A_n(x, y) &= \min_{z \in M} A(x, z) + A_{n-1}(z, y) \\ &= \min_{z_1, \dots, z_{n-1} \in M} A(x, z_1) + A(z_1, z_2) + \dots + A(z_{n-1}, y). \end{aligned}$$

**Lemma 9.** *The functions  $A_n$  are equicontinuous. In addition, there exists a real number  $l$  and a positive constant  $C$  such that*

$$|A_n(x, y) - ln| \leq C$$

for all  $n \in \mathbb{N}$  and all  $x$  and  $y$  in  $M$ .

*Proof.* The function  $A$  is continuous, hence uniformly continuous, hence there exists a modulus of continuity  $\delta : ]0, \infty) \rightarrow ]0, \infty)$  such that  $\lim_{\epsilon \rightarrow 0} \delta(\epsilon) = 0$  and such that

$$|A(x, y) - A(X, Y)| \leq \delta(d(x, X)) + \delta(d(y, Y))$$

for all  $x, y, X, Y$  in  $M$ . Let us prove by recurrence that  $\delta$  is a modulus of continuity for all the functions  $A_n$ , i. e. that

$$|A_n(x, y) - A_n(X, Y)| \leq \delta(d(x, X)) + \delta(d(y, Y)).$$

Assuming this inequality, we have

$$\begin{aligned} A_{n+1}(x, y) - A_{n+1}(X, Y) &= \min_z \max_Z A_n(x, z) + A(z, y) - A_n(X, Z) - A(Z, Y) \\ &\leq \max_Z A_n(x, Z) - A_n(X, Z) + A(Z, y) - A(Z, Y) \leq \delta(d(x, X)) + \delta(d(y, Y)). \end{aligned}$$

Since the same inequality holds for the difference  $A_{n+1}(X, Y) - A_{n+1}(x, y)$ , we have proved that  $\delta$  is a modulus of continuity for  $A_{n+1}$ . This ends the proof of equicontinuity of the functions  $A_n$ .

Let us define the sequences  $M_n := \max_{(x,y) \in M^2} A_n(x, y)$  and  $m_n := \min_{(x,y) \in M^2} A_n(x, y)$ . It is clear that the sequence  $M_n$  is subadditive, *i. e.* that  $M_{n+k} \leq M_n + M_k$  for all  $n$  and  $k$  in  $\mathbb{N}$ . In order to check this claim, we take  $x$  and  $y$  in  $M$  such that  $A_{n+k}(x, y) = M_{n+k}$ . Then there exists a point  $z$  in  $M$  such that

$$M_{n+k} = A_{n+k}(x, y) = A_n(x, z) + A_k(z, y) \leq M_n + M_k.$$

Similarly, the sequence  $m_n$  is super-additive, *i. e.*  $m_{n+k} \geq m_n + m_k$ . On the other hand, on view of the equicontinuity of  $A_n$ , there exists a constant  $C$  such that  $M_n - m_n \leq C$ . Applying a standard result on subadditive sequences, we obtain that  $M_n/n$  converges to its infimum  $M$ , and that  $m_n/n$  converges to its supremum  $m$ . Then for each  $x$  and  $y$ ,

$$nM - C \leq M_n - C \leq m_n \leq A_n(x, y) \leq M_n \leq m_n + C \leq nm + C$$

which implies that  $M = m$  and proves the Lemma.  $\square$

We make, on the function  $A$ , the hypothesis

$$(A1) \quad l = 0.$$

Note that this hypothesis implies that  $A(x, x) \geq 0$  for all  $x$ , and more generally that  $A_n(x, x) \geq 0$  for all  $x$ . Then, we can define a cost function  $c$  by the expression

$$(5) \quad c(x, y) = \liminf_{n \rightarrow \infty} A_n(x, y).$$

In view of Lemma 9, the function  $c$  takes finite values and is continuous. We have  $c(x, x) \geq 0$  and, by Lemma 11 below,  $c(x, y) + c(y, x) \geq c(x, x) \geq 0$  for all  $x$  and  $y$  in  $M$ .

