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COXETER ORBITS AND MODULAR REPRESENTATIONS

CÉDRIC BONNAFÉ AND RAPHAËL ROUQUIER

Abstract. We study the modular representations of finite groups of Lie type arising in the cohomology of certain quotients of Deligne-Lusztig varieties associated with Coxeter elements. These quotients are related to Gelfand-Graev representations and we present a conjecture on the Deligne-Lusztig restriction of Gelfand-Graev representations. We prove the conjecture for restriction to a Coxeter torus. We deduce a proof of Broué's conjecture on equivalences of derived categories arising from Deligne-Lusztig varieties, for a split group of type A_n and a Coxeter element. Our study is based on Lusztig's work in characteristic 0 [Lu2].

Introduction

In [Lu2], Lusztig proved that the Frobenius eigenspaces on the $\bar{\mathbf{Q}}_l$-cohomology spaces of the Deligne-Lusztig variety X associated to a Coxeter element are irreducible unipotent representations. We give here a partial modular analog of this result. Except in type A, we treat only part of the representations (those occurring in Harish-Chandra induced Gelfand-Graev representations). Note that we need to consider non-unipotent representations at the same time. We show Broué's conjecture on derived equivalences coming from Deligne-Lusztig varieties, in type A, and for Coxeter elements (Theorem 4.6). These are the first non-trivial examples with varieties of dimension ≥ 2.

Recall that Broué's abelian defect group conjecture [Br1] predicts that an ℓ-block of a finite group G is derived equivalent to the corresponding block of the normalizer H of a defect group D, when D is abelian. When the group under consideration is a finite group of Lie type and ℓ is not the defining characteristic (and not too small), then Broué further conjectures [Br1] that the complex of cohomology of a certain Deligne-Lusztig variety Y should provide a complex realising an equivalence. The variety Y has a natural action of $G \times C_G(D)^{\mathrm{opp}}$, which does not extend in general to an action of $G \times H^{\mathrm{opp}}$. Nevertheless, instead of an action of the relative Weyl group $H/C_G(D)$, it is expected [Br2] that there will be an action of its associated braid monoid, which will induce an action of the corresponding Hecke algebra in $\mathrm{End}_{D^b(\mathbf{Z}_\ell G)}(C)$, where $C = R\Gamma_c(Y, \mathbf{Z}_\ell)$ is the complex of proper support cohomology of Y. Finally, the Hecke algebra should be isomorphic to the group algebra of the relative Weyl group. When $H/C_G(D)$ is cyclic, the required action should be provided by the Frobenius endomorphism F.

In [Rou2], the case where Y is a curve was dealt with, and the key point was a (modular version of the) disjunction property of cohomology spaces for the G-action, which was a consequence of specific properties of curves.

Here, the key step is a disjunction property for the action of $T \rtimes F$. Our approach uses very little information on the modular representations of G and might apply to other situations. The crucial geometrical part is an explicit description of the quotient of the Deligne-Lusztig variety

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Y by $D(\mathbf{U}_0)^F$ (§3.2), the rational points of the derived subgroup of the unipotent radical of an F-stable Borel subgroup (Lusztig had shown that $\mathbf{U}_0^F\backslash Y/T$ is a product of $\mathbf{G}_m$'s). We deduce a disjunction property for the action of $T \rtimes F$ on the complex of cohomology of $D(\mathbf{U}_0)^F\backslash Y$ (Corollary 3.15).

Considering the complex of cohomology of $D(\mathbf{U}_0)^F\backslash Y$ amounts to tensoring the complex of Y by the sum of the Harish-Chandra induced Gelfand-Graev representations. For GL_n, such a sum is a progenerator and this gives the expected disjunction property for Y.

We conjecture (§2.3) that the Deligne-Lusztig restriction of a Gelfand-Graev representation is a shifted Gelfand-Graev representation. Such a result is known at the level of K_0 and in the case of Harish-Chandra induction. We deduce the truth of the conjecture for Coxeter tori, as a consequence of our geometric study (Theorem 3.10).

This is our second paper (after [BonRou1]) attempting to extend some of the fundamental results of Lusztig to the modular setting. We dedicate this paper to Professor Lusztig on his sixtieth birthday.

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1. Preliminaries

The results of this chapter are mostly classical.

1.1. **Definitions.**

1.1.1. Let R be a commutative ring. Given d a positive integer, we denote by $\mu_d(R)$ the subgroup of $R^\times$ of elements of order dividing d.

Given a set I and a group G acting on I, we denote by $[G\backslash I]$ a subset of I of representatives of orbits.

Let A be an R-algebra. We denote by A^{opp} the opposite algebra to A. Similarly, if G is a group, we denote by G^{opp} the opposite group to G and we put $G^\# = G - \{1\}$. Given M an R-module, we put $AM = A \otimes_R M$. We denote by A-Mod the category of A-modules and by A-mod the category of finitely generated A-modules. Given $\mathcal{C}$ an additive category, we denote by $K^b(\mathcal{C})$ the homotopy category of bounded complexes of objects of $\mathcal{C}$. When $\mathcal{C}$ is an abelian category, we denote by $D^b(\mathcal{C})$ its bounded derived category. We put $K^b(A) = K^b(A\text{-mod})$ and $D^b(A) = D^b(A\text{-mod})$ (when A-mod is an abelian category).

Given C and D two complexes of A-modules, we denote by $\mathrm{Hom}^\bullet_A(C, D)$ the total complex of the double complex $(\mathrm{Hom}_A(C^i, D^j))_{i,j}$ and by $R\,\mathrm{Hom}^\bullet_A(C, D)$ the corresponding derived version.

1.1.2. Let ℓ be a prime number. Let K be a finite extension of the field of ℓ-adic numbers, let $\mathcal{O}$ be the normal closure of the ring of ℓ-adic integers in K and let k be the residue field of $\mathcal{O}$. We assume K is big enough for the finite groups under consideration (*i.e.*, K contains the e-th roots of unity where e is the exponent of one of the finite groups considered).

Let H be a finite group. We denote by $\mathrm{Irr}(H)$ the set of irreducible characters of H with values in K and we put $\mathrm{Irr}(H)^\# = \mathrm{Irr}(H) - \{1\}$. Given $\chi \in \mathrm{Irr}(H)$, we put $e_\chi = \frac{\chi(1)}{|H|} \sum_{h\in H} \chi(h)h^{-1}$. If $\chi(1) = 1$ and $R = K$, $\mathcal{O}$ or k, we denote by R_χ the RH-module R on which H acts via χ.

1.1.3. Let p be a prime number distinct from ℓ. We denote by $\bar{\mathbf{F}}_p$ an algebraic closure of $\mathbf{F}_p$. Given q a power of p, we denote by $\mathbf{F}_q$ the subfield of $\bar{\mathbf{F}}_p$ with q elements. Given d a positive integer, we put $\mu_d = \mu_d(\bar{\mathbf{F}}_p)$.

1.1.4. Let X be an algebraic variety over $\bar{\mathbf{F}}_p$ acted on by a finite group H. Let R be a ring amongst K, $\mathcal{O}$ and k. We denote by $\tilde{R}\Gamma_c(X, R)$ the object of $K^b(RH\text{-Mod})$ representing the cohomology with compact support of X in R, as defined in [Ri1, Rou2]. It is a bounded complex of RH-modules which are direct summands of permutation modules (not finitely generated). If X is equipped with a Frobenius endomorphism, then that endomorphism induces an invertible operator of $\tilde{R}\Gamma_c(X, R)$. Note finally that, as a complex of RH-modules, $\tilde{R}\Gamma_c(X, R)$ is homotopy equivalent to a bounded complex of finitely generated modules which are direct summands of permutation modules.

We will denote by $R\Gamma_c(X, R)$ the corresponding "classical" object of $D^b(RH\text{-Mod})$. If X is equipped with a Frobenius endomorphism, then, $R\Gamma_c(X, R)$, viewed as a complex with an action of H and of the Frobenius endomorphism, is quasi-isomorphic to a bounded complex of modules which have finite rank over R.

1.2. **Algebraic groups.**

1.2.1. Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbf{F}}_p$, with an endomorphism F. We assume there is a positive integer f such that F^f is a Frobenius endomorphism of $\mathbf{G}$.

Let $\mathbf{B}_0$ be an F-stable Borel subgroup of $\mathbf{G}$, let $\mathbf{T}_0$ be an F-stable maximal torus of $\mathbf{B}_0$, let $\mathbf{U}_0$ denote the unipotent radical of $\mathbf{B}_0$, and let $W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ denote the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}_0$. Given $w \in W$, we denote by $\dot{w}$ a representative of w in $N_{\mathbf{G}}(\mathbf{T}_0)$. If moreover $w \in W^{F^m}$ for some positive integer m, then $\dot{w}$ is chosen in $N_{\mathbf{G}^{F^m}}(\mathbf{T}_0)$.

Let Φ denote the root system of $\mathbf{G}$ relative to $\mathbf{T}_0$, Φ^+ the set of positive roots of Φ corresponding to $\mathbf{B}_0$ and Δ the basis of Φ contained in Φ^+. We denote by $\phi : \Phi \to \Phi$ the bijection such that $F(\alpha)$ is a positive multiple of $\phi(\alpha)$. We denote by d the order of ϕ. Note that F^d is a Frobenius endomorphism of $\mathbf{G}$ defining a split structure over a finite field with q^d elements, where q is a positive real number.

If $\alpha \in \Phi$, we denote by s_α the reflection with respect to α, by $\alpha^\vee$ the associated coroot, by q_α° the power of p such that $F(\alpha) = q_\alpha^\circ \phi(\alpha)$ and by d_α the smallest natural number such that $F^{d_\alpha}(\alpha)$ is a multiple of α. In other words, d_α is the length of the orbit of α under the action of ϕ. We have $d = \mathrm{lcm}(\{d_\alpha\}_{\alpha\in\Phi})$. Let $q_\alpha = q_\alpha^\circ q_{\phi(\alpha)}^\circ \cdots q_{\phi^{d_\alpha-1}(\alpha)}^\circ$. We have $q_\alpha = q^{d_\alpha}$. Note that $d_{\phi(\alpha)} = d_\alpha$ and $q_{\phi(\alpha)} = q_\alpha$. Let $\mathbf{U}_\alpha$ denote the one-dimensional unipotent subgroup of $\mathbf{G}$ corresponding to α and let $x_\alpha : \bar{\mathbf{F}}_p \to \mathbf{U}_\alpha$ be an isomorphism of algebraic groups. We may, and we will, choose the family $(x_\alpha)_{\alpha\in\Phi}$ such that $x_\alpha(\xi^{q_\alpha^\circ}) = F(x_\alpha(\xi))$ for all $\alpha \in \Phi$ and $\xi \in \bar{\mathbf{F}}_p$. In particular, $F^{d_\alpha}(x_\alpha(\xi)) = x_\alpha(\xi^{q_\alpha})$ and we deduce an isomorphism $\mathbf{U}_\alpha^{F^{d_\alpha}} \xrightarrow{\sim} \mathbf{F}_{q_\alpha}$.

We put $G = \mathbf{G}^F$, $T_0 = \mathbf{T}_0^F$, and $U_0 = \mathbf{U}_0^F$.

1.2.2. Let $I \subset \Delta$. We denote by

- W_I the subgroup of W generated by $(s_\alpha)_{\alpha\in I}$,
- W^I the set of elements $w \in W$ which are of minimal length in $W_I w$,
- w_I the longest element of W_I,
- $\mathbf{P}_I$ the parabolic subgroup $\mathbf{B}_0 W_I \mathbf{B}_0$ of $\mathbf{G}$,
- $\mathbf{L}_I$ the unique Levi complement of $\mathbf{P}_I$ containing $\mathbf{T}_0$,
- $\mathbf{V}_I$ the unipotent radical of $\mathbf{P}_I$,
- $\mathbf{B}_I$ the Borel subgroup $\mathbf{B} \cap \mathbf{L}_I$ of $\mathbf{L}_I$,
- $\mathbf{U}_I$ the unipotent radical of $\mathbf{B}_I$.

In particular, $\mathbf{U} = \mathbf{U}_I \ltimes \mathbf{V}_I$. If I is ϕ-stable, then W_I, $\mathbf{P}_I$, $\mathbf{L}_I$, $\mathbf{V}_I$, $\mathbf{B}_I$, and $\mathbf{U}_I$ are F-stable. Note that $W_\Delta = W$, $\mathbf{P}_\Delta = \mathbf{L}_\Delta = \mathbf{G}$, $\mathbf{B}_\Delta = \mathbf{B}_0$, $\mathbf{U}_\Delta = \mathbf{U}_0$ and $\mathbf{V}_\Delta = 1$. On the other hand, $W_\emptyset = 1$, $\mathbf{P}_\emptyset = \mathbf{B}_0$, $\mathbf{L}_\emptyset = \mathbf{B}_\emptyset = \mathbf{T}_0$, $\mathbf{U}_\emptyset = 1$, and $\mathbf{V}_\emptyset = \mathbf{U}_0$.

We put $L_I = \mathbf{L}_I^F$, etc...

1.2.3. Let $D(\mathbf{U}_0)$ be the derived subgroup of $\mathbf{U}_0$. For any total order on Φ_+, the product map $\prod_{\alpha\in\Phi^+\setminus\Delta} \mathbf{U}_\alpha \to D(\mathbf{U}_0)$ is an isomorphism of varieties. It is not in general an isomorphism of algebraic groups. The canonical map $\prod_{\alpha\in\Delta} \mathbf{U}_\alpha \to \mathbf{U}_0/D(\mathbf{U}_0)$ is an isomorphism of algebraic groups commuting with F. We deduce an isomorphism from $(\mathbf{U}_0/D(\mathbf{U}_0))^F$ (which is canonically isomorphic to $\mathbf{U}_0^F/D(\mathbf{U}_0)^F$) to $\prod_{\alpha\in[\Delta/\phi]} \mathbf{U}_\alpha^{F^{d_\alpha}}$, which identifies with $\prod_{\alpha\in[\Delta/\phi]} \mathbf{F}_{q_\alpha}$.

