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Effects of parametric noise on a nonlinear oscillator

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Abstract

We study a model of a nonlinear oscillator with a random frequency and derive the asymptotic behavior of the probability distribution function when the noise is white. In the small damping limit, we show that the physical observables grow algebraically with time before the dissipative time scale is reached, and calculate the associated anomalous diffusion exponents. In the case of colored noise, with a nonzero but arbitrarily small correlation time, the characteristic exponents are modified. We determine their values thanks to a self-consistent Ansatz.

Key words: Langevin dynamics, multiplicative noise, nonlinear oscillations
PACS: 05.10.Gg, 05.40.-a, 05.45.-a

1 Introduction

A simple model of a nonlinear stochastic system is obtained by including a non-quadratic confining potential in the classical Langevin equation [1,2]. In this work, we address the problem of a nonlinear oscillator with random linear frequency. Due to the continuous injection of energy by the random force, observables such as the oscillator's mechanical energy, position and velocity, grow algebraically with time in the absence of dissipation. In Section 2, we

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calculate the associated growth exponents in the case of Gaussian white noise, derive the long time asymptotic behavior of the probability distribution function (P.D.F.) in phase space and match it with the (non-Gibbsean) stationary P.D.F. in presence of small dissipation. In Section 3, we prove that if the noise has a non-vanishing correlation time, the dynamics of the nonlinear oscillator is drastically modified: colored noise yields anomalous diffusion exponents equal to half the values found for white noise.

2 White multiplicative noise

We consider a nonlinear oscillator of amplitude $x(t)$, submitted to parametric noise and confined by a potential $\mathcal{U}(x)$. For $|x| \to \infty$, a suitable rescaling permits us to write $\mathcal{U} \sim \frac{x^{2n}}{2n}$ with $n \geq 1$. Thus, for large amplitudes, the stochastic differential equation that governs the system becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + x(t)^{2n-1} = \xi(t)x\,, \quad \text{with} \quad \langle\xi(t)\xi(t')\rangle = \mathcal{D}\,\delta(t-t'), \tag{1}$$

where γ is the damping rate and $\xi(t)$ is a Gaussian white noise of zero mean-value and of autocorrelation $\mathcal{D}$. This equation is interpreted according to the rules of Stratonovich Calculus [2]. For a quadratic potential (*i.e.* $n = 1$) and for small γ, the energy of the oscillator grows exponentially with time [3]. The effect of nonlinear restoring forces must be taken into account to avoid this exponential amplification.

We shall analyse the motion of the nonlinear stochastic oscillator following the method explained in [4,5]. Defining the energy and the angle variables as

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n} \quad , \quad \text{and} \quad \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int\limits_0^{x/E^{1/2n}} \frac{\mathrm{d}u}{\sqrt{1-\frac{u^{2n}}{2n}}}\,, \tag{2}$$

we transform the coordinates in phase space from position and velocity to energy and angle. Introducing an auxiliary variable Ω, with $\Omega = (2n)^{\frac{n+1}{2n}}\,E^{\frac{n-1}{2n}}$, we rewrite Eq. (1) as

$$\dot{\Omega} = -\gamma\,\frac{n-1}{(2n)^{\frac{1}{n}}}\,\mathcal{S}_n'(\phi)^2\,\Omega \,+\, (n-1)\,\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)\,\xi(t)\,, \tag{3}$$

$$\dot{\phi} = \gamma\,\frac{\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)}{(2n)^{\frac{1}{n}}} \,+\, \frac{\Omega}{(2n)^{\frac{1}{n}}} - \frac{\mathcal{S}_n(\phi)^2}{\Omega}\,\xi(t)\,, \tag{4}$$

where the hyperelliptic function $\mathcal{S}_n$ satisfies $\mathcal{S}_n(\phi) = x/E^{1/(2n)}$. These equations are rigorous and have been derived without any hypothesis on the noise term. In the small damping limit $\gamma \to 0$, we deduce from Eqs. (3,4) that $\Omega \sim t$ and $\phi \sim t^2$, *i.e.* ϕ is a fast variable as compared to Ω. Averaging the dynamics over ϕ yields the statistical equipartition identities [4]:

$$\langle E \rangle = \frac{n+1}{2n} \langle \dot{x}^2 \rangle \quad \text{and} \quad \langle x^{2n} \rangle = \langle \dot{x}^2 \rangle \,. \tag{5}$$