**Lemma 10.** *For each  $n \in \mathbb{N}$ , we have*

$$c(x, y) = \min_{z \in M} c(x, z) + A_n(z, y) = \min_{z \in M} A_n(x, z) + c(z, y).$$

*Proof.* Let us fix  $n$ . Passing at the liminf ( $m \rightarrow \infty$ ) in the inequality

$$A_{m+n}(x, y) \leq A_m(x, z) + A_n(z, y),$$

we obtain

$$c(x, y) \leq c(x, z) + A_n(z, y).$$

For the opposite inequality, let us notice that, for each  $m$ , there exists a point  $z_m$  in  $M$  such that

$$A_{m+n}(x, y) = A_m(x, z_m) + A_n(z_m, y).$$

Let us consider an increasing sequence of integers  $m_k$  such that the subsequence  $z_{m_k}$  has a limit  $z$  and  $\lim_{k \rightarrow \infty} A_{m_k+n}(x, y) = c(x, y)$ . At the liminf, we get, taking advantage of the equicontinuity of the functions  $A_n$ ,

$$c(x, y) \geq c(x, z) + A_n(z, y).$$

This proves that

$$c(x, y) = \min_z c(x, z) + A_n(z, y).$$

The proof of the second equality of the statement is similar.  $\square$

**Lemma 11.** *The cost function  $c$  satisfies (C1) and (C3).*

*Proof.* The triangle inequality is easily deduced from Lemma 10. Let us now prove (C3). We first prove that, given  $x$  and  $y$  in  $M$ , there exists a point  $z$  in  $M$  such that  $c(x, y) = c(x, z) + c(z, y)$ . Indeed, for each  $n$  in  $\mathbb{N}$ , there exists a point  $z_n$  such that  $c(x, y) = c(x, z_n) + A_n(z_n, y)$ . Considering an increasing sequence of integers  $n_k$  such that the subsequence  $z_{n_k}$  has a limit  $z$ , we obtain at the liminf along this subsequence that  $c(x, y) \geq c(x, z) + c(z, y)$  which is then an equality.

By recurrence, there exists a sequence  $Z_n \in M$  such that, for each  $k \in \mathbb{N}$ , we have

$$c(x, y) = c(x, Z_1) + c(Z_1, Z_2) + \dots + c(Z_{k-1}, Z_k) + c(Z_k, y).$$

Note that  $\sum_{i=\ell}^m c(Z_i, Z_{i+1}) = c(Z_\ell, Z_{m+1})$  if  $0 \leq \ell < m \leq k$ , where  $Z_0 = x$  and  $Z_{k+1} = y$ .

Let  $Z$  be an accumulation point of the sequence  $Z_n$ . For each  $\epsilon > 0$ , we can suppose, by taking a subsequence in  $Z_n$ , that all the points  $Z_n$  belong to the ball of radius  $\epsilon$  centered at  $Z$ . We conclude that, for each  $k \in \mathbb{N}$ ,

$$c(x, y) \geq c(x, Z) + (k-1)c(Z, Z) + c(Z, y) - 2(k+1)\delta(\epsilon).$$

This is possible only if  $c(Z, Z) \leq 2\delta(\epsilon)$ , and since this should hold for all  $\epsilon$  we conclude that  $c(Z, Z) \leq 0$ , hence  $c(Z, Z) = 0$ . We have proved the existence of a point  $Z \in \mathcal{A}$  such that  $c(x, y) = c(x, Z) + c(Z, y)$ .  $\square$

Let us define, the two operators  $T^\pm$  on the space  $C(M, \mathbb{R})$  of continuous functions on  $M$  by the expressions

$$T^-u(x) = \min_{y \in M} u(y) + A(y, x)$$

and

$$T^+u(x) = \max_{y \in M} u(y) - A(x, y).$$

We have the following relation between the fixed points of these operators and the admissible pairs of the Kantorovich dual problem with cost  $c$ . Recall the definition  $\mathcal{A} := \{x \in M, c(x, x) = 0\} \subset M$ .

**Theorem 12.** *Let  $A$  be a function satisfying (A1), and let  $c$  be the cost defined by (5). The pair  $(\phi_0, \phi_1)$  of functions on  $M$  is a Kantorovich admissible pair (for  $c$ ) if and only if*

- the function  $\phi_0$  is a fixed point of  $T^+$ ,
- the function  $\phi_1$  is a fixed point of  $T^-$ ,
- $\phi_0 = \phi_1$  on  $\mathcal{A}$ .

Finally, for each fixed point  $\phi_1$  of  $T^-$ , there exists one and only one function  $\phi_0$  such that  $(\phi_0, \phi_1)$  is an admissible pair.

*Proof.* Let  $(\phi_0, \phi_1)$  be an admissible pair. Then we have the expression

$$\phi_1(y) = \min_{x \in M} \phi_0(x) + c(x, y).$$

We obtain that

$$\begin{aligned} T^- \phi_1(z) &= \min_{x, y \in M} \phi_0(x) + c(x, y) + A(y, z) \\ &= \min_{x \in M} \phi_0(x) + c(x, z) = \phi_1(z). \end{aligned}$$

We prove in the same way that the function  $\phi_0$  is a fixed point of  $T^+$ . Lemma 6 implies that  $\phi_0 = \phi_1$  on  $\mathcal{A}$ .

