



**HAL**  
open science

# On some completions of the space of Hamiltonian maps

Vincent Humilière

► **To cite this version:**

| Vincent Humilière. On some completions of the space of Hamiltonian maps. 2007. hal-00014000v2

**HAL Id: hal-00014000**

**<https://hal.science/hal-00014000v2>**

Preprint submitted on 11 Jan 2007 (v2), last revised 23 Nov 2007 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On some completions of the space of Hamiltonian maps

VINCENT HUMILIÈRE

Centre de Mathématiques Laurent Schwartz  
UMR 7640 du CNRS  
Ecole Polytechnique - 91128 Palaiseau, France  
vincent.humiliere@math.polytechnique.fr

## Abstract

In his paper [15], C. Viterbo defined a distance on the set of Hamiltonian diffeomorphisms of  $\mathbb{R}^{2n}$  endowed with the standard symplectic form  $\omega_0 = dp \wedge dq$ . We study the completions of this space for the topology induced by Viterbo's distance and some others derived from it, we study their different inclusions and give some of their properties.

In particular, we give a convergence criterion for these distances. It allows us to prove that the completions contain non-ordinary elements, as for example, discontinuous Hamiltonians. We also prove that some dynamical aspects of Hamiltonian systems are preserved in the completions.

## 1 Introduction.

Given an open subset  $U$  in  $\mathbb{R}^{2n}$ , we denote by  $Ham(U)$  the set of all 1-periodic time dependent Hamiltonian functions  $\mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$  whose support for fixed time is compact and contained in  $U$ . We will write  $Ham$  for  $Ham(\mathbb{R}^{2n})$ .

Given a Hamiltonian function  $H \in Ham$ , its symplectic gradient (i.e the unique vector field  $X_H$  satisfying  $dH = \iota_{X_H}\omega_0$ ) generates a Hamiltonian isotopy  $\{\phi_H^t\}$ . The set of Hamiltonian diffeomorphisms generated by an element  $H$  in  $Ham(U)$  will be denoted by  $\mathcal{H}(U) = \{\phi_H = \phi_H^1 \mid H \in Ham(U)\}$ , and we write  $\mathcal{H}$  for  $\mathcal{H}(\mathbb{R}^{2n})$ . Finally, we call  $\mathcal{L} = \{\phi(0_n) \mid \phi \in \mathcal{H}\}$ , the set of Lagrangian submanifolds obtained from the zero section  $0_n \subset T^*\mathbb{R}^n = \mathbb{R}^{2n}$ , by a Hamiltonian isotopy with compact support.

As usual, we denote Viterbo's distance on  $\mathcal{L}$  or  $\mathcal{H}$  by  $\gamma$  (see [15]). Convergence with respect to  $\gamma$  is called  $\gamma$ -convergence.

Our main goal in this paper is to understand the completion of the space  $\mathfrak{H}(U)$  for the distance  $\gamma$ , give some convergence criterion (section 3) and compare it with the convergence for Hofer's distance  $d_H$  (see [5], chapter 5 section 1).

The notion of  $C^0$  symplectic topology has been studied by many authors, starting from the work of Eliashberg and Gromov on the  $C^0$  closure of the group of symplectic diffeomorphisms, to the later results of Viterbo ([15]) and Hofer ([4]).

More recently Oh ([9]) gave a deep study of several versions of  $C^0$  Hamiltonians. However, our definition seems to differ from his, since in all his definitions, he needs the Hamiltonians to be continuous, while our study starts as we drop this assumption.

In section 3, we introduce a symplectic invariant  $\xi_\infty$  associated to any subset of  $\mathbb{R}^{2n}$ , and prove that

**Theorem 1.1.** *If the closure of  $U$  is compact and if  $K \subset U$  is such that  $\xi_\infty(K) = 0$ , then for every sequence  $(H_k)$  in  $Ham(U)$  which is Cauchy for the metric of uniform convergence on compact subsets of  $\mathbb{R} \times (U - K)$ , the sequence  $\phi_{H_k}$  is Cauchy for  $\gamma$ .*

*As a result, any Hamiltonian continuous on  $\mathbb{R} \times (U - K)$ , with compact support in  $\mathbb{R} \times U$  has a flow defined in the  $\gamma$ -completion of  $\mathcal{H}(U)$ .*

Examples of sets  $K$  with  $\xi_\infty(K) = 0$  are given by compact submanifolds of dimension  $d \leq n - 2$ .

It is possible to extend many aspects of Hamiltonian dynamics to the completions (section 4). To any element in the completion of  $Ham$ , we can associate a flow in the completion of  $\mathcal{H}$ . We can define its action on a Lagrangian submanifold. We can also associate to them a support and extend the notion of first integral.

To some of them, it is also possible to associate a solution to Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t} + H\left(t, x, \frac{\partial u}{\partial x}\right) = 0.$$

Indeed, a  $\gamma$ -Cauchy sequence of Hamiltonians gives a  $C^0$ -Cauchy sequence of solutions (section 5).

Finally, let us mention that although we developed our theory on  $\mathbb{R}^{2n}$ , we can reasonably expect similar results (except those of sections 4.1.2 and 5) on any compact symplectic manifold satisfying

$$\omega|_{\pi_2(M)} = 0 \text{ and } c_1|_{\pi_2(M)} = 0.$$

Indeed, on these manifolds, Schwarz defined in [11] a distance which is entirely analogous to Viterbo's.

**Acknowledgments.** I am grateful to my supervisor C. Viterbo for his advices. I also want to thank my friends M. Affre and N. Roy for spending hours correcting my awful english.

## 2 Symplectic invariants.

In this section, we recall the definition of Viterbo's distance, defined first for Lagrangian submanifolds with the help of generating functions, and then for Hamiltonian diffeomorphisms (see [15]). It will be convenient to use some other distances defined from  $\gamma$  by taking suspensions, namely  $\tilde{\gamma}$ ,  $\gamma_u$ ,  $\tilde{\gamma}_u$ ,  $\hat{\gamma}$ ,  $\check{\gamma}$  and  $\gamma_2$ . Relations between these metrics will be stated and proved in Appendix.

### 2.1 Generating functions quadratic at infinity.

Let  $L$  be a Lagrangian submanifold of the cotangent bundle  $T^*M$  of a smooth manifold  $M$ . We say that  $L$  admits a *generating function* if there exists an integer  $q > 0$  and a smooth function  $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$  such that  $L$  can be written

$$L = \left\{ (x, p) \in T^*M \mid \exists \xi \in \mathbb{R}^q, \frac{\partial S}{\partial \xi}(x, \xi) = 0 \text{ and } \frac{\partial S}{\partial x}(x, \xi) = p \right\}.$$

Such function  $S$  is called a *generating function quadratic at infinity* (or just "g.f.q.i") if there exists a non degenerate quadratic form  $Q$  on  $\mathbb{R}^q$  and a compact  $K \subset M \times \mathbb{R}^q$  such that,  $\forall (x, \xi) \notin K, S(x, \xi) = Q(\xi)$ .

For instance, any quadratic form on  $\mathbb{R}^q$  viewed as a function on  $M \times \mathbb{R}^q$  is a g.f.q.i of the zero section  $0_M \subset T^*M$ . J.C. Sikorav proved in [12] that the property of having a g.f.q.i is invariant by Hamiltonian isotopy with compact support. For this reason we will be interested in the set  $\mathcal{L}$  of Lagrangian submanifolds, images of the zero section by an Hamiltonian isotopy with compact support.

Furthermore, C. Viterbo and D. Théret proved that the g.f.q.i's of a given Lagrangian submanifold are essentially unique. Before stating this result, let us introduce the following definitions: For a given function  $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$ , we call a *stabilisation* of  $S$  any function  $S' : M \times \mathbb{R}^q \times \mathbb{R}^{q'} \rightarrow \mathbb{R}$  of the form  $S'(x, \xi, \xi') = S(x, \xi) + q(\xi')$ , where  $q$  is a non-degenerate quadratic form on  $\mathbb{R}^{q'}$ . In addition, two functions  $S, S' : M \times \mathbb{R}^q \rightarrow \mathbb{R}$  are said *equivalent* if there exists a diffeomorphism  $\phi$  of  $M \times \mathbb{R}^q$  and a real  $C$  such that  $S' = S \circ \phi + C$ .

**Theorem 2.1 ([15, 13]).** *Suppose  $S, S'$  are two g.f.q.i's of the same Lagrangian submanifold in  $\mathcal{L}$ . Then, up to stabilisation,  $S$  and  $S'$  are equivalent.*

This result allows to associate symplectic invariants to any element of  $\mathcal{L}$ .

## 2.2 Invariants defined by minimax and a distance on the group of Hamiltonian diffeomorphisms.

The invariants defined in this section have been introduced by C. Viterbo in [15]. We recall their construction. We first define invariants for Lagrangian submanifolds.

Let  $L$  be an element of  $\mathcal{L}$  and  $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$  be one of its g.f.q.i's. Let us denote  $S^\lambda = \{x \in M \times \mathbb{R}^q \mid S(x) \leq \lambda\}$ . Since  $S$  is quadratic at infinity, the homotopy types of the pairs  $(S^\lambda, S^\mu)$  and  $(S^\mu, S^{-\lambda})$  do not depend on  $\lambda$ , provided that  $\lambda$  is sufficiently large. Therefore, we will denote  $S^\infty$  and  $S^{-\infty}$ , instead of  $S^\lambda$  and  $S^{-\lambda}$  for  $\lambda$  large enough.

Let us introduce  $E_\infty^-$  the negative (trivial) bundle of the quadratic form which coincides with  $S$  at infinity. We denote  $B(E_\infty^-)$ ,  $S(E_\infty^-)$  the ball bundle and the sphere bundle associated to  $E_\infty^-$ . The Thom isomorphism is given by  $H^*(M) \rightarrow H^*(B(E_\infty^-), S(E_\infty^-))$ , and we also have the isomorphism  $H^*(B(E_\infty^-), S(E_\infty^-)) \simeq H^*(S^\infty, S^{-\infty})$ . We will denote by  $T$  their composition. For further informations on those isomorphisms, see [6] for example. The inclusion  $j_\lambda : S^\lambda \rightarrow S^\infty$  induces a morphism in cohomology  $j_\lambda^* : H^*(S^\infty, S^{-\infty}) \rightarrow H^*(S^\lambda, S^{-\infty})$ , for all real number  $\lambda$ . We are now ready for the following.

**Definition 2.2.** *Let  $(u, L) \in H^*(M) \times \mathcal{L}$ , with  $u \neq 0$ . Using a g.f.q.i  $S$  of  $L$ , we define a real number  $c(u, L)$  as follows:*

$$c(u, L) = \inf\{\lambda \mid j_\lambda^* \circ T(u) \in H^*(S^\lambda, S^{-\infty}) \text{ is non zero}\}. \quad (1)$$

Observe that  $c(u, L)$  is well defined, and is independent of the choice of  $S$ 's choice. Indeed, if we replace  $S$  with an equivalent or stabilized generating function, the value of  $c(u, L)$  does not change and we conclude using theorem 2.1. Even if it doesn't depend on the generating function, we sometimes use the notation  $c(u, S)$  instead of  $c(u, L)$ .

Since the cohomology of the sets  $S^\lambda$  changes when we cross the level  $c(u, L)$ , it has to be a critical value of  $S$ .

Finally, observe that the definition can be extended to classes with compact support  $u \in H_c^*(M)$ .

Then, we can use those invariants associated to Lagrangian submanifold to define other invariants associated to Hamiltonian diffeomorphisms.

Consider an Hamiltonian diffeomorphism  $\psi \in \mathcal{H}(\mathbb{R}^{2n})$ . Its graph  $\Gamma_\psi$  is a Lagrangian submanifold of  $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$  ( $= (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$ , where  $\omega_0$  is the standard symplectic structure on  $\mathbb{R}^{2n}$ ). It coincides with the diagonal  $\Delta = \{(x, x) \mid x \in \mathbb{R}^{2n}\}$ , outside the product  $B^{2n}(r) \times B^{2n}(r)$ , for  $r$  sufficiently large. When we identify  $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$  with  $T^*\Delta$  using the map,

$$(q, p, Q, P) \mapsto \left( \frac{q+Q}{2}, \frac{p+P}{2}, P-p, Q-q \right),$$

we see that the image  $\widetilde{\Gamma}_\psi$  of  $\Gamma_\psi$  is identified with the zero section of  $T^*\Delta$  outside a compact set.

Then, we can associate the previous invariant to  $\widetilde{\Gamma}_\psi$ . Let 1 be a generator of  $H^0(\mathbb{R}^{2n})$  and  $\mu$  a generator of  $H_c^n(\mathbb{R}^{2n})$ .

**Definition 2.3 (Viterbo, [15]).** *We define,*

$$\begin{aligned} c_-(\psi) &= -c(\mu, \widetilde{\Gamma}_\psi), \\ c_+(\psi) &= -c(1, \widetilde{\Gamma}_\psi), \\ \gamma(\psi) &= c_+(\psi) - c_-(\psi), \\ \gamma(\phi, \psi) &= \gamma(\psi^{-1}\phi). \end{aligned}$$

Let us describe now some properties of the numbers  $\gamma$ ,  $c_+$  and  $c_-$ .

