On the complexity of algebraic number I. Expansions in integer bases.

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Abstract. *Let $b \geq 2$ be an integer. We prove that the b-adic expansion of every irrational algebraic number cannot have low complexity. Furthermore, we establish that irrational morphic numbers are transcendental, for a wide class of morphisms. In particular, irrational automatic numbers are transcendental. Our main tool is a new, combinatorial transcendence criterion.*

1. Introduction

Let $b \geq 2$ be an integer. The b-adic expansion of every rational number is eventually periodic, but what can be said on the b-adic expansion of an irrational algebraic number? This question was addressed for the first time by Émile Borel [11], who made the conjecture that such an expansion should satisfies the same laws as do almost all real numbers. In particular, it is expected that every irrational algebraic number is normal in base b. Recall that a real number θ is called *normal in base* b if, for any positive integer n, each one of the b^n blocks of length n on the alphabet $\{0, 1, \ldots, b-1\}$ occurs in the b-adic expansion of θ with the same frequency $1/b^n$. This conjecture is reputed to be out of reach: we even do not know whether the digit 7 occurs infinitely often in the decimal expansion of $\sqrt{2}$. However, some (very) partial results have been established.

As usual, we measure the complexity of an infinite word $\mathbf{u} = u_1 u_2 \ldots$ defined on a finite alphabet by counting the number $p(n)$ of distinct blocks of length n occurring in the word $\mathbf{u}$. In particular, the b-adic expansion of every real number normal in base b satisfies $p(n) = b^n$ for any positive integer n. Using a clever reformulation of a theorem of Ridout [31], Ferenczi and Mauduit [18] established the transcendence of the real numbers whose b-adic expansion is a non eventually periodic sequence of minimal complexity, that is, which satisfies $p(n) = n + 1$ for every $n \geq 1$ (such a sequence is called a Sturmian sequence, see the seminal papers by Morse and Hedlund [26, 27]). The combinatorial criterion given in [18] has been used subsequently to exhibit further examples of transcendental numbers with low complexity [3, 6, 4, 32]. It also implies that the complexity of the b-adic expansion of every irrational algebraic number satisfies $\liminf_{n\to\infty}(p(n) - n) = +\infty$. Although this is very far away from what is expected, no better result is known.

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In 1965, Hartmanis and Stearns [19] proposed an alternative approach for the notion of complexity of real numbers, by emphasizing the quantitative aspect of the notion of calculability introduced by Turing [39]. According to them, a real number is said to be computable in time $T(n)$ if there exists a multitape Turing machine which gives the first n-th terms of its binary expansion in (at most) $T(n)$ operations. The 'simpler' real numbers in that sense, that is, the numbers for which one can choose $T(n) = O(n)$, are said to be computable in real time. Rational numbers share clearly this property. The problem of Hartmanis and Stearns, to which a negative answer is expected, is the following: do there exist irrational algebraic numbers which are computable in real time? In 1968, Cobham [13] suggested to restrict this problem to a particular class of Turing machines, namely to the case of finite automata (see Section 3 for a definition). After several attempts by Cobham [13] in 1968 and by Loxton and van der Poorten [21] in 1982, Loxton and van der Poorten [22] finally claimed to have completely solved the restricted problem in 1988. More precisely, they asserted that the b-adic expansion of every irrational algebraic number cannot be generated by a finite automaton. The proof proposed in [22], which rests on a method introduced by Mahler [23, 24, 25], contains unfortunately a rather serious gap, as explained by Becker [8] (see also [40]). Furthermore, the combinatorial criterion established in [18] is too weak to imply this statement, often referred to as the conjecture of Loxton and van der Poorten.

In the present paper, we prove new results concerning both notions of complexity. Our Theorem 1 provides a sharper lower estimate for the complexity of the b-adic expansion of every irrational algebraic number. We are still far away from proving that such an expansion is normal, but we considerably improve upon the earlier known results. We further establish (Theorem 2) the conjecture of Loxton and van der Poorten, namely that irrational automatic numbers are transcendental. Our proof yields more general statements and allows us to confirm that irrational morphic numbers are transcendental, for a wide class of morphisms (Theorems 3 and 4).

We derive Theorems 1 to 4 from a refinement (Theorem 5) of the combinatorial criterion from [18], that we obtain as a consequence of the Schmidt Subspace Theorem.

Throughout the present paper, we adopt the following convention. We use small letters (a, u, etc.) to denote letters from some finite alphabet $\mathcal{A}$. We use capital letters (U, V, W, etc.) to denote finite words. We use bold small letters ($\mathbf{a}$, $\mathbf{u}$, etc.) to denote infinite sequence of letters. We often identify the sequence $\mathbf{a} = (a_k)_{k\geq 1}$ with the infinite word $a_1a_2\ldots$, also called $\mathbf{a}$. This should not cause any confusion.

Our paper is organized as follows. The main results are stated in Section 2 and proved in Section 5. Some definitions from automata theory and combinatorics on words are recalled in Section 3. Section 4 is devoted to the new transcendence criterion and its proof. Finally, we show in Section 6 that the Hensel expansion of every irrational algebraic p-adic number cannot have a low complexity, and we conclude in Section 7 by miscellaneous remarks.

Some of the results of the present paper were announced in [2].