This averaging procedure [6,7] allows us to derive a closed equation for the stochastic evolution of the slow variable Ω. We start with the Fokker-Planck equation for the P.D.F. $P_t(\Omega, \phi)$ associated with the system (3, 4), and average it under the hypothesis that the probability density becomes uniform in ϕ when $t \to \infty$. Hence, the reduced probability density $\tilde{P}_t(\Omega)$ satisfies

$$\partial_t \tilde{P} = \gamma \, \frac{n-1}{n+1} \, \partial_\Omega \left(\Omega \tilde{P} \right) \; + \; \frac{\tilde{\mathcal{D}}}{2} \left(\partial_\Omega^2 \tilde{P} - \frac{2}{n-1} \, \partial_\Omega \frac{\tilde{P}}{\Omega} \right) \,. \tag{6}$$

This averaged Fokker-Planck equation is associated with the following effective Langevin dynamics for the variable Ω,

$$\dot{\Omega} = \frac{\tilde{\mathcal{D}}}{n-1} \, \frac{1}{\Omega} - \gamma \, \frac{n-1}{n+1} \, \Omega + \tilde{\xi}(t) \tag{7}$$

where the effective Gaussian white noise $\tilde{\xi}$ has an amplitude $\tilde{\mathcal{D}}$ given by

$$\tilde{\mathcal{D}} = \mathcal{D} \, (2n)^{\frac{2}{n}} \, \frac{(n-1)^2}{n+1} \, \frac{\Gamma\left(\frac{3}{2n}\right) \, \Gamma\left(\frac{3n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right) \, \Gamma\left(\frac{3n+3}{2n}\right)} . \tag{8}$$

If the dissipation rate γ is taken to be zero, the solution of Eq. (6) is [4]:

$$\tilde{P}_t(E) = \frac{1}{\Gamma\left(\frac{n+1}{2(n-1)}\right)} \, \frac{n-1}{nE} \, \left(\frac{(2n)^{\frac{n+1}{n}} \, E^{\frac{n-1}{n}}}{2\tilde{\mathcal{D}} t} \right)^{\frac{n+1}{2(n-1)}} \exp\left\{ -\frac{(2n)^{\frac{n+1}{n}} \, E^{\frac{n-1}{n}}}{2\tilde{\mathcal{D}} t} \right\} . \tag{9}$$

From this P.D.F., we derive analytical expressions for statistical averages of physical observables [4]. The leading scaling behaviour is given by

$$E \sim (\mathcal{D}t)^{\frac{n}{n-1}} \,, \qquad x \sim (\mathcal{D}t)^{\frac{1}{2(n-1)}} \,, \qquad \dot{x} \sim (\mathcal{D}t)^{\frac{n}{2(n-1)}} \,. \tag{10}$$

These formulae are in excellent agreement with numerical simulations [4]. We show in Fig. 1 that the averaged energy scales like $t^{\frac{n}{n-1}}$ as predicted.