**Proposition 2.4 (Viterbo, [15]).** *For all  $\psi$  in  $\mathcal{H}$ , we have*

$$c_-(\psi) \leq 0 \leq c_+(\psi).$$

*Moreover,  $c_-(\psi) = c_+(\psi) = 0$  if and only if  $\psi = \text{Id}$ . If  $\phi$  is another diffeomorphism in  $\mathcal{H}$ , then*

$$\begin{aligned} c_+(\phi \circ \psi) &\leq c_+(\phi) + c_+(\psi), \\ c_-(\phi \circ \psi) &\geq c_-(\phi) + c_-(\psi), \\ \gamma(\phi \circ \psi) &\leq \gamma(\phi) + \gamma(\psi). \end{aligned}$$

Those properties allow to prove that  $\gamma$  is a distance on  $\mathcal{H}$ . We also have a very useful property of monotonicity of  $c_+$  and  $c_-$ .

**Proposition 2.5.** *Let  $\psi_1$  and  $\psi_2$  be two Hamiltonians generated by  $H_1$  and  $H_2$ . Suppose that for all  $(t, x) \in \mathbb{R} \times \mathbb{R}^{2n}$ , we have  $H_1(t, x) \leq H_2(t, x)$ . Then,  $c_+(\psi_1) \leq c_+(\psi_2)$  and  $c_-(\psi_1) \leq c_-(\psi_2)$ .*

As a corollary, if  $H$  is a non-negative Hamiltonian, then  $c_-(\phi_H) = 0$ . If  $H$  is in addition non zero, we deduce  $c_+(\phi_H) > 0$ .

Finally, the following proposition states the continuity of  $\gamma$  with respect to  $C^0$ -topology.

**Proposition 2.6.** *Let  $H_1$  and  $H_2$  be two compactly supported hamiltonians, generating  $\psi_1$  and  $\psi_2$ . Let  $\|\cdot\|$  be the usual norm on  $C^0(\mathbb{R}^{2n} \times [0, 1], \mathbb{R})$ . If  $\|H_1 - H_2\| \leq \varepsilon$ , then  $|\gamma(\psi_1) - \gamma(\psi_2)| \leq \varepsilon$ .*

### 2.3 Other distances derived from $\gamma$ .

Let us now introduce other distances, derived from  $\gamma$ . Those definitions are sometimes more convenient than the original one.

**Definition 2.7.** *For all Hamiltonian diffeomorphisms  $\phi, \psi \in \mathcal{H}$ , we define*

$$\tilde{\gamma}(\phi, \psi) = \sup\{\gamma(\psi^{-1}\phi(L) - L) \mid L \in \mathcal{L}\},$$

where  $\gamma(L) = c(\mu, L) - c(1, L)$ ,  $\forall L \in \mathcal{L}$  and  $L_1 - L_2 = \{(q, p_1 - p_2) \mid (q, p_1) \in L_1, (q, p_2) \in L_2\}$ , for  $L_1, L_2 \in \mathcal{L}$ .

We get by this way another distance on  $\mathcal{H}$  (see [15], [1]). Then, we define distances not anymore on  $\mathcal{H}$ , but on  $Ham$ .

**Definition 2.8.** *For any  $H, K \in Ham$ , we set*

$$\gamma_u(H, K) = \sup\{\gamma(\phi_H^t, \phi_K^t) \mid t \in [0, 1]\}$$

and

$$\tilde{\gamma}_u(H, K) = \sup\{\tilde{\gamma}(\phi_H^t, \phi_K^t) \mid t \in [0, 1]\}.$$

Here, the subscript “u” means “uniform”. Clearly,  $\gamma_u$  and  $\tilde{\gamma}_u$  are distances on  $Ham$ .

For the next two distances, the principle is to add two dimensions by associating to an Hamiltonian  $H$  two suspensions defined on  $\mathbb{R} \times \mathbb{R}^{2+2n}$ :

$$\hat{H}(s; t, \tau, x) = \tau + H(t, x),$$

$$\check{H}(s; t, \tau, x) = tH(st, x).$$

Here, the new time variable is  $s$ , while the former time variable  $t$  becomes a space variable (As a consequence  $\hat{H}$  is an autonomous Hamiltonian). We would like to define our distances by  $\hat{\gamma}(H, K) = \gamma(\hat{H}, \hat{K})$  and  $\check{\gamma}(H, K) = \gamma(\check{H}, \check{K})$ . But since  $\hat{H}$  and  $\check{H}$  are not compactly supported we have to be slightly more subtle.

**Definition 2.9.** Let  $\rho$  be a fixed real function defined on  $[0, +\infty)$ , supposed to be non-negative, smooth, decreasing, with support in  $[0, 1]$ , flat at 0 and such that  $\rho(0) = 1$ . For every natural integer  $\alpha$  and every real number  $t$ , we set  $\rho_\alpha(t) = 1$  if  $-\alpha \leq t \leq \alpha$ , and  $\rho_\alpha(t) = \rho(|t| - \alpha)$  otherwise.

We denote by  $\hat{H}_\alpha$  and  $\check{H}_\alpha$  the Hamiltonian functions defined on  $\mathbb{R} \times \mathbb{R}^{2+2n}$ , by

$$\hat{H}_\alpha(s; t, \tau, x) = \rho_\alpha(\tau)\tau + \rho_\alpha(t)H(t, x),$$

and

$$\check{H}_\alpha(s; t, \tau, x) = \rho_\alpha(t)tH(st, x).$$

Then, for  $H, K \in \text{Ham}$ , we set

$$\hat{\gamma}(H, K) = \limsup_{\alpha \rightarrow +\infty} \gamma_u(\hat{H}_\alpha, \hat{K}_\alpha),$$

and

$$\check{\gamma}(H, K) = \limsup_{\alpha \rightarrow +\infty} \gamma(\phi_{\check{H}_\alpha}, \phi_{\check{K}_\alpha}).$$

Remark that  $\hat{\gamma}(H, K)$  and  $\check{\gamma}(H, K)$  are finite. Indeed, if we denote by  $B$  a ball containing both supports of  $H$  and  $K$ , then  $\hat{H}_\alpha, \hat{K}_\alpha, \check{H}_\alpha$  and  $\check{K}_\alpha$  have support in  $\mathbb{R}^2 \times B$ , for any integer  $\alpha$ . Hence  $\gamma(\phi_{\check{H}_\alpha}, \phi_{\check{K}_\alpha}) \leq 2c(\mathbb{R}^2 \times B) \leq 2c^\infty(B)$  (See section 2.4 for notations). It shows that the limsup in the definition of  $\check{\gamma}$  is finite. The same proof shows that  $\hat{\gamma}(H, K)$  is also finite.

The triangle inequality for  $\hat{\gamma}$  and  $\check{\gamma}$  is a direct consequence of the triangle inequality axiom for  $\gamma$ . The separation axiom is obtained from the separation axiom for  $\gamma$  and proposition 2.10.

For convenience, we will not write the subscript  $\alpha$  anymore. In the following, we will denote  $\hat{H}$  for  $\hat{H}_\alpha$ , and  $\check{H}$  for  $\check{H}_\alpha$ .

*Remarks.* By repeating these constructions several times (i.e., by taking suspensions of suspensions), we can construct new distances. For example, we will use in section 5 the distance  $\gamma_2 = \limsup_{\alpha \rightarrow +\infty} \check{\gamma}(\hat{H}_\alpha, \hat{K}_\alpha)$ .

Using the invariance of  $\gamma$ , it is easy to verify that the suspended distances  $\hat{\gamma}$ ,  $\check{\gamma}$  and  $\gamma_2$  are invariant under the action of  $\mathcal{H}$ . Namely, for  $H, K$  Hamiltonians and  $\varphi$  Hamiltonian diffeomorphism, we have:

$$\hat{\gamma}(H \circ \varphi, K \circ \varphi) = \hat{\gamma}(H, K),$$

$$\check{\gamma}(H \circ \varphi, K \circ \varphi) = \check{\gamma}(H, K),$$

$$\gamma_2(H \circ \varphi, K \circ \varphi) = \gamma_2(H, K).$$

The following proposition gives inequalities between the distances. It will be proved in Appendix. It is based on the reduction inequality (Proposition A.1).



**Proposition 2.10.**

$$\begin{aligned}\tilde{\gamma} &\leq \gamma, \\ \tilde{\gamma}_u &\leq \gamma_u \leq \min(\hat{\gamma}, \check{\gamma}).\end{aligned}$$

## 2.4 Two symplectic capacities on $\mathbb{R}^{2n}$ .

We start this section by reminding the reader of the definition of a symplectic capacity.

**Definition 2.11 (Ekeland-Hofer).** *A symplectic capacity on  $(\mathbb{R}^{2n}, \omega_0)$  is a map associating to each subset  $U \subset \mathbb{R}^{2n}$  a number  $c(U) \in [0, +\infty]$  satisfying*

1.  $U \subset V \Rightarrow c(U) \leq c(V)$  (monotony),
2.  $c(\phi(U)) = c(U)$  for all Hamiltonian diffeomorphism  $\phi \in \mathcal{H}$  (symplectic invariance),
3.  $c(\lambda U) = \lambda^2 c(U)$  for all real  $\lambda > 0$  (homogeneity),
4.  $c(B^{2n}(1)) = c(B^2 \times \mathbb{R}^{2(n-1)}) = \pi$ , where  $B^{2n}(1)$  is the unit ball of  $\mathbb{R}^{2n}$  (normalisation).

The invariants defined in the previous section allow to define two symplectic capacities as follows ([15]).

**Definition 2.12.** 1. *For any compact subset  $K \subset \mathbb{R}^{2n}$ , we denote by  $\gamma(K)$  the number defined by*

$$\gamma(K) = \inf\{\gamma(\phi) \mid \phi(K) \cap K = \emptyset\}.$$

*If  $V$  is not compact, we set*

$$\gamma(V) = \sup\{\gamma(K) \mid K \subset V\}.$$

2. *For any open subset  $U \subset \mathbb{R}^{2n}$ , we denote by  $c(U)$  the number defined by*

$$c(U) = \sup\{c_+(\phi_H) \mid \text{Supp}(H) \subset U\}.$$

*If  $V$  is not an open set, we set*

$$c(V) = \inf\{c(U) \mid V \subset U\}.$$

The maps  $c$  and  $\gamma$  are symplectic capacities and moreover  $c \leq \gamma$ . We remind the reader of the definition of the displacement energy

$$d(U) = \inf\{d_H(\phi, Id) \mid \phi(U) \cap U = \emptyset\},$$

where  $d_H$  is Hofer's distance defined by

$$d_H(\phi, \psi) = \inf\{\|H - K\| \mid H \text{ generates } \phi \text{ and } K \text{ generates } \psi\},$$

whith  $\|H\| = \int_0^1 (\max_x H(t, x) - \min_x H(t, x)) dt$ .

We are going to define a new symplectic capacity derived from  $\gamma$ , but before we need the following lemma.

**Lemma 2.13.** *We consider a subset  $V \subset \mathbb{R}^{2n}$  and  $\mathbb{R}^2 \times V \subset \mathbb{R}^{2+2n}$ . Then,*

$$c(\mathbb{R}^2 \times V) \geq c(V).$$

That lemma follows from the reduction inequality of Proposition A.1. We postpone its proof to Appendix.

**Definition 2.14.** *For any open subset  $U \subset \mathbb{R}^{2n}$ , we set*

$$c^\infty(U) = \lim_{N \rightarrow \infty} c(U \times \mathbb{R}^{2N}),$$

and if  $V$  is not an open subset,

$$c^\infty(V) = \inf\{c^\infty(U) \mid V \subset U\}.$$

We obtain a symplectic capacity that satisfies  $c^\infty(V) = c^\infty(V \times \mathbb{R}^2)$  for all subset  $V$ , and  $c \leq c^\infty$ . Moreover, since  $d(U) = d(U \times \mathbb{R}^{2k})$  and  $c \leq d$ , we have  $c^\infty \leq d$ . To summarize the inequalities between capacities we have,

**Proposition 2.15.**  $c \leq \gamma \leq d$  and  $c \leq c^\infty \leq d$ .

### 3 The convergence criterion.

This is the central section of our paper. We give the proof of theorem 1.1.

#### 3.1 A sufficient condition for c-convergence to zero.

We start this section with some formulas concerning Hamiltonian flows. They can be obtained by direct computation (see [5], page 144).