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2. Main results

As mentioned in the first part of the Introduction, we measure the complexity of a real number written in some integral base $b \geq 2$ by counting, for any positive integer n, the number $p(n)$ of distinct blocks of n digits (on the alphabet $\{0, 1, \ldots, b-1\}$) occurring in its b-adic expansion. The fonction p is commonly called the *complexity function*. It follows from results of Ferenczi and Mauduit [18] (see also [4], Théorème 3) that the complexity function p of every irrational algebraic number satisfies

$$\liminf_{n\to\infty} (p(n) - n) = +\infty. \tag{1}$$

As far as we are aware, no better result is known, although it has been proved [3, 6, 32] that some special real numbers with linear complexity are transcendental.

Our first result is a considerable improvement of (1).

Theorem 1. *Let $b \geq 2$ be an integer. The complexity function of the b-adic expansion of every irrational algebraic number satisfies*

$$\liminf_{n\to\infty} \frac{p(n)}{n} = +\infty.$$

It immediately follows from Theorem 1 that every irrational real number with sub-linear complexity (i.e., such that $p(n) = O(n)$) is transcendental. However, Theorem 1 is slightly sharper, as is illustrated by an example due to Ferenczi [17]: he established the existence of a sequence on a finite alphabet whose complexity function p satisfies

$$\liminf_{n\to\infty} \frac{p(n)}{n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{p(n)}{n^t} = +\infty \quad \text{for any } t > 1.$$

Most of the previous attempts towards a proof of the conjecture of Loxton and van der Poorten have been made via the Mahler method [21, 22, 8, 29]. We stress that Becker [8] established that, for any given non-eventually periodic automatic sequence $\mathbf{u} = u_1 u_2 \ldots$, the real number $\sum_{k\geq 1} u_k b^{-k}$ is transcendental, provided that the integer b is sufficiently large (in terms of $\mathbf{u}$). Since the complexity function p of any automatic sequence satisfies $p(n) = O(n)$ (see Cobham [14]), Theorem 1 confirms straightforwardly this conjecture.

Theorem 2. *Let $b \geq 2$ be an integer. The b-adic expansion of any irrational algebraic number cannot be generated by a finite automaton. In other words, irrational automatic numbers are transcendental.*

Although Theorem 2 is a direct consequence of Theorem 1, we give in Section 5 a short proof of it, that rests on another result of Cobham [14].

Theorem 2 establishes a particular case of the following widely believed conjecture (see e.g. [5]). The definitions of morphism, recurrent morphism, and morphic number are recalled in Section 3.

Conjecture. *Irrational morphic numbers are transcendental.*

Our method allows us to confirm this conjecture for a wide class of morphisms.

Theorem 3. *Binary algebraic irrational numbers cannot be generated by a morphism.*

As observed by Allouche and Zamboni [6], it follows from [18] combined with a result of Berstel and Séébold [9] that binary irrational numbers which are fixed point of a primitive morphism or of a morphism of constant length ≥ 2 are transcendental. Our Theorem 3 is much more general.

Recently, by a totally different method, Bailey, Borwein, Crandall, and Pomerance [7] established new, interesting results on the density of the digits in the binary expansion of algebraic numbers.

For b-adic expansions with $b \geq 3$, we obtain a similar result as in Theorem 3, but an additional assumption is needed.

Theorem 4. *Let $b \geq 3$ be an integer. The b-adic expansion of an algebraic irrational number cannot be generated by a recurrent morphism.*

Unfortunately, we are unable to prove that ternary algebraic numbers cannot be generated by a morphism. Consider for instance the fixed point

$$\mathbf{u} = 0121221222122221222221222222221222\ldots$$

of the morphism defined by $0 \to 012$, $1 \to 12$, $2 \to 2$, and set $\alpha = \sum_{k \geq 1} u_k 3^{-k}$. Our method does not apply to show the transcendence of α. Let us mention that this α is known to be transcendental: this is a consequence of deep transcendence results proved in [10] and in [15], concerning the values of theta series at algebraic points.

The proofs of Theorems 1 to 4 are given in Section 5. The key point for them is a new transcendence criterion, derived from the Schmidt Subspace Theorem, and stated in Section 4. Actually, we are able to deal also, under some conditions, with non-integer bases (see Theorems 5 and 5A). Given a real number $\beta > 1$, we can expand in base β every real number ξ in $(0, 1)$ thanks to the greedy algorithm: we then get the β-expansion of ξ, introduced by Renyi [30]. Using Theorem 5, we easily see that the conclusions of Theorems 1 to 4 remain true with the expansion in base b replaced by the β-expansion when β is a Pisot or a Salem number. Recall that a Pisot (resp. Salem) number is a real algebraic integer > 1, whose complex conjugates lie inside the open unit disc (resp. inside the closed unit disc, with at least one of them on the unit circle). In particular, any integer $b \geq 2$ is a Pisot number. For instance, we get the following result.

Theorem 1A. *Let $\beta > 1$ be a Pisot or a Salem number. The complexity function of the β-expansion of every algebraic number in $(0, 1) \setminus \mathbf{Q}(\beta)$ satisfies*

$$\liminf_{n \to \infty} \frac{p(n)}{n} = +\infty.$$

Likewise, we can also state Theorems 2A, 3A, and 4A accordingly: Theorems 1 to 4 deal with algebraic irrational numbers, while Theorems 1A to 4A deal with algebraic numbers in $(0, 1)$ which do not lie in the number field generated by β.

Moreover, our method also allows us to prove that p-adic irrational numbers whose Hensel expansions have low complexity are transcendental, see Section 6.

3. Finite automata and morphic sequences

In this Section, we gather classical definitions from automata theory and combinatorics on words.