Conversely, let  $(\phi_0, \phi_1)$  satisfy the three conditions of the statement. We first observe that the functions  $\phi_0$  and  $\phi_1$  are  $c$ -Lipschitz. Indeed, we have, for each  $n$ ,

$$\phi_i(y) - \phi_i(x) \leq A_n(x, y).$$

When  $n = 1$ , this is a direct consequence of the fact that  $\phi_i$  is a fixed point of  $T^\pm$ , and the general case is proved by induction. We get

$$\phi_i(y) - \phi_i(x) \leq \liminf_{n \rightarrow \infty} A_n(x, y) = c(x, y).$$

Let us call  $\phi$  the common value of  $\phi_0$  and  $\phi_1$  on  $\mathcal{A}$ . The function  $\phi_1$  being a fixed point of  $T^-$ , for each  $n \in \mathbb{N}$ , there exists a point  $y_n$  in  $M$  such that  $\phi_1(x) = \phi_1(y_n) + A_n(y_n, x)$ . Indeed, we can find successively  $y_1, y_2, \dots$  such that

$$\begin{aligned} \phi_1(x) &= \phi_1(y_1) + A(y_1, x) = \phi_1(y_2) + A(y_2, y_1) + A(y_1, x) \\ &= \dots = \phi_1(y_n) + A(y_n, y_{n-1}) + \dots + A(y_1, x). \end{aligned}$$

By definition of  $A_n$ , we get  $\phi_1(x) \geq \phi_1(y_n) + A_n(y_n, x)$ . The reverse inequality has just been proved above.

Let  $n_k$  be a subsequence such that  $y_{n_k}$  has a limit  $y$ . At the limit, we obtain the inequality

$$\phi_1(x) \geq \phi_1(y) + c(y, x),$$

which is then an equality. We have proved that

$$\phi_1(x) = \min_{y \in M} \phi_1(y) + c(y, x).$$

But then, in view of (C3), we have

$$\begin{aligned} \phi_1(x) &= \min_{y \in M, a \in \mathcal{A}} \phi_1(y) + c(y, a) + c(a, x) \\ &= \min_{a \in \mathcal{A}} \phi_1(a) + c(a, x). \end{aligned}$$

One can prove in a similar way that

$$\phi_0(x) = \max_{a \in \mathcal{A}} \phi_0(a) - c(x, a) = \max_{a \in \mathcal{A}} \phi_1(a) - c(x, a).$$

We conclude that  $(\phi_0, \phi_1)$  is an admissible pair by Proposition 8. This also proves the uniqueness claim.

In order to prove the last part of the statement, let us consider a fixed point  $\phi_1$  of  $T^-$ . Let us define the function  $\phi_0$  by

$$\phi_0(x) = \max_{a \in \mathcal{A}} \phi_1(a) - c(x, a).$$

Since the function  $\phi_1$  is  $c$ -Lipschitz (as seen above), we have  $\phi_0 \leq \phi_1$ . On the other hand, it is clear that  $\phi_1 \leq \phi_0$  on  $\mathcal{A}$ . As a consequence, we have  $\phi_0 = \phi_1$  on  $\mathcal{A}$ . By Lemma 10, we have for all  $z \in M$  that

$$\begin{aligned} \max_{x \in M} \phi_0(x) - A(z, x) &= \max_{x \in M, a \in \mathcal{A}} \phi_1(a) - c(x, a) - A(z, x) \\ &= \max_{a \in \mathcal{A}} \phi_1(a) - c(z, a) = \phi_0(z). \end{aligned}$$

Hence the function  $\phi_0$  is a fixed point of  $T^+$  and, as a consequence, the pair  $(\phi_0, \phi_1)$  is an admissible pair.  $\square$

#### §4. Dynamics

Let us define the subset

$$D := \{(x, y) \in M \times M \text{ s. t. } A(x, y) + c(y, x) = 0\} \subset \mathcal{A} \times \mathcal{A}$$

(see Lemma 10). We shall explain in two different ways that the Borel probability measures  $\eta$  on  $M \times M$  which are supported on  $D$  and satisfy  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$  can be seen in a natural way as the analog of Mather minimizing measures in our setting.

##### 4.1. Construction via Kantorovich pairs

We first expose a construction based on Kantorovich pairs.

**Theorem 13.** *Under the assumption (A1), we have*

$$\min_{\eta} \int_{M \times M} A(x, y) d\eta(x, y) = 0,$$

where the minimum is taken on the set of Borel probability measures  $\eta$  on  $M \times M$  such that  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$ . The minimizing measures are those which are supported on  $D$ .