**Lemma 3.1.** *For all Hamiltonians  $H$  and  $K$ , with compact support, we have:*

$$\begin{aligned}\phi_{\overline{H}}^t &= (\phi_H^t)^{-1}, & \text{where } \overline{H}(t, x) &= -H(t, \phi^t(x)) \\ \phi_{H\sharp K}^t &= \phi_H^t \circ \phi_K^t, & \text{where } (H\sharp K)(t, x) &= H(t, x) + K(t, (\phi^t)^{-1}(x)) \\ \phi_{\overline{H}\sharp K}^t &= (\phi_H^t)^{-1} \circ \phi_K^t.\end{aligned}$$

*Remark.*  $(\overline{H}\sharp K)(t, x) = (K - H)(t, \phi^t(x))$ .

The following proposition shows that if a sequence of Hamiltonians  $(H_n)$  converges to zero uniformly on every compact set contained in the complement of a set whose capacity is zero, then  $(\phi_{H_n})$  converges to  $Id$  for  $\gamma$ .

**Proposition 3.2.** *Let  $H$  be a Hamiltonian on  $\mathbb{R}^{2n}$  with compact support. If  $U$  is an open subset of  $\mathbb{R}^{2n}$ , such that  $c(U) \leq \varepsilon$  and  $|H(t, x)| \leq \varepsilon$  for all  $t \in [0, 1]$  and all  $x \in \mathbb{R}^{2n} - U$ , then  $\gamma(\phi_H) \leq 4\varepsilon$ .*

*Proof.* Let  $K_1, K_2$  be Hamiltonians with compact support, such that  $0 \leq K_i \leq 1$ ,  $i = 1, 2$ ,  $K_1$  equals 1 on the support of  $H$  and  $K_2$  equals 1 on the support of  $K_1$ . Denote  $\psi_{1,\varepsilon}$  the diffeomorphism generated by  $H - \varepsilon K_1$ , and  $\psi_{2,\varepsilon}$  the diffeomorphism generated by  $\varepsilon K_2$ . Then we have  $H \leq \varepsilon K_2 + (H - \varepsilon K_1)$ . As  $(\psi_{2,\varepsilon})^{-1}$  coincides with  $Id$  on the support of  $H - \varepsilon K_1$ , the lemma 3.1 shows that  $K_2 + H - \varepsilon K_1$  is the Hamiltonian that generates  $\psi_{2,\varepsilon} \circ \psi_{1,\varepsilon}$ . The propositions 2.5, 2.4 and 2.6 then give

$$c_+(\phi) \leq c_+(\psi_{2,\varepsilon} \circ \psi_{1,\varepsilon}) \leq c_+(\psi_{2,\varepsilon}) + c_+(\psi_{1,\varepsilon}) \leq \varepsilon + c_+(\psi_{1,\varepsilon}).$$

Denote by  $\widetilde{\psi}_{1,\varepsilon}$  the diffeomorphism generated by a non-negative Hamiltonian, with support in  $U$ , and greater than  $H - \varepsilon K_1$ . Then by Proposition 2.5,  $c_+(\psi_{1,\varepsilon}) \leq c_+(\widetilde{\psi}_{1,\varepsilon})$ . Finally, since  $Supp(\widetilde{\psi}_{1,\varepsilon}) \subset U$ , we get  $c_+(\widetilde{\psi}_{1,\varepsilon}) \leq c(U) \leq \varepsilon$ .

Using the inequality  $H \geq -\varepsilon K_2 + (H + \varepsilon K_1)$ , we obtain the same type of inequality for  $c_-$ .  $\square$

For example, if  $K$  is a compact submanifold of dimension lower or equal than  $n - 1$ , then  $d(K) = 0$  (and hence  $c(K) = 0$ ).

## 3.2 What about non-zero limits?

Unfortunately, the previous result cannot be straightforwardly generalised to obtain a general convergence criterions when the limit is not zero. Indeed, we can find two Hamiltonians that are  $C^0$ -close out of a null-capacity set, but not  $\gamma$ -close.

*Example.* It is well known that the capacities  $c$  and  $\gamma$  of the unit sphere  $\mathcal{S} = \{x \in \mathbb{R}^{2n} \mid \|x\| = 1\}$  are  $\pi$ . It is also true for  $c^\infty$ . Then, for all  $\varepsilon > 0$ ,

there exists a Hamiltonian  $H$  with support in a small neighbourhood  $U$  of  $\mathcal{S}$ , and such that  $c_+(\phi_H) > \pi - \varepsilon$ . Because of proposition 2.5,  $H$  can be chosen non-negative. We set  $U^+$  a neighbourhood of  $\{x \in \mathcal{S} \mid x_1 \geq 0\}$  and  $U^-$  a neighbourhood of  $\{x \in \mathcal{S} \mid x_1 < 0\}$ , such that  $U = U^+ \cup U^-$ . If  $U$ ,  $U^+$  and  $U^-$  are chosen small enough, we have  $d(U^\pm) < \varepsilon$  and by proposition 2.15  $c^\infty(U^\pm) < \varepsilon$ . Using some partition of unity associated to the decomposition  $U = U^+ \cup U^-$ , we get two functions  $H^\pm$ , with support in  $U^\pm$  and such that  $H = H^+ + H^-$ .

Now, we see that  $H^+$  coincides with  $H$  outside  $U^-$ , whose capacity verifies  $c^\infty(U^\pm) < \varepsilon$ , but on the other hand,

$$\|\gamma(\phi_H) - \gamma(\phi_{H^+})\| \geq \gamma(\phi_H, \phi_{H^+}) \geq \pi - \varepsilon - \gamma(\phi_{H^-}) \geq \pi - 2\varepsilon.$$

It shows that the previous statement is false when the limit is not zero.

Nevertheless, we can introduce a new invariant, in order to extend the result of proposition 3.2.

**Definition 3.3.** For any subset  $U$  and any Hamiltonian  $H \in \text{Ham}$ , we define

$$\xi^H(U) = c^\infty \left( \bigcup_{t \in [0,1]} \phi_H^t(U) \right).$$

We may then set

$$\xi_\lambda(U) = \sup \xi^H(U), \text{ for } 0 < \lambda \leq \infty,$$

where the supremum is over all Hamiltonian functions  $H$  with  $\gamma_u(H) \leq \lambda$ .

**Theorem 3.4.** Let  $H_1$  and  $H_2$  be Hamiltonians on  $\mathbb{R}^{2n}$  with compact support. Let  $U$  be a subset of  $\mathbb{R}^{2n}$ , satisfying one of the two following conditions:

1.  $\xi_\infty(U) \leq \varepsilon$ .
2.  $\exists \lambda > 0, \xi_\lambda(U) = 0$

If  $|H_1(t, x) - H_2(t, x)| \leq \varepsilon$  for all  $t \in [0, 1]$  and all  $x \notin U$ , then we have  $\gamma(\phi_{H_1}, \phi_{H_2}) \leq 4\varepsilon$ .

*Proof.* Consider the Hamiltonian  $H(t, x) = H_1(t, \phi_2^t(x)) - H_2(t, \phi_2^t(x))$ . By assumption,  $|H(t, x)| \leq \varepsilon$ , for all  $(t, x)$  with  $x \notin \phi_2^{-t}(U)$  and hence for all  $t$  and all  $x \notin \bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U)$ . Each condition on  $U$  implies

$$c \left( \bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U) \right) \leq c^\infty \left( \bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U) \right) \leq \varepsilon.$$

By proposition 3.2 and lemma 3.1, we get  $\gamma(\phi_{H_1}, \phi_{H_2}) = \gamma(\phi_H) \leq 4\varepsilon$ .  $\square$

*Remark.* In the proof of theorem 3.4, we see that the important condition is in fact  $\xi^{H_2}(U) \leq \varepsilon$ , which is of course implied by both conditions  $\xi_\infty(U) \leq \varepsilon$  and  $\xi_\lambda(U) = 0$ .

**Corollary 3.5.** *The conclusion of theorem 3.4 still holds if we replace  $\gamma$  with  $\tilde{\gamma}$ ,  $\tilde{\gamma}_u$ ,  $\gamma_u$ ,  $\hat{\gamma}$  or  $\check{\gamma}$ .*

*Proof.* By proposition 2.10, we just have to prove this result for  $\hat{\gamma}$  and  $\check{\gamma}$ . Then remark that under the hypothesis of theorem 3.4, we have  $|\hat{H}_1(s; t, \tau, x) - \hat{H}_2(s; t, \tau, x)| \leq \varepsilon$  and  $|\check{H}_1(s; t, \tau, x) - \check{H}_2(s; t, \tau, x)| \leq \varepsilon$  for all integer  $\alpha$ , all  $s \in [0, 1]$ , and all  $(t, \tau, x) \notin \mathbb{R}^2 \times U$ .

Unfortunately, even if  $U$  satisfies one of the conditions of proposition 3.4, it is not in general the case for  $\mathbb{R}^2 \times U$ . However, by the above remark, it is sufficient to show that for all real number  $\delta > 0$  and all integer  $\alpha$  large enough,  $\xi^{\check{H}_2}(\mathbb{R}^2 \times U) \leq \varepsilon + \delta$  and  $\xi^{\hat{H}_2}(\mathbb{R}^2 \times U) \leq \varepsilon + \delta$ . By letting  $\delta$  tend to zero and taking limsup with respect to  $\alpha$ , we obtain  $\hat{\gamma}(H, K) \leq 4\varepsilon$  and  $\check{\gamma}(H, K) \leq 4\varepsilon$  as required.

Let us denote  $F$  for  $\check{H}_2$  or  $\hat{H}_2$ . The inequalities on  $\xi^F$  come directly from the expression of  $\phi_{\check{H}_2}$  and  $\phi_{\hat{H}_2}$  (see computations in Appendix A.2). Indeed, in both cases,

$$\bigcup_{s \in [0, 1]} (\phi_F^{-1})^s([- \alpha, \alpha]^2 \times U) \subset \mathbb{R}^2 \times \bigcup_{s \in [0, 1]} (\psi^{-1})^s(U),$$

where  $\psi^s$  is an Hamiltonian isotopy that appears in last coordinate when we compute  $\phi_F$ . Therefore, since  $\xi^F(\mathbb{R}^2 \times U) = \lim_{\alpha \rightarrow +\infty} \xi^F([- \alpha, \alpha]^2 \times U)$ , we get for any  $\delta > 0$  and any  $\alpha$  large enough:

$$\begin{aligned} \xi^F(\mathbb{R}^2 \times U) &\leq \delta + c^\infty \left( \mathbb{R}^2 \times \bigcup_{s \in [0, 1]} (\psi^{-1})^s(U) \right) \\ &= \delta + c^\infty \left( \bigcup_{s \in [0, 1]} (\psi^{-1})^s(U) \right) \leq \delta + \varepsilon. \end{aligned}$$

That concludes the proof.  $\square$

**Corollary 3.6.** *Let  $(H_k)$  be a sequence of Hamiltonians in  $\text{Ham}$ , whose supports are contained in a fixed compact set. Suppose there exist an Hamiltonian  $H \in \text{Ham}$  and a compact set  $K \in \mathbb{R}^{2n}$  with  $\xi_\infty(K) = 0$ , such that  $(H_k)$  converges uniformly to  $H$  on every compact set of  $\mathbb{R} \times (\mathbb{R}^{2n} - K)$ . Then  $(\phi_{H_k})$  converges to  $\phi_H$  for  $\tilde{\gamma}$ ,  $\gamma$ , and  $(H_k)$  converges to  $H$  for  $\tilde{\gamma}_u$ ,  $\gamma_u$ ,  $\hat{\gamma}$ ,  $\check{\gamma}$ .*

*Proof.* For  $\tilde{\gamma}$ ,  $\gamma$ , it is a direct consequence of the remark that follows theorem 3.4. We just have to verify that for all  $\varepsilon > 0$ , there exists a small neighbourhood  $U$  of  $K$  such that  $\xi^H(U) \leq \varepsilon$ . This is true because for every neighbourhood  $V$  of  $\bigcup_{t \in [0,1]} \phi_H(K)$ , we can choose a neighbourhood  $U$  of  $K$  such that

$$\bigcup_{t \in [0,1]} \phi_H(U) \subset V.$$

Since  $c^\infty(\bigcup_{t \in [0,1]} \phi_H(K)) = 0$  and  $\bigcup_{t \in [0,1]} \phi_H(K)$  is compact, we can choose  $V$  such that  $c^\infty(V) \leq \varepsilon$ , and obtain  $c^\infty(\bigcup_{t \in [0,1]} \phi_H(U)) \leq \varepsilon$  as required.

For  $\hat{\gamma}$  and  $\check{\gamma}$ , we have to verify that for all  $\varepsilon > 0$  and all  $\delta > 0$ , there exists a small neighbourhood  $U$  of  $K$  such that for all  $\alpha$  large enough  $\xi^\phi(U) \leq \varepsilon + \delta$ , where  $F$  is either  $\hat{H}$  or  $\check{H}$ . The proof made above for  $\phi_H$  shows that we can find  $U$  such that  $\xi^f(U) \leq \varepsilon$ , where  $f$  generates the isotopy  $\psi^s$  defined as in the proof of corollary 3.5. Therefore we have for all  $\delta$  and all  $\alpha$  large enough,  $\xi^F(\mathbb{R}^2 \times U) \leq \xi^F([- \alpha, \alpha]^2 \times U) + \delta \leq \xi^f(U) + \delta \leq \varepsilon + \delta$ .