1.3. **Deligne-Lusztig induction and restriction.**

1.3.1. *Harish-Chandra induction and restriction.* Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$, let $\mathbf{L}$ be an F-stable Levi complement of $\mathbf{P}$ and let $\mathbf{V}$ denote the unipotent radical of $\mathbf{P}$. We put $L = \mathbf{L}^F$, $P = \mathbf{P}^F$, and $V = \mathbf{V}^F$. The *Harish-Chandra restriction* is the functor

$$\begin{aligned} {}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathcal{O}G\text{-mod} &\to \mathcal{O}L\text{-mod} \\ M &\mapsto M^V. \end{aligned}$$

The *Harish-Chandra induction* is the functor

$$\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathcal{O}L\text{-mod} \to \mathcal{O}G\text{-mod}$$
$$M \mapsto \operatorname{Ind}_P^G \circ \operatorname{Res}_P^L M,$$

where Res_P^L is defined through the canonical surjective morphism $P \to L$. These functors are left and right adjoint to each other (note that $|V|$ is invertible in $\mathcal{O}$).

1.3.2. *Definition.* Let $\mathbf{P}$ be a parabolic subgroup of G, let $\mathbf{L}$ be a Levi complement of $\mathbf{P}$ and let $\mathbf{V}$ denote the unipotent radical of $\mathbf{P}$. We assume that $\mathbf{L}$ is F-stable. Let

$$Y_{\mathbf{V}} = Y_{\mathbf{V},\mathbf{G}} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V}\cdot F(\mathbf{V})\}$$
$$\text{and} \quad X_{\mathbf{V}} = X_{\mathbf{V},\mathbf{G}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}\cdot F(\mathbf{P})\}.$$

Let $\pi_{\mathbf{V}} : Y_{\mathbf{V}} \to X_{\mathbf{V}}$, $g\mathbf{V} \mapsto g\mathbf{P}$ be the canonical map. The group G acts on $Y_{\mathbf{V}}$ and $X_{\mathbf{V}}$ by left multiplication and L acts on $Y_{\mathbf{V}}$ by right multiplication. Moreover, $\pi_{\mathbf{V}}$ is G-equivariant and it is the quotient morphism by L. Note that $Y_{\mathbf{V}}$ and $X_{\mathbf{V}}$ are smooth.

Let us consider $R\Gamma_c(Y_{\mathbf{V}}, \mathcal{O})$, an element of $D^b((\mathcal{O}G)\otimes(\mathcal{O}L)^{\mathrm{opp}})$, which is perfect for $\mathcal{O}G$ and for $(\mathcal{O}L)^{\mathrm{opp}}$. We have associated left and right adjoint functors between the derived categories $D^b(\mathcal{O}L)$ and $D^b(\mathcal{O}G)$:

$$\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = R\Gamma_c(Y_{\mathbf{V}}, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}L} - : D^b(\mathcal{O}L) \to D^b(\mathcal{O}G)$$
$$\text{and} \quad {}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = R\operatorname{Hom}^{\bullet}_{\mathcal{O}G}(R\Gamma_c(Y_{\mathbf{V}}, \mathcal{O}), -) : D^b(\mathcal{O}G) \to D^b(\mathcal{O}L).$$

Tensoring by K, they induce adjoint morphisms between Grothendieck groups:

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathrm{K}_0(KL) \to \mathrm{K}_0(KG) \text{ and } {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathrm{K}_0(KG) \to \mathrm{K}_0(KL).$$

Note that when $\mathbf{P}$ is F-stable, then the Deligne-Lusztig functors are induced by the corresponding Harish-Chandra functors between module categories.

1.3.3. *Reductions.* We describe here some relations between Deligne-Lusztig varieties of groups of the same type.

Let $\tilde{\mathbf{G}}$ be a connected reductive algebraic group over $\bar{\mathbf{F}}_p$ endowed with an endomorphism F and assume a non-trivial power of F is a Frobenius endomorphism. Let $i : \mathbf{G} \to \tilde{\mathbf{G}}$ be a morphism of algebraic groups commuting with F and such that the kernel Z of i is a central subgroup of $\mathbf{G}$ and the image of i contains $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}]$.

Let $\tilde{\mathbf{P}}$ be a parabolic subgroup of $\tilde{\mathbf{G}}$ with an $\tilde{F}$-stable Levi complement $\tilde{\mathbf{L}}$. Let $\mathbf{P} = i^{-1}(\tilde{\mathbf{P}})$ and $\mathbf{L} = i^{-1}(\tilde{\mathbf{L}})$, a parabolic subgroup of $\mathbf{G}$ with an F-stable Levi complement. Let $\tilde{\mathbf{V}}$ (resp. $\mathbf{V}$) be the unipotent radical of $\tilde{\mathbf{P}}$ (resp. $\mathbf{P}$).

Let $N = \{(x,l) \in \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}} | xl^{-1} \in i(\mathbf{G}^F)\}$ and consider its action on $\tilde{\mathbf{G}}^F$ given by $(x,l)\cdot g = xgl^{-1}$. Then, N stabilizes $i(\mathbf{G}^F)$. The morphism i induces a canonical morphism $Y_{\mathbf{V},\mathbf{G}} \to Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}}$. Its image is isomorphic to $(Z^F \times (Z^F)^{\mathrm{opp}})\backslash Y_{\mathbf{V},\mathbf{G}} = Y_{\mathbf{V},\mathbf{G}}/Z^F = Z^F\backslash Y_{\mathbf{V},\mathbf{G}}$ and it is stable under the action of N.

Proposition 1.1. *The canonical map induces a radicial* $(\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}})$*-equivariant map*

$$\operatorname{Ind}_N^{\tilde{\mathbf{G}}^F\times(\tilde{\mathbf{L}}^F)^{\mathrm{opp}}}(Y_{\mathbf{V},\mathbf{G}}/Z^F) \to Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}},$$

i.e., *we have radicial maps*

$$\tilde{\mathbf{G}}^F \times_{i(\mathbf{G}^F)} Y_{\mathbf{V},\mathbf{G}} \leftarrow Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}} \rightarrow Y_{\mathbf{V},\mathbf{G}} \times_{i(\mathbf{L}^F)} \tilde{\mathbf{L}}^F$$

and we deduce a canonical isomorphism in $K^b(\mathcal{O}(\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}))$

$$\mathrm{Ind}_N^{\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}} \tilde{R}\Gamma_c(Y_{\mathbf{V},\mathbf{G}}, \mathcal{O})^{Z^F} \xrightarrow{\sim} \tilde{R}\Gamma_c(Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}}, \mathcal{O}).$$

Proof. Since $Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}}$ is smooth, it is enough to prove that the map is bijective on (closed) points to deduce that it is radicial (cf e.g. [Bor, AG.18.2]).

We can factor i as the composition of three maps: $\mathbf{G} \to \mathbf{G}/Z^0 \to \mathbf{G}/Z \to \tilde{\mathbf{G}}$. The finite group Z/Z^0 has an F-invariant filtration $1 = L^0 \subset L^1 \subset \cdots L^n = Z/Z^0$ where F acts trivially on L^i/L^{i+1} for $0 \leq i < n$ and the Lang map is an isomorphism on L^n/L^{n-1}. This allows us to reduce the proof of the Proposition to one of the following three cases:

(1) i is surjective and the Lang map on Z is surjective
(2) i is surjective and $Z^F = Z$
(3) i is injective.

The first and third cases are solved as in [Bon, Lemma 2.1.2]. Let us consider the second case (cf [Bon, Proof of Proposition 2.2.2]). Let $\tau : \tilde{\mathbf{G}}^F/i(\mathbf{G}^F) \xrightarrow{\sim} Z$ be the canonical isomorphism: given $g \in \tilde{\mathbf{G}}^F$, let $h \in \mathbf{G}$ with $i(h) = g$. Then, $\tau(g) = h^{-1}F(h)$.

We need to show that $Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}} = \coprod_{f \in \tilde{\mathbf{G}}^F/i(\mathbf{G}^F)} f \cdot i(Y_{\mathbf{V},\mathbf{G}})$. Let $y \in Y_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}}$. Let $g \in \mathbf{G}$ such that $y = i(g)$. There is $z \in Z$ such that $g^{-1}F(g) \in z\mathbf{V} \cdot F(\mathbf{V})$. Let now $f \in \tilde{\mathbf{G}}^F$ with $\tau(f) = z^{-1}$ and $g' \in \mathbf{G}$ with $f = i(g')$. We have $g'g \in Y_{\mathbf{V},\mathbf{G}}$, hence $fy \in i(Y_{\mathbf{V},\mathbf{G}})$ and we are done.

The assertion about $\tilde{R}\Gamma_c$ follows from the fact that this complex depends only on the étale site and a radicial morphism induces an equivalence of étale sites and from the fact that $\tilde{R}\Gamma_c$ of a quotient is isomorphic to the fixed points on $\tilde{R}\Gamma_c$ (cf [Ri1, Theorem 4.1] or [Rou2, Théorème 2.28]). □

Note that i induces an isomorphism $X_{\mathbf{V},\mathbf{G}} \xrightarrow{\sim} X_{\tilde{\mathbf{V}},\tilde{\mathbf{G}}}$ and the isomorphisms of Proposition 1.1 are compatible with that isomorphism.

1.3.4. *Local study.* We keep the notation of §1.3.2. Recall ([Rou2], [Ri1]) that $\tilde{R}\Gamma_c(Y_{\mathbf{V}}, \mathcal{O})$ is a bounded complex of $\mathcal{O}(G \times L^{\mathrm{opp}})$-modules that is homotopy equivalent to a bounded complex C' of finitely generated modules which are direct summands of permutation modules. We can furthermore assume that C' has no direct summand homotopy equivalent to 0. Let S be the Sylow ℓ-subgroup of $Z(L)$. Let b be the sum of the block idempotents of $\mathcal{O}G$ whose defect group contains (up to conjugacy) S.

Lemma 1.2. *If* $C_{\mathbf{G}}(S) = \mathbf{L}$, *then the canonical morphism* $b\mathcal{O}G \to \mathrm{End}_{K^b(\mathcal{O}L^{\mathrm{opp}})}(bC')$ *is a split injection of* $(b\mathcal{O}G \otimes (b\mathcal{O}G)^{\mathrm{opp}})$*-modules.*

Proof. We have a canonical isomorphism $\mathrm{End}^\bullet_{\mathcal{O}L^{\mathrm{opp}}}(bC') \xrightarrow{\sim} bC' \otimes_{\mathcal{O}L} \mathrm{Hom}^\bullet_{\mathcal{O}}(bC', \mathcal{O})$ and a canonical map $C' \otimes_{\mathcal{O}L} \mathrm{Hom}^\bullet_{\mathcal{O}}(bC', \mathcal{O}) \to b\mathcal{O}G$. We will show that the composition

$$b\mathcal{O}G \to \mathrm{End}^\bullet_{\mathcal{O}L^{\mathrm{opp}}}(bC') \to b\mathcal{O}G$$

is an isomorphism, which will suffice, since $\mathrm{End}_{K^b(\mathcal{O}L^{\mathrm{opp}})}(bC') = H^0(\mathrm{End}^\bullet_{\mathcal{O}L^{\mathrm{opp}}}(bC'))$.

Let e be a block idempotent of $b\mathcal{O}G$. We will show (and this will suffice) that the composition above is an isomorphism after multiplication by e and after tensoring by k

$$ekG \xrightarrow{\alpha} \mathrm{End}^{\bullet}_{kL^{\mathrm{opp}}}(ekC') \xrightarrow{\beta} ekG.$$

The composition $\beta\alpha$ is an endomorphism of (ekG, ekG)-bimodules of ekG, *i.e.*, an element of the local algebra $Z(ekG)$. So, it suffices to show that $\beta\alpha$ is not nilpotent. This will follow from the non-nilpotence of its image under $\mathrm{Br}_{\Delta S}$, where $\Delta S = \{(x, x^{-1}) | x \in S\} \subset G \times L^{\mathrm{opp}}$.

By [Ri2, Proofs of Lemma 4.2 and Theorem 4.1], there is a commutative diagram

$$\begin{tikzcd}
\mathrm{br}_S(e)kL \arrow[r, "\mathrm{Br}_{\Delta S}(\alpha)"] \arrow[dr, "a"'] & \mathrm{Br}_{\Delta S}(\mathrm{End}^{\bullet}_{kL^{\mathrm{opp}}}(eC')) \arrow[r, "\mathrm{Br}_{\Delta S}(\beta)"] & \mathrm{br}_S(e)kL \\
& \mathrm{End}^{\bullet}_{kL^{\mathrm{opp}}}(\mathrm{Br}_{\Delta S}(eC')) \arrow[u] \arrow[ur, "b"'] &
\end{tikzcd}$$

where a and b are the natural maps coming from the left action of $\mathrm{br}_S(e)kL$ on $\mathrm{Br}_{\Delta S}(eC') = \mathrm{br}_S(e)\,\mathrm{Br}_{\Delta S}(C')$. Note that $\mathrm{br}_S(e) \neq 0$.

We have $\mathrm{Br}_{\Delta S}(C') \simeq kL$ (cf [Rou2, proof of Theorem 4.5], the key point is that $Y_{\mathbf{V}}^{\Delta S} = L$). So, a and b are isomorphisms. $\square$

1.4. **Alternative construction for tori.** It will be convenient to use an alternative description of Deligne-Lusztig functors in the case of tori (as in [BonRou1, §11.2] for example).