Finite automata and automatic sequences. Let k be an integer with $k \geq 2$. We denote by Σ_k the set $\{0, 1, \ldots, k-1\}$. A k-automaton is defined as a 6-tuple

$$A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau),$$

where Q is a finite set of states, Σ_k is the input alphabet, $\delta : Q \times \Sigma_k \to Q$ is the transition function, q_0 is the initial state, Δ is the output alphabet and $\tau : Q \to \Delta$ is the output function.

For a state q in Q and for a finite word $W = w_1 w_2 \ldots w_n$ on the alphabet Σ_k, we define recursively $\delta(q, W)$ by $\delta(q, W) = \delta(\delta(q, w_1 w_2 \ldots w_{n-1}), w_n)$. Let $n \geq 0$ be an integer and let $w_r w_{r-1} \ldots w_1 w_0$ in $(\Sigma_k)^r$ be the k-ary expansion of n; thus, $n = \sum_{i=0}^{r} w_i k^i$. We denote by W_n the word $w_0 w_1 \ldots w_r$. Then, a sequence $\mathbf{a} = (a_n)_{n \geq 0}$ is said to be k-automatic if there exists a k-automaton A such that $a_n = \tau(\delta(q_0, W_n))$ for all $n \geq 0$.

A classical example of a 2-automatic sequence is given by the binary Thue-Morse sequence $\mathbf{a} = (a_n)_{n \geq 0} = 0110100110010\ldots$ This sequence is defined as follows: a_n is equal to 0 (resp. to 1) if the sum of the digits in the binary expansion of n is even (resp. is odd). It is easy to check that this sequence can be generated by the 2-automaton

$$A = \big(\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{0, 1\}, \tau\big),$$

where

$$\delta(q_0, 0) = \delta(q_1, 1) = q_0, \qquad \delta(q_0, 1) = \delta(q_1, 0) = q_1,$$

and $\tau(q_0) = 0$, $\tau(q_1) = 1$.

Morphisms. For a finite set $\mathcal{A}$, we denote by $\mathcal{A}^*$ the free monoid generated by $\mathcal{A}$. The empty word ε is the neutral element of $\mathcal{A}^*$. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets. An application from $\mathcal{A}$ to $\mathcal{B}^*$ can be uniquely extended to an homomorphism between the free monoids $\mathcal{A}^*$ and $\mathcal{B}^*$. We call morphism from $\mathcal{A}$ to $\mathcal{B}$ such an homomorphism.

Sequences generated by a morphism. A morphism ϕ from $\mathcal{A}$ into itself is said to be prolongable if there exists a letter a such that $\phi(a) = aW$, where W is a non-empty word such that $\phi^k(W) \neq \varepsilon$ for every $k \geq 0$. In that case, the sequence of finite words $(\phi^k(a))_{k \geq 1}$ converges in $\mathcal{A}^{\mathbf{N}}$ (endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$) to an infinite word $\mathbf{a}$. This infinite word is clearly a fixed point for ϕ and we say that $\mathbf{a}$ is generated by the morphism ϕ. If, moreover, every letter occurring in $\mathbf{a}$ occurs at

least twice, then we say that $\mathbf{a}$ is generated by a recurrent morphism. If the alphabet $\mathcal{A}$ has two letters, then we say that $\mathbf{a}$ is generated by a binary morphism. More generally, an infinite sequence $\mathbf{a}$ in $\mathcal{A}^{\mathbf{N}}$ is said to be morphic if there exist a sequence $\mathbf{u}$ generated by a morphism defined over an alphabet $\mathcal{B}$ and a morphism from $\mathcal{B}$ to $\mathcal{A}$ such that $\mathbf{a} = \phi(\mathbf{u})$.

For instance, the Fibonacci morphism σ defined from the alphabet $\{0, 1\}$ into itself by $\sigma(0) = 01$ and $\sigma(1) = 0$ is a binary, recurrent morphism which generates the Fibonacci infinite word

$$\mathbf{a} = \lim_{n\to\infty} \sigma^n(0) = 010010100100101001\ldots$$

This infinite word is an example of a Sturmian sequence and its complexity function satisfies thus $p(n) = n + 1$ for every positive integer n.

Automatic and morphic real numbers. Following the previous definitions, we say that a real number α is automatic (respectively, generated by a morphism, generated by a recurrent morphism, or morphic) if there exists an integer $b \geq 2$ such that the b-adic expansion of α is automatic (respectively, generated by a morphism, generated by a recurrent morphism, or morphic).

A classical example of binary automatic number is given by

$$\sum_{n\geq 1} \frac{1}{2^{2^n}}$$

which is transcendental, as proved by Kempner [20].

4. A transcendence criterion for stammering sequences

First, we need to introduce some notation. Let $\mathcal{A}$ be a finite set. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. For any positive integer ℓ, we write W^ℓ for the word $W\ldots W$ (ℓ times repeated concatenation of the word W). More generally, for any positive real number x, we denote by W^x the word $W^{\lfloor x\rfloor}W'$, where W' is the prefix of W of length $\lceil (x - \lfloor x\rfloor)|W|\rceil$. Here, and in all what follows, $\lfloor y\rfloor$ and $\lceil y\rceil$ denote, respectively, the integer part and the upper integer part of the real number y. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be a sequence of elements from $\mathcal{A}$, that we identify with the infinite word $a_1a_2\ldots$ Let $w > 1$ be a real number. We say that $\mathbf{a}$ satisfies Condition $(*)_w$ if $\mathbf{a}$ is not eventually periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $U_nV_n^w$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|U_n|/|V_n|)_{n\geq 1}$ is bounded from above;

(iii) The sequence $(|V_n|)_{n\geq 1}$ is increasing.