By proposition 2.10, corollary 3.6 is also true for  $\tilde{\gamma}_u$  and  $\gamma_u$ .  $\square$

*Remark.* Similar proofs give that theorem 3.4 and corollary 3.6 still hold for  $\gamma_2$ .

### 3.3 Example of a non trivial $\xi$ -small set.

**Proposition 3.7.** *Let  $U$  be a closed submanifold of  $\mathbb{R}^{2n}$  whose dimension  $d$  verifies  $d \leq n - 2$ . Then  $\xi_\infty(U) = 0$ .*

*Proof.* Let  $H \in \text{Ham}$ . The problem is that  $\bigcup_{t \in [0,1]} \phi_H^t(U)$  is not in general a manifold. To avoid that problem, we are going to add two dimensions and make a suspension in this way. We denote by  $\Phi$  the Hamiltonian diffeomorphism on  $\mathbb{R}^{2+2n} = \{(t, \tau, x)\}$  generated by the Hamiltonian

$$[0, 1] \times \mathbb{R}^{2+2n} \rightarrow \mathbb{R}, \quad (s; t, \tau, x) \mapsto tH(ts, x).$$

We also set  $V = \Phi([0, 1] \times [-1, 1] \times U)$ . The computation of  $\Phi$  gives

$$\Phi(t, \tau, x) = (t, \tau - H(t, x), \phi^t(x)).$$

We see that  $\bigcup_{t \in [0,1]} \phi_H^t(U)$  can be obtained from  $V$  by symplectic reduction by the coisotropic manifold  $\{\tau = 0\}$ . So we are going to look for a Hamiltonian diffeomorphism  $\phi_K$  that displaces  $V$  and preserves  $\{\tau = 0\}$  at the same time. If the Hamiltonian does not depend on  $t$ , the second condition is verified. Since  $V$  is compact, it is sufficient for  $K$  to verify

$$\forall v \in V, \quad \mathbb{R}X_K(v) \cap T_v V = \{0\},$$

which is equivalent to

$$\forall v \in V, \ker dK(v) \oplus T_v V^\perp = \mathbb{R}^{2+2n}$$

and to

$$\forall v \in V, T_v V^\perp \not\subset \ker dK(v).$$

That makes us consider the 1-jet bundle  $J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})$  and its submanifold

$$W = \{(s, q; \sigma, p; z) \mid (s, q) \in V, z \in \mathbb{R}, T_{(s,q)} V^\perp \subset \ker(\sigma, p)\}.$$

The dimension of  $W$  is exactly  $2n + 1$ . Indeed, the vector space  $\{(\sigma, p) \in \mathbb{R}^{2n+2*} \mid T_{(s,q)} V^\perp \subset \ker(\sigma, p)\}$  has dimension  $2n + 2 - \dim(T_{(s,q)} V^\perp) = n$ .

By Thom transversality theorem (see [3] for example), there exists a function  $L$  whose 1-jet verifies  $j^1 L \pitchfork W$ . But  $j^1 L$  can be seen as a function  $\mathbb{R} \times \mathbb{R}^{1+2n} \rightarrow J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})$ , and by lemma 4.6 page 53 in [3], we have for a generic choice of  $s \in \mathbb{R}$ ,  $j^1 L(s, \cdot) \pitchfork W$ . We fix  $s$  as previously and we denote  $K : \mathbb{R}^{2+2n} \rightarrow \mathbb{R}$ ,  $K(t, \cdot) = L(s, \cdot)$

Then, notice that for every  $s, q, p, z$ , the set of all  $\sigma$  such that  $(s, q; \sigma, p; z) \in W$  is either  $\emptyset$  or  $\mathbb{R}$ . It can be shown by direct computation of  $T V^\perp$ , whose first component appears to be always  $\{0\}$ . As a consequence, we get  $j^1 K \pitchfork W$  ( $j^1 K$  differs from  $j^1 L(s, \cdot)$  just by its  $\sigma$ -component which is  $\{0\}$  instead of  $\frac{\partial L}{\partial s}(s, \cdot)$  for  $j^1 L(s, \cdot)$ ).

Now, since  $(2n+2) + (2n+1) = \dim(j^1 K(\mathbb{R}^{2+2n})) + \dim(W) < \dim(J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})) = 4n + 5$ , we get  $j^1 K(\mathbb{R}^{2+2n}) \cap W = \emptyset$ . It follows that  $K$  satisfies the two conditions: it preserves  $\{\tau = 0\}$  and it satisfies

$$\forall v \in V, \mathbb{R} X_K(v) \cap T_v V = \{0\}.$$

As  $V$  is compact, for  $\varepsilon$  small enough, since  $\phi_{\varepsilon K} = \phi_K^\varepsilon$ , we have  $\phi_{\varepsilon K}(V) \cap V = \emptyset$ . In addition  $\varepsilon K$  can be made as  $C^0$ -small as we want.

We are now ready for the reduction by  $\{\tau = 0\}$ . Since it preserves  $\{\tau = 0\}$ ,  $\varepsilon K$  induces a Hamiltonian on the reduction  $\mathbb{R}^{2n}$ . This Hamiltonian is  $C^0$ -small and generates a diffeomorphism  $\psi$  whose Hofer's distance to identity  $d_H(\psi, id)$  is small, and that satisfies

$$\psi \left( \bigcup_{t \in [0,1]} \phi_H^t(U) \right) \cap \bigcup_{t \in [0,1]} \phi_H^t(U) = \emptyset.$$

This Hamiltonian is not compactly supported, but any Hamiltonian with compact support which coincides with it on a sufficiently large ball, would have the same properties. That proves  $d\left(\bigcup_{t \in [0,1]} \phi_H^t(U)\right) = 0$ , and since  $c^\infty \leq d$ , we get  $\xi^H(U) = 0$ .  $\square$

## 4 Completions

In this section we start the study of the completions of  $Ham$  and  $\mathcal{H}$  for the different distances.

### 4.1 Extension of Hamiltonian dynamics

In this section, we introduce the completions and give the first properties of their elements: the existence of a flow that acts on Lagrangian submanifold, the notion of first integral and the existence of a support. The full section 5 will be devoted to another property related to Hamilton-Jacobi equation.

#### 4.1.1 Notations, inclusions and definition

Let us denote respectively  $\mathfrak{H}$ ,  $\mathfrak{H}_u$ ,  $\tilde{\mathfrak{H}}$ ,  $\tilde{\mathfrak{H}}_u$ ,  $\hat{\mathfrak{H}}$ ,  $\check{\mathfrak{H}}$  and  $\mathfrak{H}_2$  the completions of  $(\mathcal{H}, \gamma)$ ,  $(Ham, \gamma_u)$ ,  $(\mathcal{H}, \tilde{\gamma})$ ,  $(Ham, \tilde{\gamma}_u)$ ,  $(Ham, \hat{\gamma})$ ,  $(Ham, \check{\gamma})$  and  $(Ham, \gamma_2)$ .

The inequalities proved in Proposition 2.10 induce inclusions between the completions which may be summarized by the following diagram. Here,  $\mathfrak{H}_H$  denotes the completion of  $\mathcal{H}$  for Hofer's distance  $d_H$  (which satisfies  $d_H \leq \gamma$ ) and  $C_c$  the set of continuous Hamiltonians with compact support. The arrows  $\mathfrak{H}_u \rightarrow \tilde{\mathfrak{H}}$  and  $\tilde{\mathfrak{H}}_u \rightarrow \tilde{\mathfrak{H}}$  are induced by the map  $H \mapsto \phi_H^1$ .

$$\begin{array}{ccccccc}
 C_c & \hookrightarrow & \tilde{\mathfrak{H}} & & & & \\
 \downarrow & & \searrow & & & & \\
 \mathfrak{H}_2 & \hookrightarrow & \hat{\mathfrak{H}} & \hookrightarrow & \mathfrak{H}_u & \hookrightarrow & \tilde{\mathfrak{H}}_u \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathfrak{H}_H & \hookrightarrow & \tilde{\mathfrak{H}}
 \end{array}$$

The map  $(H, t) \mapsto \phi_H^t$ ,  $Ham \times \mathbb{R} \rightarrow \mathcal{H}$  induces maps  $\mathfrak{H}_u \times \mathbb{R} \rightarrow \tilde{\mathfrak{H}}$  and  $\tilde{\mathfrak{H}}_u \times \mathbb{R} \rightarrow \tilde{\mathfrak{H}}$ . Therefore, any element  $h$  in  $\mathfrak{H}_u$ ,  $\tilde{\mathfrak{H}}_u$ ,  $\hat{\mathfrak{H}}$ ,  $\check{\mathfrak{H}}$  or  $\mathfrak{H}_2$  can be associated a path in either  $\tilde{\mathfrak{H}}$ , or  $\tilde{\mathfrak{H}}$ . This path of course has the semi-group property (the group laws on  $\tilde{\mathfrak{H}}$  and  $\tilde{\mathfrak{H}}$  are those naturally induced by the group law on  $\mathcal{H}$ ).

**Definition 4.1.** *Such a path will be called the generalized Hamiltonian flow generated by  $h$ .*

#### 4.1.2 Action on Lagrangian submanifolds

Recall that the set  $\mathcal{L}$  of Lagrangian submanifolds isotopic to the zero section by compactly supported Hamiltonian isotopy, can be endowed with Viterbo's



distance, also denoted  $\gamma$  (see definition 2.7). Let us denote  $\mathfrak{L}$  the completion of  $\mathcal{L}$  with respect to this distance.

**Proposition 4.2.** *The groups  $\mathfrak{H}$  and  $\tilde{\mathfrak{H}}$  naturally act on the set  $\mathfrak{L}$ . This action extends the action of  $\mathcal{H}$  on  $\mathcal{L}$ .*

*Proof.* It is a simple consequence of the inequality  $\tilde{\gamma} \leq \gamma$  (Proposition 2.10 proved in Appendix).

Let  $L \in \mathfrak{L}$  represented by a sequence  $(L_k)$  and  $\phi$  in  $\mathfrak{H}$  (proof is the same for  $\tilde{\mathfrak{H}}$ ), represented by a sequence  $(\phi_k)$ . We are going to show that  $(\phi_k(L_k))$  defines an element of  $\mathfrak{L}$  that we will denote  $\phi(L)$ .

This follows easily from the fact that for  $\phi, \psi \in \mathcal{H}$  and  $L, M \in \mathcal{L}$ ,

$$\begin{aligned} \gamma(\phi(L) - \psi(M)) &\leq \gamma(\phi(L) - \psi(L)) + \gamma(\psi(L) - \psi(K)) \\ &\leq \gamma(\psi^{-1}\phi(L) - L) + \gamma(L - K) \\ &\leq \tilde{\gamma}(\phi, \psi) + \gamma(L - K). \quad \square \end{aligned}$$

*Remark.* A consequence of Proposition 4.2 is that we can define what is a Lagrangian submanifold *invariant* under a generalized flow.

That leads us to another question which is: Can we define what is an invariant hypersurface of a generalized flow?

A (partial) answer to this question is that we can define what is a first integral of a generalized Hamiltonian flow.

### 4.1.3 Notion of first integral

This property has been first mentioned in [1], in the definition (3.3) of the so-called  $c$ -commuting Hamiltonians. Let us restate it with our notations.

An element in one of the completions  $\mathfrak{H}_u$ ,  $\tilde{\mathfrak{H}}_u$ ,  $\hat{\mathfrak{H}}$ ,  $\check{\mathfrak{H}}$  and  $\mathfrak{H}_2$  will be said *autonomous* if it can be represented by a Cauchy sequence of time-independent Hamiltonian functions.

**Definition 4.3.** *Let  $H, K$  be two elements in one of the above completions, generating two respective generalized flows  $\phi_H^t$  and  $\phi_K^t$ . Then we will say that  $H$  and  $K$  commute, or that  $K$  is a first integral of  $H$  if  $\phi_K^s \phi_H^t \phi_K^{-s} \phi_H^{-t} = Id$ .*

*In other words,  $K$  is a first integral of  $H$  if there exists two Cauchy sequences  $(H_n)$  and  $(K_n)$  representing  $H$  and  $K$ , such that for all  $s$  and  $t$ ,  $\phi_{K_n}^s \phi_{H_n}^t \phi_{K_n}^{-s} \phi_{H_n}^{-t}$   $c$ -converges to  $Id$ .*

It is proved in [1] that this definition extends the usual definition of commuting Hamiltonian functions.

#### 4.1.4 Existence of a support

In this section, we state a lemma which makes it possible to define a support for the elements of the different completions.