*Proof.* Let us first prove that there exists a measure  $\eta$  on  $M \times M$  which is supported on  $D$  and such that  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$ . By Lemma 10, for each  $x_0 \in \mathcal{A}$ , there exists a point  $x_1$  in  $\mathcal{A}$  such that  $(x_0, x_1) \in D$ . Hence there exists a sequence  $x_0, x_1, x_2, \dots, x_n, \dots$  of points of  $\mathcal{A}$  such that  $(x_n, x_{n+1}) \in D$  for each  $n$ . Let us now consider the sequence

$$\eta_n = \frac{\delta_{(x_0, x_1)} + \delta_{(x_1, x_2)} + \dots + \delta_{(x_{n-1}, x_n)}}{n}$$

of probability measures on  $\mathcal{A} \times \mathcal{A}$ . Every accumulation point (for the weak topology) of the sequence  $\eta_n$  satisfies the desired property. Since the set of probability measures on  $M \times M$  is compact for the weak topology, such accumulation points exist.

Consider a measure  $\eta$  on  $M \times M$  which is supported on  $D$  and such that  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$ . We have

$$\int A(x, y) d\eta(x, y) = \int -c(y, x) d\eta(y, x) \leq \int \phi(x) - \phi(y) d\eta(x, y) = 0,$$

where  $\phi$  is any  $c$ -Lipschitz function.

On the other hand, let  $\eta$  be a probability measure on  $M \times M$  such that  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$ . Consider a function  $\phi$  which is  $A$ -Lipschitz. Such

functions exist, for example, take a fixed point of  $T^+$ . We have

$$(6) \quad 0 = \int \phi(y) - \phi(x) d\eta(x, y) \leq \int A(x, y) d\eta(x, y).$$

We have proved that the minimum in the statement is indeed zero, and that the measures supported on  $D$  are minimizing. There remains to prove that every minimizing measure is supported on  $D$ .

It is clear that a measure  $\eta$  is minimizing if and only if, for each  $A$ -Lipschitz function  $\phi$ , there is equality in (6), which means that the measure  $\eta$  is supported on the set

$$D_1 = \{(x, y) \in M^2 \mid \phi(y) - \phi(x) = A(x, y) \text{ for all } A\text{-Lipschitz functions } \phi\}.$$

Let  $D_\infty$  be the set of pairs  $(x_0, x_1)$  such that there exists a sequence  $x_i$ ,  $i \in \mathbb{Z}$  satisfying  $(x_i, x_{i+1}) \in D_1$  for all  $i \in \mathbb{Z}$  (and of course with the given points  $x_0$  and  $x_1$ ).

We claim that  $D_\infty = D$ . In order to prove this claim, let  $\phi$  be  $A$ -Lipschitz. Observe that  $\phi$  is  $A_n$ -Lipschitz for all  $n \in \mathbb{N}$  and  $c$ -Lipschitz. If  $(x_0, x_1)$  is a point in  $D_\infty$ , then there exists a sequence  $x_i$ ,  $i \in \mathbb{Z}$  such that

$$\phi(x_j) - \phi(x_i) = A_{j-i}(x_i, x_j)$$

for each  $i < j$  in  $\mathbb{Z}$ . If  $\alpha$  is an accumulation point of the sequence  $x_i$  at  $-\infty$ , and  $\omega$  is an accumulation point at  $+\infty$ , then we get the equalities

$$\phi(\omega) - \phi(\alpha) = c(\alpha, \omega) \quad \text{and} \quad \phi(x_i) - \phi(\alpha) = c(\alpha, x_i)$$

for each  $i \in \mathbb{Z}$ . We can take in particular  $\phi = c(\alpha, \cdot)$  and we get that  $c(\alpha, \alpha) = 0$ , hence  $\alpha \in \mathcal{A}$ . Let  $(\phi_0, \phi_1)$  be a Kantorovich pair for  $c$ , so that both  $\phi_0$  and  $\phi_1$  are  $A$ -Lipschitz (see Theorem 12). We get  $\phi_1(\alpha) = \phi_0(\alpha)$  (because  $\alpha \in \mathcal{A}$ , see Theorem 12) hence  $\phi_1(x_i) = \phi_0(x_i)$ . Since this holds for all Kantorovich pairs, we get that  $x_i \in \mathcal{A}$  (see Lemma 2). In other words, we have proved that  $D_\infty \in \mathcal{A} \times \mathcal{A}$ . Now let  $(x_0, x_1)$  be a point of  $D_\infty$ . We have  $x_1 \in \mathcal{A}$ , and, since the function  $c(x_1, \cdot)$  is  $A$ -Lipschitz, we have the equality  $c(x_1, x_1) - c(x_1, x_0) = A(x_0, x_1)$ . Recalling that  $c(x_1, x_1) = 0$ , we get  $c(x_1, x_0) + A(x_0, x_1) = 0$ , hence  $(x_0, x_1) \in D$ . The proof of the Theorem then follows from the next Lemma.  $\square$

