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On some completions of the space of Hamiltonian maps

Vincent Humilière

Abstract

In his paper [12], C. Viterbo defined a distance on the set of Hamiltonian diffeomorphisms of \mathbb{R}^{2n} endowed with the standard symplectic form $\omega_0 = dp \wedge dq$. We study the completions of this space for the topology induced by Viterbo's distance and some others derived from it, we study their different inclusions and give some of their properties. In particular, we give a convergence criterion for these distances. This allows us to prove that the completions contain non-ordinary elements, as for example, discontinuous Hamiltonians.

1 Introduction.

Given an open subset U in \mathbb{R}^{2n} , we denote by $Ham(U)$ the set of all 1-periodic time dependent Hamiltonian functions $\mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ whose support for fixed time is compact and contained in U . We will write Ham for $Ham(\mathbb{R}^{2n})$.

Given a Hamiltonian function $H \in Ham$, its symplectic gradient (i.e the unique vector field X_H satisfying $dH = \iota_{X_H} \omega_0$) generates a Hamiltonian isotopy $\{\phi_H^t\}$. The set of Hamiltonians diffeomorphisms generated by an element H in $Ham(U)$ will be denoted by $\mathcal{H}(U) = \{\phi_H = \phi_H^1 \mid H \in Ham(U)\}$, and we write \mathcal{H} for $\mathcal{H}(\mathbb{R}^{2n})$. Finally, we call $\mathcal{L} = \{\phi(0_n) \mid \phi \in \mathcal{H}\}$, the set of Lagrangian submanifolds obtained from the zero section $0_n \subset T^*\mathbb{R}^n = \mathbb{R}^{2n}$, by a Hamiltonian isotopy with compact support.

As usual, we denote Viterbo's distance on \mathcal{L} or \mathcal{H} by γ (see [12]). Convergence with respect to γ is called c-convergence.

Our main goal in this paper is to understand the completion of the space $\mathfrak{H}(U)$ for the distance γ , give some convergence criterion (section 4) and compare it with the convergence for Hofer's distance d_H (see [4], chapter 5 section 1).

The notion of C^0 symplectic topology has been studied by many authors, starting from the work of Eliashberg and Gromov on the C^0 closure of the

group of symplectic diffeomorphisms, to the later results of Viterbo ([12]), Hofer ([3])

More recently Y.G. Oh ([7]) gave a deep study of several versions of C^0 Hamiltonians. However our definition seems to differ from his, since in all his definitions, he needs the Hamiltonians to be continuous, while our study starts as we drop this assumption.

In section 4, we define a symplectic invariant ξ_∞ and prove that

Theorem 1.1. *If the closure of U is compact and if $K \subset U$ is such that $\xi_\infty(K) = 0$, then for every sequence (H_k) in $\text{Ham}(U)$ which is Cauchy for the metric of uniform convergence on compact subsets of $U - K$, the sequence ϕ_{H_k} is Cauchy for γ .*

As a result, any Hamiltonian continuous on $U - K$, with compact support in $\mathbb{R} \times (U - K)$ has a flow defined in the γ -completion of $\mathcal{H}(U)$.

Examples of sets K with $\xi_\infty(K) = 0$ are given by compact submanifolds of dimension $d \leq n - 2$.

The elements of the completion have some other properties. We can associate (section 5) to them a support by defining $\text{supp}(\phi)$ as

$$\bigcap \{ \overline{U} \subset \mathbb{R}^{2n} \mid \text{there exists } (\phi_n) \text{ representing } \phi \text{ such that } \forall n, \text{supp}(\phi_n) \subset U \}.$$

To some of them, it is also possible to associate a solution to Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t} + H \left(t, x, \frac{\partial u}{\partial x} \right) = 0.$$

Indeed, a γ -Cauchy sequence of Hamiltonians gives a C^0 -Cauchy sequence of solutions (section 6).

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2 Symplectic invariants.

In this section, we recall the definition of Viterbo's distance, defined first for Lagrangian submanifold with the help of generating functions, and then for Hamiltonian diffeomorphisms (see [12]). It will be convenient to use some other distances defined from γ by taking suspensions, namely $\tilde{\gamma}$, γ_u , $\tilde{\gamma}_u$, $\hat{\gamma}$, $\check{\gamma}$ and γ_2 . Relations between these metrics will be stated and proved in section 3.

2.1 Generating functions quadratic at infinity.

Let L be a Lagrangian submanifold of the cotangent bundle T^*M of a smooth manifold M . We say that L admits a *generating function* if there exists an integer $q > 0$ and a smooth function $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$ such that L can be written

$$L = \left\{ (x, p) \in T^*M \mid \exists \xi \in \mathbb{R}^q, \frac{\partial S}{\partial \xi}(x, \xi) = 0 \text{ and } \frac{\partial S}{\partial x}(x, \xi) = p \right\}.$$

Such function S is called a *generating function quadratic at infinity* (or just “g.f.q.i”) if there exists a non degenerate quadratic form Q on \mathbb{R}^q and a compact $K \subset M \times \mathbb{R}^q$ such that, $\forall (x, \xi) \notin K, S(x, \xi) = Q(\xi)$.

For instance, any quadratic form on \mathbb{R}^q viewed as a function on $M \times \mathbb{R}^q$ is a g.f.q.i of the zero section $0_M \subset T^*M$. J.C. Sikorav proved in [9] that the property of having a g.f.q.i is invariant by Hamiltonian isotopy with compact support. For this reason we will be interested in the set \mathcal{L} of Lagrangian submanifolds, images of the zero section by an Hamiltonian isotopy with compact support.

Furthermore, C. Viterbo and D. Th  ret proved that the g.f.q.i’s of a given Lagrangian submanifold are essentially unique. Before stating this result, let us introduce the following definitions: For a given function $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$, we call a *stabilisation* of S any function $S' : M \times \mathbb{R}^q \times \mathbb{R}^{q'} \rightarrow \mathbb{R}$ of the form $S'(x, \xi, \xi') = S(x, \xi) + q(\xi')$, where q is a non-degenerate quadratic form on $\mathbb{R}^{q'}$. In addition, two functions $S, S' : M \times \mathbb{R}^q \rightarrow \mathbb{R}$ are said *equivalent* if there exists a diffeomorphism ϕ of $M \times \mathbb{R}^q$ and a real C such that $S' = S \circ \phi + C$.

Theorem 2.1 ([12, 10]). *Suppose S, S' are two g.f.q.i’s of the same Lagrangian submanifold in \mathcal{L} . Then, up to stabilisation, S and S' are equivalent.*

This result allows to associate symplectic invariants to any element of \mathcal{L} .

2.2 Invariants defined by minimax and a distance on the group of Hamiltonian diffeomorphisms.

The invariants defined in this section have been introduced by C. Viterbo in [12]. We recall their construction. We first define invariants for Lagrangian submanifolds.

Let L be an element of \mathcal{L} and $S : M \times \mathbb{R}^q \rightarrow \mathbb{R}$ be one of its g.f.q.i’s. Let us denote $S^\lambda = \{x \in M \times \mathbb{R}^q \mid S(x) \leq \lambda\}$. Since S is quadratic at infinity, the homotopy types of the pairs (S^λ, S^μ) and $(S^\mu, S^{-\lambda})$ do not depend on λ , provided that λ is sufficiently large. Therefore, we will denote S^∞ and $S^{-\infty}$, instead of S^λ and $S^{-\lambda}$ for λ large enough.

Let us introduce E_∞^- the negative (trivial) bundle of the quadratic form which coincides with S at infinity. We denote $B(E_\infty^-)$, $S(E_\infty^-)$ the ball bundle and the sphere bundle associated to E_∞^- . The Thom isomorphism is given by $H^*(M) \rightarrow H^*(B(E_\infty^-), S(E_\infty^-))$, and we also have the isomorphism $H^*(B(E_\infty^-), S(E_\infty^-)) \simeq H^*(S^\infty, S^{-\infty})$. We will denote by T their composition. For further informations on those isomorphisms, see [5] for example. The inclusion $j_\lambda : S^\lambda \rightarrow S^\infty$ induces a morphism in cohomology $j_\lambda^* : H^*(S^\infty, S^{-\infty}) \rightarrow H^*(S^\lambda, S^{-\infty})$, for all real number λ . We are now ready for the following.

Definition 2.2. *Let $(u, L) \in H^*(M) \times \mathcal{L}$, with $u \neq 0$. Using a g.f.q.i S of L , we define a real number $c(u, L)$ as follows:*

$$c(u, L) = \inf\{\lambda \mid j_\lambda^* \circ T(u) \in H^*(S^\lambda, S^{-\infty}) \text{ is non zero}\}. \quad (1)$$

Observe that $c(u, L)$ is well defined, and is independent of the choice of S 's choice. Indeed, if we replace S with an equivalent or stabilized generating function, the value of $c(u, L)$ does not change and we conclude using theorem 2.1. Even if it doesn't depend on the generating function, we sometimes use the notation $c(u, S)$ instead of $c(u, L)$.

Since the cohomology of the sets S^λ changes when we cross the level $c(u, L)$, it has to be a critical value of S .

Finally, observe that the definition can be extended to classes with compact support $u \in H_c^*(M)$.

Then, we can use those invariants associated to Lagrangian submanifold to define other invariants associated to Hamiltonian diffeomorphisms.

Consider an Hamiltonian diffeomorphism $\psi \in \mathcal{H}$. Its graph Γ_ψ is a Lagrangian submanifold of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ ($= (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$, where ω_0 is the standard symplectic structure on \mathbb{R}^{2n}). It coincides with the diagonal $\Delta = \{(x, x) \mid x \in \mathbb{R}^{2n}\}$, outside the product $B^{2n}(r) \times B^{2n}(r)$, for r sufficiently large. When we identify $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ with $T^*\Delta$ using the map,

$$(q, p, Q, P) \mapsto \left(\frac{q+Q}{2}, \frac{p+P}{2}, P-p, Q-q \right),$$

we see that the image $\widetilde{\Gamma}_\psi$ of Γ_ψ is identified with the zero section of $T^*\Delta$ outside a compact set.

Then, we can associate the previous invariant to $\widetilde{\Gamma}_\psi$. Let 1 be a generator of $H^0(\mathbb{R}^{2n})$ and μ a generator of $H_c^n(\mathbb{R}^{2n})$.