**Lemma 4.4. a.** *Let  $(\phi_n)$  be a sequence in  $\mathcal{H}$  converging to a Hamiltonian diffeomorphism  $\phi$ , with respect to  $\gamma$  or  $\tilde{\gamma}$ . Assume that there exists a set  $U$  such that  $\text{supp}(\phi_n) \subset U$ . Then  $\text{supp}(\phi) \subset \bar{U}$ .*

**b.** *Let  $(H_n)$  be a sequence in  $\text{Ham}$  converging to a smooth Hamiltonian function  $H$ , with respect to  $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}, \check{\gamma}$ , etc. Assume that there exists a set  $U$  such that  $\text{supp}(H_n) \subset U$ . Then  $\text{supp}(H) \subset \bar{U}$ .*

*Proof.* **a.** Thanks to Proposition 2.10, we just have to prove the assertion in the case of  $\tilde{\gamma}$ . Suppose  $\text{supp}(\phi) \not\subset \bar{U}$ . Then there exists an  $x$  in  $\mathbb{R}^{2n} - \bar{U}$  such that  $\phi(x) \neq x$ . Let  $\psi$  be a Hamiltonian diffeomorphism whose support is included in  $\mathbb{R}^{2n} - \bar{U}$  and which does not contain  $\phi(x)$ . Suppose in addition that  $\psi(x) \neq x$ . Then, since the supports of  $\phi_n$  and  $\psi$  are disjoint, we have  $\psi \circ \phi_n^{-1} \circ \psi^{-1} \circ \phi_n = \text{Id}$ , for all integer  $n$ . Taking limit, we get on one hand  $\psi \circ \phi^{-1} \circ \psi^{-1} \circ \phi = \text{Id}$ . But on the other hand, we have by construction,  $\psi \circ \phi^{-1} \circ \psi^{-1} \circ \phi(x) = \psi(x) \neq x$ , which is contradictory.

**b.** We use the first part of the lemma to conclude that for all time  $t$ ,  $\text{supp}(\phi^t) \subset \bar{U}$ . This implies that  $\text{supp}(H) \subset \bar{U}$ .  $\square$

*Remark.* A similar argument shows that the property of letting globally invariant any sphere centered at 0, is invariant by taking  $\gamma$  or  $\tilde{\gamma}$  limits. Similarly, a  $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$  or  $\check{\gamma}$  limit of radial Hamiltonians is radial.

**Definition 4.5.** *Let  $\eta$  be an element of  $\mathfrak{H}, \mathfrak{H}_u, \tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}_u, \hat{\mathfrak{H}}, \check{\mathfrak{H}}$  or  $\mathfrak{H}_2$ . Then we define  $\text{support}(\eta)$  as*

$$\bigcap \{ \bar{U} \mid U \text{ open set, such that there exists } (\eta_n) \text{ representing } \eta \text{ such that}$$

$$\forall n, \text{supp}(\eta_n) \subset U \},$$

where “supp” denotes the usual notions of support for smooth Hamiltonians and for Hamiltonian diffeomorphisms.

This new notion of support coincides with the usual one for smooth Hamiltonians and Hamiltonian diffeomorphisms. Indeed, let  $\eta$  be either an Hamiltonian diffeomorphism viewed as an element of  $\mathfrak{H}$  or  $\tilde{\mathfrak{H}}$ , or a smooth Hamiltonian seen as an element of  $\mathfrak{H}_u, \tilde{\mathfrak{H}}_u, \hat{\mathfrak{H}}, \check{\mathfrak{H}}$  or  $\mathfrak{H}_2$ . Let  $(\eta_n)$  be a sequence representing  $\eta$ , and  $U$  an open set with  $\text{supp}(\eta_n) \subset U$  for all  $n$ . Then lemma 4.4 gives  $\text{supp}(\eta) \subset \bar{U}$ . Hence  $\text{supp}(\eta) \subset \text{support}(\eta)$ . Conversely, for any

neighbourhood  $U$  of  $\text{supp}(\eta)$  the constant sequence  $(\eta)$  converges to  $\eta$  and has support in  $U$ . Therefore  $\text{support}(\eta) \subset \bigcap_{\mathcal{V}} \overline{U}$ , where the intersection is over the set  $\mathcal{V}$  of all open neighbourhoods of  $\text{supp}(\eta)$ . Then, it is easy to see that  $\bigcap_{\mathcal{V}} \overline{U} = \bigcap_{\mathcal{V}} U = \text{supp}(\eta)$ .

## 4.2 Description of some elements of the completions

In this section, we show that a large set of elements of the different completions can be associated to discontinuous Hamiltonians. An immediate consequence of this is the non-completeness of all our distances. It also makes possible to understand those elements in a more geometrical way.

**Definition 4.6.** *We denote by  $\mathfrak{F}$  the set of all functions  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R} \cup \{+\infty\}$  such that:*

- (i)  $K = H^{-1}(\{+\infty\})$  satisfies  $c^\infty(K) = 0$ ,
- (ii)  $H$  vanishes at infinity:  $\forall \varepsilon > 0, \exists r, (|x| > r \Rightarrow |H(x)| < \varepsilon)$ ,
- (iii)  $H$  is continuous.

We also set  $\mathfrak{F}^\infty = \{H \in \mathfrak{F} \mid H \text{ is smooth on } \mathbb{R}^{2n} - H^{-1}(\{+\infty\})\}$ .

**Proposition 4.7.** *Suppose  $H$  is an element of  $\mathfrak{F}$  and  $K = H^{-1}(\{+\infty\})$ . Then there exists a sequence of smooth autonomous Hamiltonians  $(H_k) \in \text{Ham}$  with the following properties:*

- a.  $(H_k)$  converges to  $H$  uniformly on every compact subset of  $\mathbb{R}^{2n} - K$ .
- b.  $(H_k)$  is Cauchy for  $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$  and  $\check{\gamma}$ .

Moreover, if  $H \in \mathfrak{F}^\infty$ , then any sequence  $(H_k)$  that converges to  $H$  uniformly on the compact subsets of  $\mathbb{R}^{2n} - K$ , does not converge in  $\text{Ham}$ , for none of the distances  $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$  and  $\check{\gamma}$ .

*Remark.* In the case of non-autonomous Hamiltonians, we would have a similar result, but with a stronger assumption on the set  $K$  of discontinuities. Namely, the conclusions of Proposition 4.7 hold for a time-dependent function  $H$  on  $\mathbb{R} \times \mathbb{R}^{2n}$ , smooth on  $\mathbb{R} \times (\mathbb{R}^{2n} - K)$ , where  $K$  is a compact set satisfying  $\xi^\infty(K) = 0$ .

This result is a simple application of theorem 3.4. Therefore, we choose to give the details not in that case, but in a more restrictive, but also more interesting case. Indeed, for the elements of  $\mathfrak{F}^\infty$ , the set of discontinuity is somehow "stable" under the Hamiltonian flow. This property allows to

consider functions with a larger discontinuity set ( $c^\infty(K) = 0$  instead of  $\xi^\infty(K) = 0$ ).

*Proof.* Fix  $k > 0$ . Properties (ii) and (iii) in Definition 4.6 imply that  $K$  is compact. Since  $c^\infty(K) = 0$ , there exists an open neighbourhood  $U$  of  $K$  such that  $c^\infty(U) \leq \frac{1}{k}$ . Then, if we denote  $H^{>A} = \{x \mid H(x) > A\}$ , we have for  $A$  large enough,  $K \subset H^{>A} \subset U$ . Indeed, if it was not true, then for all integer  $l > 0$ , there would exist a point  $a_l$  in  $H^{>l}$ , but not in  $U$ . Then, the sequence  $(a_l)$  would take values in  $H^{\geq 1} \cap (\mathbb{R}^{2n} - U)$  which is compact, and hence it would have a subsequence that would converge to an element of  $K \cap (\mathbb{R}^{2n} - U)$ , which contradicts our assumption. Let us fix a real number  $A_k$  such that  $H^{>A_k} \subset U$ .

Now, let  $H_k$  be a smooth function with compact support such that  $|H_k - H| < \frac{1}{k}$  on  $\mathbb{R}^{2n} - H^{>A_k + \frac{2}{k}}$ , and such that  $|H_k - A_k - \frac{2}{k}| < \frac{1}{k}$  on  $H^{>A_k + \frac{2}{k}}$ . The sequence  $(H_k)$  clearly converges to  $H$  uniformly on every compact subset of  $\mathbb{R}^{2n} - K$ . Let us see why it is Cauchy.

By Proposition 2.10, we just have to prove it for  $\hat{\gamma}$  and  $\tilde{\gamma}$ . We write  $F_k$  for either  $\check{H}_k$  or  $\hat{H}_k$ . We also denote, as in the proof of Corollary 3.5,  $\psi_k$  for the third coordinate of  $\phi_{F_k}$ . Since  $H_k$  is an autonomous Hamiltonian, its flow  $\phi_{H_k}^t$  preserves its level sets. Hence, the isotopy  $\psi_k^s$  preserves the level sets of  $H_k$  (see the computations in Appendix A.2). Therefore, since by construction  $H^{>A_k + \frac{2}{k}} \subset H_k^{>A_k + \frac{1}{k}}$ , we have

$$\bigcup_{t \in [0,1]} \psi_k^t(H^{>A_k + \frac{2}{k}}) \subset H_k^{>A_k + \frac{1}{k}}.$$

Let  $\delta > 0$  and suppose  $\alpha$  is sufficiently large. Then, as in the proof of Corollary 3.5,

$$\begin{aligned} \xi^{F_k}(\mathbb{R}^2 \times H^{>A_k + \frac{2}{k}}) &\leq \delta + c^\infty\left(\mathbb{R}^2 \times \bigcup_{s \in [0,1]} (\psi_k^{-1})^s(H^{>A_k + \frac{2}{k}})\right) \\ &\leq \delta + c^\infty\left(\bigcup_{s \in [0,1]} (\psi_k^{-1})^s(H^{>A_k + \frac{2}{k}})\right) \\ &\leq \delta + c^\infty(H_k^{>A_k + \frac{1}{k}}). \end{aligned}$$

Since  $H_k^{>A_k + \frac{1}{k}} \subset H^{>A_k} \subset U$  and  $c^\infty(U) \leq \frac{1}{k}$ , we obtain  $\xi^{F_k}(H^{>A_k + \frac{2}{k}}) \leq \frac{1}{k} + \delta$ .

Now, pick an integer  $l \geq k$ . If  $l$  and  $k$  are large enough, then we have  $|\hat{H}_k - \hat{H}_l| \leq \frac{1}{k}$  and  $|\check{H}_k - \check{H}_l| \leq \frac{1}{k}$  on  $\mathbb{R}^{2+2n} - (\mathbb{R}^2 \times H^{>A_k + \frac{2}{k}})$ . Therefore, by the

remark that follows Theorem 3.4, we get  $\hat{\gamma}(H_l, H_k) \leq \frac{4}{k}$  and  $\check{\gamma}(H_l, H_k) \leq \frac{4}{k}$ , after taking limsup with respect to  $\alpha$ . It proves that  $(H_k)$  is a Cauchy sequence for  $\tilde{\gamma}_u$ ,  $\gamma_u$ ,  $\hat{\gamma}$ , and  $\check{\gamma}$ .

Suppose now that  $H$  is smooth on  $\mathbb{R}^{2n} - K$ . Then we can choose  $H_k$  such that it coincides with  $H$  on  $B_k - H^{>A_k + \frac{2}{k}}$ , where  $B_k$  is the ball of radius  $k$ , centered at 0. Suppose that  $(H_k)$  converges to a Hamiltonian  $L \in Ham$  for  $\tilde{\gamma}_u$ ,  $\gamma_u$ ,  $\hat{\gamma}$ , and  $\check{\gamma}$ . Then for any integer  $k$ ,  $\overline{H_k} \sharp H_l$  converges to  $\overline{H_k} \sharp L$  while  $l$  tends to infinity for  $\tilde{\gamma}_u$  (see Lemma 3.1 for notations). According to Lemma 4.4, since  $\overline{H_k} \sharp H_l$  has support in the complementary of  $B_k - H^{>A_k + \frac{2}{k}}$ ,  $\overline{H_k} \sharp L$  has support in its closure and hence  $L$  coincides with  $H$  on  $B_k - H^{>A_k + \frac{2}{k}}$ . Since it is true for any  $k$ ,  $L$  has to coincide with  $H$  on  $\mathbb{R}^{2n} - K$ . Therefore  $L$  cannot belong to  $Ham$ , which contradicts our assumptions.

Finally, if  $(L_k)$  is another sequence of Hamiltonians that converges to  $H$  uniformly on the compact subsets of  $\mathbb{R}^{2n} - K$ , then, similarly as in the above proof that  $(H_k)$  is Cauchy, we obtain that  $\hat{\gamma}(L_k, H_k)$  and  $\check{\gamma}(L_k, H_k)$  converge to 0, where  $H_k$  is the particular sequence defined in the previous paragraph. Since  $(H_k)$  does not converge,  $(L_k)$  does not converge either.  $\square$

**Corollary 4.8.** *The metric spaces  $(\mathcal{H}, \gamma)$ ,  $(Ham, \gamma_u)$ ,  $(\mathcal{H}, \hat{\gamma})$ ,  $(Ham, \tilde{\gamma}_u)$ ,  $(Ham, \hat{\gamma})$  and  $(Ham, \check{\gamma})$  are not complete.*

*Proof.* For  $(Ham, \gamma_u)$ ,  $(Ham, \hat{\gamma})$ ,  $(Ham, \tilde{\gamma}_u)$ , and  $(Ham, \check{\gamma})$ , it is exactly what we proved in Proposition 4.7. We are now going to prove it in the case of  $\gamma$  and  $\tilde{\gamma}$ . Since the proof is exactly the same in both cases, we will only do it for  $\gamma$ .