**Lemma 14.** *If  $\eta$  is a probability measure on  $M \times M$  which is supported on  $D_1$  and such that  $\pi_{0\sharp}(\eta) = \pi_{1\sharp}(\eta)$ , then  $\eta$  is concentrated on  $D_\infty$ .*



*Proof.* Let us set  $\mu = \pi_{0\#}(\eta) = \pi_{1\#}(\eta)$  and let

$$X_1 = \pi_0(D_1) \cap \pi_1(D_1) \subset M$$

be the set of points  $x_0 \in M$  such that a sequence  $x_{-1}, x_0, x_1$  exists, with  $(x_{-1}, x_0) \in D_1$  and  $(x_0, x_1) \in D_1$ . Clearly, we have  $\mu(\pi_0(D_1)) = \mu(\pi_1(D_1)) = 1$  hence  $\mu(X_1) = 1$ . Let

$$D_2 = D_1 \cap (X_1 \times X_1)$$

be the set of pairs  $(x_0, x_1) \in M^2$  such that there exist  $x_{-1}, x_0, x_1, x_2$  with  $(x_i, x_{i+1}) \in D_1$  for  $i = -1, 0, 1$ . Let

$$X_2 = \pi_0(D_2) \cap \pi_1(D_2) \subset M$$

be the set of points  $x_0 \in M$  such that a sequence  $x_{-2}, x_{-1}, x_0, x_1, x_2$  exists, with  $(x_i, x_{i+1}) \in D_1$  for all  $-2 \leq i \leq 1$ . Since  $\mu(X_1) = 1$ , we have  $\eta(D_2) = 1$ , hence  $\mu(X_2) = 1$ . By recurrence, we build a sequences  $D_n \subset M \times M$  and  $X_n \subset M$  such that

$$D_n = D_1 \cap (X_{n-1} \times X_{n-1})$$

and

$$X_n = \pi_0(D_n) \cap \pi_1(D_n) \subset M.$$

By recurrence, we see that  $\eta(D_n) = 1$  and that  $\mu(X_n) = 1$ . Now we have

$$D_\infty = \bigcap_{n \in \mathbb{Z}} D_n$$

hence  $\eta(D_\infty) = 1$ . □

#### 4.2. Ergodic Construction

It is worth explaining that the preceding construction could have been performed in a quite different way, which does not use our theory of Kantorovich pairs, but relies on Ergodic theory, as the first papers of Mather.

Consider  $X = M^{\mathbb{Z}}$  endowed with the product topology, so that  $X$  is a metrizable compact space. We shall denote by  $\mathcal{M}_T(X)$  the set of Borelian probability measures on  $X$  which are invariant by translation. More precisely, we denote by  $T : X \rightarrow X$  the translation map

$$T(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots) = (\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots)$$

with  $b_i = a_{i+1}$  for all  $i \in \mathbb{Z}$ , so that  $\mathcal{M}_T(X)$  is the set of probability measures  $\nu$  on  $X$  such that  $T_{\#}\nu = \nu$ .

**Theorem 15.** *We have*

$$\min_{\nu \in \mathcal{M}_T(X)} \int_X A(x_0, x_1) d\nu(x) = 0.$$

*The measure  $\nu$  is minimizing if and only if its marginal  $\eta = (\pi_0 \times \pi_1)_\# \nu$  is concentrated on  $D$ .*

Note that Theorem 15 is equivalent to Theorem 13 in view of the following:

**Lemma 16.** *Let  $\eta$  be a Borelian probability measure on  $M^2$  such that  $\pi_{0\#}(\eta) = \pi_{1\#}(\eta)$ . Then there exists a Borelian measure  $\nu$  on  $X$  that is  $T$ -invariant and such that  $\eta$  is its push-forward by the map  $X \ni x \rightarrow (x_0, x_1) \in M^2$ .*

*Proof.* This follows from the Hahn-Kolmogorov extension theorem (see e.g. Theorem 0.1.5 in [11], Lemma 10.2.4 in [6] and Theorem 12.1.2 in [6]). Let  $\Omega$  be the algebra of subsets  $G$  of  $X$  of the type  $G = \prod_{i \in \mathbb{Z}} G_i$  where  $G_i \neq M$  for at most a finite number of indices  $i$  (the number depending on  $G$ ) and every  $G_i$  is a Borelian subset of  $M$ . We first define the  $T$ -invariant probability measure  $\nu$  on  $\Omega$  and then apply the Hahn-Kolmogorov extension theorem, which provides an unique extension to the Borel  $\sigma$ -algebra (by uniqueness, the extension is  $T$ -invariant).