Definition 2.3 (Viterbo, [12]). *We define,*

$$c_-(\psi) = -c(\mu, \widetilde{\Gamma}_\psi),$$

$$\begin{aligned}
c_+(\psi) &= -c(1, \widetilde{\Gamma}_\psi), \\
\gamma(\psi) &= c_+(\psi) - c_-(\psi), \\
\gamma(\phi, \psi) &= \gamma(\psi^{-1}\phi).
\end{aligned}$$

Let us describe now some properties of the numbers γ , c_+ and c_- .

Proposition 2.4 (Viterbo, [12]). *For all ψ in \mathcal{H} , we have*

$$c_-(\psi) \leq 0 \leq c_+(\psi).$$

Moreover, $c_-(\psi) = c_+(\psi) = 0$ if and only if $\psi = \text{Id}$.
If ϕ is another diffeomorphism in \mathcal{H} , then

$$\begin{aligned}
c_+(\phi \circ \psi) &\leq c_+(\phi) + c_+(\psi), \\
c_-(\phi \circ \psi) &\geq c_-(\phi) + c_-(\psi), \\
\gamma(\phi \circ \psi) &\leq \gamma(\phi) + \gamma(\psi).
\end{aligned}$$

Those properties allows to prove that γ is a distance on \mathcal{H} . We also have a very useful property of monotonicity of c_+ and c_- .

Proposition 2.5. *Let ψ_1 and ψ_2 be two Hamiltonians generated by H_1 and H_2 . Suppose that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^{2n}$, we have $H_1(t, x) \leq H_2(t, x)$. Then, $c_+(\psi_1) \leq c_+(\psi_2)$ and $c_-(\psi_1) \leq c_-(\psi_2)$.*

As a corollary, we get that if H is a non-negative Hamiltonian, then $c_-(\phi_H) = 0$. If H is in addition non zero, we deduce $c_+(\phi_H) > 0$.

Finally, the following proposition states the continuity of γ with respect to C^0 -topology.

Proposition 2.6. *Let H_1 and H_2 be two compactly supported hamiltonians, generating ψ_1 and ψ_2 . Let $\|\cdot\|$ be the usual norm on $C^0(\mathbb{R}^{2n} \times [0, 1], \mathbb{R})$. If $\|H_1 - H_2\| \leq \varepsilon$, then $|\gamma(\psi_1) - \gamma(\psi_2)| \leq \varepsilon$.*

2.3 Other distances derived from γ .

Let us now introduce other distances, derived from γ . Those definitions are sometimes more convenient than the original one.

Definition 2.7. *For all Hamiltonian diffeomorphisms $\phi, \psi \in \mathcal{H}$, we define*

$$\tilde{\gamma}(\phi, \psi) = \inf\{\gamma(\psi^{-1}\phi(L) - L) \mid L \in \mathcal{L}\},$$

where $\gamma(L) = c(\mu, L) - c(1, L)$, $\forall L \in \mathcal{L}$ and $L_1 - L_2 = \{(q, p_1 - p_2) \mid (q, p_1) \in L_1, (q, p_2) \in L_2\}$, for $L_1, L_2 \in \mathcal{L}$.

We get by this way another distance on \mathcal{H} (see [12], [1]). Then, we define distances not anymore on \mathcal{H} , but on Ham .

Definition 2.8. For any $H, K \in Ham$, we set

$$\gamma_u(H, K) = \sup\{\gamma(\phi_H^t, \phi_K^t) \mid t \in [0, 1]\}$$

and

$$\tilde{\gamma}_u(H, K) = \sup\{\tilde{\gamma}(\phi_H^t, \phi_K^t) \mid t \in [0, 1]\}.$$

Here, the subscript “u” means “uniform”. Clearly, γ_u and $\tilde{\gamma}_u$ are distances on Ham .

The two next distances are constructed with the help of suspensions. We already explained the principle in the introduction, we now give the precise definition.

Definition 2.9. Let ρ be a fixed real function defined on $[0, +\infty)$, supposed to be non-negative, smooth, decreasing, with support in $[0, 1]$, flat at 0 and such that $\rho(0) = 1$. For every natural integer α and every real number t , we set $\rho_\alpha(t) = 1$ if $-\alpha \leq t \leq \alpha$, and $\rho_\alpha(t) = \rho(|t| - \alpha)$ otherwise.

We denote by \hat{H}_α and \check{H}_α the Hamiltonian functions defined on $\mathbb{R} \times \mathbb{R}^{2+2n}$, by

$$\hat{H}_\alpha(s; t, \tau, x) = \rho_\alpha(\tau)\tau + \rho_\alpha(t)H(t, x),$$

and

$$\check{H}_\alpha(s; t, \tau, x) = \rho_\alpha(t)tH(st, x).$$

Then, for $H, K \in Ham$, we set

$$\hat{\gamma}(H, K) = \limsup_{\alpha \rightarrow +\infty} \gamma_u(\hat{H}_\alpha, \hat{K}_\alpha),$$

and

$$\tilde{\gamma}(H, K) = \limsup_{\alpha \rightarrow +\infty} \gamma(\phi_{\check{H}_\alpha}, \phi_{\check{K}_\alpha}).$$

Remark that $\hat{\gamma}(H, K)$ and $\tilde{\gamma}(H, K)$ are finite. Indeed, if we denote by B a ball containing both supports of H and K , then $\hat{H}_\alpha, \hat{K}_\alpha, \check{H}_\alpha$ and \check{K}_α have support in $\mathbb{R}^2 \times B$, for any integer α . Hence $\gamma(\phi_{\check{H}_\alpha}, \phi_{\check{K}_\alpha}) \leq 2c(\mathbb{R}^2 \times B) \leq 2c^\infty(B)$ (See section 2.4 for notations). It shows that the limsup in the definition of $\tilde{\gamma}$ is finite. The same proof shows that $\hat{\gamma}(H, K)$ is also finite.

The triangle inequality for $\hat{\gamma}$ and $\tilde{\gamma}$ is a direct consequence of the triangle inequality axiom for γ . The separation axiom is obtained from the separation axiom for γ and proposition 3.1.

By repeating these constructions several times (i.e. by taking suspensions of suspensions), we can construct new distances. For example, we will use in section 6 the distance $\gamma_2 = \limsup_{\alpha \rightarrow +\infty} \tilde{\gamma}(\hat{H}_\alpha, \hat{K}_\alpha)$.

For convenience, we will not write the subscript α anymore. In the following, we will denote \hat{H} for \hat{H}_α , and \check{H} for \check{H}_α .

2.4 Two symplectic capacities on \mathbb{R}^{2n} .

We start this section by reminding the reader of the definition of a symplectic capacity.

Definition 2.10 (Ekeland-Hofer). *A symplectic capacity on $(\mathbb{R}^{2n}, \omega_0)$, is a map associating to each subset $U \subset \mathbb{R}^{2n}$, a number $c(U) \in [0, +\infty]$, satisfying*

1. $U \subset V \Rightarrow c(U) \leq c(V)$ (monotony),
2. $c(\phi(U)) = c(U)$ for all Hamiltonian diffeomorphism $\phi \in \mathcal{H}$ (symplectic invariance),
3. $c(\lambda U) = \lambda^2 c(U)$ for all real $\lambda > 0$ (homogeneity),
4. $c(B^{2n}(1)) = c(B^2 \times \mathbb{R}^{2(n-1)}) = \pi$, where $B^{2n}(1)$ is the unit ball of \mathbb{R}^{2n} (normalisation).

The invariants defined in the previous section allow to define two symplectic capacities [12]. Moreover, we have an inequality between them.

Definition 2.11. 1. *For any compact subset $K \subset \mathbb{R}^{2n}$, we denote by $\gamma(K)$ the number defined by*

$$\gamma(K) = \inf\{\gamma(\phi) \mid \phi(K) \cap K = \emptyset\}.$$

If V is not compact, we set

$$\gamma(V) = \sup\{\gamma(K) \mid K \subset V\}.$$

2. *For any open subset $U \subset \mathbb{R}^{2n}$, we denote by $c(U)$ the number defined by*

$$c(U) = \sup\{c_+(\phi_H) \mid \text{Supp}(H) \subset U\}.$$

If V is not an open set, we set

$$c(V) = \inf\{c(U) \mid V \subset U\}.$$

The maps c and γ are symplectic capacities and moreover, $c \leq \gamma$. We remind the reader of the definition of the displacement energy

$$d(U) = \inf\{d_H(\phi, Id) \mid \phi(U) \cap U = \emptyset\},$$

where d_H is Hofer's distance (we already recalled its definition in the introduction).

We are going to define a new symplectic capacity derived from γ , but before we need the following lemma.

Lemma 2.12. *We consider a subset $V \subset \mathbb{R}^{2n}$ and $\mathbb{R}^2 \times V \subset \mathbb{R}^{2+2n}$. Then,*

$$c(\mathbb{R}^2 \times V) \geq c(V).$$

We will prove that lemma at the end of section 3.

Definition 2.13. *For any open subset $U \subset \mathbb{R}^{2n}$, we set*

$$c^\infty(U) = \lim_{N \rightarrow \infty} c(U \times \mathbb{R}^{2N}),$$

and if V is not an open subset,

$$c^\infty(V) = \inf\{c^\infty(U) \mid V \subset U\}.$$

We obtain a symplectic capacity that satisfies $c^\infty(V) = c^\infty(V \times \mathbb{R}^2)$ for all subset V , and $c \leq c^\infty$. Moreover, since $d(U) = d(U \times \mathbb{R}^{2k})$, we have also $c^\infty \leq d$. To summarize the inequalities between capacities we have,

Proposition 2.14. $c \leq \gamma \leq d$ and $c \leq c^\infty \leq d$.