We consider a decreasing function  $h : [0, +\infty) \rightarrow [0, +\infty)$ , with support in  $[-1, 1]$ , and equal to 1 on  $[-3/4, 3/4]$ . Then we define  $H_k(x) = \sum_{i=1}^k h(2^i |x|^2)$ , for  $x \in \mathbb{R}^{2n}$ . Suppose that  $(\phi_{H_k})$  converges to  $\phi \in \mathcal{H}$  for  $\gamma$ . We are going to prove that  $\phi$  cannot be smooth at 0. Lemma 4.4 implies that  $\phi$  coincides with  $\phi_{H_k}$  out of  $B_{2^{-k}}$ . So, we can compute the explicit form of  $\phi$ . In polar coordinates, we obtain:

$$\phi(r, \theta) = (\theta - r^2 f'(r^2), r),$$

for  $r > 0$  where  $f(s) = \sum_{i>0} h(2^i s)$  (for any  $s$ , all the terms in this sum are 0 except maybe one). If we denote by  $(q, p)$  the coordinates in  $\mathbb{R}^{2n}$ , and by  $\phi_1$  the projection of  $\phi$  on  $\mathbb{R}^n \times \{0\}$ , we have for  $q \in 2^{-i}[1/2, 1]$ ,

$$\frac{\partial \phi_1}{\partial q}(q, 0) = \cos(q^2 2^i h'(2^i q^2)) - 2(q^4 2^{2i} h''(2^i q^2) + q^3 2^i h'(2^i q^2)) \sin(q^2 2^i h'(2^i q^2)). \quad (2)$$

Suppose that  $h$  is chosen so that there exists  $q_1$  and  $q'_1$  in  $[1/2, 1]$  such that  $\frac{\partial \phi_1}{\partial q}(q_1, 0) \neq \frac{\partial \phi_1}{\partial q}(q'_1, 0)$  (we denote by  $A$  their difference), and define  $q_i =$

$\sqrt{2^{-i}}q_1$  and  $q'_i = \sqrt{2^{-i}}q'_1$ . Then,  $(q_i)$  and  $(q'_i)$  vanish, but from (2) we see that  $\frac{\partial\phi_1}{\partial q}(q_i, 0) - \frac{\partial\phi_1}{\partial q}(q'_i, 0)$  converges to  $A$ . Therefore  $\phi$  cannot be smooth at 0.  $\square$

*Remark.* In the previous example, the sequence of diffeomorphisms  $(\phi_{H_k})$  was converging almost everywhere to a homeomorphism (which was not a diffeomorphism). Therefore, one could think that the class of  $(\phi_{H_k})$  in the completion  $\mathfrak{H}$  can be represented by a homeomorphism. However, with the help of Proposition 4.7, we can show that it is not true in general, at least in dimension  $2n \geq 4$ .

Indeed, consider  $H : \mathbb{R}^2 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ ,

$$(x_1, x_2) \mapsto \frac{1}{\|x_1\|^2 - 1 + \|x_2\|^2} \chi(\|(x_1, x_2)\|),$$

where  $\chi$  is smooth with compact support and equals 1 on the ball of radius 2 centered at zero. Clearly,  $H \in \mathfrak{F}^\infty$  (because  $K = H^{-1}(\{+\infty\}) = \mathbb{S}^1 \times \{0\}$  satisfies  $c^\infty(K) = 0$  as required). Consider the sequence  $(H_k)$  constructed in the proof of Proposition 4.7. Since  $(H_k)$  is Cauchy for  $\gamma_u$ ,  $(\phi_{H_k})$  is Cauchy for  $\gamma$ . Suppose it converges to an element  $\phi$ . Then, Lemma 4.4 implies that for any neighbourhood  $U$  of  $K$  and for  $k$  large enough,  $\phi$  coincides with  $\phi_{H_k}$  on  $\mathbb{R}^{2+2n} - U$ . Therefore, we can compute the explicit form of  $\phi$  on  $\mathbb{R}^{2+2n} - K$ .

In polar coordinates  $(s_1, \theta_1, s_2, \theta_2)$  with  $s_1 = \|x_1\|^2$  and  $s_2 = \|x_2\|^2$ , we get for  $s_1 < 1$ :

$$\phi(s_1, 0, 0, 0) = \left( s_1, \frac{s_1}{(1 - s_1)^2} \right).$$

If we let  $s_1$  converge to 1, we see that  $\phi$  is not continuous and thus is not a homeomorphism.

**Corollary 4.9.** *The set  $\mathfrak{F}^\infty$  can be embedded into each completion  $\mathfrak{H}_u$ ,  $\tilde{\mathfrak{H}}_u$ ,  $\hat{\mathfrak{H}}$  and  $\check{\mathfrak{H}}$ .*

*Proof.* Since  $\hat{\mathfrak{H}} \subset \mathfrak{H}_u \subset \tilde{\mathfrak{H}}_u$  and  $\check{\mathfrak{H}} \subset \mathfrak{H}_u \subset \tilde{\mathfrak{H}}_u$ , it is enough to prove it for  $\hat{\gamma}$  and  $\check{\gamma}$ . We will make the proof for  $\hat{\gamma}$  and the proof for  $\check{\gamma}$  is exactly the same. Let  $J$  be the function that associates to any  $H \in \mathfrak{F}^\infty$  the element of  $\hat{\mathfrak{H}}$  represented by any sequence  $(H_k)$  that converges uniformly to  $H$  on the compact sets of  $\mathbb{R}^{2n} - H^{-1}(\{+\infty\})$ . As we noticed at the end of the proof of Proposition 4.7, two such sequences are equivalent and hence  $J$  is well-defined.

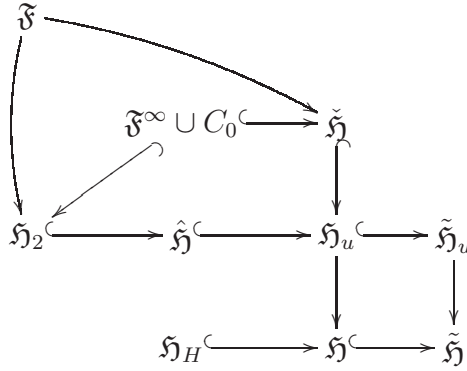
Let us now prove that  $J$  is one-to-one. Let  $H, G \in \mathfrak{F}^\infty$  and let  $(H_k), (G_k)$  be two sequences respectively associated to them, precisely constructed as in the last but one paragraph of the previous proof. Suppose that  $G \neq H$ , we

are going to show that  $\gamma(H_k, G_k)$  does not converge to zero, that will imply that  $\hat{\gamma}(H_k, G_k)$  does not converge to zero.

We can define almost everywhere the flows  $\phi_G^t, \phi_H^t$  and  $\psi^t = \phi_G^{-t} \circ \phi_H^t$ . Let  $\psi_k = \phi_{G_k}^{-1} \circ \phi_{H_k}$ . Since  $G \neq H$ , there exists a point  $x$  such that  $\psi(x) \neq x$ . Hence, there exists a small ball  $B$  around  $x$  such that  $\psi(B) \cap B = \emptyset$ . Let  $K$  be a compact neighbourhood of  $\bigcup_t \psi^t(B)$ . For  $k$  large enough,  $H_k$  and  $G_k$  coincide respectively with  $H$  and  $G$  on  $K$ , and thus  $\psi_k(B) \cap B = \emptyset$  too. Since  $\gamma(H_k, G_k) = \gamma(\psi_k) \geq \gamma(B) > 0$ ,  $\gamma(H_k, G_k)$  cannot converge to zero.  $\square$

*Remark.* As usual, the results of Proposition 4.7 and Corollary 4.8 and 4.9 still hold for  $\gamma_2$ .

Now, if we denote by  $C_0$  the set of continuous Hamiltonians that vanish at infinity, we can improve the diagram of section 4.1.1:



## 5 Application to Hamilton-Jacobi Equation.

Let  $H$  be a smooth Hamiltonian function on  $\mathbb{R} \times \mathbb{R}^{2n}$ . We consider the evolution Hamilton-Jacobi equation ( $HJ$ ):

$$\frac{\partial u}{\partial t} + H\left(t, x, \frac{\partial u}{\partial x}\right) = 0,$$

where  $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $(t, x) \mapsto u(t, x)$  satisfies an initial condition  $u(0, x) = u_0(x)$ . First, we remind the reader of the construction of a variational solution of ( $HJ$ ) (see for example [14] or [10]).

### 5.1 Recall on variational solutions of ( $HJ$ ).

Let us denote by  $\Lambda_0$  the graph of  $du_0$  and call it the initial submanifold. In fact, the following construction can be made for any Lagrangian submanifold  $\Lambda_0 \subset \mathbb{R}^{2n}$ . We consider  $\Sigma = \hat{H}^{-1}(\{0\}) \subset \mathbb{R}^{2+2n}$ . A geometric solution of

$(HJ)$  is a Lagrangian submanifold  $L$  that satisfies  $\Lambda_0 \leq L \leq \Sigma$ . For example, the graph of the differential of a smooth function  $u$  is a geometric solution if and only if  $u$  itself is solution of  $(HJ)$ .

With the help of the flow  $\phi_{\hat{H}}^t$ , we can construct a geometrical solution  $L_H = \bigcup_{t \in I} \phi_{\hat{H}}^t(\Lambda_0)$ , where  $I$  is an open interval containing  $[0, 1]$  and such that  $\rho_\alpha = 1$  on  $I$ . The Lagrangian submanifold  $L_H$  obtained is an element of  $\mathcal{L}(\mathbb{R}^{2+2n})$ .

For any element  $L \in \mathcal{L}(\mathbb{R}^{2k})$ , we can associate a function  $u_L$  on  $\mathbb{R}^{2k}$  by the following method.

Let  $S : \mathbb{R}^k \times \mathbb{R}^q \rightarrow \mathbb{R}$  be a g.f.q.i of  $L$ . Denote by  $1_z$  the fundamental class in  $H^0(z)$ . Then, we define  $u_L$  by

$$u_L(z) = c(1_z, S|_{z \times \mathbb{R}^q}),$$

with notations of section 2. The function  $u_L$  is everywhere  $C^0$ , and it is proved in [10], that  $u_L$  is  $C^k$  on a dense open set, for  $k \geq 1$ . Moreover, when it is defined, we have  $(x, du_L(x)) \in L$ . Therefore, the function  $u_{L_H}$  is a solution of  $(HJ)$  on any open set on which it is smooth.

We are now going to prove an interesting property of the elements of  $\mathfrak{H}_2$ , which is the fact that we can extend to them the construction of a variational solution of  $(HJ)$ .

## 5.2 Extension to the completion

**Proposition 5.1.** *Let  $H$  and  $K$  be two Hamiltonian functions, and  $u_{L_H}, u_{L_K}$  the solution obtained by the above method with the same initial submanifold  $\Lambda_0$ . Then,*

$$\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma_2(H, K).$$

That leads us to the following definition.

**Definition 5.2.** *Let  $H \in \mathfrak{H}_2$ . A continuous function  $u$  will be called generalized variational solution of  $(HJ)$  for  $H$ , if there exists a Cauchy sequence  $(H_k)$  in Ham representing  $H$  and such that the sequence of solutions  $(u_{L_{H_k}})$   $C^0$ -converges to  $u$ .*

Therefore, proposition 5.1 implies the following statement:

**Theorem 5.3.** *For each initial condition  $u_0$ , any element  $H$  in the completion  $\mathfrak{H}_2$  admits a unique generalized variational solution  $u_H$ . Moreover, the so constructed map  $\mathfrak{H}_2 \rightarrow C^0$  is continuous.*



In particular, any Hamiltonian function in  $\mathfrak{F}^\infty$  (see definition 4.6) admits a unique generalized variational solution.

*Proof.* Let  $(H_k) \in Ham$  be a Cauchy sequence for  $\gamma_2$  representing an element  $H \in \mathfrak{H}_2$ . Then, proposition 5.1 implies that  $(u_{L_{H_k}})$  is a Cauchy sequence in  $C^0$  and hence converges to a continuous function  $u$ . Moreover, if  $(H_k)$  and  $(F_k)$  are two equivalent Cauchy sequences for  $\gamma_2$ , then proposition 5.1 also implies that  $(u_{L_{H_k}})$  and  $(u_{L_{F_k}})$  are equivalent, and hence converge to the same limit. It gives the existence and the unicity.