1.4.1. Let $w \in W$. We define

$$\begin{aligned} Y(\dot{w}) = Y_{\mathbf{G},F}(\dot{w}) &= \{g\mathbf{U}_0 \in \mathbf{G}/\mathbf{U}_0 \mid g^{-1}F(g) \in \mathbf{U}_0\dot{w}\mathbf{U}_0\} \\ \text{and} \quad X(w) = X_{\mathbf{G},F}(w) &= \{g\mathbf{B}_0 \in \mathbf{G}/\mathbf{B}_0 \mid g^{-1}F(g) \in \mathbf{B}_0 w\mathbf{B}_0\} \end{aligned}$$

Let $\pi_w : Y(\dot{w}) \to X(w)$, $g\mathbf{U}_0 \mapsto g\mathbf{B}_0$. The group G acts on $Y(\dot{w})$ and $X(w)$ by left multiplication while $\mathbf{T}_0^{wF}$ acts on $Y(\dot{w})$ by right multiplication. Moreover, π_w is G-equivariant and it is isomorphic to the quotient morphism by $\mathbf{T}_0^{wF}$. The complex $R\Gamma_c(Y(\dot{w}), \mathcal{O})$, viewed in $D^b((\mathcal{O}G) \otimes (\mathcal{O}\mathbf{T}_0^{wF})^{\mathrm{opp}})$, induces left and right adjoint functors between the derived categories $D^b(\mathcal{O}\mathbf{T}_0^{wF})$ and $D^b(\mathcal{O}G)$:

$$\begin{aligned} \mathcal{R}_{\dot{w}} &= R\Gamma_c(Y(\dot{w}), \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}\mathbf{T}_0^{wF}} - : D^b(\mathcal{O}\mathbf{T}_0^{wF}) \to D^b(\mathcal{O}G) \\ \text{and} \quad {}^*\mathcal{R}_{\dot{w}} &= R\,\mathrm{Hom}^{\bullet}_{\mathcal{O}G}(R\Gamma_c(Y(\dot{w}), \mathcal{O}), -) : D^b(\mathcal{O}G) \to D^b(\mathcal{O}\mathbf{T}_0^{wF}). \end{aligned}$$

Let $g \in \mathbf{G}$ be such that $g^{-1}F(g) = \dot{w}$. Let $\mathbf{T}_w = g\mathbf{T}_0g^{-1}$, $\mathbf{B}_w = g\mathbf{B}_0g^{-1}$ and $\mathbf{U}_w = g\mathbf{U}_0g^{-1}$. Then, $\mathbf{T}_w$ is an F-stable maximal torus of the Borel subgroup $\mathbf{B}_w$ of $\mathbf{G}$, conjugation by g induces an isomorphism $\mathbf{T}_0^{wF} \simeq \mathbf{T}_w^F$, and multiplication by g^{-1} on the right induces an isomorphism $Y(\dot{w}) \xrightarrow{\sim} Y_{\mathbf{U}_w}$ which is equivariant for the actions of G and $\mathbf{T}_0^{wF} \simeq \mathbf{T}_w^F$. This provides an identification between the Deligne-Lusztig induction functors $\mathcal{R}_{\mathbf{T}_w \subset \mathbf{B}_w}^{\mathbf{G}}$ and $\mathcal{R}_{\dot{w}}$ (and similarly for the Deligne-Lusztig restriction functors).

1.4.2. Let us consider here the case of a product of groups cyclically permuted by F (cf [Lu2, §1.18], [DiMiRou, Proposition 2.3.3], whose proofs for the varieties X extend to the varieties Y).

Proposition 1.3. *Assume $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_s$ with $F(\mathbf{G}_i) = \mathbf{G}_{i+1}$ for $i < s$ and $F(\mathbf{G}_s) = \mathbf{G}_1$. Let $\mathbf{B}_1$ be an F^s-stable Borel subgroup of $\mathbf{G}_1$ and $\mathbf{T}_1$ an F^s-stable maximal torus of $\mathbf{B}_1$. Let $\mathbf{B}_i = F^i(\mathbf{B}_1)$ and $\mathbf{T}_i = F^i(\mathbf{T}_1)$. We assume that $\mathbf{B}_0 = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$ and $\mathbf{T}_0 = \mathbf{T}_1 \times \cdots \times \mathbf{T}_s$. Let W_i be the Weyl group of $\mathbf{G}_i$ relative to $\mathbf{T}_i$. We identify W with $W_1 \times \cdots \times W_s$. Let $w = (w_1, w_2, \ldots, w_s) \in W$ and let $v = F^{1-s}(w_s)F^{2-s}(w_{s-1}) \cdots w_1$. The first projection gives isomorphisms $\mathbf{G}^F \xrightarrow{\sim} \mathbf{G}_1^{F^s}$ and $\mathbf{T}_0^{wF} \xrightarrow{\sim} \mathbf{T}_1^{vF^s}$ and we identify the groups via these isomorphisms.*

If $l(w_1) + \cdots + l(w_s) = l(v)$, then there is a canonical $(\mathbf{G}^F \times (\mathbf{T}_0^{wF})^{\mathrm{opp}})$-equivariant radicial map

$$Y_{\mathbf{G},F}(\dot{w}) \xrightarrow{\sim} Y_{\mathbf{G}_1,F^s}(\dot{w}_1).$$

Remark 1.4. In general, let $\hat{\mathbf{G}}$ be the simply connected covering of $[\mathbf{G}, \mathbf{G}]$. This provides a morphism $i : \hat{\mathbf{G}} \to \mathbf{G}$ for which Proposition 1.1 applies. This reduces the study of the variety $Y(\dot{w})$ to the case of a simply connected group. Assume now $\mathbf{G}$ is simply connected. Then, there is a decomposition $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_s$ where each $\mathbf{G}_i$ is quasi-simple and F permutes the components. A variety $Y(\dot{w})$ for $\mathbf{G}$ decomposes as a product of varieties for the products of groups in each orbit of F. So, the study of $Y(\dot{w})$ is further reduced to the case where F permutes transitively the components. Proposition 1.3 reduces finally the study to the case where $\mathbf{G}$ is quasi-simple, when w is of the special type described in Proposition 1.3.

1.4.3.

Proposition 1.5. *Let $w \in W$. Assume that $\mathbf{G}$ is semisimple and simply-connected. Then, $Y(\dot{w})$ is irreducible if and only if $X(w)$ is irreducible.*

Proof. First, since $\pi_w : Y(\dot{w}) \to X(w)$ is surjective, we see that if $Y(\dot{w})$ is irreducible, then $X(w)$ is irreducible.

Let us assume now that $X(w)$ is irreducible. Since $X(w) = Y(\dot{w})/\mathbf{T}_0^{wF}$ is irreducible, it follows that $\mathbf{T}_0^{wF}$ permutes transitively the irreducible components of $Y(\dot{w})$. Note that $\mathbf{G}^F \backslash Y(\dot{w})$ is also irreducible, hence $\mathbf{G}^F$ permutes transitively the irreducible components of $Y(\dot{w})$. Let Y° be an irreducible component of $Y(\dot{w})$, H_1 its stabilizer in $\mathbf{T}_0^{wF}$ and H_2 its stabilizer in $\mathbf{G}^F$. Since the actions of $\mathbf{T}_0^{wF}$ and $\mathbf{G}^F$ on $Y(\dot{w})$ commute, it follows that H_2 is a normal subgroup of $\mathbf{G}^F$ and $\mathbf{G}^F/H_2 \simeq \mathbf{T}_0^{wF}/H_1$. But $G^F/[G^F, G^F]$ is a p-group because $\mathbf{G}$ is semi-simple and simply connected [St, Theorem 12.4]. On the other hand, $\mathbf{T}_0^{wF}/H_1$ is a p'-group. So $H_2 = \mathbf{G}^F$ and we are done. □

Remark 1.6. One can actually show that $X(w)$ is irreducible if and only if w is not contained in a proper standard F-stable parabolic subgroup of W (see [BonRou2]).

2. Gelfand-Graev representations

2.1. **Definitions.**

2.1.1. Let $\psi : U_0 \to \Lambda^\times$ be a linear character trivial on $D(\mathbf{U}_0)^F$ and let $\alpha \in [\Delta/\phi]$. We denote by $\psi_\alpha : \mathbf{F}_{q_\alpha} \to \Lambda^\times$ the restriction of ψ through $\mathbf{F}_{q_\alpha} \xrightarrow{\sim} \mathbf{U}_\alpha^{F^{d_\alpha}} \hookrightarrow \mathbf{U}_0^F/D(\mathbf{U}_0)^F$. Conversely, given $(\psi_\alpha : \mathbf{F}_{q_\alpha} \to \Lambda^\times)_{\alpha\in[\Delta/\phi]}$ a family of linear characters, we denote by $\boxtimes_{\alpha\in[\Delta/\phi]}\psi_\alpha$ the linear character $U_0 \to \Lambda^\times$ trivial on $D(\mathbf{U}_0)^F$ such that $(\boxtimes_{\alpha\in[\Delta/\phi]}\psi_\alpha)_\beta = \psi_\beta$ for every $\beta \in [\Delta/\phi]$.

A linear character ψ is *regular* (or G-regular if we need to emphasize the ambient group) if it is trivial on $D(\mathbf{U}_0)^F$ and if ψ_α is non-trivial for every $\alpha \in [\Delta/\phi]$.

2.1.2. *Levi subgroups.* Let ψ be a linear character of U_0 which is trivial on $D(\mathbf{U}_0)^F$. If I is a ϕ-stable subset of Δ, we set $\psi_I = \mathrm{Res}_{U_I}^{U_0}\psi$. If ψ' is a linear character of U_I which is trivial on $D(\mathbf{U}_I)^F$, we denote by $\tilde{\psi}'$ the restriction of ψ' through the morphism $U_0 \to U_0/V_I \xrightarrow{\sim} U_I$. This is a linear character of U_0 which is trivial on $D(\mathbf{U}_0)^F$.

2.1.3. Since U_0 is a p-group, given $R = K$ or $R = k$, the canonical map $\mathrm{Hom}(U_0, \mathcal{O}^\times) \to \mathrm{Hom}(U_0, R^\times)$ is an isomorphism and these groups will be identified. Given $\psi \in \mathrm{Hom}(U_0, \mathcal{O}^\times)$, we set

$$\Gamma_\psi = \mathrm{Ind}_{U_0}^G \mathcal{O}_\psi.$$

If the ambient group is not clear from the context, Γ_ψ will be denoted by $\Gamma_{\psi,G}$. Note that Γ_ψ is a projective $\mathcal{O}G$-module. If ψ is regular, then Γ_ψ is a *Gelfand-Graev module* of G and the character of $K\Gamma_\psi$ is called a *Gelfand-Graev character.*

2.2. **Induction and restriction.** The results in this §2.2 are all classical.

2.2.1. *Harish-Chandra restriction.* The next Proposition shows the Harish-Chandra restriction of a Gelfand-Graev module is a Gelfand-Graev module, a result of Rodier over K (see [Ca, Proposition 8.1.6]). It must be noticed that, in [Ca, Chapter 8], the author works under the hypothesis that the centre is connected. However, the proof of Rodier's result given in [Ca, Proof of Proposition 8.1.6] remains valid when the centre is not connected (see for instance [DiLeMi2, Theorem 2.9]). In both cases, the proof proposed is character-theoretic. Since we want to work with modular representations, we present here a module-theoretic argument, which is only the translation of the previous proofs.

Theorem 2.1. *Let $\psi : U_0 \to \mathcal{O}^\times$ be a regular linear character. Let I be a ϕ-stable subset of Δ. Let $\psi_I = \mathrm{Res}_{U_I}^{U_0}\psi$. Then ψ_I is an L_I-regular linear character and*

$${}^*\mathcal{R}_{\mathbf{L}_I\subset\mathbf{P}_I}^{\mathbf{G}}\Gamma_{\psi,G} \simeq \Gamma_{\psi_I,L_I}.$$

Proof. As explained above, this result is known over K using scalar products of characters. The result is an immediate consequence, since ${}^*\mathcal{R}_{\mathbf{L}_I}^{\mathbf{G}}\Gamma_{\psi,G}$ is projective and two projective modules with equal characters are isomorphic. We provide nevertheless here a direct module-theoretic proof — it shows there is a canonical isomorphism.

First, it is clear from the definition that ψ_I is a regular linear character of U_I. Moreover,

$${}^*\mathcal{R}_{\mathbf{L}_I\subset\mathbf{P}_I}^{\mathbf{G}}\Gamma_{\psi,G} \simeq \left(\mathrm{Res}_{P_I}^G \mathrm{Ind}_{U_0}^G \mathcal{O}_\psi\right)^{V_I}.$$

The map $(W^I)^F \to P_I\backslash G/U_0$, $w \mapsto P_I w U_0$ is bijective. Thus, by the Mackey formula for classical induction and restriction, we have

$${}^*\mathcal{R}_{\mathbf{L}_I\subset\mathbf{P}_I}^{\mathbf{G}}\Gamma_{\psi,G} \simeq \bigoplus_{w\in(W^I)^F}\left(\mathrm{Ind}_{P_I\cap {}^wU_0}^{P_I}\mathrm{Res}_{P_I\cap {}^wU_0}^{{}^wU_0}\mathcal{O}_{{}^{\dot{w}}\psi}\right)^{V_I}.$$

Fix $w \in (W^I)^F$. Then, $\mathbf{P}_I \cap {}^w\mathbf{U}_0 = (\mathbf{L}_I \cap {}^w\mathbf{U}_0) \cdot (\mathbf{V}_I \cap {}^w\mathbf{U}_0)$ (see for instance [DiLeMi1, 2.9.3]). So, if the restriction of ${}^w\psi$ to $V_I \cap {}^wU_0$ is non-trivial, then

$$\left(\mathrm{Ind}_{P_I \cap {}^wU_0}^{P_I} \mathrm{Res}_{P_I \cap {}^wU_0}^{{}^wU_0} \mathcal{O}_{\dot{w}\psi}\right)^{V_I} = 0.$$

By [DiLeMi1, Page 163], this happens unless $w = w_I w_\Delta$. On the other hand, $\mathbf{P}_I \cap {}^{w_I w_\Delta}\mathbf{U}_0 = \mathbf{U}_I$. Therefore,

$${}^*\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} \Gamma_{\psi,G} = \left(\mathrm{Ind}_{U_I}^{P_I} \mathrm{Res}_{U_I}^{{}^{w_I w_\Delta}U_0} \mathcal{O}_{\dot{w}_I \dot{w}_\Delta \psi}\right)^{V_I}.$$

In other words,

$${}^*\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} \Gamma_{\psi,G} \simeq \mathrm{Ind}_{U_I}^{L_I} \mathrm{Res}_{U_I}^{{}^{w_I w_\Delta}U_0} \mathcal{O}_{\dot{w}_I \dot{w}_\Delta \psi}.$$

Since ψ_I is T_0-conjugate to $\mathrm{Res}_{U_I}^{{}^{w_I w_\Delta}U_0}(\dot{w}_I \dot{w}_\Delta \psi)$ by [DiLeMi1, 2.9.6], we get the result. □

2.2.2. *Harish-Chandra induction.* Let I be a ϕ-stable subset of Δ and let $\psi : U_I \to \mathcal{O}^\times$ be a linear character of U_I. Then

$$\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} \Gamma_{\psi,L_I} \simeq \Gamma_{\tilde{\psi},G},$$

where $\tilde{\psi} : U_0 \to \mathcal{O}^\times$ is the lift of ψ through the canonical map $U_0 \to U_0/V_I \xrightarrow{\sim} U_I$.