As suggested to us by Guy Barat, a sequence satisfying Condition $(*)_w$ for some $w > 1$ may be called a stammering sequence.

Theorem 5. *Let $\beta > 1$ be a Pisot or a Salem number. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be a bounded sequence of rational integers. If there exists a real number $w > 1$ such that $\mathbf{a}$ satisfies Condition $(*)_w$, then the real number*

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$$

either belongs to $\mathbf{Q}(\beta)$, or is transcendental.

The proof of Theorem 5 rests on the Schmidt Subspace Theorem [37] (see also [38]), and more precisely on a p-adic generalization due to Schlickewei [34, 35] and Evertse [16]. Note that the particular case when β is an integer ≥ 2 was proved in [2]. Note also that Adamczewski [1] proved that, under a stronger assumption on the sequence $(a_k)_{k\geq 1}$, the number α defined in the statement of Theorem 5 is transcendental.

Remarks.

• Theorem 5 is considerably stronger than the criterion of Ferenczi and Mauduit [18]: our assumption $w > 1$ replaces their assumption $w > 2$. This type of condition is rather flexible, compared with the Mahler method, for which a functional equation is needed. For instance, the conclusion of Theorem 5 also holds if the sequence $\mathbf{a}$ is an unbounded sequence of integers that does not increase too rapidly. Nevertheless, one should acknowledge that, when it can be applied, the Mahler method gives the transcendence of the infinite series $\sum_{k=1}^{+\infty} a_k \beta^{-k}$ for every algebraic number β such that this series converges.

• We emphasize that if a sequence $\mathbf{u}$ satisfies Condition $(*)_w$ and if ϕ is a non-erasing morphism (that is, if the image by ϕ of any letter has length at least 1), then $\phi(\mathbf{u})$ satisfies Condition $(*)_w$, as well. This observation is used in the proof of Theorem 2.

• If β is an algebraic number which is neither a Pisot, nor a Salem number, it is still possible to get a transcendence criterion using the approach followed for proving Theorem 5. However, the assumption $w > 1$ should then be replaced by a weaker one, involving the Mahler measure of β and $\limsup_{n\to\infty} |U_n|/|V_n|$. Furthermore, the same approach shows that the full strength of Theorem 5 holds when β is a Gaussian integer. More details will be given in a subsequent work.

Before beginning the proof of Theorem 5, we quote a version of the Schmidt Subspace Theorem, as formulated by Evertse [16].

We normalize absolute values and heights as follows. Let $\mathbf{K}$ be an algebraic number field of degree d. Let $M(\mathbf{K})$ denote the set of places on $\mathbf{K}$. For x in $\mathbf{K}$ and a place v in $M(\mathbf{K})$, define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/d}$ if v corresponds to the embedding $\sigma : \mathbf{K} \hookrightarrow \mathbf{R}$;

(ii) $|x|_v = |\sigma(x)|^{2/d} = |\overline{\sigma}(x)|^{2/d}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \overline{\sigma} : \mathbf{K} \hookrightarrow \mathbf{C}$;

(iii) $|x|_v = (N\mathfrak{p})^{-ord_{\mathfrak{p}}(x)/d}$ if v corresponds to the prime ideal $\mathfrak{p}$ of $O_{\mathbf{K}}$.

These absolute values satisfy the product formula

$$\prod_{v \in M(\mathbf{K})} |x|_v = 1 \qquad \text{for } x \text{ in } \mathbf{K}^*.$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be in $\mathbf{K}^n$ with $\mathbf{x} \neq \mathbf{0}$. For a place v in $M(\mathbf{K})$, put

$$|\mathbf{x}|_v = \left(\sum_{i=1}^{n} |x_i|_v^{2d} \right)^{1/(2d)} \quad \text{if } v \text{ is real infinite;}$$

$$|\mathbf{x}|_v = \left(\sum_{i=1}^{n} |x_i|_v^{d} \right)^{1/d} \quad \text{if } v \text{ is complex infinite;}$$

$$|\mathbf{x}|_v = \max\{|x_1|_v, \ldots, |x_n|_v\} \quad \text{if } v \text{ is finite.}$$

Now define the *height* of $\mathbf{x}$ by

$$H(\mathbf{x}) = H(x_1, \ldots, x_n) = \prod_{v \in M(\mathbf{K})} |\mathbf{x}|_v.$$

We stress that $H(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the number field $\mathbf{K}$ containing the coordinates of $\mathbf{x}$, see e.g. [16].

We use the following formulation of the Subspace Theorem over number fields. In the sequel, we assume that the algebraic closure of $\mathbf{K}$ is $\overline{\mathbf{Q}}$. We choose for every place v in $M(\mathbf{K})$ a continuation of $|\cdot|_v$ to $\overline{\mathbf{Q}}$, that we denote also by $|\cdot|_v$.

Theorem E. *Let $\mathbf{K}$ be an algebraic number field. Let $m \geq 2$ be an integer. Let S be a finite set of places on $\mathbf{K}$ containing all infinite places. For each v in S, let $L_{1,v}, \ldots, L_{m,v}$ be linear forms with algebraic coefficients and with*

$$\mathit{rank}\,\{L_{1,v}, \ldots, L_{m,v}\} = m.$$

Let ε be real with $0 < \varepsilon < 1$. Then, the set of solutions $\mathbf{x}$ in $\mathbf{K}^m$ to the inequality

$$\prod_{v \in S} \prod_{i=1}^{m} \frac{|L_{i,v}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq H(\mathbf{x})^{-m-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbf{K}^m$.