3 Inequalities between the new distances

The goal of this section is to relate the different distances. We will prove:

Proposition 3.1.

$$\begin{aligned} \tilde{\gamma} &\leq \gamma, \\ \tilde{\gamma}_u &\leq \gamma_u \leq \min(\hat{\gamma}, \tilde{\gamma}). \end{aligned}$$

The inequality $\gamma_u \leq \tilde{\gamma}$ implies $\hat{\gamma} \leq \gamma_2$, and it is known (see [12]) that $\gamma \leq d_H$. The results on the completions may be summarized by the following diagram. Here, \mathfrak{H}_H denotes the completion of \mathcal{H} for Hofer's distance, and C_c the set of continuous Hamiltonians with compact support. The arrows $\mathfrak{H}_u \rightarrow \mathfrak{H}$ and $\tilde{\mathfrak{H}}_u \rightarrow \tilde{\mathfrak{H}}$ are induced by the map $H \mapsto \phi_H^1$.

$$\begin{array}{ccccccc} C_c & \hookrightarrow & \check{\mathfrak{H}} & & & & \\ \downarrow & & \searrow & & & & \\ \mathfrak{H}_2 & \hookrightarrow & \hat{\mathfrak{H}} & \hookrightarrow & \mathfrak{H}_u & \hookrightarrow & \tilde{\mathfrak{H}}_u \\ & & & & \downarrow & & \downarrow \\ & & \mathfrak{H}_H & \hookrightarrow & \mathfrak{H} & \hookrightarrow & \tilde{\mathfrak{H}} \end{array}$$

3.1 Inequality between $\tilde{\gamma}$ and γ .

We wish to prove the first inequality of proposition 3.1, i.e. $\gamma \geq \tilde{\gamma}$.

Let φ be a Hamiltonian diffeomorphism, and $L \in \mathcal{L}$. We wish to show that $\gamma(\varphi(L) - L) \leq \gamma(\varphi)$. If we denote by N the zero section of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, there exists a Hamiltonian isotopy ψ^t such that $L = \psi^1(N)$. Therefore, we just need to prove $\gamma(\varphi(N)) \leq \gamma(\varphi)$. Indeed, if we assume this inequality, then $\gamma(\varphi(L) - L) = \gamma(\varphi \circ \psi^1(N) - \psi^1(N)) = \gamma(\psi^{-1} \circ \varphi \circ \psi^1(N) - N)$ by symplectic invariance of γ . Then, by assumption we get $\gamma(\varphi(L) - L) \leq \gamma(\psi^{-1} \circ \varphi \circ \psi^1) = \gamma(\varphi)$.

Let us prove now that $\gamma(\varphi(N)) \leq \gamma(\varphi)$. We denote by Δ_p the diagonal in $\mathbb{R}^p \times \mathbb{R}^p$, and by Φ the symplectic identification $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \rightarrow T^*\Delta_{2n}$. Recall that $\widetilde{\Gamma}_\varphi$ is by definition the image of the graph Γ_φ of φ . Clearly, $\varphi(N)$ is identified to the symplectic reduction of $N \times \Gamma_\varphi \subset \mathbb{R}^{6n}$ by the coisotropic linear subspace $\Delta_{2n} \times \mathbb{R}^{2n}$. It is therefore identified to the reduction of $N \times \widetilde{\Gamma}_\varphi$ by $W = (Id_{\mathbb{R}^{2n}} \times \Phi)(\Delta_{2n} \times \mathbb{R}^{2n})$. One can easily show that for all $L \in \mathcal{L}$, $\gamma(N \times L) = \gamma(L)$. In particular, $\gamma(\varphi) = \gamma(N \times \widetilde{\Gamma}_\varphi)$, and the proof will be achieved if we prove the following proposition.

Proposition 3.2. *For all Lagrangian submanifold L in \mathbb{R}^{2n} and all linear coisotropic subspace W of \mathbb{R}^{2n} , we have $\gamma(L) \geq \gamma(L_W)$, where L_W denotes the image of L by reduction by W .*

We first prove the following lemma.

Lemma 3.3. *Let L be a Lagrangian submanifold in a cotangent bundle of the form $T^*M = T^*B \times \mathbb{R}^{2k}$. Consider the two coisotropic submanifolds $X = T^*B \times \{x_0\} \times \mathbb{R}^n$ and $Y = T^*B \times \mathbb{R}^n \times \{0\}$. Denote by L_X and L_Y the reductions of L by respectively X and Y . Then*

$$c(1, L_X) \geq c(1, L) = c(1, L_Y),$$

$$c(\mu_B, L_X) \leq c(\mu_M, L) = c(\mu_B, L_Y).$$

Proof. We start the proof by showing that $c(1, L_X) \geq c(1, L)$. Let us fix $\lambda \in \mathbb{R}$ and consider the inclusion $i : B \simeq \{0\} \times B \rightarrow M$. Let S be a g.f.q.i. of L defined on a bundle $\pi : E \rightarrow M$. Then the function $S_X = S|_{\pi^{-1}(B \times \{x_0\})}$ is a generating function for L_X . Since S_X is a restriction of S , we have an inclusion of the sublevels $S_X^\lambda \subset S^\lambda$, which induces a morphism $i_\lambda : H^*(S^\lambda, S^{-\infty}) \rightarrow H^*(S_X^\lambda, S_X^{-\infty})$. The naturality of Thom isomorphism and the fact that all different inclusions commute make the following diagram

commutative.

$$\begin{array}{ccccc}
H^*(B) & \xrightarrow{T} & H^*(S_X^\infty, S_X^{-\infty}) & \xrightarrow{j_{X,\lambda}^*} & H^*(S_X^\lambda, S_X^{-\infty}) \\
i^* \uparrow & & \uparrow i_\infty & & \uparrow i_\lambda \\
H^*(M) & \xrightarrow{T} & H^*(S^\infty, S^{-\infty}) & \xrightarrow{j_\lambda^*} & H^*(S^\lambda, S^{-\infty})
\end{array}$$

Suppose now that $j_{X,\lambda}^* \circ T(1) \neq 0$. Then $i_\lambda \circ j_\lambda^* \circ T(1) = j_{X,\lambda}^* \circ T \circ i^*(1) = j_{X,\lambda}^* \circ T(1) \neq 0$ hence $j_\lambda^* \circ T(1) \neq 0$. That proves $c(1, L_X) \geq c(1, L)$.

In the case of L_Y , we also have an explicit generating function, constructed as follows. Since \mathbb{R}^k is contractible we can suppose that the fibers of π do not depend on the second coordinate of M . Denote by $i : B \simeq B \times \{0\} \rightarrow E$ the inclusion and by $\tau : B \times \mathbb{R}^k \rightarrow B$ the trivial bundle of rank k over B . Consider the vector bundle over B , $\rho = \tau \oplus i^*\pi$ whose total space is $F = \pi^{-1}(B \times \{0\}) \times \mathbb{R}^n$. Then, the function S_Y , defined for all $v \in B$ and $(x, \xi) \in \rho^{-1}(v)$ by $S_Y(v; x, \xi) = S(v, x; \xi)$ is a g.f.q.i for L_Y . The map $f : E \rightarrow F$, $(v, x; \xi) \mapsto (v, x, \xi)$ is a diffeomorphism and satisfies $S_Y \circ f = S$. Therefore, we have $S_Y^\lambda = f(S^\lambda)$, an isomorphism $H^*(S^\lambda, S^{-\infty}) \simeq H^*(S_Y^\lambda, S_Y^{-\infty})$ and a commutative diagram

$$\begin{array}{ccccc}
H^*(B) & \xrightarrow{T} & H^*(S_Y^\infty, S_Y^{-\infty}) & \xrightarrow{j_{Y,\lambda}^*} & H^*(S_Y^\lambda, S_Y^{-\infty}) \\
i^* \uparrow & & \uparrow \simeq & & \uparrow \simeq \\
H^*(M) & \xrightarrow{T} & H^*(S^\infty, S^{-\infty}) & \xrightarrow{j_\lambda^*} & H^*(S^\lambda, S^{-\infty})
\end{array}$$

The previous argument gives $c(1, L_Y) \geq c(1, L)$. The reverse inequality is obtained from the same diagram with the inclusion i replaced by the projection $p : M \rightarrow B$ (which reverses vertical arrows).

Finally, $c(\mu_B, L_N) \leq c(\mu_M, L) = c(\mu_B, L_Y)$ is obtained from $c(1, L_X) \geq c(1, L) = c(1, L_Y)$ by Poincaré duality, by noticing that $\overline{L_X} = \overline{L_X}$ and $\overline{L_Y} = \overline{L_Y}$. \square

Lemma 3.4. *Let W be a coisotropic linear subspace of \mathbb{R}^{2n} . Denote by N the zero section of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$. Then there exists a decomposition in linear isotropic subspaces $\mathbb{R}^{2n} = N_1 \oplus V_1 \oplus N_2 \oplus V_2 \oplus N_3 \oplus V_3$, where $N = N_1 \oplus N_2 \oplus N_3$ and each $N_i \oplus V_i$, $i = 1, 2, 3$ is a symplectic subspace, such that $W = N_1 \oplus V_1 \oplus N_2 \oplus V_3$.*

Proof. Let us first recall that if W is coisotropic with symplectic orthogonal $W^\omega \subset W$, any subspace F such that $F \oplus W^\omega = W$ is symplectic. Indeed, since $F \subset W$, $F \cap F^\omega = F \cap F^\omega \cap W = F \cap (F \oplus W^\omega)^\omega = F \cap W^\omega = \{0\}$.

If there exists a decomposition as in the lemma, then $W^\omega = N_2 \oplus V_3$. Therefore we set $N_2 = W^\omega \cap N$. Then, we define N_1 as one complementary of N_2 in $W \cap N$, and F_1 as one complementary of W^ω in W , containing N_1 . By the above remark, F_1 is symplectic, and we can choose V_1 as one Lagrangian complementary of N_1 in F_1 .

Then, we define V_3 as a complementary of N_2 in W^ω . Since $W \cap N = N_1 \oplus N_2$, $V_3 \cap N = 0$, and we can define N_3 as a complementary of $N_1 \oplus N_2$ in N . Then, $F_3 = N_3 \oplus V_3$ is symplectic since it is a complementary of $(N_1 \oplus N_2 \oplus F_3)^\omega$ in $N_1 \oplus N_2 \oplus F_3$.