2.3. **Deligne-Lusztig restriction.**

2.3.1. *Restriction to Levi subgroups.* Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with an F-stable Levi complement $\mathbf{L}$. Let $\mathbf{V}$ be the unipotent radical of $\mathbf{P}$. We denote by $\mathbf{U}_\mathbf{L}$ the unipotent radical of some F-stable Borel subgroup of $\mathbf{L}$. The proof of the next result is due to Digne, Lehrer and Michel. If the centre of $\mathbf{L}$ is disconnected, then the proof is given in [DiLeMi2]: it requires the theory of character sheaves. This explains why the scope of validity of this result is not complete, but it is reasonable to hope that it holds in general. If the centre of $\mathbf{L}$ is connected, then see [DiLeMi1].

Theorem 2.2 (Digne-Lehrer-Michel)**.** *Assume that one of the following holds:*

(1) *p is good for G, q is large enough and F is a Frobenius endomorphism;*
(2) *the center of $\mathbf{L}$ is connected.*

Let $\psi : U_0 \to \mathcal{O}^\times$ be a G-regular linear character. Then there exists an L-regular linear character ψ_L of U_L such that

$${}^*R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}}[K\Gamma_{\psi,G}] = (-1)^{\dim Y_\mathbf{V}}[K\Gamma_{\psi_L,L}].$$

We have good evidences that a much stronger result holds:

Conjecture 2.3. *Let $\psi : U_0 \to \mathcal{O}^\times$ be a G-regular linear character. Then there exists an L-regular linear character ψ_L of U_L such that*

$${}^*\mathcal{R}_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}} \Gamma_{\psi,G} \simeq \Gamma_{\psi_L,L}[-\dim Y_\mathbf{V}].$$

It is immediate that Conjecture 2.3 is compatible with Theorem 2.2. If $\mathbf{P}$ is F-stable, then the conjecture holds by Theorem 2.1. As we will see in Theorem 3.10, the conjecture holds if $\mathbf{L}$ is a maximal torus and $(\mathbf{P}, F(\mathbf{P}))$ lies in the $\mathbf{G}$-orbit of $(\mathbf{B}_0, {}^w\mathbf{B}_0)$, where w is a product of simple reflections lying in different F-orbits.

Note that Conjecture 2.3 is also compatible with the Jordan decomposition [BonRou1, Theorem B'].

Remark 2.4. Let us examine the consequences of Conjecture 2.3 at the level of KG-modules. An irreducible character of G is called *regular* if it is a component of a Gelfand-Graev character of G (for instance, the Steinberg character is regular). Then, Conjecture 2.3 over K is equivalent to the statement that regular characters appear only in degree $\dim Y_{\mathbf{V}}$ in the cohomology of the Deligne-Lusztig variety $Y_{\mathbf{V}}$. This is known for the Steinberg character, if $\mathbf{L}$ is a maximal torus [DiMiRou, Proposition 3.3.15].

2.3.2. *Restriction to tori.* We now restate Conjecture 2.3 in the case of tori:

Conjecture 2.5. *Let $w \in W$ and let $\psi : U_0 \to \mathcal{O}^\times$ be a regular linear character. Then*

$$^*\mathcal{R}_{\dot{w}}\Gamma_{\psi,G} \simeq \mathcal{O}\mathbf{T}_0^{wF}[-l(w)].$$

Remark 2.6. Through the identification of §1.4.1, Conjecture 2.5 is equivalent to Conjecture 2.3 for tori. Indeed, $\mathcal{O}\mathbf{T}_w^F \simeq \mathcal{O}\mathbf{T}_0^{wF}$ is the unique Gelfand-Graev module of $\mathbf{T}_w^F$.

We now propose a conjecture which refines Conjecture 2.5. We first need some notation. Given $x, w \in W$, we put

$$\begin{aligned} Y_x(\dot{w}) &= \{g\mathbf{U}_0 \in \mathbf{B}_0 x\mathbf{U}_0/\mathbf{U}_0 \mid g^{-1}F(g) \in \mathbf{U}_0\dot{w}\mathbf{U}_0\} \\ \text{and} \quad X_x(w) &= \{g\mathbf{B}_0 \in \mathbf{B}_0 x\mathbf{B}_0/\mathbf{B}_0 \mid g^{-1}F(g) \in \mathbf{B}_0 w\mathbf{B}_0\}. \end{aligned}$$

Then $(Y_x(\dot{w}))_{x\in W}$ (resp. $(X_x(w))_{x\in W}$) is a stratification of $Y(\dot{w})$ (resp. $X(w)$). Note that some strata might be empty. Note also that $Y_x(\dot{w})$ and $X_x(w)$ are stable under left multiplication by B_0. Moreover, $Y_x(\dot{w})$ is stable by right multiplication by $\mathbf{T}_0^{wF}$ and the natural map $Y_x(\dot{w}) \to X_x(w)$, $g\mathbf{U}_0 \to g\mathbf{B}_0$ is the quotient morphism for this action.

Conjecture 2.7. *Let $\psi : U_0 \to \mathcal{O}^\times$ be a G-regular linear character. Then, there are isomorphisms in $D^b(\mathcal{O}\mathbf{T}_0^{wF})$:*

$$R\operatorname{Hom}^\bullet_{\mathcal{O}U_0}(R\Gamma_c(Y_x(\dot{w}), \mathcal{O}), \mathcal{O}_\psi) \simeq \begin{cases} \mathcal{O}\mathbf{T}_0^{wF} & \text{if } x = w_\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $^*\mathcal{R}_{\dot{w}}\Gamma_{\psi,G} \simeq R\operatorname{Hom}^\bullet_{\mathcal{O}U_0}(R\Gamma_c(Y(\dot{w}), \mathcal{O}), \mathcal{O}_\psi)$, hence Conjecture 2.7 implies Conjecture 2.5.

Remark 2.8. The proof of Theorem 2.1 shows that that Conjecture 2.7 holds for $w = 1$. More generally, if I is a ϕ-stable subset of Δ and if $w \in W_I$, then Conjecture 2.7 holds for $(\mathbf{G}, w)$ if and only if it holds for $(\mathbf{L}_I, w)$.

3. Coxeter orbits

Notation: *Let $r = |\Delta/\phi|$. We write $[\Delta/\phi] = \{\alpha_1, \dots, \alpha_r\}$. Let $w = s_{\alpha_1} \dots s_{\alpha_r}$, a twisted Coxeter element. We put $T = \mathbf{T}_0^{wF}$.*

The aim of this section is to study the complex of cohomology $C = R\Gamma_c(Y(\dot{w}), \mathcal{O})$ of the Deligne-Lusztig variety $Y(\dot{w})$. As a consequence, we get that Conjecture 2.7 holds for w (see Theorem 3.10).

3.1. The variety $D(\mathbf{U}_0)^F \backslash X(w)$.

3.1.1. The variety $X(w)$ has been studied by Lusztig [Lu2, §2]. Before summarizing some of his results, we need some notation. Let

$$X'(w) = X'_{\mathbf{G}}(w) = \{u \in \mathbf{U}_0 \mid u^{-1}F(u) \in \mathbf{U}^{\#}_{-w_\Delta(\alpha_1)} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_r)}\}.$$

Then, we have [Lu2, Corollary 2.5, Theorem 2.6 and Proposition 4.8]:

Theorem 3.1 (Lusztig)**.** *We have:*

(a) $X(w) \subset \mathbf{B}_0 w_\Delta \mathbf{B}_0/\mathbf{B}_0$.
(b) *The map* $X'(w) \to X(w)$, $u \mapsto uw_\Delta\mathbf{B}_0$ *is an isomorphism of varieties.*
(c) *The variety* $X(w)$ *is irreducible.*

The isomorphism (b) above is B_0-equivariant (U_0 acts on $\mathbf{U}_0$ by left multiplication and T_0 acts on $\mathbf{U}_0$ by conjugation).

The Lang map $\mathbf{U}_0 \to \mathbf{U}_0$, $u \mapsto u^{-1}F(u)$, induces an isomorphism

$$U_0\backslash X'(w) \xrightarrow{\sim} \mathbf{U}^{\#}_{-w_\Delta(\alpha_1)} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_r)} \xrightarrow{\sim} (\mathbf{G}_m)^r.$$

Let

$$X''(w) = \{uD(\mathbf{U}_0) \in \mathbf{U}_0/D(\mathbf{U}_0) \mid u^{-1}F(u) \in \mathbf{U}^{\#}_{-w_\Delta(\alpha_1)} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_r)} D(\mathbf{U}_0)\}.$$

Then, $X''(w)$ is an open subvariety of $\mathbf{U}_0/D(\mathbf{U}_0)$. Moreover, the canonical map $X'(w) \to X''(w)$, $u \mapsto uD(\mathbf{U}_0)$ factorizes through $D(\mathbf{U}_0)^F\backslash X'(w)$. In fact:

Proposition 3.2. *The induced map* $D(\mathbf{U}_0)^F\backslash X'(w) \to X''(w)$ *is a* U_0*-equivariant isomorphism of varieties.*

Proof. Let $Z = \mathbf{U}^{\#}_{-w_\Delta(\alpha_1)} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_r)}$ and consider the morphism $\tau : X'(w) \to X''(w)$, $u \mapsto uD(\mathbf{U}_0)$. First, let us show that the fibers of τ are $D(\mathbf{U}_0)^F$-orbits. Let u and u' be two elements of $X'(w)$ such that $\tau(u) = \tau(u')$. There exists $v \in D(\mathbf{U}_0)$ such that $u' = vu$. Therefore,

$$u'^{-1}F(u') = u^{-1}(v^{-1}F(v))u \cdot u^{-1}F(u).$$

Recall that $u'^{-1}F(u')$ and $u^{-1}F(u)$ belong to Z and the map $Z \to \mathbf{U}_0/D(\mathbf{U}_0)$ is injective. Therefore, $u^{-1}(v^{-1}F(v))u = 1$. In other words, $F(v) = v$.

Let us now show that τ is surjective. Let $u \in \mathbf{U}_0$ with $u^{-1}F(u) \in ZD(\mathbf{U}_0)$. Write $u^{-1}F(u) = yx$, with $x \in Z$ and $y \in D(\mathbf{U}_0)$. By Lang's Theorem applied to the group $D(\mathbf{U}_0)$ and the isogeny $D(\mathbf{U}_0) \to D(\mathbf{U}_0)$, $v \mapsto xF(v)x^{-1}$, there exists $v \in D(\mathbf{U}_0)$ such that $vxF(v)^{-1}x^{-1} = y$. Then, $(uv)^{-1}F(uv) = x$. So $uv \in X'(w)$ and $\tau(uv) = uD(\mathbf{U}_0)$. So τ is surjective.

Therefore, is it sufficient to show that τ is étale. Since the maps $X'(w) \to U_0\backslash X'(w)$ and $X''(w) \to U_0\backslash X''(w) = (\mathbf{U}_0^F/D(\mathbf{U}_0)^F)\backslash X''(w)$ are étale, it is sufficient to show that the map $\tau' : U_0\backslash X'(w) \to U_0\backslash X''(w)$ induced by τ is an isomorphism. Via the canonical isomorphisms $U_0\backslash X'(w) \xrightarrow{\sim} Z$ and $U_0\backslash X''(w) \xrightarrow{\sim} ZD(\mathbf{U}_0)/D(\mathbf{U}_0)$, then τ' is the canonical map $Z \to ZD(\mathbf{U}_0)/D(\mathbf{U}_0)$, which is clearly an isomorphism. □

3.1.2. Let $i \in \{1, \ldots, r\}$, let $w_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}} s_{\alpha_{i+1}} \cdots s_{\alpha_r}$, and let I be the complement of the ϕ-orbit of α_i in Δ. Let

$$X'_i(w) = \{u \in \mathbf{U}_0 \mid u^{-1}F(u) \in \mathbf{U}^{\#}_{-w_\Delta(\alpha_1)} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_{i-1})} \mathbf{U}^{\#}_{-w_\Delta(\alpha_{i+1})} \cdots \mathbf{U}^{\#}_{-w_\Delta(\alpha_r)}\}.$$

Let $\bar{X}'_i(w) = X'(w) \cup X'_i(w) \subset \mathbf{U}_0$ (a disjoint union). Let $X(w \cup w_i) = X(w) \cup X(w_i)$ (a disjoint union). This is a partial compactification of $X(w)$, with divisor a union of disconnected

irreducible components and $X_i'(w)$ is obtained by removing some of these components, as shows the following Proposition.