Let  $\eta = \int_M \eta_{x_1} d\mu(x_1)$  be the disintegration of  $\eta$  with respect to the projection  $M^2 \ni (x_0, x_1) \rightarrow x_1 \in M$ . In particular  $\mu = \pi_{1\#}(\eta)$  (see e.g. Theorem 5.3.1 in [2] for the disintegration theorem). Define for  $m < n$

$$\begin{aligned} & \nu(\dots \times M \times M \times G_m \times \dots \times G_n \times M \times M \times \dots) \\ &= \int_{G_m \times \dots \times G_n} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) d\mu(x_n). \end{aligned}$$

This is well defined because if  $G_{m-1} = M$  then

$$\begin{aligned} & \int_{G_{m-1} \times G_m \times \dots \times G_n} d\eta_{x_m}(x_{m-1}) d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) d\mu(x_n) \\ &= \int_{G_m \times \dots \times G_n} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) d\mu(x_n) \end{aligned}$$

and if  $G_{n+1} = M$  then

$$\begin{aligned}
&= \int_{G_m \times \dots \times G_n \times G_{n+1}} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) d\eta_{x_{n+1}}(x_n) d\mu(x_{n+1}) \\
&= \int_{G_n \times M} \left\{ \int_{G_m \times \dots \times G_{n-1}} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) \right\} d\eta_{x_{n+1}}(x_n) d\mu(x_{n+1}) \\
&= \int_{G_n \times M} \left\{ \int_{G_m \times \dots \times G_{n-1}} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) \right\} d\eta(x_n, x_{n+1}) \\
&= \int_{G_n} \left\{ \int_{G_m \times \dots \times G_{n-1}} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) \right\} d\mu(x_n) \\
&= \int_{G_m \times \dots \times G_n} d\eta_{x_{m+1}}(x_m) \dots d\eta_{x_n}(x_{n-1}) d\mu(x_n)
\end{aligned}$$

because  $\mu = \pi_{1\sharp}(\eta) = \pi_{0\sharp}(\eta)$ .

Clearly  $\nu(X) = 1$  and  $\nu$  is  $T$ -invariant on  $\Omega$ .  $\square$

Although we have proved the equivalence between Theorem 15 and Theorem 13 we shall, as announced, detail another proof of Theorem 15.

For  $x \in X$  and every Borelian subset  $B$ , we define

$$\tau_B(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \text{card}\{0 \leq j \leq n-1 \mid T^j(x) \in B\}$$

(when the notation is used, it is understood that the limit exists). A Borelian probability  $\nu$  on  $X$  is ergodic if and only if, for every Borelian subset  $B \subset X$ , there holds  $\tau_B(x) = \nu(B)$   $\nu$ -almost surely.

Following Section II.6 in the book by Mañé [11], there exists a Borelian set  $\Sigma \subset X$  such that  $\nu(\Sigma) = 1$  for each  $\nu \in \mathcal{M}_T(X)$ , and, for each  $x \in \Sigma$ , the measure

$$\nu_x := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$$

is well defined and ergodic, where the limit is understood in the sense of the weak topology, that is

$$\forall f \in C(X, \mathbb{R}) \quad \int_X f d\nu_x = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^n f(T^j(x)).$$

Moreover  $\nu_x \in \mathcal{M}_T(X)$  and  $x$  belongs to the support of  $\nu_x$  for all  $x \in \Sigma$ . In addition, still following [11], we have that the function  $x \mapsto \int f d\nu_x$

is  $\nu$ -integrable and  $T$ -invariant, and that

$$(7) \quad \int_X \left( \int_X f d\nu_x \right) d\nu = \int_X f d\nu.$$

holds for every  $f \in \mathcal{L}^1(X, \nu)$ . Note that the measure  $\nu_x$  is the conditional probability measure of  $\nu$  with respect to the  $\sigma$ -algebra of  $T$ -invariant sets.

Let us prove that  $\alpha \leq 0$ . We define the continuous function  $\Gamma : X \rightarrow \mathbb{R}$  by  $\Gamma(x) = A(x_0, x_1)$ . By standard convexity arguments, the following minimum is reached:

$$\alpha = \min_{\nu \in \mathcal{M}_T(X)} \int_X \Gamma(x) d\nu(x).$$

Fix  $x_0 \in M$ . For all  $\epsilon > 0$ , we can find  $n \geq 1$  and  $x_1, \dots, x_n \in M$  such that

$$x_n = x_0 \quad \text{and} \quad \frac{1}{n} \sum_{j=0}^{n-1} A(x_j, x_{j+1}) < \epsilon.$$

Let  $x = (\dots, x_0, \dots, x_n, \dots) \in X$  have periodic components with period  $n$  and define  $\nu \in \mathcal{M}_T(X)$  by

$$\nu = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$$

where  $\delta_{T^j(x)}$  is the Dirac measure at  $T^j(x)$ . Then  $\int_X \Gamma d\nu < \epsilon$ , which proves that  $\alpha \leq 0$  (because  $\epsilon$  can be chosen arbitrarily small).