Finally, we define F_2 as a complementary of $F_1 \oplus F_3$ in \mathbb{R}^{2n} . Then, F_2 is symplectic for a similar reason as F_3 , and we can define V_2 as a Lagrangian complementary of N_2 in F_2 . The decomposition $\mathbb{R}^{2n} = N_1 \oplus V_1 \oplus N_2 \oplus V_2 \oplus N_3 \oplus V_3$ satisfies all the requirements of lemma 3.4. \square

Proof of proposition 3.2. Since the linear symplectic group acts transitively on the set of all pairs of complementary Lagrangian subspaces (see proposition 7.4 in Chapter 1 of [6]), and since the space of Lagrangian subspaces which are complementary to the zero section N is path connected, there exists a symplectic isotopy Ψ^t of \mathbb{R}^{2n} such that $\Psi^0 = Id$ and that Ψ^1 lets all the elements of N invariant and maps V on $V_1 \oplus V_2 \oplus V_3$. Since \mathbb{R}^{2n} is simply connected, that isotopy is Hamiltonian.

The reduction of L by W is identified with the reduction of $\Psi^1(L)$ by $\Psi^1(W)$. Therefore, applying twice the lemma 3.4, we get $\gamma(L_W) \leq \gamma(\Psi^1(L))$. But, by proposition 2.6 in [12], we have $\gamma(L) = \gamma(\Psi^1(L))$. That concludes the proof of proposition 3.2. \square

Remark. Note that in the end of the previous proof, lemma 3.4 also implies $c(1, L_W) \geq c(1, L)$. That will be useful in the proof of lemma 2.12.

3.2 Inequalities involving the ‘‘suspended distances’’.

We now prove the inequality $\gamma_u(H, K) \leq \hat{\gamma}(H, K)$, for any H, K Hamiltonian functions. It is sufficient to prove that for all Hamiltonian functions H, K , all s in $[0, 1]$, and all α large enough, $\gamma(\phi_K^{-s} \phi_H^s) \leq \gamma(\phi_{\hat{K}}^{-s} \phi_{\hat{H}}^s)$. We will prove that the graph of $\phi_K^{-s} \phi_H^s$ can be obtained by reduction of the graph of $\phi_{\hat{K}}^{-s} \phi_{\hat{H}}^s$, and then use proposition 3.2.

We denote by $\hat{\Phi}^s$ the flow at time s of the Hamiltonian $\hat{H} : (s; t, \tau, x) \mapsto \rho_\alpha(\tau)\tau + \rho_\alpha(t)H(t; x)$. By direct computation, we get

$$\hat{\Phi}^s(t, \tau, x) = (t(s), \tau(s), x(s)),$$

with

$$\begin{aligned} t(s) &= t + \int_0^s (\rho'_\alpha(\tau(\sigma))\tau(\sigma) + \rho_\alpha(\tau(\sigma)))d\sigma \\ \tau(s) &= \tau - \int_0^s (\rho'_\alpha(t(\sigma))H(t(\sigma), x(\sigma) + \rho_\alpha(t(\sigma))\frac{\partial H}{\partial t}(t(\sigma), x(\sigma))))d\sigma \end{aligned}$$

and $x(s)$ solution of $\dot{x}(s) = \rho_\alpha(t(s))X_H(t(s), x(s))$. If we denote $M = \max(\|\rho_\alpha\|_{C^1}, \|H\|_{C^1})$, we see that $\tau(s) \in [\tau - |s|M^2, \tau + |s|M^2]$. Suppose $\tau \in [-M^2 - 2M, M^2 + 2M]$ and α is large enough, then $\rho_\alpha(\tau(s)) = 1$ and $t(s) = t + s$. Hence $x(s) = (\phi_H)_t^{t+s}(x)$. We set

$$\begin{aligned} I_H(s, t, x) &= - \int_0^s (\rho'_\alpha(t(\sigma))H(t(\sigma), x(\sigma) + \rho_\alpha(t(\sigma))\frac{\partial H}{\partial t}(t(\sigma), x(\sigma))))d\sigma \\ &= H(t, x) - H(t + s, \phi_t^{t+s}(x)), \end{aligned}$$

and $J(s, t, x) = I_H(s, t, x) + I_K(-s, t + s, (\phi_H)_t^{t+s}(x))$. Then, we can write the expression of the composition:

$$\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s(t, \tau, x) = (t, \tau + J(s, t, x), (\phi_K)_t^{t-s}(\phi_H)_t^{t+s}(x)).$$

We can now compute the intersection of the graph $\Gamma_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s}$ with the set $U = [-1, 1] \times \mathbb{R} \times [-M^2, M^2] \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, and its image by the natural identification $\Psi : \mathbb{R}^{4+4n} \rightarrow T^*\Delta_{2+2n}$. We get

$$\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s} \cap \Psi(U) = \{(t, J(s, t, x), \tau + \frac{1}{2}J(s, t, x), 0, z(x)) \mid$$

$$(t, \tau, x) \in [0, 1] \times [-M^2, M^2] \times \mathbb{R}^{2n}, z(x) \in \tilde{\Gamma}_{(\phi_K)_t^{t-s}(\phi_H)_t^{t+s}}\}.$$

Consider the coisotropic submanifold $W = \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \mathbb{R}^{4n}$. Since $\tau + \frac{1}{2}J(s, t, x) = 0$ implies $\tau \in [-M^2 - 2M, M^2 + 2M]$, and since $W \subset \Psi(U)$, we see that $\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s}$ is obtained from $\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s}$ by reduction by W . By proposition 3.2, we get $\gamma(\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s}) \leq \gamma(\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s})$ and hence $\gamma(\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s) \leq \gamma(\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s)$. \square

We are now going to prove $\gamma_u(H, K) \leq \tilde{\gamma}(H, K)$. The idea of the proof is the same as the previous one: we show that for any $s \in [0, 1]$, $\tilde{\Gamma}_{\phi_{\tilde{K}}^{-s}\phi_{\tilde{H}}^s}$ is obtained by reduction of $\tilde{\Gamma}_{\phi_{\tilde{K}}^{-1}\phi_{\tilde{H}}}$, for α large enough.

Recall that by definition, $\tilde{H}(s; t, \tau, x) = \rho_\alpha(t)tH(st; x)$. As above, we compute the flow : $\phi_{\tilde{H}}^s(t, \tau, x) = (t(s), \tau(s), x(s))$, and we obtain

$$\begin{aligned} t(s) &= t \\ \tau(s) &= \tau + I_H(s, t, x) \end{aligned}$$

where $I_H(s, t, x) = \rho_\alpha(t)sH(st, x(s)) - \rho'_\alpha(t)t \int_0^s H(\sigma t, x(\sigma))d\sigma$ and $x(s)$ is solution of $\dot{x}(s) = \rho_\alpha(t)tX_H(st, x(s))$. For $t \in [-1, 1]$ and $\alpha \geq 1$, it gives $x(s) = \phi^{ts}(x)$.

Similarly as above, we set $J(s, t, x) = I_H(s, t, x) + I_K(-s, t + s, (\phi_H)^{ts}(x))$, the set $U = [-1, 1] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and the identification $\Psi : \mathbb{R}^{4+4n} \rightarrow T^*\Delta_{2+2n}$. The graph can be written this way:

$$\begin{aligned} \tilde{\Gamma}_{\phi_{\bar{K}}^{-s}\phi_{\bar{H}}^s} \cap \Psi(U) &= \{(t, J(s, t, x), \tau + \frac{1}{2}J(s, t, x), 0, z(x)) \mid \\ &\quad (t, \tau, x) \in [0, 1] \times \mathbb{R} \times \mathbb{R}^{2n}, z(x) \in \tilde{\Gamma}_{\phi_{\bar{K}}^{-st}\phi_{\bar{H}}^{st}}\}. \end{aligned}$$

Now, we see that $\tilde{\Gamma}_{\phi_{\bar{K}}^{-t}\phi_{\bar{H}}^t}$ is the reduction of $\tilde{\Gamma}_{\phi_{\bar{K}}^{-1}\phi_{\bar{H}}}$ by the coisotropic manifold $W = \{t\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \mathbb{R}^{4n}$. Using lemma 3.3 twice, we conclude that for all $t \in [0, 1]$, $\gamma(\phi_{\bar{K}}^{-t}\phi_{\bar{H}}^t) \leq \gamma(\phi_{\bar{K}}^{-1}\phi_{\bar{H}})$. \square

Proof of lemma 2.12. It is sufficient to show that $c(V) \leq c(\mathbb{R}^2 \times V)$ for all open subset $V \in \mathbb{R}^{2n}$. Let H be an Hamiltonian function with support in V . We just have to find an Hamiltonian function K with support in $V \times \mathbb{R}^2$ satisfying the inequality $c_+(H) \leq c_+(K)$. If we set $K = \tilde{H}_\alpha$ for α large enough, K has support in $\mathbb{R}^2 \times V$, and we saw in particular in the previous proof that $\tilde{\Gamma}_{\phi_H^1}$ is the reduction of $\tilde{\Gamma}_{\phi_{\bar{H}}}$. Therefore, by the remark that ends section 3.1, we have $c_+(H) \leq c_+(K)$ as required. \square

4 The convergence criterion.

This is the central section of our paper. We give the proof of theorem 1.1.

4.1 A sufficient condition for c-convergence to zero.

We start this section with some formulas concerning Hamiltonian flows. They can be obtained by direct computation (see [4], page 144).

Lemma 4.1. *For all Hamiltonians H and K , with compact support, we have:*

$$\begin{aligned} \phi_H^t &= (\phi_H^t)^{-1}, & \text{where } \overline{H}(t, x) &= -H(t, \phi^t(x)) \\ \phi_{H\sharp K}^t &= \phi_H^t \circ \phi_K^t, & \text{where } (H\sharp K)(t, x) &= H(t, x) + K(t, (\phi^t)^{-1}(x)) \\ \phi_{\overline{H}\sharp K}^t &= (\phi_H^t)^{-1} \circ \phi_K^t. \end{aligned}$$

Remark. $(\overline{H}\sharp K)(t, x) = (K - H)(t, \phi^t(x))$.

The following proposition shows that if a sequence of Hamiltonians (H_n) converges to zero uniformly on every compact set contained in the complement of a set whose capacity is zero, then (ϕ_{H_n}) c-converges to Id .