For a proof of Theorem E, the reader is directed to [16], where a quantitative version is established (in the sense that one bounds explicitly the number of exceptional subspaces).

We now turn to the proof of Theorem 5. Keep the notation and the assumptions of this theorem. Assume that the parameter $w > 1$ is fixed, as well as the sequences $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ occurring in the definition of Condition $(*)_w$. Set also $r_n = |U_n|$ and $s_n = |V_n|$ for any $n \geq 1$. We aim to prove that the real number

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$$

either lies in $\mathbf{Q}(\beta)$ or is transcendental. The key fact is the observation that α admits infinitely many good approximants in the number field $\mathbf{Q}(\beta)$ obtained by truncating its expansion and completing it by periodicity. Precisely, for any positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$\begin{aligned} b_h^{(n)} &= a_h \quad \text{for } 1 \leq h \leq r_n + s_n, \\ b_{r_n+h+js_n}^{(n)} &= a_{r_n+h} \quad \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0. \end{aligned}$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is eventually periodic, with preperiod U_n and with period V_n. Set

$$\alpha_n = \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k},$$

and observe that

$$\alpha - \alpha_n = \sum_{k=r_n+\lceil ws_n\rceil+1}^{+\infty} \frac{a_k - b_k^{(n)}}{\beta^k}. \tag{2}$$

Lemma 1. *For any integer n, there exists an integer polynomial $P_n(X)$ of degree at most $r_n + s_n - 1$ such that*

$$\alpha_n = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}.$$

Further, the coefficients of $P_n(X)$ are bounded in absolute value by $2\max_{k\geq 1} |a_k|$.

proof. By definition of α_n, we get

$$\begin{aligned} \alpha_n &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=r_n+1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} = \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{+\infty} \frac{b_{r_n+k}^{(n)}}{\beta^k} \\ &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^k} \left(\sum_{j=0}^{+\infty} \frac{1}{\beta^{js_n}} \right) \\ &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^{k+r_n-s_n}(\beta^{s_n} - 1)} = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}, \end{aligned}$$

where we have set

$$P_n(X) = \sum_{k=1}^{r_n} a_k X^{r_n-k}(X^{s_n} - 1) + \sum_{k=1}^{s_n} a_{r_n+k} X^{s_n-k}.$$

The last assertion of the lemma is clear. □

Set $\mathbf{K} = \mathbf{Q}(\beta)$ and denote by d the degree of $\mathbf{K}$. We assume that α is algebraic, and we consider the following linear forms, in three variables and with algebraic coefficients. For

the place v corresponding to the embedding of $\mathbf{K}$ defined by $\beta \hookrightarrow \beta$, set $L_{1,v}(x,y,z) = x$, $L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = \alpha x + \alpha y + z$. It follows from (2) and Lemma 1 that

$$|L_{3,v}(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))|_v = |\alpha(\beta^{r_n}(\beta^{s_n}-1)) - P_n(\beta)|^{1/d} \ll \frac{1}{\beta^{(w-1)s_n/d}}, \quad (3)$$

where we have chosen the continuation of $|\cdot|_v$ to $\overline{\mathbf{Q}}$ defined by $|x|_v = |x|^{1/d}$. Here and throughout this Section, the constants implied by the Vinogradov symbol $\ll$ depend (at most) on α, β, and $\max_{k\geq 1}|a_k|$, but are independent of n.

Denote by S'_∞ the set of all other infinite places on $\mathbf{K}$ and by S_0 the set of all finite places on $\mathbf{K}$ dividing β. Observe that S_0 is empty if β is an algebraic unit. For any v in $S_0 \cup S'_\infty$, set $L_{1,v}(x,y,z) = x$, $L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = z$. Denote by S the union of S_0 and the infinite places on $\mathbf{K}$. Clearly, for any v in S, the forms $L_{1,v}$, $L_{2,v}$ and $L_{3,v}$ are linearly independent.

To simplify the exposition, set

$$\mathbf{x}_n = (\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta)).$$

We wish to estimate the product

$$\Pi := \prod_{v\in S}\prod_{i=1}^{3} \frac{|L_{i,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v} = \prod_{v\in S} |\beta^{r_n+s_n}|_v\, |\beta^{r_n}|_v \, \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3}$$

from above. By the product formula and the definition of S, we immediately get that

$$\Pi = \prod_{v\in S} \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3}. \quad (4)$$

Since the polynomial $P_n(X)$ has integer coefficients and since β is an algebraic integer, we have $|L_{3,v}(\mathbf{x}_n)|_v = |P_n(\beta)|_v \leq 1$ for any place v in S_0. Furthermore, as the conjugates of β have moduli at most 1, we have for any infinite place v in S'_∞

$$|L_{3,v}(\mathbf{x}_n)|_v \ll (r_n+s_n)^{d_v/d},$$

where $d_v = 1$ or 2 according as v is real infinite or complex infinite, respectively. Together with (3) and (4), this gives

$$\begin{aligned}\Pi &\ll (r_n+s_n)^{(d-1)/d}\, \beta^{-(w-1)s_n/d} \prod_{v\in S} |\mathbf{x}_n|_v^{-3} \\ &\ll (r_n+s_n)^{(d-1)/d}\, \beta^{-(w-1)s_n/d}\, H(\mathbf{x}_n)^{-3},\end{aligned}$$

since $|\mathbf{x}_n|_v = 1$ if v does not belong to S.