Proposition 3.3. *We have a commutative diagram*

$$\begin{tikzcd}
 & \bar{X}_i'(w) \arrow[r, hook, "u\mapsto uw_\Delta \mathbf{B}_0"] & X(w\cup w_i) \\
 & X_i'(w) \arrow[u, hook] & X(w_i) \arrow[u, hook] \\
 & V_I\times X'_{\mathbf{L}_I}(w_i) \arrow[u, "(v,u)\mapsto vw_\Delta w_I u w_I w_\Delta", "\sim"'] \arrow[d, "(v,u)\mapsto(vw_\Delta w_I V_I, uw_I\mathbf{B}_I)", "\sim"'] & \sim \arrow[u, "(gV_I,h\mathbf{B}_i)\mapsto gh\mathbf{B}_0"] \\
 & P_I w_\Delta P_I/V_I\times_{L_I} X_{\mathbf{L}_I}(w_i) \arrow[r, hook] & G/V_I\times_{L_I} X_{\mathbf{L}_I}(w_i)
\end{tikzcd}$$

Proof. The map $G/V_I\times_{L_I} X_{\mathbf{L}_I}(w_i)\to X(w_i)$ is an isomorphism [Lu1, Lemma 3]. The map $X'_{\mathbf{L}_I}(w_i)\to X_{\mathbf{L}_I}(w_i),\ u\mapsto uw_I\mathbf{B}_I$ is an isomorphism by Theorem 3.1 (b). We are left with proving that the map

$$V_I\times X'_{\mathbf{L}_I}(w_i)\to X_i'(w),\ (v,u)\mapsto vw_\Delta w_I u w_I w_\Delta$$

is an isomorphism. Let $u=u_1u_2$ with $u_1\in\mathbf{V}_I$ and $u_2\in\mathbf{U}_I$. We have $u^{-1}F(u)=u_2^{-1}F(u_2)v$ where $v=F(u_2)^{-1}u_1^{-1}F(u_1)F(u_2)\in\mathbf{V}_I$. So, $u\in X_i'(w)$ if and only if $u_1\in V_I$ and $u_2\in X_i'(w)\cap\mathbf{U}_I=w_Iw_\Delta X'_{\mathbf{L}_I}(w)w_\Delta w_I$ and we are done. □

3.1.3. Let us describe now the variety $X''(w)$. Given q' a power of p, we denote by $\mathcal{L}_{q'}:\mathbf{A}^1\to\mathbf{A}^1,\ x\mapsto x^{q'}-x$ the Lang map. This is an étale Galois covering with group $\mathbf{F}_{q'}$ (*Artin-Schreier covering*). Given $\alpha\in[\Delta/\phi]$, we set $q^*_\alpha=1$ and given $1\le j\le d_\alpha-1$, we define inductively $q^*_{\phi^j(\alpha)}=q^\circ_{\phi^{j-1}(\alpha)}q^*_{\phi^{j-1}(\alpha)}$. We define

$$\gamma:\prod_{i=1}^r\mathcal{L}^{-1}_{q_{\alpha_i}}(\mathbf{G}_m)\to X''(w)$$

$$(\xi_1,\dots,\xi_r)\mapsto\prod_{i=1}^r\Big(\prod_{j=0}^{d_{\alpha_i}-1}x_{\phi^j(\alpha_i)}(\xi_i^{q^*_{\phi^j(\alpha_i)}})\Big)D(\mathbf{U}_0)$$

Note that the group $\prod_{i=1}^r\mathbf{F}_{q_{\alpha_i}}\simeq U_0^F/D(\mathbf{U}_0)^F$ acts on $\prod_{i=1}^r\mathcal{L}^{-1}_{q_{\alpha_i}}(\mathbf{G}_m)$ by addition on each component. Finally, define $\mathcal{L}:\prod_{i=1}^r\mathcal{L}^{-1}_{q_{\alpha_i}}(\mathbf{G}_m)\to(\mathbf{G}_m)^r,\ (\xi_1,\dots,\xi_r)\mapsto(\mathcal{L}_{q_{\alpha_1}}(\xi_1),\dots,\mathcal{L}_{q_{\alpha_r}}(\xi_r))$. This is the quotient map by $\prod_{i=1}^r\mathbf{F}_{q_{\alpha_i}}$.

The next proposition is immediately checked:

Proposition 3.4. *The map γ is an isomorphism of varieties. Through the isomorphism $\prod_{i=1}^r\mathbf{F}_{q_{\alpha_i}}\xrightarrow{\sim}U_0/D(\mathbf{U}_0)^F$, it is $U_0/D(\mathbf{U}_0)^F$-equivariant. Moreover, we have a commutative*

diagram

$$\begin{tikzcd}
\prod_{i=1}^r \mathcal{L}_{q_{\alpha_i}}^{-1}(\mathbf{G}_m) \arrow[r, "\gamma", "\sim"'] \arrow[d, "\mathcal{L}"'] & X''(w) \arrow[d, "\mathrm{can}"] \\
(\mathbf{G}_m)^r \arrow[r, "\mathrm{can}", "\sim"'] & \mathbf{U}_0^F\backslash X'(w)
\end{tikzcd}$$

3.2. **The variety $D(\mathbf{U}_0)^F\backslash Y(\dot{w})$.** We describe the abelian covering $D(\mathbf{U}_0)^F\backslash Y(\dot{w})$ of the variety $U_0\backslash X(w) \simeq \mathbf{G}_m^r$ by the group $(U_0/D(\mathbf{U}_0)^F) \times T$.

3.2.1. Composing the isomorphism (b) of Theorem 3.1 and those of Propositions 3.2 and 3.4, we get an isomorphism $D(\mathbf{U}_0)^F\backslash X(w) \xrightarrow{\sim} \prod_{i=1}^r \mathcal{L}_{q_{\alpha_i}}^{-1}(\mathbf{G}_m)$ and a commutative diagram whose squares are cartesian

$$\begin{tikzcd}
Y(\dot{w}) \arrow[r] \arrow[d, "\pi_w"'] & D(\mathbf{U}_0)^F\backslash Y(\dot{w}) \arrow[r] \arrow[d, "\pi'_w"'] & \mathbf{U}_0^F\backslash Y(\dot{w}) \arrow[d, "\pi''_w"'] \\
X(w) \arrow[r] & \prod_{i=1}^r \mathcal{L}_{q_{\alpha_i}}^{-1}(\mathbf{G}_m) \arrow[r, "\mathcal{L}"'] & (\mathbf{G}_m)^r
\end{tikzcd} \tag{1}$$

Here, π_w, π'_w and π''_w are the quotient maps by the right action of $\mathbf{T}_0^{wF}$.

3.2.2. Let $i : \hat{\mathbf{G}} \to \mathbf{G}$ be the semisimple simply-connected covering of the derived group of $\mathbf{G}$. There exists a unique isogeny on $\hat{\mathbf{G}}$ (still denoted by F) such that $i \circ F = F \circ i$. Let $Y^\circ(\dot{w})$ denote the image of the Deligne-Lusztig variety $Y_{\hat{\mathbf{G}}}(\dot{w})$ through i. By Proposition 1.5 and Theorem 3.1(c), $Y^\circ(\dot{w})$ is connected. Its stabilizer in G contains $i(\hat{\mathbf{G}}^F)$, so in particular it is stabilized by U_0. It is also F^d-stable. Let H be the stabilizer of $Y^\circ(\dot{w})$ in T (we have $H = i(i^{-1}(\mathbf{T}_0)^{wF})$).

We have a canonical $G \times (T \rtimes \langle F^d \rangle)^{\mathrm{opp}}$-equivariant isomorphism $Y^\circ(\dot{w}) \times_H T \xrightarrow{\sim} Y(\dot{w})$.

Recall that the tame fundamental group of $(\mathbf{G}_m)^r$ is the r-th power of the tame fundamental group of $\mathbf{G}_m$. There exists positive integers $m_1, \dots, m_r$ dividing $|H|$ and an étale Galois covering $\pi'' : (\mathbf{G}_m)^r \to U_0\backslash Y^\circ(\dot{w})$ with Galois group N satisfying the following properties:

- $\pi''_w \circ \pi'' : (\mathbf{G}_m)^r \to (\mathbf{G}_m)^r$ sends $(t_1, \dots, t_r)$ to $(t_1^{m_1}, \dots, t_r^{m_r})$
- The restriction $\phi_i : \mu_{m_i} \to H$ of the canonical map $\prod_{i=1}^r \mu_{m_i} \to H$ to the i-th factor is injective.

So, N is a subgroup of $\prod_{i=1}^r \mu_{m_i}$ and we have a canonical isomorphism $(\prod_{i=1}^r \mu_{m_i})/N \xrightarrow{\sim} H$. Moreover, π'' induces an isomorphism $(\mathbf{G}_m)^r/N \xrightarrow{\sim} U_0\backslash Y^\circ(\dot{w})$.

Let us recall some constructions related to tori and their characters. Let $Y(\mathbf{T}_0)$ be the cocharacter group of $\mathbf{T}_0$. Let c be a positive integer divisible by d such that $(wF)^c = F^c$. Let ζ be a generator of $\mathbf{F}_{q^c}^\times$. We consider the surjective morphism of groups

$$N_w : Y(\mathbf{T}_0) \to T,\ \lambda \mapsto N_{F^c/wF}(\lambda)(\zeta).$$

We put $\beta_i^\vee = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i^\vee)$.

Proposition 3.5. *The subgroup $\phi_i(\mu_{m_i})$ of T is generated by $N_w(\beta_i^\vee)$.*

Proof. We identify $X(w)$ with $X'(w)$ and $U_0\backslash X(w)$ with $\mathbf{G}_m^r$ via the canonical isomorphisms.

Let $T_i = \mathbf{T}^{w_iF}$. Let $j_i : X(w) \to X(w \cup w_i)$ and $k_i : X(w_i) \to X(w \cup w_i)$ be the canonical immersions. Let $\theta \in \mathrm{Irr}(KT)$. Let $\mathcal{F}_\theta = (\pi_{w*}K) \otimes_{KT} K_\theta$. Let $\mathcal{G}_\theta = k_i^* j_{i*}\mathcal{F}_\theta$. From [BonRou1, Proposition 7.3], we deduce the following:

(case 1) if $\theta(N_w(\beta_i^\vee)) \neq 1$, then, $\mathcal{G}_\theta = 0$

(case 2) otherwise, there is a character θ_i of T_i such that $\mathcal{G}_\theta \simeq \mathcal{F}_{\theta_i}$, where $\mathcal{F}_{\theta_i} = (\pi_{w_i*}K) \otimes_{KT_i} K_{\theta_i}$.

Assume we are in case 2. Let $q_i : X(w_i) \to U_0\backslash X(w_i)$ be the quotient map. Then, $(q_{i*}\mathcal{F}_{\theta_i})^{U_0}$ is a non-zero locally constant sheaf $\mathcal{L}_i$ of constant rank on $U_0\backslash X(w_i)$. Via the isomorphisms of Proposition 3.3, its restriction to $U_0\backslash X_i'(w) \simeq \mathbf{G}_m^{i-1} \times \mathbf{G}_m^{r-i}$ is a non-zero locally constant sheaf. On the other hand, in case 1, then the restriction of $(q_{i*}\mathcal{G}_\theta)^{U_0}$ to $\mathbf{G}_m^{i-1} \times \mathbf{G}_m^{r-i}$ is 0.

Let $\bar{\mathcal{F}}_\theta$ be the rank 1 locally constant sheaf on $\mathbf{G}_m^r$ corresponding, via the covering π_w'', to the character θ. Let $q : X'(w) \to \mathbf{G}_m^r$ be the canonical map. Then, $\bar{\mathcal{F}}_\theta \simeq (q_*\mathcal{F}_\theta)^{U_0}$. Denote by $j_i' : \mathbf{G}_m^r \to \mathbf{G}_m^{i-1} \times \mathbf{A}^1 \times \mathbf{G}_m^{r-i}$ and $k_i' : \mathbf{G}_m^{i-1} \times \mathbf{G}_m^{r-i} \to \mathbf{G}_m^{i-1} \times \mathbf{A}^1 \times \mathbf{G}_m^{r-i}$ the canonical immersions. Via base change, we deduce that $k_i'^* j_{i*}'\bar{\mathcal{F}}_\theta$ is 0 in case 1 and non-zero in case 2. This means that the restriction of θ to μ_{m_i} is trivial in case 1 but not in case 2.

We summarize the constructions in the following commutative diagram

$$\begin{tikzcd}
Y(\dot{w}) \arrow[r, "\pi_w"] \arrow[d] & X(w) \simeq X'(w) \arrow[r, hook] \arrow[d, "q"'] & \bar{X}_i'(w) \arrow[d] & X_i'(w) \arrow[l, hook'] \arrow[d] \\
U_0\backslash Y(\dot{w}) \arrow[r] & U_0\backslash X(w) \simeq \mathbf{G}_m^r \arrow[r, hook, "j_i'"'] & \mathbf{G}_m^{i-1} \times \mathbf{A}^1 \times \mathbf{G}_m^{r-i} & \mathbf{G}_m^{i-1} \times \mathbf{G}_m^{r-i} \arrow[l, hook', "k_i'"]
\end{tikzcd}$$

□

Let $Y' = (D(\mathbf{U}_0)^F\backslash Y^\circ(\dot{w})) \times_{\mathbf{U}_0^F\backslash Y^\circ(\dot{w})} (\mathbf{G}_m)^r$. The cartesian square (1) gives an isomorphism $Y' \xrightarrow{\sim} \prod_{i=1}^r Y_{q_{\alpha_i},m_i}$ where, given q' is a power of p and s a positive integer, we set

$$Y_{q',s} = \{(\xi, t) \in \mathbf{A}^1 \times \mathbf{G}_m \mid \xi^{q'} - \xi = t^s\}.$$

This variety has an action of $\mathbf{F}_{q'}$ by addition on the first coordinate and an action of μ_s by multiplication on the second coordinate. We consider the quotient maps $\tau_{q',s} : Y_{q',s} \to \mathbf{G}_m$, $(\xi, t) \mapsto t$ and $\rho_{q',s} : Y_{q',s} \to \mathbf{A}^1 - \mathbf{A}^1(\mathbf{F}_{q'})$, $(\xi, t) \mapsto \xi$.