Proposition 4.2. *Let H be a Hamiltonian on \mathbb{R}^{2n} with compact support. If U is an open subset of \mathbb{R}^{2n} , such that $c(U) \leq \varepsilon$ and $|H(t, x)| \leq \varepsilon$ for all $t \in [0, 1]$ and all $x \in \mathbb{R}^{2n} - U$, then $\gamma(\phi_H) \leq 4\varepsilon$.*

Proof. Let K_1, K_2 be Hamiltonians with compact support, such that $0 \leq K_i \leq 1$, $i = 1, 2$, K_1 equals 1 on the support of H and K_2 equals 1 on the support of K_1 . Denote $\psi_{1,\varepsilon}$ the diffeomorphism generated by $H - \varepsilon K_1$, and $\psi_{2,\varepsilon}$ the diffeomorphism generated by εK_2 . Then we have $H \leq \varepsilon K_2 + (H - \varepsilon K_1)$. As $(\psi_{2,\varepsilon})^{-1}$ coincides with Id on the support of $H - \varepsilon K_1$, the lemma 4.1 shows that $K_2 + H - \varepsilon K_1$ is the Hamiltonian that generates $\psi_{2,\varepsilon} \circ \psi_{1,\varepsilon}$. The propositions 2.5, 2.4 and 2.6 then give

$$c_+(\phi) \leq c_+(\psi_{2,\varepsilon} \circ \psi_{1,\varepsilon}) \leq c_+(\psi_{2,\varepsilon}) + c_+(\psi_{1,\varepsilon}) \leq \varepsilon + c_+(\psi_{1,\varepsilon}).$$

Denote by $\widetilde{\psi_{1,\varepsilon}}$ the diffeomorphism generated by a non-negative Hamiltonian, with support in U , and greater than $H - \varepsilon K_1$. Then by Proposition 2.5, $c_+(\psi_{1,\varepsilon}) \leq c_+(\widetilde{\psi_{1,\varepsilon}})$. Finally, since $Supp(\widetilde{\psi_{1,\varepsilon}}) \subset U$, we get $c_+(\widetilde{\psi_{1,\varepsilon}}) \leq c(U) \leq \varepsilon$.

Using the inequality $H \geq -\varepsilon K_2 + (H + \varepsilon K_1)$, we obtain the same type of inequality for c_- . \square

For example, if K is a compact submanifold of dimension $d \leq n - 1$, then $\gamma(K) = 0$ (and hence $c(K) = 0$).

4.2 What about non-zero limits?

Unfortunately, the previous result is false in general when the limit is not zero, even if we replace the symplectic capacity c by c^∞ . What follows is a counter example.

Example. It is well known that the capacities c and γ of the unit sphere $\mathcal{S} = \{x \in \mathbb{R}^{2n} \mid \|x\| = 1\}$ are π . It is also true for c^∞ . Then, for all $\varepsilon > 0$, there exists a Hamiltonian H with support in a small neighbourhood U of \mathcal{S} , and such that $c_+(\phi_H) > \pi - \varepsilon$. Because of proposition 2.5, H can be chosen non-negative. We set U^+ a neighbourhood of $\{x \in \mathcal{S} \mid x_1 \geq 0\}$ and U^- a neighbourhood of $\{x \in \mathcal{S} \mid x_1 < 0\}$, such that $U = U^+ \cup U^-$. If U, U^+ and U^- are chosen small enough, we have $d(U^\pm) < \varepsilon$ and by proposition 2.14 $c^\infty(U^\pm) < \varepsilon$. Using some partition of unity associated to the decomposition $U = U^+ \cup U^-$, we get two functions H^\pm , with support in U^\pm and such that $H = H^+ + H^-$.

Now, we see that H^+ coincides with H outside U^- , whose capacity verifies $c^\infty(U^\pm) < \varepsilon$, but on the other hand,

$$\|\gamma(\phi_H) - \gamma(\phi_{H^+})\| \geq \gamma(\phi_H, \phi_{H^+}) \geq \pi - \varepsilon - \gamma(\phi_{H^-}) \geq \pi - 2\varepsilon.$$

It shows that the previous statement is false when the limit is not zero.

Nevertheless, we can introduce a new invariant, in order to extend the result of proposition 4.2.

Definition 4.3. For any subset U and any Hamiltonian $H \in \text{Ham}$, we define

$$\xi^H(U) = c^\infty \left(\bigcup_{t \in [0,1]} \phi_H^t(U) \right).$$

We may then set

$$\xi_\lambda(U) = \sup \xi^H(U), \text{ for } 0 < \lambda \leq \infty,$$

where the supremum is over all Hamiltonian functions H with $\gamma_u(H) \leq \lambda$.

Theorem 4.4. Let H_1 and H_2 be Hamiltonians on \mathbb{R}^{2n} with compact support. Let U be a subset of \mathbb{R}^{2n} , satisfying one of the two following conditions:

1. $\xi_\infty(U) \leq \varepsilon$.
2. $\exists \lambda > 0, \xi_\lambda(U) = 0$

If $|H_1(t, x) - H_2(t, x)| \leq \varepsilon$ for all $t \in [0, 1]$ and all $x \notin U$, then we have $\gamma(\phi_{H_1}, \phi_{H_2}) \leq 4\varepsilon$.

Proof. Consider the Hamiltonian $H(t, x) = H_1(t, \phi_2^t(x)) - H_2(t, \phi_2^t(x))$. By assumption, $|H(t, x)| \leq \varepsilon$, for all (t, x) with $x \notin \phi_2^{-t}(U)$ and hence for all t and all $x \notin \bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U)$. Each condition on U implies

$$c \left(\bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U) \right) \leq c^\infty \left(\bigcup_{t \in [0,1]} (\phi_2^{-1})^t(U) \right) \leq \varepsilon.$$

By proposition 4.2 and lemma 4.1, we get $\gamma(\phi_{H_1}, \phi_{H_2}) = \gamma(\phi_H) \leq 4\varepsilon$. \square

Remark. In the proof of theorem 4.4, we see that the important condition is in fact $\xi^{H_2}(U) \leq \varepsilon$, which is of course implied by both conditions $\xi_\infty(U) \leq \varepsilon$ and $\xi_\lambda(U) = 0$.

Corollary 4.5. The conclusion of theorem 4.4 still holds if we replace γ with $\tilde{\gamma}$, $\tilde{\gamma}_u$, γ_u , $\hat{\gamma}$ or $\check{\gamma}$.

Proof. By proposition 3.1, we just have to prove this result for $\hat{\gamma}$ and $\check{\gamma}$. Then remark that under the hypothesis of theorem 4.4, we have $|\hat{H}_1(s; t, \tau, x) - \check{H}_2(s; t, \tau, x)| \leq \varepsilon$ and $|\check{H}_1(s; t, \tau, x) - \check{H}_2(s; t, \tau, x)| \leq \varepsilon$ for all integer α , all $s \in [0, 1]$, and all $(t, \tau, x) \notin \mathbb{R}^2 \times U$.

Unfortunately, even if U satisfies one of the conditions of proposition 4.4, it is not in general the case for $\mathbb{R}^2 \times U$. However, by the above remark, it is sufficient to show that for all real number $\delta > 0$ and all integer α large enough, $\xi^{\check{H}_2}(\mathbb{R}^2 \times U) \leq \varepsilon + \delta$ and $\xi^{\hat{H}_2}(\mathbb{R}^2 \times U) \leq \varepsilon + \delta$. By letting δ tend to zero and taking limsup with respect to α , we obtain $\hat{\gamma}(H, K) \leq 4\varepsilon$ and $\check{\gamma}(H, K) \leq 4\varepsilon$ as required.

Let us denote F for \check{H}_2 or \hat{H}_2 . The inequalities on ξ^F come directly from the expression of $\phi_{\check{H}_2}$ and $\phi_{\hat{H}_2}$ (see computations of section 3). Indeed, in both cases,

$$\bigcup_{s \in [0,1]} (\phi_F^{-1})^s([- \alpha, \alpha]^2 \times U) \subset \mathbb{R}^2 \times \bigcup_{s \in [0,1]} (\psi^{-1})^s(U),$$

where ψ^s is an Hamiltonian isotopy that appears in last coordinate when we compute ϕ_F . Therefore, since $\xi^F(\mathbb{R}^2 \times U) = \lim_{\alpha \rightarrow +\infty} \xi^F([- \alpha, \alpha]^2 \times U)$, we get for any $\delta > 0$ and any α large enough:

$$\begin{aligned} \xi^F(\mathbb{R}^2 \times U) &\leq \delta + c^\infty \left(\mathbb{R}^2 \times \bigcup_{s \in [0,1]} (\psi^{-1})^s(U) \right) \\ &= \delta + c^\infty \left(\bigcup_{s \in [0,1]} (\psi^{-1})^s(U) \right) \leq \delta + \varepsilon. \end{aligned}$$

That concludes the proof. \square

Corollary 4.6. *Let (H_k) be a sequence of Hamiltonians in Ham , whose supports are contained in a fixed compact set. Suppose there exist an Hamiltonian $H \in Ham$ and a compact set $K \in \mathbb{R}^{2n}$ with $\xi_\infty(K) = 0$, such that (H_k) converges uniformly to H on every compact set of $\mathbb{R} \times (\mathbb{R}^{2n} - K)$. Then (ϕ_{H_k}) converges to ϕ_H for $\tilde{\gamma}$, γ , and (H_k) converges to H for $\tilde{\gamma}_u$, γ_u , $\hat{\gamma}$, $\check{\gamma}$.*

Proof. For $\tilde{\gamma}$, γ , it is a direct consequence of the remark that follows theorem 4.4. We just have to verify that for all $\varepsilon > 0$, there exists a small neighbourhood U of K such that $\xi^H(U) \leq \varepsilon$. This is true because for every neighbourhood V of $\bigcup_{t \in [0,1]} \phi_H(K)$, we can choose a neighbourhood U of K such that

$$\bigcup_{t \in [0,1]} \phi_H(U) \subset V.$$

Since $c^\infty(\bigcup_{t \in [0,1]} \phi_H(K)) = 0$ and $\bigcup_{t \in [0,1]} \phi_H(K)$ is compact, we can choose V such that $c^\infty(V) \leq \varepsilon$, and obtain $c^\infty(\bigcup_{t \in [0,1]} \phi_H(U)) \leq \varepsilon$ as required.