Furthermore, it follows from Lemma 1 and from the fact that the moduli of the complex conjugates of β are at most 1 that

$$H(\mathbf{x}_n) \ll (r_n+s_n)^d\, \beta^{(r_n+s_n)/d}.$$

Consequently, we infer from Condition $(*)_w$ that

$$\prod_{v\in S}\prod_{i=1}^{3}\frac{|L_{i,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v} \ll (r_n+s_n)^{dw}\, H(\mathbf{x}_n)^{-(w-1)s_n/(r_n+s_n)}\, H(\mathbf{x}_n)^{-3}$$
$$\ll H(\mathbf{x}_n)^{-3-\varepsilon},$$

for some positive real number ε.

It then follows from Theorem E that the points $(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))$ lie in a finite number of proper subspaces of $\mathbf{K}^3$. Thus, there exist a non-zero triple (x_0, y_0, z_0) in $\mathbf{K}^3$ and infinitely many integers n such that

$$x_0 - y_0\frac{\beta^{r_n}}{\beta^{r_n+s_n}} - z_0\frac{P_n(\beta)}{\beta^{r_n+s_n}} = 0.$$

Taking the limit along this subsequence of integers and noting that $(s_n)_{n\geq 1}$ tends to infinity, we get that $x_0 = z_0\alpha$. Thus, α belongs to $\mathbf{K} = \mathbf{Q}(\beta)$, as asserted. □

Let us restrict our attention to the case when β is a Pisot number. Dealing with the β-expansions of real numbers (instead of arbitrary power series in β) allows us to improve the conclusion of Theorem 5.

Theorem 5A. *Let $\beta > 1$ be a Pisot number. Let α be in $(0,1)$, and consider its β-expansion*

$$\alpha := \sum_{k=1}^{+\infty}\frac{a_k}{\beta^k}.$$

If $(a_k)_{k\geq 1}$ satisfies Condition $(*)_w$ for some real number $w > 1$, then α is transcendental.

Proof. By a result of K. Schmidt [36], we know that the β-expansion of every element of $\mathbf{Q}(\beta)\cap(0,1)$ is eventually periodic. Thus, it does not satisfy Condition $(*)_w$. We conclude by applying Theorem 5. □

Note that for a Salem number β, it is an important open problem to decide whether every element of $\mathbf{Q}(\beta)\cap(0,1)$ has an eventually periodic β-expansion.

5. Proofs of Theorems 1 to 4

We begin by a short proof of Theorem 2.

Proof of Theorem 2. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be a non-eventually periodic automatic sequence defined on a finite alphabet $\mathcal{A}$. Recall that a morphism is called uniform if the images of each letter have the same length. Following Cobham [14], there exist a morphism ϕ from an alphabet $\mathcal{B} = \{1, 2, \ldots, r\}$ to the alphabet $\mathcal{A}$ and an uniform morphism σ from $\mathcal{B}$ into itself such that $\mathbf{a} = \phi(\mathbf{u})$, where $\mathbf{u}$ is a fixed point for σ. Observe first that the sequence $\mathbf{a}$

satisfies Condition $(*)_w$ if this is the case for $\mathbf{u}$. Further, by the Dirichlet *Schubfachprinzip*, the prefix of length $r+1$ of $\mathbf{u}$ can be written under the form $W_1uW_2uW_3$, where u is a letter and W_1, W_2, W_3 are (possibly empty) finite words. We check that the assumptions of Theorem 1 are satisfied by $\mathbf{u}$ with the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ defined for any $n \geq 1$ by $U_n = \sigma^n(W_1)$ and $V_n = \sigma^n(uW_2)$. Indeed, since σ is a morphism of constant length, we get, on the one hand, that

$$\frac{|U_n|}{|V_n|} \leq \frac{|W_1|}{1+|W_2|} \leq r-1$$

and, on the other hand, that $\sigma^n(u)$ is a prefix of V_n of length at least $1/r$ times the length of V_n. It follows that Condition $(*)_{1+1/r}$ is satisfied by the sequence $\mathbf{u}$, and thus by our sequence $\mathbf{a}$ (here, we use the observation we made in Section 4). Let $b \geq 2$ be an integer. By applying Theorem 5 with $\beta = b$, we conclude that the automatic number $\sum_{k=1}^{+\infty} a_k b^{-k}$ is transcendental. □

Proof of Theorem 1. Let α be an irrational number. Without any loss of generality, we assume that α is in $(0,1)$ and we denote by $0.u_1u_2\dots u_k\dots$ its b-adic expansion. The sequence $(u_k)_{k\geq 1}$ takes its values in $\{0,1,\dots,b-1\}$ and is not eventually periodic. We assume that there exists an integer $\kappa \geq 2$ such that the complexity function p of $(u_k)_{k\geq 1}$ satisfies

$$p(n) \leq \kappa n \qquad \text{for infinitely many integers } n \geq 1,$$

and we shall derive that Condition $(*)_w$ is then fulfilled by the sequence $(u_k)_{k\geq 1}$ for a suitable $w > 1$. By Theorem 5, this will imply that α is transcendental.

Let n_k be an integer with $p(n_k) \leq \kappa n_k$. Denote by $U(\ell)$ the prefix of $\mathbf{u} := u_1u_2\dots$ of length ℓ. By the Dirichlet *Schubfachprinzip*, there exists (at least) one word M_k of length n_k which has (at least) two occurrences in $U((\kappa+1)n_k)$. Thus, there are (possibly empty) words A_k, B_k, C_k and D_k, such that

$$U((\kappa+1)n_k) = A_kM_kC_kD_k = A_kB_kM_kD_k \qquad \text{and} \qquad |B_k| \geq 1.$$

We observe that $|A_k| \leq \kappa n_k$. We have to distinguish three cases:

(i) $|B_k| > |M_k|$;

(ii) $\lceil |M_k|/3 \rceil \leq |B_k| \leq |M_k|$;

(iii) $1 \leq |B_k| < \lceil |M_k|/3 \rceil$.