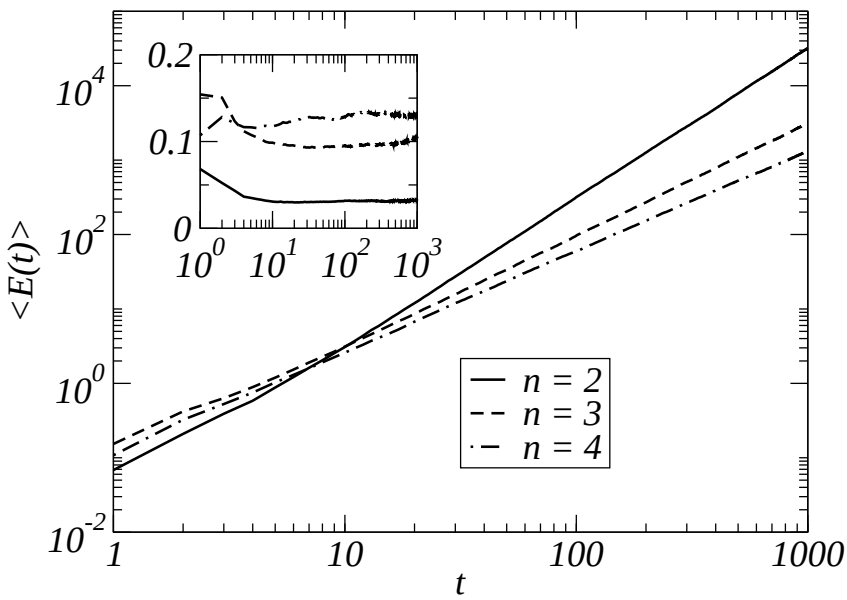

Fig. 1. Scaling behavior of $\langle E(t)\rangle$ for white, multiplicative noise. Inset: the ratio $\langle E\rangle/t^{n/(n-1)}$ is plotted vs. time. Eq. (1) is integrated numerically for $\gamma = 0$, $\mathcal{D} = 1$, with a timestep δt, and averaged over 10^4 realizations for $n = 2$, $\delta t = 5\,10^{-4}$; $n = 3$, $\delta t = 5\,10^{-4}$; $n = 4$, $\delta t = 10^{-4}$.

The amplitude x undergoes an anomalous diffusion with an exponent that diverges as $n \to 1$, which is consistent with the exponential growth of the linear oscillator.

In the presence of dissipation, the stationary solution of the Fokker-Planck equation (6) is

$$\tilde{P}_{\rm st}(E) = \frac{1}{\Gamma\left(\frac{n+1}{2(n-1)}\right)}\,\frac{n-1}{nE}\left(\frac{(2n)^{\frac{n+1}{n}}\,E^{\frac{n-1}{n}}}{\frac{n+1}{\gamma(n-1)}\tilde{\mathcal{D}}}\right)^{\frac{n+1}{2(n-1)}}\exp\left\{-\frac{(2n)^{\frac{n+1}{n}}\,E^{\frac{n-1}{n}}}{\frac{n+1}{\gamma(n-1)}\tilde{\mathcal{D}}}\right\} \quad (11)$$

This measure is *not* the canonical Boltzmann-Gibbs distribution. The crossover time t_c from the asymptotic distribution function for the undamped oscillator (9) to the stationary measure in presence of damping (11) is $t_c = \frac{1}{2\gamma}\frac{n+1}{n-1}$. Thus, Eq. (9), although derived for a non-dissipative system, is physically relevant: it provides an intermediate time asymptotic behavior [8] for the P.D.F. before the stationary state is reached; the energy of the system is distributed according to the time-dependent P.D.F. (9) for times t such that $1 \ll t \ll t_c$ and then follows the stationary law (11) when $t \gg t_c$.

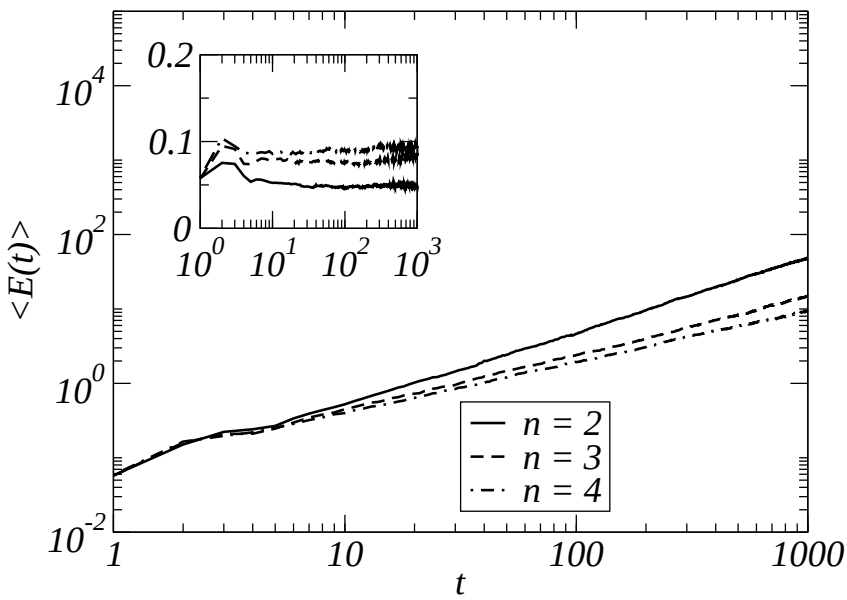

Fig. 2. Scaling behavior of $\langle E(t)\rangle$ for colored Ornstein-Uhlenbeck multiplicative noise. Inset: the ratio $\langle E\rangle/t^{n/(2(n-1))}$ is plotted vs. time. Eqs. (12)-(13) are integrated numerically for $\mathcal{D}=1$, $\tau=1$, with a timestep δt, and averaged over 10^3 realizations for $n=2$, $\delta t=10^{-5}$; $n=3$, $\delta t=5\,10^{-6}$; $n=4$, $\delta t=2\,10^{-6}$.