For $\hat{\gamma}$ and $\check{\gamma}$, we have to verify that for all $\varepsilon > 0$ and all $\delta > 0$, there exists a small neighbourhood U of K such that for all α large enough $\xi^\phi(U) \leq \varepsilon + \delta$, where F is either \hat{H} or \check{H} . The proof made above for ϕ_H shows that we can find U such that $\xi^f(U) \leq \varepsilon$, where f generates the isotopy ψ^s defined as in the proof of corollary 4.5. Therefore we have for all δ and all α large enough, $\xi^F(\mathbb{R}^2 \times U) \leq \xi^F([- \alpha, \alpha]^2 \times U) + \delta \leq \xi^f(U) + \delta \leq \varepsilon + \delta$.

By proposition 3.1, corollary 4.6 is also true for $\tilde{\gamma}_u$ and γ_u . \square

Remark. Similar proofs give that theorem 4.4 and corollary 4.6 still hold for γ_2 .

4.3 Example of a non trivial ξ -small set.

Proposition 4.7. *Let U be a closed submanifold of \mathbb{R}^{2n} whose dimension d verifies $d \leq n - 2$. Then $\xi_\infty(U) = 0$.*

Proof. Let $H \in Ham$. The problem is that $\bigcup_{t \in [0,1]} \phi_H^t(U)$ is not in general a manifold. To avoid that problem, we are going to add two dimensions and make a suspension in this way. We denote by Φ the Hamiltonian diffeomorphism on $\mathbb{R}^{2+2n} = \{(t, \tau, x)\}$ generated by the Hamiltonian

$$[0, 1] \times \mathbb{R}^{2+2n} \rightarrow \mathbb{R}, (s; t, \tau, x) \mapsto tH(ts, x).$$

We also set $V = \Phi([0, 1] \times [-1, 1] \times U)$. The computation of Φ gives

$$\Phi(t, \tau, x) = (t, \tau - H(t, x), \phi^t(x)).$$

We see that $\bigcup_{t \in [0,1]} \phi_H^t(U)$ can be obtained from V by symplectic reduction by the coisotropic manifold $\{\tau = 0\}$. So we are going to look for a Hamiltonian diffeomorphism ϕ_K that displaces V and preserves $\{\tau = 0\}$ at the same time. If the Hamiltonian does not depend on t , the second condition is verified. Since V is compact, it is sufficient for K to verify

$$\forall v \in V, \mathbb{R}X_K(v) \cap T_v V = \{0\},$$

which is equivalent to

$$\forall v \in V, \ker dK(v) \oplus T_v V^\perp = \mathbb{R}^{2+2n}$$

and to

$$\forall v \in V, T_v V^\perp \not\subset \ker dK(v).$$

That makes us consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})$ and its submanifold

$$W = \{(s, q; \sigma, p; z) \mid (s, q) \in V, z \in \mathbb{R}, T_{(s,q)}V^\perp \subset \ker(\sigma, p)\}.$$

The dimension of W is exactly $2n + 1$. Indeed, the vector space $\{(\sigma, p) \in \mathbb{R}^{2n+2*} \mid T_{(s,q)}V^\perp \subset \ker(\sigma, p)\}$ has dimension $2n + 2 - \dim(T_{(s,q)}V^\perp) = n$.

By Thom transversality theorem (see [2] for example), there exists a function L whose 1-jet verifies $j^1L \pitchfork W$. But j^1L can be seen as a function $\mathbb{R} \times \mathbb{R}^{1+2n} \rightarrow J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})$, and by lemma 4.6 page 53 in [2], we have for a generic choice of $s \in \mathbb{R}$, $j^1L(s, \cdot) \pitchfork W$. We fix s as previously and we denote $K : \mathbb{R}^{2+2n} \rightarrow \mathbb{R}$, $K(t, \cdot) = L(s, \cdot)$

Then, notice that for every s, q, p, z , the set of all σ such that $(s, q; \sigma, p; z) \in W$ is either \emptyset or \mathbb{R} . It can be shown by direct computation of TV^\perp , whose first component appears to be always $\{0\}$. As a consequence, we get $j^1K \pitchfork W$ (j^1K differs from $j^1L(s, \cdot)$ just by its σ -component which is $\{0\}$ instead of $\frac{\partial L}{\partial s}(s, \cdot)$ for $j^1L(s, \cdot)$).

Now, since $(2n+2) + (2n+1) = \dim(j^1K(\mathbb{R}^{2+2n})) + \dim(W) < \dim(J^1(\mathbb{R} \times \mathbb{R}^{1+2n}, \mathbb{R})) = 4n + 5$, we get $j^1K(\mathbb{R}^{2+2n}) \cap W = \emptyset$. It follows that K satisfies the two conditions: it preserves $\{\tau = 0\}$ and it satisfies

$$\forall v \in V, \mathbb{R}X_K(v) \cap T_vV = \{0\}.$$

As V is compact, for ε small enough, since $\phi_{\varepsilon K} = \phi_K^\varepsilon$, we have $\phi_{\varepsilon K}(V) \cap V = \emptyset$. In addition εK can be made as C^0 -small as we want.

We are now ready for the reduction by $\{\tau = 0\}$. Since it preserves $\{\tau = 0\}$, εK induces a Hamiltonian on the reduction \mathbb{R}^{2n} . This Hamiltonian is C^0 -small and generates a diffeomorphism ψ whose Hofer's distance to identity $d_H(\psi, id)$ is small, and that satisfies

$$\psi \left(\bigcup_{t \in [0,1]} \phi_H^t(U) \right) \cap \bigcup_{t \in [0,1]} \phi_H^t(U) = \emptyset.$$

This Hamiltonian is not compactly supported, but any Hamiltonian with compact support which coincides with it on a sufficiently large ball, would have the same properties. That proves $d \left(\bigcup_{t \in [0,1]} \phi_H^t(U) \right) = 0$, and since $c^\infty \leq d$, we get $\xi^H(U) = 0$. \square

5 Non-completeness.

In this section we prove the non-completeness of the topologies described above. We give an example of a non-converging Cauchy sequence. Before

constructing it, we state a lemma which makes it possible to define a support for the elements of the different completions.

Lemma 5.1. a. *Let (ϕ_n) be a sequence in \mathcal{H} converging to a Hamiltonian diffeomorphism ϕ , with respect to γ or $\tilde{\gamma}$. Assume that there exists a set U such that $\text{supp}(\phi_n) \subset U$. Then $\text{supp}(\phi) \subset \bar{U}$.*

b. *Let (H_n) be a sequence in Ham converging to a smooth Hamiltonian function H , with respect to $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}, \check{\gamma}$, etc. Assume that there exists a set U such that $\text{supp}(H_n) \subset U$. Then $\text{supp}(H) \subset \bar{U}$.*

Proof. **a.** Because of proposition 3.1, we just have to prove the assertion in the case of $\tilde{\gamma}$. Suppose $\text{supp}(\phi) \not\subset \bar{U}$. Then there exists an x in $\mathbb{R}^{2n} - \bar{U}$ such that $\phi(x) \neq x$. Let ψ be a Hamiltonian diffeomorphism whose support is included in $\mathbb{R}^{2n} - \bar{U}$, and does not contain $\phi(x)$. Suppose in addition that $\psi(x) \neq x$. Then, since the supports of ϕ_n and ψ are disjoint, we have $\psi \circ \phi_n^{-1} \circ \psi^{-1} \circ \phi_n = Id$, for all integer n . Taking limit, we get on one hand $\psi \circ \phi^{-1} \circ \psi^{-1} \circ \phi = Id$. But on the other hand, we have by construction, $\psi \circ \phi^{-1} \circ \psi^{-1} \circ \phi(x) = \psi(x) \neq x$, which is contradictory.

b. We use the first part of the lemma to conclude that for all time t , $\text{supp}(\phi^t) \subset \bar{U}$. This implies that $\text{supp}(H) \subset \bar{U}$. \square

Remark. A similar argument shows that the property of letting globally invariant any sphere centered at 0, is invariant by taking γ or $\tilde{\gamma}$ limits. Similarly, a $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$ or $\check{\gamma}$ limit of radial Hamiltonians is radial.

Definition 5.2. *Let η be an element of $\mathfrak{H}, \mathfrak{H}_u, \tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}_u, \hat{\mathfrak{H}}, \check{\mathfrak{H}}$ or \mathfrak{H}_2 . Then we define $\text{support}(\eta)$ as*

$$\bigcap \{ \bar{U} \mid U \text{ open set, such that there exists } (\eta_n) \text{ representing } \eta \text{ such that}$$

$$\forall n, \text{supp}(\eta_n) \subset U \},$$

where “*supp*” denotes the usual notions of support for smooth Hamiltonians and for Hamiltonian diffeomorphisms.

This new notion of support coincides with the usual one for smooth Hamiltonians and Hamiltonian diffeomorphisms. Let us prove it. Let η be either an Hamiltonian diffeomorphism seen as an element of \mathfrak{H} or $\tilde{\mathfrak{H}}$, or a smooth Hamiltonian seen as an element of $\mathfrak{H}_u, \tilde{\mathfrak{H}}_u, \hat{\mathfrak{H}}, \check{\mathfrak{H}}$ or \mathfrak{H}_2 . Let (η_n) be a sequence representing η , and U be an open set with $\text{supp}(\eta_n) \subset U$ for all n . Then lemma 5.1 gives $\text{supp}(\eta) \subset \bar{U}$. Hence $\text{supp}(\eta) \subset \text{support}(\eta)$. Conversely, for any neighbourhood U of $\text{support}(\eta)$ the constant sequence (η) converges to η and has support in U . Hence $\text{support}(\eta) \subset \bigcap_{\gamma} \bar{U}$, where the intersection is

over the set \mathcal{V} of all open neighbourhoods of $\text{supp}(\eta)$. Then, it is easy to see that $\bigcap_{\mathcal{V}} \overline{U} = \bigcap_{\mathcal{V}} U = \text{supp}(\eta)$.

We are now ready to prove the non-completeness property. Let us introduce the following family of Hamiltonians.

Definition 5.3. We denote by \mathfrak{F} the set of all functions $H : \mathbb{R}^{2n} \rightarrow \mathbb{R} \cup \{+\infty\}$, such that:

- (i) $K = H^{-1}(\{+\infty\})$ satisfies $c^\infty(K) = 0$,
- (ii) H vanishes at infinity: $\forall \varepsilon > 0, \exists r, (|x| > r \Rightarrow |H(x)| < \varepsilon)$,
- (iii) H is continuous.