Let $\tau = \prod_{i=1}^r \tau_{q_{\alpha_i},m_i} : \prod_{i=1}^r Y_{q_{\alpha_i},m_i} \to (\mathbf{G}_m)^r$, $\pi' : \prod_{i=1}^r Y_{q_{\alpha_i},m_i} \to D(\mathbf{U}_0)^F\backslash Y^\circ(\dot{w})$ the canonical map, $\rho' = \prod_{i=1}^r \rho_{q_{\alpha_i},m_i} : \prod_{i=1}^r Y_{q_{\alpha_i},m_i} \to \prod_{i=1}^r \mathcal{L}_{q_{\alpha_i}}^{-1}(\mathbf{G}_m)$ and $\rho'' = \pi_w'' \circ \pi''$. We have a commutative diagram all of whose squares are cartesian

$$\begin{tikzcd}
\prod_{i=1}^r Y_{q_{\alpha_i},m_i} \arrow[r, "\tau"] \arrow[d, "\pi'"'] \arrow[dd, bend right=60, "\rho'"'] & (\mathbf{G}_m)^r \arrow[d, "\pi''"'] \arrow[dd, bend left=60, "\rho''"] \\
D(\mathbf{U}_0)^F\backslash Y^\circ(\dot{w}) \arrow[r] \arrow[d, "\pi_w'"'] & \mathbf{U}_0^F\backslash Y^\circ(\dot{w}) \arrow[d, "\pi_w''"'] \\
\prod_{i=1}^r \mathcal{L}_{q_{\alpha_i}}^{-1}(\mathbf{G}_m) \arrow[r, "\mathcal{L}"] & (\mathbf{G}_m)^r
\end{tikzcd}$$

3.3. **Cohomology of $D(\mathbf{U}_0)^F \setminus Y(\dot{w})$.** In this subsection, we shall describe the action of $\mathcal{O}T$ on $R\Gamma_c(D(\mathbf{U}_0)^F \setminus Y(\dot{w}), \mathcal{O})$ and the action of F^d on its cohomology. For this, we determine first the cohomology of the curves $Y_{q',s}$ in order to use the previous diagram (§3.3.1). We also need a result on Koszul complexes (§3.3.2).

3.3.1. *Cohomology of the curves $Y_{q',s}$.* Let $s \in \mathbf{Z}_{>0}$ be prime to p and let q' be a power of p. In §3.2.2, we introduced a smooth affine connected closed curve $Y_{q',s}$ in $\mathbf{A}^1 \times \mathbf{G}_m$. This is a variety defined over $\mathbf{F}_{q'}$ and we denote by F' the corresponding Frobenius endomorphism. Note that $Y_{q',s}$ is an open subvariety of a variety considered by Laumon [Lau, §3.2]. The variety $Y_{q',2}$ was considered by Lusztig [Lu3, p.18]. We will describe the complex of cohomology for the finite group actions and the cohomology for the additional action of the Frobenius endomorphism.

Let g be a generator of μ_s. We put $Z = 0 \to \mathcal{O}\mu_s \xrightarrow{g-1} \mathcal{O}\mu_s \to 0$, a complex of $\mathcal{O}\mu_s$-modules, with non-zero terms in degrees 0 and 1. Note that, up to isomorphism of complexes of $\mathcal{O}\mu_s$-modules, Z does not depend on the choice of g.

Lemma 3.6. *Let $\psi \in \mathrm{Irr}(\mathbf{F}_{q'})$. We have*

$$e_\psi R\Gamma_c(Y_{q',s}, \mathcal{O}) \simeq \begin{cases} \mathcal{O}\mu_s[-1] & \text{if } \psi \neq 1 \\ Z[-1] & \text{if } \psi = 1 \end{cases}$$

in $D^b(\mathcal{O}\mu_s)$.

Proof. We put $Y = Y_{q',s}$ in the proof. Since Y is a smooth affine curve, it follows that $H^i_c(Y, \mathcal{O}) = 0$ for $i \neq 1, 2$ and $H^*_c(Y, \mathcal{O})$ is free over $\mathcal{O}$. Note further that the action of μ_s is free on Y, so $R\Gamma_c(Y, \mathcal{O})$ is a perfect complex of $\mathcal{O}\mu_s$-modules. Choose $R\Gamma_c(Y, \mathcal{O}) \in D^b(\mathcal{O}(\mathbf{F}_{q'} \times \mu_s))$ the unique (up to isomorphism) bounded complex of projective modules with no non-zero direct summand homotopy equivalent to 0. Then, $R\Gamma_c(Y, \mathcal{O})$ is a complex of finitely generated projective $\mathcal{O}(\mathbf{F}_{q'} \times \mu_s)$-modules with non-zero terms only in degrees 1 and 2. Since Y is irreducible, we have $H^2_c(Y, \mathcal{O}) \simeq \mathcal{O}$, with a trivial action of $\mathbf{F}_{q'} \times \mu_s$.

Given $\psi \in \mathrm{Irr}(\mathbf{F}_{q'})$, let $C_\psi = e_\psi R\Gamma_c(Y, \mathcal{O})$. We have $R\Gamma_c(Y, \mathcal{O}) = \bigoplus_{\psi \in \mathrm{Irr}(\mathbf{F}_{q'})} C_\psi$. Assume $\psi \neq 1$. The complex C_ψ has non-zero homology only in degree 1 and the $\mathcal{O}\mu_s$-module $M = H^1(C_\psi)$ is projective. As in [DeLu, §3.4], one sees that the character of M vanishes outside 1, hence it is a free $\mathcal{O}\mu_s$-module. We have $M^{\mu_s} \simeq e_\psi H^1_c(Y/\mu_s, \mathcal{O}) \simeq \mathcal{O}$. It follows that M is a free $\mathcal{O}\mu_s$-module of rank 1.

We have $C_1 \simeq R\Gamma_c(Y/\mathbf{F}_{q'}, \mathcal{O})$. So, $H^1(C_1) \simeq H^2(C_1) \simeq \mathcal{O}$ with trivial action of μ_s. It follows that C_1^2 is a projective cover of $\mathcal{O}$ and C_1 is isomorphic to $0 \to \mathcal{O}\mu_s \xrightarrow{g-1} \mathcal{O}\mu_s \to 0$. □

Let $\theta \in \mathrm{Irr}(\mu_s)$. Let c_θ be the smallest positive integer such that F'^{c_θ} fixes θ. We denote by $\tilde{\theta}$ the extension of θ to $\mu_s \rtimes \langle F'^{c_\theta} \rangle$ on which F'^{c_θ} acts trivially and we put $L_\theta = \mathrm{Ind}_{\mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mu_s \rtimes \langle F' \rangle} \tilde{\theta}$.

Given $\lambda \in K^\times$, we denote by $K(\lambda)$ the vector space K with action of F' given by multiplication by λ. Let $Q = \{\alpha \in \bar{\mathbf{F}}_p^\times | \alpha^s \in \mathbf{F}_{q'}^\times\}$.

Lemma 3.7. *Fix $\psi \in \mathrm{Irr}(\mathbf{F}_{q'})^\#$. There is $\lambda : \mathrm{Irr}(\mu_s)^\#/F' \to \mathcal{O}^\times$ such that*

$$H^1_c(Y_{q',s}, K) \simeq \big(K\mathbf{F}_{q'} \otimes K(1)\big) \oplus \bigoplus_{\substack{\alpha \in Q/\mu_s \\ \theta \in [\mathrm{Irr}(\mu_s)^\#/F']}} \left(\alpha^s(\psi) \otimes L_\theta \otimes K\left(\lambda(\theta) \cdot \sqrt[c_\theta]{\theta(\alpha^{q'^{c_\theta}-1})}\right)\right)$$

as $K(\mathbf{F}_{q'} \times \mu_s \rtimes \langle F' \rangle)$-modules.

Furthermore, we have $H^2_c(Y_{q',s}, K) \simeq K(q')$.

Proof. Note first that the statement about $H^2_c(Y_{q',s}, K)$ follows immediately from Lemma 3.6.

We have an action of Q on $Y_{q',s}$ extending the action of μ_s: an element $\alpha \in Q$ acts by $(\xi, t) \mapsto (\alpha^s \xi, \alpha t)$. This provides $Y_{q',s}$ with an action of $(\mathbf{F}_{q'} \rtimes Q) \rtimes \langle F' \rangle$. The action of Q on $\mathbf{F}_{q'}$ is given by $\alpha : (\mathbf{F}_{q'} \ni a \mapsto \alpha^s a)$.

Let $L = H^1_c(Y_{q',s}, K)/H^1_c(Y_{q',s}, K)^{\mu_s}$. Given $\theta \in \mathrm{Irr}(\mu_s)$, we put $V_\theta = \mathrm{Ind}_{\mathbf{F}_{q'} \times \mu_s}^{\mathbf{F}_{q'} \rtimes Q}(\psi \otimes \theta)$ (this is independent of the choice of ψ up to isomorphism). Then, $L \simeq \bigoplus_{\theta \in \mathrm{Irr}(\mu_s)^\#} V_\theta$ is the decomposition into irreducible $K(\mathbf{F}_{q'} \rtimes Q)$-modules: it is uniquely determined by its restriction to $\mathbf{F}_{q'} \times \mu_s$ which is described in Lemma 3.6.

Let $\theta \in \mathrm{Irr}(\mu_s)^\#$. We put $L'_\theta = \mathrm{Ind}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mathbf{F}_{q'} \rtimes Q \rtimes \langle F' \rangle}(\psi \otimes \tilde{\theta})$. This is an irreducible representation which extends $\bigoplus_{i=0}^{c_\theta - 1} V_{F'^i(\theta)}$. It follows that there are scalars $\lambda(\theta) \in \mathcal{O}^\times$ such that

$$L \simeq \bigoplus_{\theta \in \mathrm{Irr}(\mu_s)^\#/F'} L'_\theta \otimes K(\lambda(\theta)).$$

We have

$$\mathrm{Res}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F' \rangle}^{\mathbf{F}_{q'} \rtimes Q \rtimes \langle F' \rangle} L'_\theta \simeq \mathrm{Ind}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F' \rangle} \mathrm{Res}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mathbf{F}_{q'} \rtimes Q \rtimes \langle F'^{c_\theta} \rangle} L''_\theta$$

where $L''_\theta = \mathrm{Ind}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mathbf{F}_{q'} \rtimes Q \rtimes \langle F'^{c_\theta} \rangle}(\psi \otimes \tilde{\theta})$. Now,

$$\mathrm{Res}_{\mathbf{F}_{q'} \times \mu_s \rtimes \langle F'^{c_\theta} \rangle}^{\mathbf{F}_{q'} \rtimes Q \rtimes \langle F'^{c_\theta} \rangle} L''_\theta \simeq \bigoplus_{\alpha \in Q/\mu_s} \alpha^s(\psi) \otimes \tilde{\theta} \otimes K'(\theta(\alpha^{q'^{c_\theta} - 1}))$$

where $K'(\lambda)$ is the one dimensional representation of $\langle F'^{c_\theta} \rangle$ where F'^{c_θ} acts by λ. The Lemma follows. □

3.3.2. *Koszul complexes.* Let H be a finite abelian group. Given $H' \leq H$ a cyclic subgroup and g a generator of H', we put $Z_H(H') = 0 \to \mathcal{O}H \xrightarrow{g-1} \mathcal{O}H \to 0$, where the non-zero terms are in degrees 0 and 1. This is a complex of $\mathcal{O}H$-modules whose isomorphism class is independent of g. Given $H_1, \ldots, H_m$ a collection of cyclic subgroups of H, we put $Z_H(H_1, \ldots, H_m) = Z_H(H_1) \otimes_{\mathcal{O}H} \cdots \otimes_{\mathcal{O}H} Z_H(H_m)$, a Koszul complex.

Lemma 3.8. *Let $H_1, \ldots, H_m$ be finite cyclic subgroups of H. Then, the cohomology of the complex $Z_H(H_1, \ldots, H_m)$ is free over $\mathcal{O}$.*

Proof. We prove the Lemma by induction on m. Let $\bar{H} = H/H_n$ and $\bar{H}_i = H_i/(H_i \cap H_n)$. We have a distinguished triangle in $D^b(\mathcal{O}H)$

$$H^0 Z_H(H_n) \to Z_H(H_n) \to (H^1 Z_H(H_n))[-1] \rightsquigarrow .$$

We have an isomorphism $\mathcal{O}\bar{H} \xrightarrow{\sim} H^1 Z_H(H_n)$ induced by the identity and an isomorphism $\mathcal{O}\bar{H} \xrightarrow{\sim} H^0 Z_H(H_n)$ induced by $1 \mapsto \sum_{h \in H_n} h$. We have an isomorphism of complexes of $\mathcal{O}\bar{H}$-modules

$$\mathcal{O}\bar{H} \otimes_{\mathcal{O}H} Z_H(H_1, \ldots, H_{n-1}) \xrightarrow{\sim} Z_{\bar{H}}(\bar{H}_1, \ldots, \bar{H}_{n-1})$$

and we deduce that there is a distinguished triangle in $D^b(\mathcal{O}H)$

$$Z_{\bar{H}} \to Z_H \to Z_{\bar{H}}[-1] \rightsquigarrow$$

where $Z_H = Z_H(H_1, \dots, H_n)$ and $Z_{\bar{H}} = Z_{\bar{H}}(\bar{H}_1, \dots, \bar{H}_{n-1})$. This induces a long exact sequence

$$0 \to H^0 Z_{\bar{H}} \to H^0 Z_H \to 0 \to H^1 Z_{\bar{H}} \to H^1 Z_H \to H^0 Z_{\bar{H}} \to H^2 Z_{\bar{H}} \to \cdots$$

Note that $KZ_H(H') \simeq K(H/H') \oplus K(H/H')[-1]$, hence $H^i(KZ_H) \simeq K(H/H_1 \cdots H_n)^{\binom{n}{i}}$ and similarly $H^i(KZ_{\bar{H}}) \simeq K(H/H_1 \cdots H_n)^{\binom{n-1}{i}}$. We deduce that the connection maps $H^i Z_{\bar{H}} \to H^{i+2} Z_{\bar{H}}$ vanish over K. By induction, these $\mathcal{O}$-modules are free over $\mathcal{O}$, hence these maps vanish over $\mathcal{O}$ as well and it follows that $H^* Z_H$ is free over $\mathcal{O}$. □

3.3.3. *Action of T, Conjecture 2.7.* We deduce from §3.2.2 isomorphisms in $D^b(\mathcal{O}T)$:

$$\begin{aligned} R\Gamma_c(D(\mathbf{U}_0)^F \backslash Y(\dot{w}), \mathcal{O}) &\xrightarrow{\sim} R\Gamma_c(\prod_{i=1}^r Y_{q_{\alpha_i}, m_i}, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}(\prod_{i=1}^r \mu_{m_i})} \mathcal{O}T \\ &\xrightarrow{\sim} \left(R\Gamma_c(Y_{q_{\alpha_1}, m_1}, \mathcal{O}) \otimes_{\mathcal{O}\mu_{m_1}} \mathcal{O}T\right) \otimes^{\mathbf{L}}_{\mathcal{O}T} \cdots \otimes^{\mathbf{L}}_{\mathcal{O}T} \left(R\Gamma_c(Y_{q_{\alpha_r}, m_r}, \mathcal{O}) \otimes_{\mathcal{O}\mu_{m_1}} \mathcal{O}T\right). \end{aligned}$$

Let $a \in \mathcal{O}T$. We put $Z(a) = 0 \to \mathcal{O}T \xrightarrow{a} \mathcal{O}T \to 0$, with non-zero terms in degrees 0 and 1. Given $a_1, \dots, a_m \in \mathcal{O}T$, we define the Koszul complex $Z(\{a_1, \dots, a_m\}) = Z(a_1) \otimes_{\mathcal{O}T} \cdots \otimes_{\mathcal{O}T} Z(a_m)$. We also put $Z(\emptyset) = \mathcal{O}T$. Given $g \in T$, we have $Z(g-1) = Z_T(\langle g \rangle)$ with the notations of §3.3.2.