3 Colored multiplicative noise

We now consider a system without damping and submitted to a multiplicative noise with non-zero correlation time τ. In [4], we infered from dimensional analysis that the white noise scalings are modified. Here, we shall prove this conjecture by calculating the anomalous exponents from a self-consistent approach.

We study the equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+x(t)^{2n-1}=\eta(t)\,x(t)\,, \tag{12}$$

where $\eta(t)$ is a colored, Ornstein-Uhlenbeck, Gaussian noise, obeying

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t}=-\frac{1}{\tau}\eta(t)+\frac{1}{\tau}\xi(t). \tag{13}$$

When $t,t'\gg\tau$, we find:

$$\langle\eta(t)\rangle=0 \;\text{ and }\; \langle\eta(t)\eta(t')\rangle=\frac{\mathcal{D}}{2\,\tau}\,\mathrm{e}^{-|t-t'|/\tau}\,. \tag{14}$$

As stated earlier, Eqs. (3) and (4) remain valid if $\xi(t)$ is replaced by $\eta(t)$. We

assume that, in the long time limit, Ω grows algebraically with time with a scaling exponent α, *i.e.*, $\Omega \sim t^{\alpha}$. We deduce from Eq. (4) that $\phi \sim t^{\nu}$ with $\nu = \alpha + 1$. Substituting this scaling of ϕ in Eq. (3), we obtain (leaving aside all proportionality constants):

$$\Omega \sim \int_0^t \mathrm{dz}\, \mathcal{S}_{\mathrm{n}}(\mathrm{z}^{\nu})\, \mathcal{S}'_{\mathrm{n}}(\mathrm{z}^{\nu})\, \eta(\mathrm{z})\,. \tag{15}$$

The asymptotic behavior of this expression can be estimated as in [5] by discretizing time. This simplification does not alter the critical exponents but modifies only the prefactors. We thus replace the colored Gaussian noise η by a discrete random variable ϵ_k which takes the values $\pm\sqrt{\mathcal{D}/(2\tau)}$ randomly during the time interval $[k\tau\,, (k+1)\tau]$. From Eq. (15), we deduce that

$$\langle \Omega^2 \rangle \sim \sum_{k=0}^{t/\tau} \left(\int_{k\tau}^{(k+1)\tau} \mathrm{dz}\, \mathcal{S}_{\mathrm{n}}(\mathrm{z}^{\nu})\, \mathcal{S}'_{\mathrm{n}}(\mathrm{z}^{\nu}) \right)^2 \sim \sum_{\mathrm{k}=1}^{\mathrm{t}/\tau} \frac{1}{\left(\mathrm{k}\tau\right)^{2\nu-2}} \sim \mathrm{t}^{3-2\nu}\,, \tag{16}$$

where the integral is evaluated by integrating by parts and retaining only the leading terms [5]. Assuming that the variable Ω is not multifractal, we also have $\Omega^2 \sim t^{2\alpha}$. We then deduce that $3 - 2\nu = 2\alpha$. Since $\nu = \alpha + 1$, we obtain $\alpha = 1/4$ and $\nu = 5/4$. Finally, the scaling exponents for colored noise are

$$E \sim t^{\frac{n}{2(n-1)}}, \quad x \sim t^{\frac{1}{4(n-1)}}, \quad \dot{x} \sim t^{\frac{n}{4(n-1)}}. \tag{17}$$

These predictions are confirmed by numerical simulations (see in Fig. 2 the scaling behavior of the averaged energy).