We also set $\mathfrak{F}^\infty = \{H \in \mathfrak{F} \mid H \text{ is smooth on } \mathbb{R}^{2n} - H^{-1}(\{+\infty\})\}$.

Proposition 5.4. Suppose H is an element \mathfrak{F} and $K = H^{-1}(\{+\infty\})$. Then there exists a sequence of smooth autonomous Hamiltonians $(H_k) \in \text{Ham}$ which

- a. converges to H uniformly on every compact subset of $\mathbb{R}^{2n} - K$.
- b. is Cauchy for $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$ and $\check{\gamma}$.

Moreover, if $H \in \mathfrak{F}^\infty$, then any sequence (H_k) that converges to H uniformly on the compact subsets of $\mathbb{R}^{2n} - K$, does not converge in Ham , for none of the distances $\gamma_u, \tilde{\gamma}_u, \hat{\gamma}$ and $\check{\gamma}$.

Proof. Fix $k > 0$. Properties (ii) and (iii) in definition 5.3 imply that K is compact. Since $c^\infty(K) = 0$, there exists an open neighbourhood U of K such that $c^\infty(U) \leq \frac{1}{k}$. Then, if we denote $H^{>A} = \{x \mid H(x) > A\}$, we have for A large enough, $K \subset H^{>A} \subset U$. Indeed, if not, for all integer $l > 0$, there would exist a point a_l in $H^{>l}$, but not in U . Then, the sequence (a_l) would take values in $H^{\geq 1} \cap (\mathbb{R}^{2n} - U)$ which is compact, and hence it would have a subsequence that would converge to an element of $K \cap (\mathbb{R}^{2n} - U)$, which contradicts our assumption. Let us fix a real number A_k such that $H^{>A_k} \subset U$.

Now, let H_k be a smooth function with compact support such that $|H_k - H| < \frac{1}{k}$ on $\mathbb{R}^{2n} - H^{>A_k + \frac{2}{k}}$, and such that $|H_k - A_k - \frac{2}{k}| < \frac{1}{k}$ on $H^{>A_k + \frac{2}{k}}$. The sequence (H_k) clearly converges to H uniformly on every compact subset of $\mathbb{R}^{2n} - K$. Let us see why it is Cauchy.

By proposition 3.1, we just have to prove it for $\hat{\gamma}$ and $\check{\gamma}$. We denote F_k for either \check{H}_k or \hat{H}_k . We also denote, as in the proof of corollary 4.5, F_k for the third coordinate of ϕ_{F_k} . Since H_k is an autonomous Hamiltonian, its

flow $\phi_{H_k}^t$ preserves its levels. Hence, the isotopy ψ_k^s preserves the levels of H_k (see computations of section 3). Therefore, since by construction $H^{>A_k+\frac{2}{k}} \subset H_k^{>A_k+\frac{1}{k}}$, we have

$$\bigcup_{t \in [0,1]} \psi_k^t(H^{>A_k+\frac{2}{k}}) \subset H_k^{>A_k+\frac{1}{k}}.$$

Let $\delta > 0$, and suppose α is sufficiently large. Then, as in the proof of corollary 4.5,

$$\begin{aligned} \xi^{F_k}(\mathbb{R}^2 \times H^{>A_k+\frac{2}{k}}) &\leq \delta + c^\infty \left(\mathbb{R}^2 \times \bigcup_{s \in [0,1]} (\psi_k^{-1})^s(H^{>A_k+\frac{2}{k}}) \right) \\ &\leq \delta + c^\infty \left(\bigcup_{s \in [0,1]} (\psi_k^{-1})^s(H^{>A_k+\frac{2}{k}}) \right) \\ &\leq \delta + c^\infty(H_k^{>A_k+\frac{1}{k}}). \end{aligned}$$

Since $H_k^{>A_k+\frac{1}{k}} \subset H^{>A_k} \subset U$ and $c^\infty(U) \leq \frac{1}{k}$, we get $\xi^{F_k}(H^{>A_k+\frac{2}{k}}) \leq \frac{1}{k} + \delta$.

Now, pick an integer $l \geq k$. If l and k are large enough, then we have $|\hat{H}_k - \hat{H}_l| \leq \frac{1}{k}$ and $|\check{H}_k - \check{H}_l| \leq \frac{1}{k}$ on $\mathbb{R}^{2+2n} - (\mathbb{R}^2 \times H^{>A_k+\frac{2}{k}})$. Therefore, by the remark that follows theorem 4.4, we get $\hat{\gamma}(H_l, H_k) \leq \frac{4}{k}$ and $\check{\gamma}(H_l, H_k) \leq \frac{4}{k}$, after taking limsup with respect to α . It proves that (H_k) is a Cauchy sequence for $\tilde{\gamma}_u$, γ_u , $\hat{\gamma}$, and $\check{\gamma}$.

Suppose now that H is smooth on $\mathbb{R}^{2n} - K$. Then we can choose H_k such that it coincides with H on $B_k - H^{>A_k+\frac{2}{k}}$, where B_k is the ball of radius k , centered at 0. Suppose that (H_k) converges to an Hamiltonian $L \in Ham$ for $\tilde{\gamma}_u$, γ_u , $\hat{\gamma}$, and $\check{\gamma}$. Then for any integer k , $\overline{H_k} \# H_l$ converges to $\overline{H_k} \# L$ while l tends to infinity for $\tilde{\gamma}_u$ (see lemma 4.1 for notations). According to lemma 5.1, since $\overline{H_k} \# H_l$ has support in the complementary of $B_k - H^{>A_k+\frac{2}{k}}$, $\overline{H_k} \# L$ has support in its closure, and hence L coincides with H on $B_k - H^{>A_k+\frac{2}{k}}$. Since it is true for any k , L has to coincide with H on $\mathbb{R}^{2n} - K$. Hence L cannot belong to Ham , which contradicts our assumptions.

Finally, if (L_k) is another sequence of Hamiltonians that converges to H uniformly on the compact subsets of $\mathbb{R}^{2n} - K$, then, similarly to the above proof that (H_k) is Cauchy, we obtain that $\hat{\gamma}(L_k, H_k)$ and $\check{\gamma}(L_k, H_k)$ converge to 0, where H_k is the particular sequence defined in the previous paragraph. Since (H_k) does not converge, (L_k) does not converge either. \square

Corollary 5.5. *The metric spaces (\mathcal{H}, γ) , (Ham, γ_u) , $(\mathcal{H}, \tilde{\gamma})$, $(Ham, \tilde{\gamma}_u)$, $(Ham, \hat{\gamma})$ and $(Ham, \check{\gamma})$ are not complete.*

Proof. For (Ham, γ_u) , $(Ham, \hat{\gamma})$, $(Ham, \tilde{\gamma}_u)$, and $(Ham, \tilde{\gamma})$, it is exactly what we proved in proposition 5.4. We are now going to prove it in the case of γ and $\tilde{\gamma}$. Since the proof is exactly the same in both cases, we will only make it for γ .

We consider a function $h : [0, +\infty) \rightarrow [0, +\infty)$ decreasing, with support in $[-1, 1]$, and equal to 1 on $[-3/4, 3/4]$. Then we define $H_k(x) = \sum_{i=1}^k h(2^i|x|^2)$, for $x \in \mathbb{R}^{2n}$. Suppose that (ϕ_{H_k}) converges to $\phi \in \mathcal{H}$ for γ . We are going to prove that ϕ cannot be smooth at 0. Lemma 5.1 implies that ϕ coincides with ϕ_{H_k} out of $B_{2^{-k}}$. So, we can compute the explicit form of ϕ . In polar coordinates, we get:

$$\phi(r, \theta) = (\theta - r^2 f'(r^2), r),$$

for $r > 0$ where $f(s) = \sum_{i>0} h(2^i s)$ (for any s , all the terms in this sum are 0 except maybe one). If we denote by (q, p) the coordinates in \mathbb{R}^{2n} , and by ϕ_1 the projection of ϕ on $\mathbb{R}^n \times \{0\}$, we have for $q \in 2^{-i}[1/2, 1]$,

$$\frac{\partial \phi_1}{\partial q}(q, 0) = \cos(q^2 2^i h'(2^i q^2)) - 2(q^4 2^{2i} h''(2^i q^2) + q^3 2^i h'(2^i q^2)) \sin(q^2 2^i h'(2^i q^2)). \quad (2)$$

Suppose that h is chosen so that there exists q_1 and q'_1 in $[1/2, 1]$ such that $\frac{\partial \phi_1}{\partial q}(q_1, 0) \neq \frac{\partial \phi_1}{\partial q}(q'_1, 0)$ (we denote by A their difference), and define $q_i = \sqrt{2^{-i}} q_1$ and $q'_i = \sqrt{2^{-i}} q'_1$. Then, (q_i) and (q'_i) converge to 0, but from (2) we see that $\frac{\partial \phi_1}{\partial q}(q_i, 0) - \frac{\partial \phi_1}{\partial q}(q'_i, 0)$ converges to A . Therefore ϕ cannot be smooth at 0. \square

Remark. In the previous example, the sequence of diffeomorphisms (ϕ_{H_k}) was converging almost everywhere to a homeomorphism (which was not a diffeomorphism). Therefore, we can say that the class of (ϕ_{H_k}) in the completion \mathfrak{H} can be represented by a homeomorphism. However, with the help of proposition 5.4, we can show that it is not in general true, at least in dimension $2n \geq 4$.

Indeed, consider $H : \mathbb{R}^2 \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$,

$$(x_1, x_2) \mapsto \frac{1}{\| \|x_1\|^2 - 1 \| + \|x_2\|^2} \chi(\|(x_1, x_2)\|),$$

where χ is smooth with compact support and equals 1 on the ball of radius 2 centered at zero. Clearly, $H \in \mathfrak{F}^\infty$ ($K = H^{-1}(\{+\infty\}) = \mathbb{S}^1 \times \{0\}$ satisfies $c^\infty(K) = 0$ as required). Consider the sequence (H_k) as constructed in the proof of proposition 5.4. Since (H_k) is Cauchy for γ_u , (ϕ_{H_k}) is Cauchy for γ . Suppose it converges to an element ϕ . Then, lemma 5.1 implies that for

any neighbourhood U of K and for k large enough, ϕ coincides with ϕ_{H_k} on $\mathbb{R}^{2+2n} - U$. Therefore, we can compute the explicit form of ϕ on $\mathbb{R}^{2+2n} - K$.