If ψ is a linear character of U_0 trivial on $D(\mathbf{U}_0)^F$, we put $I(\psi) = \{i \in \{1, 2, \dots, r\} \mid \psi_{\alpha_i} \neq 1\}$. It is easily checked that, if $t \in T_0$, then $I({}^t\psi) = I(\psi)$.

From the above isomorphism, Proposition 3.5, and Lemmas 3.6 and 3.8, we deduce

Lemma 3.9. *Let ψ be a linear character of U_0 trivial on $D(\mathbf{U}_0)^F$. We have an isomorphism in $D^b(\mathcal{O}T)$*

$$e_\psi C[r] \simeq Z(\{N_w(\beta_i^\vee) - 1\}_{i \notin I(\psi)}).$$

The cohomology of $e_\psi C$ is free over $\mathcal{O}$.

Conjecture 2.7 for twisted Coxeter elements of Levi subgroups now follows easily from this lemma:

Theorem 3.10. *Let I be a ϕ-stable subset of Δ. Let $[I/\phi] = \{\beta_1, \dots, \beta_{r'}\}$ and $w' = s_{\beta_1} \cdots s_{\beta_{r'}}$. Then, Conjecture 2.7 holds for $(\mathbf{G}, w')$.*

Proof. Thanks to Remark 2.8, we can and will assume that $I = \Delta$. In other words, we may assume that $w' = w$. We have $Y_x(\dot{w}) = \emptyset$ for $x \neq w_\Delta$ by Theorem 3.1 (a), so it is enough to prove Conjecture 2.5.

Let ψ be a regular linear character of U_0. By Lemma 3.9, we have $e_\psi R\Gamma_c(Y(\dot{w}), \mathcal{O}) \simeq Z(\emptyset)[-r] = \mathcal{O}T[-r]$. □

3.3.4. *Action of F^d.* Consider the endomorphism of $\prod_{i=1}^r Y_{q_{\alpha_i}, m_i}$ given by elevation to the power $q_{\alpha_i}^{d/d_{\alpha_i}}$ on the i-th component. Then, the morphism π' is equivariant with respect to the action of that endomorphism and of F^d on $D(\mathbf{U}_0)^F \backslash Y^\circ(\dot{w})$.

Let $\theta \in \mathrm{Irr}(T)$. Given $i \in \{1, \dots, r\}$, we denote by θ_i the restriction of θ to $\phi_i(\mu_{m_i}) =< N_w(\beta_i^\vee) >$. Let $I_\theta = \{i \in \{1, 2, \dots, r\} \mid \theta_i \neq 1\}$. It is easily checked that $I_{\theta \circ F^d} = I_\theta$. Let c_θ be the smallest positive integer such that F^{dc_θ} fixes θ. We denote by $\tilde{\theta}$ the extension of θ to $T \rtimes \langle F^{dc_\theta} \rangle$ on which F^{dc_θ} acts trivially and we put $V_\theta = \mathrm{Ind}_{T \rtimes \langle F^{dc_\theta} \rangle}^{T \rtimes \langle F^d \rangle} \tilde{\theta}$. Given $\lambda \in K^\times$, we denote by $K(\lambda)$ the vector space K with an action of F^d given by multiplication by λ.

Lemma 3.11. *Let ψ be a linear character of U_0 trivial on $D(\mathbf{U}_0)^F$. Then there is a map $\nu : \mathrm{Irr}(T)/F^d \to \mathcal{O}^\times$ such that for any $j \in \mathbf{Z}$, we have an isomorphism of $K(T \rtimes \langle F^d \rangle)$-modules*

$$H^{r+j}(e_\psi KC) \simeq \bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ I_\theta \subset I(\psi)}} \big(V_\theta \otimes K(q^{dj}\nu(\theta))\big)^{\oplus \binom{r-|I(\psi)|}{j}}$$

for $0 \leq j \leq r - |I(\psi)|$. Furthermore, $H^{r+j}(e_\psi KC) = 0$ if $j < 0$ or $j > r - |I(\psi)|$.

Proof. We use Lemma 3.7. There are scalars $\lambda(i,\theta) \in \mathcal{O}^\times$ such that for $i \in I(\psi)$, we have

$$e_{\psi_{\alpha_i}} H^1_c(Y_{q_{\alpha_i},m_i}, K) \otimes_{K\mu_{m_i}} KT \simeq \Big(\bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ \theta_i \neq 1}} V_\theta \otimes K(\lambda(i,\theta)) \Big) \oplus \Big(\bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ \theta_i = 1}} V_\theta \Big).$$

Note that $e_{\psi_{\alpha_i}} H^j_c(Y_{q_{\alpha_i},m_i}, K) = 0$ for $j \neq 1$. On the other hand, we have

$$e_1 H^1_c(Y_{q_{\alpha_i},m_i}, K) \otimes_{K\mu_{m_i}} KT \simeq \bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ \theta_i = 1}} V_\theta$$

and

$$e_1 H^2_c(Y_{q_{\alpha_i},m_i}, K) \otimes_{K\mu_{m_i}} KT \simeq \bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ \theta_i = 1}} V_\theta \otimes K(q^d).$$

Note that $e_1 H^j_c(Y_{q_{\alpha_i},m_i}, K) = 0$ for $j \neq 1, 2$.

Given $\theta \in \mathrm{Irr}(T)$, we have $V_\theta \otimes_{KT} V_\theta \simeq V_\theta$. So,

$$H^{r+j}(e_\psi KC) \simeq \left(\bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ I_\theta \subset I(\psi)}} V_\theta \otimes K\left(q^{dj} \prod_{i \in I_\theta} \lambda(i,\theta) \right) \right)^{\oplus \binom{r-|I(\psi)|}{j}}$$

for $0 \leq j \leq r - |I(\psi)|$ and $H^{r+j}(e_\psi KC) = 0$ otherwise. The result follows. □

From now on, and until the end of this section, we fix a regular linear character ψ of U_0. We put

$$e(\psi) = \sum_{\substack{I \subset \Delta \\ \phi(I) = I}} e_{\tilde{\psi}_I}.$$

For the definition of $\tilde{\psi}_I$, see §2.1.2.

Remark 3.12. If the center of $\mathbf{G}$ is connected, then $\{\tilde{\psi}_I \mid I \subset \Delta,\ \phi(I) = I\}$ is a set of representatives of T_0-orbits of linear characters of U_0 which are trivial on $D(\mathbf{U}_0)^F$. This follows easily from [DiLeMi1, Theorem 2.4] and from the fact that the centre of any Levi subgroup of $\mathbf{G}$ is also connected [DiLeMi1, Lemma 1.4].

Proposition 3.13. *There is a map $\nu : \mathrm{Irr}(T)/F^d \to \mathcal{O}^\times$ such that for all j, we have an isomorphism of $K(T \rtimes \langle F^d \rangle)$-modules*

$$H^{r+j}(e(\psi)KC) \simeq \bigoplus_{\substack{I \subset \{1,2,\dots,r\} \\ |I| \leq r-j}} \Big(\bigoplus_{\substack{\theta \in [\mathrm{Irr}(T)/F^d] \\ I_\theta \subset I}} \big(V_\theta \otimes K(q^{dj}\nu(\theta))\big)^{\oplus \binom{r-|I|}{j}} \Big).$$

Proof. This follows easily from the proof of Lemma 3.11, the scalars $\lambda(i,\theta)$ (which are defined whenever $i \in I(\psi) \cap I_\theta$) depending only on θ and ψ_{α_i}. □

3.3.5. *Endomorphism algebra.* Let $E = R\,\mathrm{End}^\bullet_{\mathcal{O}T}(e(\psi)C)$, an object of $D^b(\mathcal{O}T)$.

Proposition 3.14. *We have an isomorphism in $D^b(\mathcal{O}T)$*

$$E \simeq \bigoplus_{\substack{I \subset \{1,\dots,r\} \\ j \in \mathbf{Z}}} Z(\{N_w(\beta_i^\vee) - 1\}_{i \in I})[j]^{m_{I,j}}$$

for some non-negative integers $m_{I,j}$. The cohomology of E is free over $\mathcal{O}$ and $H^j(E) = 0$ for $|j| > r$.

Let $j \in \mathbf{Z}$. We have an isomorphism of $K(T \rtimes \langle F^d \rangle)$-modules

$$H^j(KE) \simeq \bigoplus_{\theta \in [\mathrm{Irr}(T)/F^d]} \big(V_\theta \otimes K(q^{dj})\big)^{m(\theta,j)}$$

for some integers $m(\theta,j)$. We have $m(\theta,j) = 0$ if $r - |I_\theta| < |j|$.

Proof. Given $a \in \mathcal{O}T$, we have isomorphisms

$$\mathrm{Hom}^\bullet_{\mathcal{O}T}(Z(a), \mathcal{O}T) \simeq Z(a)[1] \text{ and } Z(a) \otimes_{\mathcal{O}H} Z(a) \simeq Z(a) \oplus Z(a)[-1].$$

We deduce from Lemma 3.9 that $e(\psi)C \simeq \bigoplus_{I \subset \{1,\dots,r\}} Z(\{N_w(\beta_i^\vee) - 1\}_{i \in I})[-r]$. Hence,

$$E \simeq \bigoplus_{I,J \subset \{1,\dots,r\}} \bigoplus_{k=0}^{|I \cap J|} Z(\{N_w(\beta_i^\vee) - 1\}_{i \in I \cup J})[|J| - k]^{\binom{|I \cap J|}{k}}.$$

The freeness of the cohomology follows from Lemma 3.8. The second assertion is a direct consequence of Proposition 3.13. □

From now on, and until the end of this section, we assume that $\mathbf{G}$ is quasi-simple. Let h be the Coxeter number of $\mathbf{G}$ relative to F (cf [Lu2, §1.13]). In other words, $h = |W^{wF}|$. Let ν be a non-negative integer with $\ell^\nu \equiv 1 \pmod h$. We put $\tilde{F} = F^{d\ell^\nu}$.

Corollary 3.15. *Assume that $\mathbf{G}$ is quasi-simple and that $\ell \not| h$. Then, the canonical map $k \otimes_{\mathcal{O}} H^0(E)^{\tilde{F}} \to H^0(k \otimes_{\mathcal{O}} E)^{\tilde{F}}$ is an isomorphism.*

Assume furthermore that for all $\theta \in \mathrm{Irr}(T)$ and all $j \in \{1,\dots,r - |I_\theta|\}$, then $q^{djc_\theta} \neq 1 \pmod \ell$. Then, $H^i(E)^{\tilde{F}} = H^i(k \otimes_{\mathcal{O}} E)^{\tilde{F}} = 0$ for $i \neq 0$.

Proof. Note that the canonical map $kH^i(E) \to H^i(kE)$ is an isomorphism since $H^i(E)$ is free over $\mathcal{O}$. Now, $H^i(E)^{\tilde{F}}$ coincides with the generalized eigenspace of $\tilde{F}$ for the eigenvalue 1. The eigenvalues of $\tilde{F}$ on $H^0(E)$ are h-th roots of unity, hence their reductions modulo ℓ remain distinct. It follows that the generalized 1-eigenspace of $\tilde{F}$ on $kH^0(E)$ is the image of the generalized 1-eigenspace on $H^0(E)$.

Also, the eigenvalues of $\tilde{F}$ on $H^j(E)$ and $H^j(kE)$ are of the form $q^{\ell^\nu dj}\zeta_{c_\theta}$, where $\zeta_{c_\theta}^{c_\theta} = 1$ and the result follows. □

4. Groups of type A

Hypothesis: *In this section, and only in this section, we assume that* $\mathbf{G}$ *is of type* A_{n-1} *(for some non-zero natural number* n*) and that* F *is a split Frobenius endomorphism of* $\mathbf{G}$ *(i.e.* $d = 1$*). We keep the notation of Section 3, i.e.* w *is a Coxeter element of* W*.*

Note in particular that $r = n - 1$, $W \simeq \mathfrak{S}_n$, w is a cycle of length n, and $h = n$.

4.1. **A progenerator for $\mathcal{O}G$.** The following result is probably classical.