(i). Under this assumption, there exists a word E_k such that

$$U((\kappa+1)n_k) = A_kM_kE_kM_kD_k.$$

Since $|E_k| \leq (\kappa-1)|M_k|$, the word $A_k(M_kE_k)^s$ with $s = 1+1/\kappa$ is a prefix of $\mathbf{u}$. Furthermore, we observe that

$$|M_kE_k| \geq |M_k| \geq \frac{|A_k|}{\kappa}.$$

(ii). Under this assumption, there exist two words E_k and F_k such that

$$U((\kappa+1)n_k) = A_k M_k^{1/3} E_k M_k^{1/3} E_k F_k.$$

Thus, the word $A_k(M_k^{1/3}E_k)^2$ is a prefix of $\mathbf{u}$. Furthermore, we observe that

$$|M_k^{1/3}E_k| \geq \frac{|M_k|}{3} \geq \frac{|A_k|}{3\kappa}.$$

(iii). In the present case, B_k is clearly a prefix of M_k, and we infer from $B_k M_k = M_k C_k$ that B_k^t is a prefix of M_k, where t is the integer part of $|M_k|/|B_k|$. Observe that $t \geq 3$. Setting $s = \lfloor t/2 \rfloor$, we see that $A_k(B_k^s)^2$ is a prefix of $\mathbf{u}$ and

$$|B_k^s| \geq \frac{|M_k|}{4} \geq \frac{|A_k|}{4\kappa}.$$

In each of the three cases above, we have proved that there are finite words U_k, V_k such that $U_k V_k^{1+1/\kappa}$ is a prefix of $\mathbf{u}$ and:

- $|U_k| \leq \kappa n_k$;
- $|V_k| \geq n_k/4$;
- $w \geq 1 + 1/\kappa > 1$.

Consequently, the sequence $(|U_k|/|V_k|)_{k\geq 1}$ is bounded from above by 4κ. Furthermore, it follows from the lower bound $|V_k| \geq n_k/4$ that we may assume that the sequence $(|V_k|)_{k\geq 1}$ is strictly increasing. This implies that the sequence $\mathbf{u}$ satisfies Condition $(*)_{1+1/\kappa}$. By applying Theorem 5 with $\beta = b$, we conclude that α is transcendental. □

Proof of Theorem 3. Let $\mathbf{a}$ be a sequence generated by a morphism ϕ defined on a finite alphabet $\mathcal{A}$. For any positive integer n, there exists a letter a_n satisfying

$$|\phi^n(a_n)| = \max\{|\phi^n(j)| : j \in \mathcal{A}\}.$$

This implies the existence of a letter a in $\mathcal{A}$ and of a strictly increasing sequence of positive integers $(n_k)_{k\geq 1}$ such that for every $k \geq 1$ we have

$$|\phi^{n_k}(a)| = \max\{|\phi^{n_k}(j)| : j \in \mathcal{A}\}.$$

Assume from now on that $\mathcal{A}$ has two elements. Since the sequence $\mathbf{a}$ is not eventually periodic there exist at least two occurrences in $\mathbf{a}$ of the two elements of $\mathcal{A}$. In particular, there exist at least two occurrences of the letter a in the sequence $\mathbf{a}$. We can thus find two (possibly empty) finite words W_1 and W_2 such that $W_1 a W_2 a$ is a prefix of $\mathbf{a}$. We check that the assumptions of Theorem 5 are satisfied by $\mathbf{a}$ with the sequences $(U_k)_{k\geq 1}$

and $(V_k)_{k\geq 1}$ defined by $U_k = \phi^{n_k}(W_1)$ and $V_k = \phi^{n_k}(aW_2)$ for any $k \geq 1$. Indeed, by definition of a, we have

$$\frac{|U_k|}{|V_k|} \leq |W_1|$$

and $\phi^{n_k}(a)$ is a prefix of V_k of length at least $1/(|W_2|+1)$ times the length of V_k. It follows that Condition $(*)_w$ is satisfied by the sequence $\mathbf{a}$ with $w = 1+1/(|W_2|+1)$. We conclude by applying Theorem 5. □

Proof of Theorem 4. Let $\mathbf{a}$ be a sequence generated by a recurrent morphism ϕ defined on an alphabet $\mathcal{A}$. As we have already noticed in the beginning of the proof of Theorem 3, there exist a letter a and a strictly increasing sequence of positive integers $(n_k)_{k\geq 1}$ such that for every $k \geq 1$ we have

$$|\phi^{n_k}(a)| = \max\{|\phi^{n_k}(j)| : j \in \mathcal{A}\}.$$

Since by assumption the sequence $\mathbf{a}$ is recurrent there exist at least two occurrences of the letter a. We then apply the same trick as in the proof of Theorem 3, and we again conclude by applying Theorem 5. □

6. Transcendence of p-adic numbers

Let p be a prime number. As usual, we denote by $\mathbf{Q}_p$ the field of p-adic numbers. We call algebraic (resp. transcendental) any element of $\mathbf{Q}_p$ which is algebraic (resp. transcendental) over $\mathbf{Q}$. A suitable version of the Schmidt Subspace Theorem, due to Schlickewei [33], can be applied to derive a lower bound for the complexity of the Hensel expansion of every irrational algebraic number in $\mathbf{Q}_p$.