In polar coordinates $(s_1, \theta_1, s_2, \theta_2)$ with $s_1 = \|x_1\|^2$ and $s_2 = \|x_2\|^2$, we get for $s_1 < 1$:

$$\phi(s_1, 0, 0, 0) = \left(s_1, \frac{s_1}{(1 - s_1)^2} \right).$$

If we make s_1 converge to 1, we see that ϕ is not continuous and hence is not a homeomorphism.

Corollary 5.6. *The set \mathfrak{F}^∞ can be embedded into each completion $\mathfrak{H}_u, \tilde{\mathfrak{H}}_u, \hat{\mathfrak{H}}$ and $\check{\mathfrak{H}}$.*

Proof. Since $\hat{\mathfrak{H}} \subset \mathfrak{H}_u \subset \tilde{\mathfrak{H}}_u$ and $\check{\mathfrak{H}} \subset \mathfrak{H}_u \subset \tilde{\mathfrak{H}}_u$, we just have to prove it for $\hat{\gamma}$ and $\check{\gamma}$. We will make the proof for $\hat{\gamma}$. The proof for $\check{\gamma}$ is exactly the same. Let J be the function that associates to any $H \in \mathfrak{F}^\infty$ the element of $\hat{\mathfrak{H}}$ represented by any sequence (H_k) that converges uniformly to H on the compact sets of $\mathbb{R}^{2n} - H^{-1}(\{+\infty\})$. As we noticed at the end of the proof of proposition 5.4, two such sequences are equivalent and hence J is well defined.

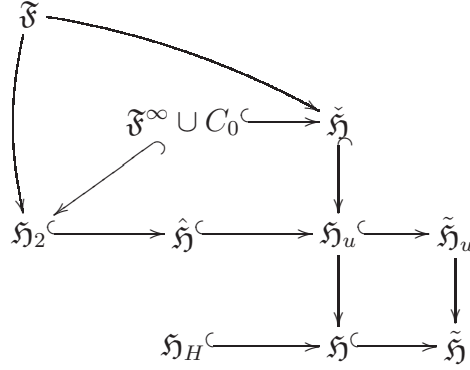
Let us now prove that J is one-to-one. Let $H, G \in \mathfrak{F}^\infty$ and let $(H_k), (G_k)$ be two sequences respectively associated to them, precisely constructed as in the last but one paragraph of the previous proof. Suppose that $G \neq H$, we are going to show that $\gamma(H_k, G_k)$ does not converge to zero, that will imply that $\hat{\gamma}(H_k, G_k)$ does not converge to zero.

We can define almost everywhere the flows ϕ_G^t, ϕ_H^t and $\psi^t = \phi_G^{-t} \circ \phi_H^t$. Let $\psi_k = \phi_{G_k}^{-1} \circ \phi_{H_k}$. Since $G \neq H$, there exists a point x such that $\psi(x) \neq x$. Hence, there exists a small ball B around x such that $\psi(B) \cap B = \emptyset$. Let K be a compact neighbourhood of $\bigcup_t \psi^t(B)$. For k large enough, H_k and G_k coincide respectively with H and G on K , and hence $\psi_k(B) \cap B = \emptyset$ too. Since $\gamma(H_k, G_k) = \gamma(\psi_k) \geq \gamma(B) > 0$, $\gamma(H_k, G_k)$ cannot converge to zero. \square

Remark. As usual, the results of proposition 5.4 and corollary 5.5 and 5.6 still hold for γ_2 .

Now, if we denote C_0 the set of continuous Hamiltonians that vanish at

infinity, we can improve the diagram of section 3:



6 Application to Hamilton-Jacobi Equation.

Let H be a smooth Hamiltonian function on $\mathbb{R} \times \mathbb{R}^{2n}$. We consider the Hamilton-Jacobi equation (HJ):

$$\frac{\partial u}{\partial t} + H\left(t, x, \frac{\partial u}{\partial x}\right) = 0,$$

where $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $(t, x) \mapsto u(t, x)$ satisfies an initial condition $u(0, x) = u_0(x)$.

First, we remind the reader of the construction of a variational solution of (HJ) (see for example [11] or [8]). We denote by Λ_0 the graph of du_0 and call it the initial submanifold. In fact, the following construction can be made for any Lagrangian submanifold $\Lambda_0 \subset \mathbb{R}^{2n}$. We consider $\Sigma = \hat{H}^{-1}(\{0\}) \subset \mathbb{R}^{2+2n}$. A geometric solution of (HJ) is a Lagrangian submanifold L that satisfies $\Lambda_0 \leq L \leq \Sigma$. For example, the graph of the differential of a smooth function u is a geometric solution if and only if u itself is solution of (HJ).

With the help of the flow $\phi_{\hat{H}}^t$, we can construct a geometrical solution $L_H = \bigcup_{t \in I} \phi_{\hat{H}}^t(\Lambda_0)$, where I is an open interval containing $[0, 1]$ and such that $\rho_\alpha = 1$ on I . The Lagrangian submanifold L_H obtained is an element of $\mathcal{L}(\mathbb{R}^{2+2n})$.

For any element $L \in \mathcal{L}(\mathbb{R}^{2k})$, we can associate a function u_L on \mathbb{R}^{2k} by the following method.

Let $S : \mathbb{R}^k \times \mathbb{R}^q \rightarrow \mathbb{R}$ be a g.f.q.i of L . Denote by 1_z the fundamental class in $H^0(z)$. Then, we define u_L by

$$u_L(z) = c(1_z, S|_{z \times \mathbb{R}^q}),$$

with notations of section 2. The function u_L is everywhere C^0 , and it is proved in [8], that u_L is C^k on a dense open set, for $k \geq 1$. Moreover, when

it is defined, we have $(x, du_L(x)) \in L$. Therefore, the function u_{L_H} is a solution of (HJ) on any open set on which it is smooth.

We are now going to prove an interesting property of the elements of \mathfrak{H}_2 , which is the fact that we can generalize to them the construction of a solution of (HJ) .

Proposition 6.1. *Let H and K be two Hamiltonian functions, and u_{L_H}, u_{L_K} the solution obtained by the above method with the same initial submanifold Λ_0 . Then,*

$$\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma_2(H, K).$$

Let $(H_k) \in Ham$ be a Cauchy sequence for γ_2 representing an element $H \in \mathfrak{H}_2$. Then, proposition 6.1 implies that $(u_{L_{H_k}})$ is a Cauchy sequence in $C^0(I)$ and hence converges to continuous function u . Moreover, if (H_k) and (F_k) are two equivalent Cauchy sequences for γ_2 , then proposition 6.1 also implies that $(u_{L_{H_k}})$ and $(u_{L_{F_k}})$ are equivalent, and hence converge to the same limit. That allows us to give the following definition.

Definition 6.2. *Let $H \in \mathfrak{H}_2$. A continuous function u will be called generalized variational solution of (HJ) for H , if there exists a Cauchy sequence (H_k) in Ham representing H and such that the sequence of solutions $(u_{L_{H_k}})$ C^0 -converges to u .*

Therefore, proposition 6.1 implies the following statement:

Corollary 6.3. *Any element of \mathfrak{H}_2 admits a generalized variational solution. In particular, any Hamiltonian function in \mathfrak{F}^∞ (see definition 5.3) admits a generalized variational solution.*

To prove proposition 6.1, we first prove the following lemma:

Lemma 6.4. *For any $L \in \mathcal{L}$, we have*

$$\|u_L\|_{C^0} \leq \gamma(L).$$

Proof. Since L coincides with the zero section out of a compact set, u_L has a compact support. It follows that $\|u_L\|_{C^0} \leq \max(u_L) - \min(u_L)$. We will prove that $\min(u_L) \geq c(1, L)$. It will also imply that $\max(u_L) \leq c(\mu, L)$ by Poincaré duality. Indeed, using the fact that $c(\mu, L) = -c(1, \bar{L})$ and that for all $z, \mu_z = 1_z$, we have $u_L = -u_{\bar{L}}$.

Let $z \in \mathbb{R}^k$, and $S : \mathbb{R}^k \times \mathbb{R}^q \rightarrow \mathbb{R}$ be a g.f.q.i of $L \subset \mathbb{R}^{2k}$. Then, $S|_{\{z\} \times \mathbb{R}^q}$ is a g.f.q.i. of the reduction of L by the coisotropic submanifold $\{z\} \times \mathbb{R}^k \subset \mathbb{R}^{2k}$. Therefore, by lemma 3.3, we get $c(1_z, S|_{\{z\} \times \mathbb{R}^q}) \geq c(1, S)$, for all z and hence $\min(u_L) \geq c(1, L)$ as required. \square

Proof of proposition 6.1. The proposition comes from a sequence of inequalities:

$$\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma(L_H, L_K) \leq \tilde{\gamma}(\phi_{\check{H}}, \phi_{\check{K}}) \leq \gamma_2(H, K).$$

The third inequality comes from the first inequality in proposition 3.1. The second one is proved in [1]. Finally, the first one comes from the lemma 6.4 above and proposition 3.3 in [12], which states that for all $u, v \in H^*(\mathbb{R}^n)$, $c(u \cup v, L_1 + L_2) \leq c(u, L_1) + c(v, L_2)$, where $L_1 + L_2 = \{(q, p_1 + p_2) \mid (q, p_1) \in L_1, (q, p_2) \in L_2\}$.

Indeed, for $u = v = 1_{(t,x)}$, $L_1 = (L_H - L_K)|_{(t,x)}$ and $L_2 = L_K|_{(t,x)}$, we get $c(1_{(t,x)}, L_H|_{(t,x)}) - c(1_{(t,x)}, L_K|_{(t,x)}) \leq -c(1_{(t,x)}, (L_H - L_K)|_{(t,x)})$. Then, lemma 6.4 gives $-c(1_{(t,x)}, (L_H - L_K)|_{(t,x)}) \leq \gamma(L_H - L_K) = \gamma(L_H, L_K)$. By exchanging H and K and taking the supremum over (t, x) , we obtain $\|u_{L_H} - u_{L_K}\|_{C^0} \leq \gamma(L_H, L_K)$ as required. \square

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