Theorem 4.1. $\mathrm{Ind}_{D(\mathbf{U}_0)^F}^G \mathcal{O}$ *is a progenerator for* $\mathcal{O}G$*.*

Proof. Consider the situation of §1.3.3. Let $\tilde{\mathbf{U}}_0 = i(\mathbf{U}_0)$. Then i induces an isomorphism $D(\mathbf{U}_0)^F \xrightarrow{\sim} D(\tilde{\mathbf{U}}_0)^F$. So,

$$\mathrm{Ind}_{D(\tilde{\mathbf{U}}_0)^F}^{\tilde{G}} \mathcal{O} \simeq \mathrm{Ind}_G^{\tilde{G}}\big(\mathrm{Ind}_{D(\mathbf{U}_0)^F}^G \mathcal{O}\big)$$

and

$$\mathrm{Res}_G^{\tilde{G}} \mathrm{Ind}_{D(\tilde{\mathbf{U}}_0)^F}^{\tilde{G}} \mathcal{O} \simeq \Big(\mathrm{Ind}_{D(\mathbf{U}_0)^F}^G \mathcal{O}\Big)^{\oplus r},$$

where $r = |\tilde{G}/i(G)|$. It follows that $\mathrm{Ind}_{D(\mathbf{U}_0)^F}^G \mathcal{O}$ is a progenerator for $\mathcal{O}G$ if and only if $\mathrm{Ind}_{D(\tilde{\mathbf{U}}_0)^F}^{\tilde{G}} \mathcal{O}$ is a progenerator for $\mathcal{O}\tilde{G}$. This implies that we only need to prove the theorem for $\mathbf{G} = \mathrm{GL}_n$, which we assume for the rest of the proof. Let $M_I = \mathrm{Ind}_{D(\mathbf{U}_I)^F}^{L_I} \mathcal{O}$.

Let S be a simple cuspidal $\mathcal{O}G$-module. Then, there is an $\mathcal{O}G$-module $\tilde{S}$, free over $\mathcal{O}$, with $k\tilde{S} \simeq S$ and such that $K\tilde{S}$ is cuspidal (see [Dip, 4.15 and 5.23]). Now, there exists a regular linear character ψ of U_0 such that $\mathrm{Hom}_{KG}(K\Gamma_\psi, K\tilde{S}) \neq 0$, so $\mathrm{Hom}_{\mathcal{O}G}(\Gamma_\psi, \tilde{S}) \neq 0$, since Γ_ψ is projective. Hence $\mathrm{Hom}_{\mathcal{O}G}(\Gamma_\psi, S) \neq 0$. In particular, $\mathrm{Hom}_{\mathcal{O}G}(M_\Delta, S) \neq 0$, because Γ_ψ is a direct summand of M_Δ.

Let now S be an arbitrary simple $\mathcal{O}G$-module. Then, there is $I \subset \Delta$ and a simple cuspidal $\mathcal{O}L_I$-module S' such that $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} S', S) \neq 0$ and we deduce from the study above and the exactness of the Harish-Chandra induction that $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} M_I, S) \neq 0$. It follows again that $\mathrm{Hom}_{\mathcal{O}G}(M_\Delta, S) \neq 0$ for any simple $\mathcal{O}G$-module S because $\mathcal{R}_{\mathbf{L}_I \subset \mathbf{P}_I}^{\mathbf{G}} M_I$ is a direct summand of M_Δ. So we are done. □

Corollary 4.2. *Let* ψ *be a regular linear character of* U_0*. If the centre of* $\mathbf{G}$ *is connected, then* $\mathcal{O}Ge(\psi)$ *is a progenerator for* $\mathcal{O}G$*.*

Proof. This follows from Theorem 4.1 and Remark 3.12. □

4.2. **Description over K.** We identify T with T_w as in §1.4.1. From now on, and until the end of this paper, we assume that ℓ divides $|G|$ but does not divide $|G/T|$. In other words, we assume that the order of q modulo ℓ is equal to n. Note in particular that $\ell > n$. Let S be a Sylow ℓ-subgroup of T (note that S is a Sylow ℓ-subgroup of G). Let b be the sum of all block idempotents of $\mathcal{O}G$ having S as a defect group. Since $C_{\mathbf{G}}(s) = \mathbf{T}_w$ for every non-trivial ℓ-element s of T, b is the sum of all block idempotents of $\mathcal{O}G$ with non-zero defect.

Lemma 4.3. *The canonical map* $bKG \to \mathrm{End}^\bullet_{KT}(KC)^{\tilde F}$ *is an isomorphism in* $D^b(KG \otimes (KG)^{\mathrm{opp}})$.

Proof. Let $\zeta, \zeta' \in \mathrm{Irr}_K(T)$ be two characters which are not conjugate under the action of $N_G(T)$. Then, $\mathrm{Hom}_{KG}(H^*_c(Y(\dot w), K) \otimes_{KT} \zeta, H^*_c(Y(\dot w), K) \otimes_{KT} \zeta') = 0$ [DiMi2, Proposition 13.3].

Let $\zeta \in \mathrm{Irr}(T)$. Let s be a semi-simple element of the group dual to $\mathbf{G}$ whose class is dual to that of ζ. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ corresponding to the centralizer of s and containing $\mathbf{T}_w$. There is a decomposition $n = n_1 n_2$ in positive integers such that $L = \mathrm{GL}_{n_1}(\mathbf{F}_{q^{n_2}})$ and $\mathbf{T}_w$ is a Coxeter torus of $\mathbf{L}$. We denote by Y_L the Deligne-Lusztig variety associated to a Coxeter element for $\mathbf{L}$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L}$.

The character ζ is the restriction of a character $\hat\zeta \in \mathrm{Irr}_K(L)$ and we have an isomorphism of KL-modules

$$H^j_c(Y_L, K) \otimes_{KT} \zeta \simeq \hat\zeta \otimes_K H^j_c(Y_L/T, K).$$

By [Lu2, Theorem 6.1], we have

$$\mathrm{Hom}_{KL}(H^i_c(Y_L/T, K), H^{i'}_c(Y_L/T, K)) = 0 \text{ for } i \neq i',$$

hence

$$\mathrm{Hom}_{KL}(H^i_c(Y_L, K) \otimes_{KT} \zeta, H^{i'}_c(Y_L, K) \otimes_{KT} \zeta) = 0 \text{ for } i \neq i'.$$

Furthermore, if not zero, then $H^j_c(Y_L/T, K)$ is an irreducible representation of KL [Lu2]. By [BonRou1, Theorem B], the KG-modules $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\hat\zeta \otimes_K H^i_c(Y_L/T, K))$ are irreducible and distinct, when i varies. Since

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(R\Gamma_c(Y_L, K)) \simeq R\Gamma_c(Y(\dot w), K),$$

it follows that the $H^i_c(Y(\dot w), K) \otimes_K \zeta$ are irreducible and distinct, when i varies.

Note that $H^*_c(Y(\dot w), K) \otimes_{KT} \zeta$ depends on ζ only up to $N_G(T)$-conjugacy. Since the action of $\tilde F$ on T coincides with that of w, it follows that, if V is an irreducible KG-module, then $V \otimes_{KG} H^*_c(Y(\dot w), K)$ is irreducible under the right action of $T \rtimes \langle \tilde F \rangle$. The result follows now from Lemma 1.2. □

Remark 4.4. One needs actually a weaker statement than the disjunction of the cohomology groups for the G-action. We only need to know that the part of the Lefschetz character of the cohomology of $Y(\dot w)$ corresponding to a fixed eigenvalue of F is an irreducible representation of G^F. This can be deduced, in the unipotent case, from [DiMi1].

4.3. **Determination of the $(T \rtimes F)$-endomorphisms.**

Proposition 4.5. *For* $R = \mathcal{O}$ *or* k, *the canonical map* $bRG \to R\,\mathrm{End}^\bullet_{RT}(RC)^{\tilde F}$ *is an isomorphism in* $D^b(RG \otimes (RG)^{\mathrm{opp}})$.

Proof. By Proposition 1.1, it is enough to prove the Proposition for the simply connected covering of $[\mathbf{G}, \mathbf{G}]$, which is isomorphic to SL_n. Using the embedding $\mathrm{SL}_n \to \mathrm{GL}_n$, we have a further reduction to the case $\mathbf{G} = \mathrm{GL}_n$.

We choose for $\mathbf{T}_0$ the diagonal torus of GL_n. We denote by $x_i \in X(\mathbf{T}_0)$ the i-th coordinate function and by $\omega_i^\vee \in Y(\mathbf{T}_0)$ the cocharacter that sends a to the diagonal matrix with coefficients 1 at all positions except the i-th where the coefficient is a. Let ζ be a generator of $\mathbf{F}_{q^n}^\times$. We have $N_w(\omega_i^\vee) = (\zeta, \zeta^q, \dots, \zeta^{q^{n-1}})^{q^{n-i+1}}$. Furthermore, x_1 induces an isomorphism $T \xrightarrow{\sim} \mathbf{F}_{q^n}^\times$

and we identify these groups. Let $s_i = (i, i+1)$. We have $\beta_i^\vee = \omega_1^\vee - \omega_{i+1}^\vee$ and $N_w(\beta_i^\vee) = \zeta^{1-q^{n-i}}$. So, $\mu_{m_i} = \langle \zeta^{1-q^{n-i}} \rangle$ by Proposition 3.5.

Let $\theta \in \mathrm{Irr}(\mathbf{F}_{q^n}^\times)$. We have $c_\theta = \min\{a \geq 1 | \theta(\zeta^{1-q^a}) = 1\}$ and $n-1-|I_\theta| = |\{i \in \{1, \ldots, n-1\} | \theta(\zeta^{1-q^{n-i}}) = 1\}|$. It follows that $n-1-|I_\theta| = \frac{n}{c_\theta} - 1$.

Let ψ be a linear regular character of U_0. We deduce now from Corollary 3.15 that $H^i(RE)^{\tilde{F}} = 0$ for $i \neq 0$, hence using Corollary 4.2, we obtain $\mathrm{Hom}_{D^b(RT)}(RC, RC[i])^{\tilde{F}} = 0$ for $i \neq 0$.

By Lemmas 1.2 and 4.3, the canonical map $b\mathcal{O}G \to \mathrm{End}_{K^b(\mathcal{O}T)}(C)^{\tilde{F}}$ is an isomorphism. Lemma 3.15 shows that the corresponding assertion is true over k as well. □

4.4. **Broué's conjecture.**

Theorem 4.6. *Assume* $\mathbf{G}$ *has type* A_{n-1} *and* F *is a split Frobenius endomorphism over* $\mathbf{F}_q$*. Assume moreover that the order of* q *modulo* ℓ *is equal to* n*. Let* $\mathcal{O}Gb$ *be the sum of blocks of* $\mathcal{O}G$ *with non-zero defect. Let* S *be a Sylow* ℓ*-subgroup of* T*. Then,* $C_G(S) = T$*.*

The action of $\mathcal{O}G$ *on* $\tilde{R}\Gamma_c(Y(\dot{w}), \mathcal{O})$ *comes from an action of* $\mathcal{O}Gb$ *and the right action of* $\mathcal{O}T$ *extends to an action of* $\mathcal{O}N_G(T)$*. The complex thus obtained induces a splendid Rickard equivalence between* $\mathcal{O}Gb$ *and* $\mathcal{O}N_G(T)$*.*

Proof. The complex of $(\mathcal{O}G, \mathcal{O}T)$-bimodules C' (cf §1.3.4) is w-stable. It follows that it extends to a complex D of $(\mathcal{O}G, \mathcal{O}(T \rtimes \langle w \rangle))$-bimodules.

On the other hand, the complex kC is isomorphic in $D^b(k(G \times (T \rtimes F)^{\mathrm{opp}})\text{-Mod})$ to a bounded complex D' whose terms are finite dimensional. There is a positive integer N such that F^N acts trivially on D'. We take for ν a positive integer such that $\ell^\nu \geq |k|$, $\ell^\nu \equiv 1 \pmod{n}$, and $\nu \geq \nu_l(N)$ and we put $\tilde{F} = F^{\ell^\nu}$ as above. There is a positive integer t prime to ℓ such that $\tilde{F}^t$ acts trivially on D'. Let e be a block idempotent of $b\mathcal{O}G$. Then, it follows from Proposition 4.5 that $\mathrm{End}_{D^b(kG \otimes (kT \rtimes \langle \tilde{F} \rangle)^{\mathrm{opp}})}(eD') \simeq Z(ekG)$, hence eD' is isomorphic in $D^b(kG \otimes (kT \rtimes \langle \tilde{F} \rangle)^{\mathrm{opp}})$ to an indecomposable complex X. The complexes kX and ekD have quasi-isomorphic restriction to $kG \otimes kT$, hence (Clifford theory) they differ by tensor product by a one-dimensional $k\langle \tilde{F} \rangle$-module L, where we identify w with $\tilde{F}$. In particular, the canonical map

$$ekG \to \mathrm{End}^\bullet_{k(T \rtimes w)^{\mathrm{opp}}}(ekD)$$

is an isomorphism. It follows from [Ri2, Theorem 2.1] that ekD is a two-sided tilting complex, *i.e.*, the canonical map $\mathrm{br}(e)k(T \rtimes w) \to \mathrm{End}^\bullet_{ekG}(ekD)$ is an isomorphism as well. We deduce that D is a two-sided tilting complex for $b\mathcal{O}G$ and $\mathcal{O}(T \rtimes w)$ (cf *e.g.*[Ri2, Proof of Theorem 5.2]).

Let Q be an ℓ-subgroup of $G \times T^{\mathrm{opp}}$. We have $Y(\dot{w})^Q = \emptyset$ unless Q is conjugate to a subgroup of ΔS and $Y(\dot{w})^Q = T$ if Q is a non-trivial subgroup of ΔS. It follows (cf [Rou1, proof of Theorem 5.6]) that D is a splendid Rickard complex. □

Remark 4.7. For groups not untwisted of type A, the $\mathcal{O}G$-module $M = \mathrm{Ind}_{D(\mathbf{U}_0)^F}^G \mathcal{O}$ is not a progenerator in general. It might nevertheless be possible to prove that $R\Gamma_c(D(\mathbf{U}_0)^F \backslash Y(\dot{w}), \mathcal{O})$ induces a derived equivalence between $\mathrm{End}_{\mathcal{O}G}(bM)$ and a quotient of $\mathcal{O}N_G(T)$.

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