We observe that the exponents are halved when the noise is colored. Further, we expect that the colored noise scaling will be observed for large enough time as soon as the correlation time τ is nonzero. This result may seem paradoxical at first sight because it is often wrongly thought that "in the long time limit, colored noise appears white". In fact the period T of the underlying deterministic oscillator must be compared with the correlation time τ of the noise. If $\tau \ll T$, the noise is uncorrelated over a period of the deterministic oscillator and acts as if it were white: the scalings found in Eq. (10) are satisfied. But, for $T \ll \tau$, the noise remains coherent over a period and its effect is perceptible only when it is considered over a large number of periods and hence the diffusion slows down: the scalings of Eq. (17) now apply. The crossover between the two regimes is observed when $T \sim \tau$. The deterministic period T decreases with the energy as $T \sim E^{-\frac{n-1}{2n}}$ [4]. From Eq.(10), we have $E^{-\frac{n-1}{2n}} \sim (\mathcal{D}t)^{-1/2}$. Therefore, the crossover time scales as $t_c \sim (\mathcal{D}\tau^2)^{-1}$. As shown in Figure 3, numerical simulations confirm this scenario. Such a crucial

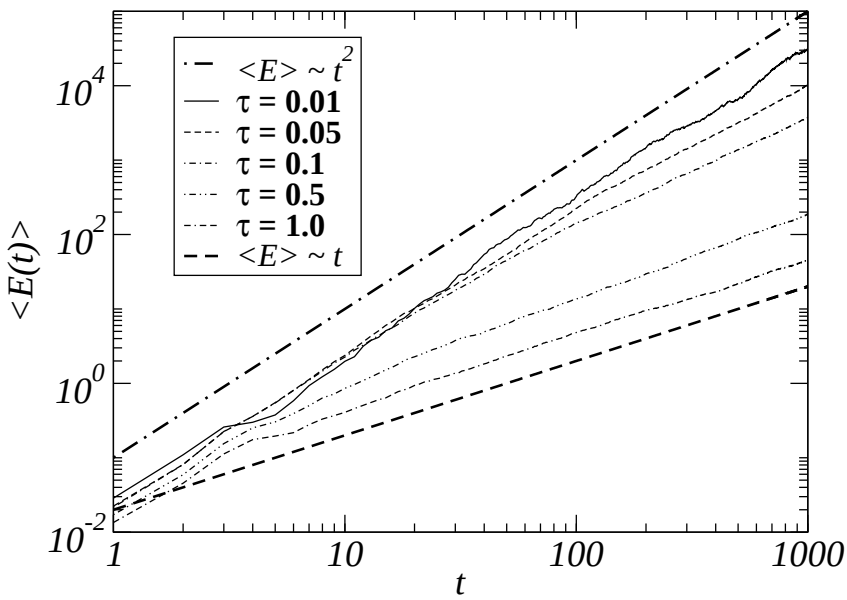

Fig. 3. Scaling behavior of $\langle E(t)\rangle$ for an oscillator in a quartic potential ($n = 3$) submitted to Ornstein-Uhlenbeck multiplicative noise with various values of the correlation time. We find a crossover between the white noise behavior observed at short times, $\langle E(t)\rangle \propto t^2$, and the colored noise behavior recovered at large enough times, $\langle E(t)\rangle \propto t$.

relevance of the correlation time of the noise was also found in the study of the bifurcation threshold of the Duffing oscillator with multiplicative noise [9].

4 Conclusion

A particle trapped in a confining potential and subject to multiplicative noise undergoes anomalous diffusion before the dissipative time scale is reached. The critical exponents associated with this behaviour are calculated exactly in the case of Gaussian white noise and an analytical expression for the intermediate time asymptotics of the P.D.F. in phase space is derived.

Our approach relies on the integrability of the underlying deterministic system: we derive exact stochastic equations in energy-angle variables and, after averaging out the phase variations, a projected dynamics for the energy variable is obtained. Our analytical predictions agree perfectly with numerical simulations. For colored multiplicative noise, the anomalous diffusion exponents have been calculated in a self-consistent manner. We find that these exponents are halved in presence of time correlations, however small the correlation time τ may be.

We observe a crossover from the white noise regime at *short* times to the colored noise scalings at *long* times. The associated crossover time scales like τ^{-2}

and corresponds to the regime when the period of the associated deterministic oscillator is of the order of τ. In the colored noise case, however, the averaging technique does not lead to conclusive results because the noise itself is averaged out to the leading order. A precise calculation in the case of colored Gaussian noise still remains to be done.

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