Let  $\nu \in \mathcal{M}_T(X)$  be any optimal measure. The equality

$$\int_X \left( \int_X \Gamma d\nu_x \right) d\nu = \int_X \Gamma d\nu = \alpha$$

shows that  $\int_X \Gamma d\nu_x = \alpha$  for  $\nu$ -almost all  $x \in \Sigma$ . For such a  $x$ , we get

$$(8) \quad 0 \geq \alpha = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Gamma(T^j(x)) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} A(x_j, x_{j+1}).$$

Assume for a while that  $x_0 \notin \mathcal{A}$ . Then there exists a neighborhood  $U$  of  $x_0$  in  $M$ ,  $\delta > 0$  and  $N \geq 1$  such that

$$A_n(y_0, z_0) > \delta > 0 \quad \text{for all } y_0, z_0 \in U \text{ and } n \geq N.$$

Setting  $\tilde{U} = \{y \in X \mid y_0 \in U\}$ , we get

$$0 < \nu_x(\tilde{U}) \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \text{card}\{0 \leq j \leq n-1 \mid x_j \in U\}.$$

We denote by  $(x_{j_k} : k \geq 0)$  the sequence of components of  $x$  in  $U$  (of non-negative index). We obtain

$$0 < \nu_x(\tilde{U}) \leq \liminf_{m \rightarrow +\infty} \frac{mN}{j_{mN}}$$

and, considering  $(x_{j_{kN}} : k \geq 0)$ , we get the contradiction

$$\liminf_{m \rightarrow \infty} \frac{1}{j_{mN}} \sum_{j=0}^{j_{mN}-1} A(x_j, x_{j+1}) \geq \liminf_{m \rightarrow \infty} \frac{m\delta}{j_{mN}} \geq \nu_x(\tilde{U})\delta/N > 0$$

(compare with (8)). This contradiction shows that  $x_0 \in \mathcal{A}$  for  $\nu$ -almost all  $x$ , that is, the marginal  $\mu = \pi_{0\#}\nu$  is concentrated on  $\mathcal{A}$ .

Let us now check that  $\alpha \geq 0$ . For contradiction, suppose  $\alpha < 0$ . Then for  $x \in \Sigma$  as above such that  $\nu_x \in \mathcal{M}_T(X)$  and  $\Gamma(\nu_x) = \alpha$ , we get

$$\begin{aligned} 0 &> \alpha = \Gamma(\nu_x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} A(x_j, x_{j+1}) \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n+1} \left( \sum_{j=0}^{n-1} A(x_j, x_{j+1}) + A(x_n, x_0) \right) \\ &\geq \limsup_{n \rightarrow +\infty} \frac{1}{n+1} A_{n+1}(x_0, x_0). \end{aligned}$$

This contradicts  $l = 0$  (see hypothesis (A1)).

We have proved that  $\alpha = 0$ , and that every minimizing  $T$ -invariant measure  $\nu$  has its marginal  $\mu = \pi_{0\#}\nu$  concentrated on  $\mathcal{A}$ . Let us now prove that every minimizing measure  $\nu \in \mathcal{M}_T(X)$  is supported on  $\{x \in X \mid A(x_0, x_1) + c(x_1, x_0) = 0\}$ . Let  $x$  belong to the support of  $\nu$  and observe that (see Lemma 10)  $c(x_1, y_1) \leq c(x_1, y_0) + A(y_0, y_1)$  for all  $y_0, y_1 \in M$ . Therefore

$$0 = \int_X c(x_1, y_1) - c(x_1, y_0) d\nu(y) \leq \int_X A(y_0, y_1) d\nu(y) = \alpha = 0$$

and

$$\int_X A(y_0, y_1) - c(x_1, y_1) + c(x_1, y_0) d\nu(y) = 0$$

where the integrand is non negative. Hence  $c(x_1, y_1) = c(x_1, y_0) + A(y_0, y_1)$  for  $\nu$ -almost all  $y$ . Since  $x$  is in the support of  $\nu$ , we get  $c(x_1, x_1) = c(x_1, x_0) + A(x_0, x_1)$ . We have just seen that  $y_0 \in \mathcal{A}$  for  $\nu$ -almost all  $y$ . By the  $T$ -invariance of  $\nu$ , we also have  $y_1 \in \mathcal{A}$  for  $\nu$ -almost all  $y$ . Since  $x$  is in the support of  $\nu$ , we therefore obtain  $x_1 \in \mathcal{A}$  and  $0 = c(x_1, x_1) = c(x_1, x_0) + A(x_0, x_1)$ .