Theorem 1B. *Let α be an irrational algebraic number in $\mathbf{Q}_p$ and denote by*

$$\alpha = \sum_{k=-m}^{+\infty} a_k p^k$$

its Hensel expansion. Then, the complexity function p of the sequence $(a_k)_{k\geq -m}$ satisfies

$$\liminf_{n\to\infty} \frac{p(n)}{n} = +\infty.$$

Likewise (see Section 2), we can also state Theorems 2B, 3B and 4B. Theorem 1B follows from Theorem 6 below, along with the arguments used in the proof of Theorem 1.

The method of proof of Theorem 5 applies to provide us with a new transcendence criterion for p-adic numbers.

Theorem 6. *Let p be a prime number and let $(a_k)_{k\geq -m}$ be a sequence taking its values in $\{0, 1, \ldots, p-1\}$. Let $w > 1$ be a real number. If the sequence $(a_k)_{k\geq 1}$ satisfies Condition $(*)_w$, then the p-adic number*

$$\alpha := \sum_{k=-m}^{+\infty} a_k p^k$$

is transcendental.

We briefly outline the proof of Theorem 6. Let p and $(a_k)_{k\geq -m}$ be as in the statement of this theorem. There exist a parameter $w > 1$ and two sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ of finite words as in the definition of Condition $(*)_w$. For any $n \geq 1$, set $r_n = |U_n|$ and $s_n = |V_n|$. To establish Theorem 6, it is enough to prove that the p-adic number

$$\alpha' := \sum_{k=1}^{+\infty} a_k p^k$$

is transcendental. As in the proof of Theorem 5, the key fact is the observation that α' admits infinitely many good rational approximants obtained by truncating its Hensel expansion and completing by periodicity. Precisely, for any positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ exactly as in Section 4, and we set

$$\alpha_n = \sum_{k=1}^{+\infty} b_k^{(n)} p^k.$$

An easy calculation shows that we have

$$|\alpha' - \alpha_n|_p \leq p^{-r_n - w s_n}, \quad \alpha_n = \frac{p_n}{p^{s_n} - 1},$$

where

$$p_n = \left(\sum_{k=1}^{r_n} a_k\, p^k\right)(p^{s_n} - 1) - \sum_{k=1}^{s_n} a_{r_n+k}\, p^{r_n+k}.$$

Assuming that α' is an algebraic number in $\mathbf{Q}_p$, we apply Theorem 4.1 of Schlickewei [33] with the linear forms $L_{1,p}(x, y, z) = x$, $L_{2,p}(x, y, z) = y$, $L_{3,p}(x, y, z) = \alpha' x + \alpha' y + z$, $L_{1,\infty}(x, y, z) = x$, $L_{2,\infty}(x, y, z) = y$, and $L_{3,\infty}(x, y, z) = z$. Setting $\mathbf{x}_n := (p^{s_n}, -1, -p_n)$, we get $|L_{1,p}(\mathbf{x}_n)|_p = p^{-s_n}$ and $|L_{3,p}(\mathbf{x}_n)|_p \leq p^{-r_n - w s_n}$. We then follow the same lines as in the proof of Theorem 5, and we end up in a contradiction. This proves that α' is transcendental. □

7. Concluding remarks

It is of interest to compare our result with a celebrated theorem of Christol, Kamae, Mendès France, and Rauzy [12] concerning algebraic elements of the field $\mathbf{F}_p((X))$. Their

result asserts that, for any given prime number p, the sequence of integers $\mathbf{u} = (u_k)_{k\geq 1}$ is p-automatic if and only if the formal power series $\displaystyle\sum_{k\geq 1} u_k X^k$ is algebraic over the field of rational functions $\mathbf{F}_p(X)$. Thanks to Theorem 2, we thus easily derive the following statement.

Theorem 7. *Let $b \geq 2$ be an integer and p be a prime number. The formal power series $\displaystyle\sum_{k\geq 1} u_k X^k$ and the real number $\displaystyle\sum_{k\geq 1} \frac{u_k}{b^k}$ are both algebraic (over $\mathbf{F}_p(X)$ and over $\mathbf{Q}$, respectively) if and only if they are rational.*

Note that Theorem 2B (see Section 6) naturally gives rise to a similar result where the real number $\displaystyle\sum_{k\geq 1} \frac{u_k}{b^k}$ is replaced by the q-adic number $\displaystyle\sum_{k\geq 1} u_k q^k$, for an arbitrary prime number q. In particular, this holds true for $q = p$.

In 1991, Morton and Mourant [28] proved the following result: If $k \geq 2$ is an integer, P is a non-zero pattern of digits in base k, and if $e_{k,P,b}(n) \in \{0, 1, \ldots, b-1\}$ counts the number of occurrences modulo b of P in the k-ary expansion of n, then the real number $\alpha(k, P, b) = \displaystyle\sum_{n=0}^{+\infty} \frac{e_{k,P,b}(n)}{b^n}$ is transcendental except when $k = 3$, $P = 1$ and $b = 2$. Moreover, we have $\alpha(3, 1, 2) = 2/3$ in this particular case.

The proof given by Morton and Mourant is based on Theorem 2 and their paper refers to the work of Loxton and van der Poorten [22]. The present work validates their result.

It is interesting to remark that the simplest case $k = 2$, $P = 1$ and $b = 2$ corresponds to the well-known Thue–Morse number, whose transcendence has been proved by Mahler [23]. The theorem of Morton and Mourant can thus be seen as a full generalisation of the Mahler result.

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