The continuity of the map  $\mathfrak{H}_2 \rightarrow C^0$  is also an immediate consequence of Proposition 5.1.  $\square$

To prove proposition 5.1, we first prove the following lemma:

**Lemma 5.4.** *For any  $L \in \mathcal{L}$ , we have*

$$\|u_L\|_{C^0} \leq \gamma(L).$$

*Proof.* Since  $L$  coincides with the zero section out of a compact set,  $u_L$  has a compact support. It follows that  $\|u_L\|_{C^0} \leq \max(u_L) - \min(u_L)$ . We will prove that  $\min(u_L) \geq c(1, L)$ . It will also imply that  $\max(u_L) \leq c(\mu, L)$  by Poincaré duality. Indeed, using the fact that  $c(\mu, L) = -c(1, \bar{L})$  and that for all  $z$ ,  $\mu_z = 1_z$ , we have  $u_L = -u_{\bar{L}}$ .

Let  $z \in \mathbb{R}^k$ , and  $S : \mathbb{R}^k \times \mathbb{R}^q \rightarrow \mathbb{R}$  be a g.f.q.i of  $L \subset \mathbb{R}^{2k}$ . Then,  $S|_{\{z\} \times \mathbb{R}^q}$  is a g.f.q.i. of the reduction of  $L$  by the coisotropic submanifold  $\{z\} \times \mathbb{R}^k \subset \mathbb{R}^{2k}$ . Therefore, by lemma A.2, we get  $c(1_z, S|_{\{z\} \times \mathbb{R}^q}) \geq c(1, S)$ , for all  $z$  and hence  $\min(u_L) \geq c(1, L)$  as required.  $\square$

*Proof of proposition 5.1.* The proposition comes from a sequence of inequalities:

$$\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma(L_H, L_K) \leq \tilde{\gamma}(\phi_{\check{H}}, \phi_{\check{K}}) \leq \gamma_2(H, K).$$

The third inequality comes from the first inequality in proposition 2.10. The second one is proved in [1]. Finally, the first one comes from the lemma 5.4 above and proposition 3.3 in [15], which states that for all  $u, v \in H^*(\mathbb{R}^n)$ ,  $c(u \cup v, L_1 + L_2) \leq c(u, L_1) + c(v, L_2)$ , where  $L_1 + L_2 = \{(q, p_1 + p_2) \mid (q, p_1) \in L_1, (q, p_2) \in L_2\}$ .

Indeed, for  $u = v = 1_{(t,x)}$ ,  $L_1 = (L_H - L_K)|_{(t,x)}$  and  $L_2 = L_K|_{(t,x)}$ , we get  $c(1_{(t,x)}, L_H|_{(t,x)}) - c(1_{(t,x)}, L_K|_{(t,x)}) \leq -c(1_{(t,x)}, (L_H - L_K)|_{(t,x)})$ . Then, lemma 5.4 gives  $-c(1_{(t,x)}, (L_H - L_K)|_{(t,x)}) \leq \gamma(L_H - L_K) = \gamma(L_H, L_K)$ . By exchanging  $H$  and  $K$  and taking the supremum over  $(t, x)$ , we obtain  $\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma(L_H, L_K)$  as required.  $\square$

*Remark and Question.* Joukovskaia proved in [7] that for Hamiltonian functions that are convex in  $p$ , variational solutions of  $(HJ)$  coincide with viscosity solutions (These are a notion of weak solution introduced by Crandall and Lions in [2] that has shown its efficiency in a lot of domains of applications including optimal control and differential games, front propagation problems, finance, image theory.... ). We are tempted to use it together with some convergence result on viscosity solutions, to prove that our generalized variational solution is a viscosity solution. This would give another interpretation of our notion of solution, and since our solution is continuous, it would also give a continuity result on viscosity solutions.

However, since we developed our theory in the context of compactly supported Hamiltonians, we cannot reason on Hamiltonian functions convex in  $p$ . That leads us to our question : Can we define a completion with similar properties for a class Hamiltonian functions convex in  $p$ ?

## A Appendix: Proof of inequalities

In this appendix we prove proposition 2.10 and lemma 2.13. All those inequalities are based on the reduction inequality stated in proposition A.1.

### A.1 Inequality between $\tilde{\gamma}$ and $\gamma$ .

We first prove the inequality  $\gamma \geq \tilde{\gamma}$ .

Let  $\varphi$  be a Hamiltonian diffeomorphism, and  $L \in \mathcal{L}$ . We wish to show that  $\gamma(\varphi(L) - L) \leq \gamma(\varphi)$ . If we denote by  $N$  the zero section of  $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ , there exists a Hamiltonian isotopy  $\psi^t$  such that  $L = \psi^1(N)$ . Therefore, we just need to prove  $\gamma(\varphi(N)) \leq \gamma(\varphi)$ . Indeed, if we assume this inequality, then  $\gamma(\varphi(L) - L) = \gamma(\varphi \circ \psi^1(N) - \psi^1(N)) = \gamma(\psi^{-1} \circ \varphi \circ \psi^1(N) - N)$ , using formula (2.1) in [1]. Then, by assumption we get  $\gamma(\varphi(L) - L) \leq \gamma(\psi^{-1} \circ \varphi \circ \psi^1) = \gamma(\varphi)$ .

Let us prove now that  $\gamma(\varphi(N)) \leq \gamma(\varphi)$ . We denote by  $\Delta_p$  the diagonal in  $\mathbb{R}^p \times \mathbb{R}^p$ , and by  $\Phi$  the symplectic identification  $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \rightarrow T^*\Delta_{2n}$ . Recall that  $\widetilde{\Gamma}_\varphi$  is by definition the image of the graph  $\Gamma_\varphi$  of  $\varphi$ . Clearly,  $\varphi(N)$  is identified to the symplectic reduction of  $N \times \Gamma_\varphi \subset \mathbb{R}^{6n}$  by the coisotropic linear subspace  $\Delta_{2n} \times \mathbb{R}^{2n}$ . It is therefore identified to the reduction of  $N \times \widetilde{\Gamma}_\varphi$  by  $W = (Id_{\mathbb{R}^{2n}} \times \Phi)(\Delta_{2n} \times \mathbb{R}^{2n})$ . One can easily show that for all  $L \in \mathcal{L}$ ,  $\gamma(N \times L) = \gamma(L)$ . In particular,  $\gamma(\varphi) = \gamma(N \times \widetilde{\Gamma}_\varphi)$ , and the proof will be achieved if we prove the following proposition.

**Proposition A.1 (Reduction Inequality).** *For every Lagrangian submanifold  $L$  in  $\mathbb{R}^{2n}$  and every linear coisotropic subspace  $W$  of  $\mathbb{R}^{2n}$ , we have  $\gamma(L) \geq \gamma(L_W)$ , where  $L_W$  denotes the image of  $L$  by reduction by  $W$ .*

We first prove the following lemma.

**Lemma A.2.** *Let  $L$  be a Lagrangian submanifold in a cotangent bundle of the form  $T^*M = T^*B \times \mathbb{R}^{2k}$ . Consider the two coisotropic submanifolds  $X = T^*B \times \{x_0\} \times \mathbb{R}^n$  and  $Y = T^*B \times \mathbb{R}^n \times \{0\}$ . Denote by  $L_X$  and  $L_Y$  the reductions of  $L$  by respectively  $X$  and  $Y$ . Then*

$$c(1, L_X) \geq c(1, L) = c(1, L_Y),$$

$$c(\mu_B, L_X) \leq c(\mu_M, L) = c(\mu_B, L_Y).$$

*Proof.* We start the proof by showing that  $c(1, L_X) \geq c(1, L)$ . Let us fix  $\lambda \in \mathbb{R}$  and consider the inclusion  $i : B \simeq \{0\} \times B \rightarrow M$ . Let  $S$  be a g.f.q.i. of  $L$  defined on a bundle  $\pi : E \rightarrow M$ . Then the function  $S_X = S|_{\pi^{-1}(B \times \{x_0\})}$  is a generating function for  $L_X$ . Since  $S_X$  is a restriction of  $S$ , we have an inclusion of the sublevels  $S_X^\lambda \subset S^\lambda$ , which induces a morphism  $i_\lambda : H^*(S^\lambda, S^{-\infty}) \rightarrow H^*(S_X^\lambda, S_X^{-\infty})$ . The naturality of Thom isomorphism and the fact that all different inclusions commute make the following diagram commutative.

$$\begin{array}{ccccc} H^*(B) & \xrightarrow{T} & H^*(S_X^\infty, S_X^{-\infty}) & \xrightarrow{j_{X,\lambda}^*} & H^*(S_X^\lambda, S_X^{-\infty}) \\ i^* \uparrow & & \uparrow i_\infty & & \uparrow i_\lambda \\ H^*(M) & \xrightarrow{T} & H^*(S^\infty, S^{-\infty}) & \xrightarrow{j_\lambda^*} & H^*(S^\lambda, S^{-\infty}) \end{array}$$

Suppose now that  $j_{X,\lambda}^* \circ T(1) \neq 0$ . Then  $i_\lambda \circ j_\lambda^* \circ T(1) = j_{X,\lambda}^* \circ T \circ i^*(1) = j_{X,\lambda}^* \circ T(1) \neq 0$  hence  $j_\lambda^* \circ T(1) \neq 0$ . That proves  $c(1, L_X) \geq c(1, L)$ .

In the case of  $L_Y$ , we also have an explicit generating function, constructed as follows. Since  $\mathbb{R}^k$  is contractible we can suppose that the fibers of  $\pi$  do not depend on the second coordinate of  $M$ . Denote by  $i : B \simeq B \times \{0\} \rightarrow E$  the inclusion and by  $\tau : B \times \mathbb{R}^k \rightarrow B$  the trivial bundle of rank  $k$  over  $B$ . Consider the vector bundle over  $B$ ,  $\rho = \tau \oplus i^*\pi$  whose total space is  $F = \pi^{-1}(B \times \{0\}) \times \mathbb{R}^n$ . Then, the function  $S_Y$ , defined for all  $v \in B$  and  $(x, \xi) \in \rho^{-1}(v)$  by  $S_Y(v; x, \xi) = S(v, x; \xi)$  is a g.f.q.i for  $L_Y$ . The map  $f : E \rightarrow F$ ,  $(v, x; \xi) \mapsto (v, x, \xi)$  is a diffeomorphism and satisfies  $S_Y \circ f = S$ . Therefore, we have  $S_Y^\lambda = f(S^\lambda)$ , an isomorphism  $H^*(S^\lambda, S^{-\infty}) \simeq H^*(S_Y^\lambda, S_Y^{-\infty})$  and a commutative diagram

$$\begin{array}{ccccc} H^*(B) & \xrightarrow{T} & H^*(S_Y^\infty, S_Y^{-\infty}) & \xrightarrow{j_{Y,\lambda}^*} & H^*(S_Y^\lambda, S_Y^{-\infty}) \\ i^* \uparrow & & \uparrow \simeq & & \uparrow \simeq \\ H^*(M) & \xrightarrow{T} & H^*(S^\infty, S^{-\infty}) & \xrightarrow{j_\lambda^*} & H^*(S^\lambda, S^{-\infty}) \end{array}$$

The previous argument gives  $c(1, L_Y) \geq c(1, L)$ . The reverse inequality is obtained from the same diagram with the inclusion  $i$  replaced by the projection  $p : M \rightarrow B$  (which reverses vertical arrows).

Finally,  $c(\mu_B, L_N) \leq c(\mu_M, L) = c(\mu_B, L_Y)$  is obtained from  $c(1, L_X) \geq c(1, L) = c(1, L_Y)$  by Poincaré duality, by noticing that  $\overline{L_X} = \overline{L_X}$  and  $\overline{L_Y} = \overline{L_Y}$ .  $\square$

**Lemma A.3.** *Let  $W$  be a coisotropic linear subspace of  $\mathbb{R}^{2n}$ . Denote by  $N$  the zero section of  $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ . Then there exists a decomposition in linear isotropic subspaces  $\mathbb{R}^{2n} = N_1 \oplus V_1 \oplus N_2 \oplus V_2 \oplus N_3 \oplus V_3$ , where  $N = N_1 \oplus N_2 \oplus N_3$  and each  $N_i \oplus V_i$ ,  $i = 1, 2, 3$  is a symplectic subspace, such that  $W = N_1 \oplus V_1 \oplus N_2 \oplus V_3$ .*

*Proof.* Let us first recall that if  $W$  is coisotropic with symplectic orthogonal  $W^\omega \subset W$ , any subspace  $F$  such that  $F \oplus W^\omega = W$  is symplectic. Indeed, since  $F \subset W$ ,  $F \cap F^\omega = F \cap F^\omega \cap W = F \cap (F \oplus W^\omega)^\omega = F \cap W^\omega = \{0\}$ .

If there exists a decomposition as in the lemma, then  $W^\omega = N_2 \oplus V_3$ . Therefore we set  $N_2 = W^\omega \cap N$ . Then, we define  $N_1$  as one complementary of  $N_2$  in  $W \cap N$ , and  $F_1$  as one complementary of  $W^\omega$  in  $W$ , containing  $N_1$ . By the above remark,  $F_1$  is symplectic, and we can choose  $V_1$  as one Lagrangian complementary of  $N_1$  in  $F_1$ .