Finally let  $\nu \in \mathcal{M}_T(X)$  be concentrated on

$$\tilde{D} = \{y \in X \mid A(y_0, y_1) + c(y_1, y_0) = 0\}$$

and let us prove that  $\int_X \Gamma d\nu = \alpha$ . By (7) applied to the characteristic function of  $\tilde{D}$ , we get that  $\nu_x(\tilde{D}) = 1$  for  $\nu$ -almost all  $x \in \Sigma$ . By (7) applied to  $\Gamma$ , we see that it suffices to check that  $\int_X \Gamma d\nu_x = \alpha$  for all  $x \in \Sigma$  such that  $\nu_x$  is concentrated on  $\tilde{D}$ . This follows from (C1):

$$\begin{aligned} 0 = \alpha &\leq \int_X A(y_0, y_1) d\nu_x(y) = - \int_X c(y_1, y_0) d\nu_x(y) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} c(x_{j+1}, x_j) \leq - \liminf_{n \rightarrow \infty} \frac{1}{n} c(x_n, x_0) = 0. \end{aligned}$$

□

## §5. Aubry-Mather theory

We now briefly explain the relations between our discussions and the literature on Aubry-Mather theory, and especially [10], [13] and [9]. From now on, the space  $M$  is a compact connected manifold and we consider a  $C^2$  Lagrangian function  $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$  as in the Introduction. In this context, we define  $A : M \times M \rightarrow \mathbb{R}$  by

$$A(x, y) = \min_{\gamma} \int_0^1 L(\gamma(t), \dot{\gamma}(t), t) dt,$$

where the minimum is taken on the set of  $C^2$  curves  $\gamma : [0, 1] \rightarrow M$  which satisfy  $\gamma(0) = x$  and  $\gamma(1) = y$ .

The function  $c$  defined by (5) is one of the central objects of Mather's theory of globally minimizing orbits, see [10]. He called it the Peierls barrier. It contains most of the information concerning the globally minimizing orbits, as was explained by Mather, see also [3]. The set  $\mathcal{A}$  of points  $x \in M$  such that  $c(x, x) = 0$  is called the projected Aubry set. It is especially important because Mather proved the existence of

a vectorfield  $X(x)$  on  $\mathcal{A}$  whose graph is invariant under the Lagrangian flow  $\varphi$ . This invariant set is called the Aubry set. The analog of the Aubry set in our general theory is the set  $D$  defined in the beginning of section 4.

The operators  $T^\pm$  have been introduced by Albert Fathi in this context, see [7],[8] and [9]. He called Weak KAM solutions the fixed points of  $T^-$ , and we call backward weak KAM solutions the fixed points of  $T^+$ . He also noticed that, for each weak KAM solution  $\phi_1$ , there exists one and only one backward weak KAM solution  $\phi_0$  which is equal to  $\phi_1$  on the projected Aubry set. This is the main part of our Theorem 12. Albert Fathi also proved Lemma 2 in this context. Our novelty in these matters consists of pointing out and using the equivalence with Kantorovich admissible pairs, which allows, for example, a strikingly simple proof of the important result of Fathi called Lemma 2 in our paper. The representation of weak KAM solutions given in Proposition 8 was obtained by Contreras in [5].

The minimizing measures of Theorem 13 are the famous Mather measures, see [10]. To be more precise, we should say that there is a natural bijection between the set of minimizing measures in Theorem 13 and the set of Mather measures. This bijection is described in [4]. In order to give the reader a clue of this bijection, let us recall that the Mather measures are probability measures on the tangent bundle  $TM$ , and that the minimizing measures of Theorem 13 are probability measures on  $M \times M$ . Denoting by  $\varphi$  the time-one Lagrangian flow, and by  $\pi : TM \rightarrow M$  the standard projection, we have a well-defined mapping  $(\pi, \pi \circ \varphi)_\#$  from the set of probability measures on  $TM$  to the set of probability measures on  $M \times M$ . This mapping induces a bijection between the set of Mather measures on  $TM$  and the set of minimizing measures of Theorem 13.

The part of Theorem 13 stating that the minimizing measures are precisely the measures supported on  $D$  is the analogous in our setting of the theorem of Mañé stating that all invariant measures supported on the Aubry set are minimizing, see [12].

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