Then, we define  $V_3$  as a complementary of  $N_2$  in  $W^\omega$ . Since  $W \cap N = N_1 \oplus N_2$ ,  $V_3 \cap N = 0$ , and we can define  $N_3$  as a complementary of  $N_1 \oplus N_2$  in  $N$ . Then,  $F_3 = N_3 \oplus V_3$  is symplectic since it is a complementary of  $(N_1 \oplus N_2 \oplus F_3)^\omega$  in  $N_1 \oplus N_2 \oplus F_3$ .

Finally, we define  $F_2$  as a complementary of  $F_1 \oplus F_3$  in  $\mathbb{R}^{2n}$ . Then,  $F_2$  is symplectic for a similar reason as  $F_3$ , and we can define  $V_2$  as a Lagrangian complementary of  $N_2$  in  $F_2$ . The decomposition  $\mathbb{R}^{2n} = N_1 \oplus V_1 \oplus N_2 \oplus V_2 \oplus N_3 \oplus V_3$  satisfies all the requirements of lemma A.3.  $\square$

*Proof of proposition A.1.* Since the linear symplectic group acts transitively on the set of all pairs of complementary Lagrangian subspaces (see proposition 7.4 in Chapter 1 of [8]), and since the space of Lagrangian subspaces which are complementary to the zero section  $N$  is path connected, there exists a symplectic isotopy  $\Psi^t$  of  $\mathbb{R}^{2n}$  such that  $\Psi^0 = Id$  and that  $\Psi^1$  lets all the elements of  $N$  invariant and maps  $V$  on  $V_1 \oplus V_2 \oplus V_3$ . Since  $\mathbb{R}^{2n}$  is simply connected, that isotopy is Hamiltonian.

The reduction of  $L$  by  $W$  is identified with the reduction of  $\Psi^1(L)$  by  $\Psi^1(W)$ . Therefore, applying twice the lemma A.3, we get  $\gamma(L_W) \leq \gamma(\Psi^1(L))$ . But, by proposition 2.6 in [15], we have  $\gamma(L) = \gamma(\Psi^1(L))$ . That concludes the proof of proposition A.1.  $\square$

*Remark.* Note that in the end of the previous proof, lemma A.3 also implies  $c(1, L_W) \geq c(1, L)$ . That will be useful in the proof of lemma 2.13.

## A.2 Inequalities involving the “suspended distances”.

We now prove the inequality  $\gamma_u(H, K) \leq \hat{\gamma}(H, K)$ , for any  $H, K$  Hamiltonian functions. It is sufficient to prove that for all Hamiltonian functions  $H, K$ , all  $s$  in  $[0, 1]$ , and all  $\alpha$  large enough,  $\gamma(\phi_K^{-s}\phi_H^s) \leq \gamma(\phi_{\hat{K}}^{-s}\phi_{\hat{H}}^s)$ . We will prove that the graph of  $\phi_K^{-s}\phi_H^s$  can be obtained by reduction of the graph of  $\phi_{\hat{K}}^{-s}\phi_{\hat{H}}^s$ , and then use proposition A.1.

We denote by  $\hat{\Phi}^s$  the flow at time  $s$  of the Hamiltonian  $\hat{H} : (s; t, \tau, x) \mapsto \rho_\alpha(\tau)\tau + \rho_\alpha(t)H(t; x)$ . By direct computation, we get

$$\hat{\Phi}^s(t, \tau, x) = (t(s), \tau(s), x(s)),$$

with

$$\begin{aligned} t(s) &= t + \int_0^s (\rho'_\alpha(\tau(\sigma))\tau(\sigma) + \rho_\alpha(\tau(\sigma)))d\sigma \\ \tau(s) &= \tau - \int_0^s (\rho'_\alpha(t(\sigma))H(t(\sigma), x(\sigma) + \rho_\alpha(t(\sigma))\frac{\partial H}{\partial t}(t(\sigma), x(\sigma))))d\sigma \end{aligned}$$

and  $x(s)$  solution of  $\dot{x}(s) = \rho_\alpha(t(s))X_H(t(s), x(s))$ . If we denote  $M = \max(\|\rho_\alpha\|_{C^1}, \|H\|_{C^1})$ , we see that  $\tau(s) \in [\tau - |s|M^2, \tau + |s|M^2]$ . Suppose  $\tau \in [-M^2 - 2M, M^2 + 2M]$  and  $\alpha$  is large enough, then  $\rho_\alpha(\tau(s)) = 1$  and  $t(s) = t + s$ . Hence  $x(s) = (\phi_H)_t^{t+s}(x)$ . We set

$$\begin{aligned} I_H(s, t, x) &= - \int_0^s (\rho'_\alpha(t(\sigma))H(t(\sigma), x(\sigma) + \rho_\alpha(t(\sigma))\frac{\partial H}{\partial t}(t(\sigma), x(\sigma))))d\sigma \\ &= H(t, x) - H(t + s, \phi_t^{t+s}(x)), \end{aligned}$$

and  $J(s, t, x) = I_H(s, t, x) + I_K(-s, t + s, (\phi_H)_t^{t+s}(x))$ . Then, we can write the expression of the composition:

$$\phi_K^{-s}\phi_{\hat{H}}^s(t, \tau, x) = (t, \tau + J(s, t, x), (\phi_K)_t^{t-s}(\phi_H)_t^{t+s}(x)).$$

We can now compute the intersection of the graph  $\Gamma_{\phi_K^{-s}\phi_{\hat{H}}^s}$  with the set  $U = [-1, 1] \times \mathbb{R} \times [-M^2, M^2] \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ , and its image by the natural identification  $\Psi : \mathbb{R}^{4+4n} \rightarrow T^*\Delta_{2+2n}$ . We get

$$\tilde{\Gamma}_{\phi_K^{-s}\phi_{\hat{H}}^s} \cap \Psi(U) = \{(t, J(s, t, x), \tau + \frac{1}{2}J(s, t, x), 0, z(x)) \mid$$

$$(t, \tau, x) \in [0, 1] \times [-M^2, M^2] \times \mathbb{R}^{2n}, \quad z(x) \in \tilde{\Gamma}_{(\phi_K)_t^{t-s}(\phi_H)_t^{t+s}}.$$

Consider the coisotropic submanifold  $W = \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \mathbb{R}^{4n}$ . Since  $\tau + \frac{1}{2}J(s, t, x) = 0$  implies  $\tau \in [-M^2 - 2M, M^2 + 2M]$ , and since  $W \subset \Psi(U)$ , we see that  $\tilde{\Gamma}_{\phi_K^{-s}\phi_H^s}$  is obtained from  $\tilde{\Gamma}_{\phi_K^{-s}\phi_H^s}$  by reduction by  $W$ . By proposition A.1, we get  $\gamma(\tilde{\Gamma}_{\phi_K^{-s}\phi_H^s}) \leq \gamma(\tilde{\Gamma}_{\phi_K^{-s}\phi_H^s})$  and hence  $\gamma(\phi_K^{-s}\phi_H^s) \leq \gamma(\phi_K^{-s}\phi_H^s)$ .  $\square$

We are now going to prove  $\gamma_u(H, K) \leq \check{\gamma}(H, K)$ . The idea of the proof is the same as the previous one: we show that for any  $s \in [0, 1]$ ,  $\tilde{\Gamma}_{\phi_K^{-s}\phi_H^s}$  is obtained by reduction of  $\tilde{\Gamma}_{\phi_K^{-1}\phi_H}$ , for  $\alpha$  large enough.

Recall that by definition,  $\tilde{H}(s; t, \tau, x) = \rho_\alpha(t)tH(st; x)$ . As above, we compute the flow :  $\phi_{\tilde{H}}^s(t, \tau, x) = (t(s), \tau(s), x(s))$ , and we obtain

$$\begin{aligned} t(s) &= t \\ \tau(s) &= \tau + I_H(s, t, x) \end{aligned}$$

where  $I_H(s, t, x) = \rho_\alpha(t)sH(st, x(s)) - \rho'_\alpha(t)t \int_0^s H(\sigma t, x(\sigma))d\sigma$  and  $x(s)$  is solution of  $\dot{x}(s) = \rho_\alpha(t)tX_H(st, x(s))$ . For  $t \in [-1, 1]$  and  $\alpha \geq 1$ , it gives  $x(s) = \phi^{ts}(x)$ .

Similarly as above, we set  $J(s, t, x) = I_H(s, t, x) + I_K(-s, t + s, (\phi_H)^{ts}(x))$ , the set  $U = [-1, 1] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$  and the identification  $\Psi : \mathbb{R}^{4+4n} \rightarrow T^*\Delta_{2+2n}$ . The graph can be written this way:

$$\begin{aligned} \tilde{\Gamma}_{\phi_K^{-s}\phi_H^s} \cap \Psi(U) &= \{(t, J(s, t, x), \tau + \frac{1}{2}J(s, t, x), 0, z(x)) \mid \\ &(t, \tau, x) \in [0, 1] \times \mathbb{R} \times \mathbb{R}^{2n}, \quad z(x) \in \tilde{\Gamma}_{\phi_K^{-st}\phi_H^{st}}\}. \end{aligned}$$

Now, we see that  $\tilde{\Gamma}_{\phi_K^{-t}\phi_H^t}$  is the reduction of  $\tilde{\Gamma}_{\phi_K^{-1}\phi_H}$  by the coisotropic manifold  $W = \{t\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \mathbb{R}^{4n}$ . Using lemma A.2 twice, we conclude that for all  $t \in [0, 1]$ ,  $\gamma(\phi_K^{-t}\phi_H^t) \leq \gamma(\phi_K^{-1}\phi_H)$ .  $\square$

### A.3 Proof of lemma 2.13.

It is sufficient to show that  $c(V) \leq c(\mathbb{R}^2 \times V)$  for all open subset  $V \in \mathbb{R}^{2n}$ . Let  $H$  be an Hamiltonian function with support in  $V$ . We just have to find an Hamiltonian function  $K$  with support in  $V \times \mathbb{R}^2$  satisfying the inequality  $c_+(H) \leq c_+(K)$ . If we set  $K = \tilde{H}_\alpha$  for  $\alpha$  large enough,  $K$  has support in  $\mathbb{R}^2 \times V$ , and we saw in particular in the previous proof that  $\tilde{\Gamma}_{\phi_H^1}$  is the reduction of  $\tilde{\Gamma}_{\phi_H}$ . Therefore, by the remark that ends section A.1, we have  $c_+(H) \leq c_+(K)$  as required.  $\square$

## References

- [1] CARDIN F. and VITERBO C. *Commuting Hamiltonians and Hamilton-Jacobi multi-time equations*. preprint, math.SG/0507418.
- [2] CRANDALL M.G and LIONS P-L. *Viscosity solutions of Hamilton-Jacobi equations*. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.
- [3] GOLUBITSKY M. and GUILLEMIN V. *Stable mappings and their singularities*. Number 14 in Graduate texts in mathematics. Berlin Springer, New York, 1973.
- [4] HOFER H. *On the topological properties of symplectic maps*. *Proc. Roy. Soc. Edinburgh Sect. A*, 115:25–38, 1990.
- [5] HOFER H. and ZEHNDER E. *Symplectic invariants and Hamiltonian dynamics*. Birkhauser, 1994.
- [6] HUSEMOLLER D. *Fiber Bundles*. Springer-Verlag, 1975.
- [7] JOUKOVSKAÏA T. *Singularités de minimax et solutions faibles d'équations aux dérivées partielles*. PhD thesis, Université Paris 7, 1993.
- [8] LIBERMANN P. and MARLE C.M. *Geométrie symplectique, Bases théorique de la mécanique, Tome I*. Publications Mathématiques de l'Université Paris VII, 1986.
- [9] OH Y.G. *The group of Hamiltonian homeomorphisms and  $C^0$  symplectic topology I*. preprint, math.SG/0402210 v2 August 2005.
- [10] OTTOLENGHI A. and VITERBO C. Solutions généralisées pour l'équation d'Hamilton-Jacobi dans le cas d'évolution. Manuscript.
- [11] SCHWARZ M. *On the action spectrum for closed symplectically aspherical manifolds*. *Pacific J. Math.*, 193:419–461, 2000.
- [12] SIKORAV J.C. Sur les immersions Lagrangiennes admettant une phase génératrice globale. *Compte-rendu de l'Académie des Sciences*, 302:119–122, 1986.
- [13] THÉRET D. A complete proof of Viterbo's uniqueness theorem on generating functions. *Topology and its Applications*, 96(3):246–266, 1999.
- [14] VITERBO C. Solutions d'équations de Hamilton-Jacobi. Séminaire X-EDP, Palaiseau, 1992.

- [15] VITERBO C. Symplectic topology as the geometry of generating functions. *Math. Annalen*, 292:685–710, 1992.