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# A note on a.s. finiteness of perpetual integral functionals of diffusions

Paavo Salminen

Åbo Akademi University,  
Mathematical Department,  
Fänriksgatan 3 B,  
FIN-20500 Åbo, Finland,  
email: phsalmin@abo.fi

Marc Yor

Université Pierre et Marie Curie,  
Laboratoire de Probabilités  
et Modèles aléatoires ,  
4, Place Jussieu, Case 188  
F-75252 Paris Cedex 05, France

## Abstract

In this note, with the help of the boundary classification of diffusions, we derive a criterion of the convergence of perpetual integral functionals of transient real-valued diffusions.

In the particular case of transient Bessel processes, we note that this criterion agrees with the one obtained via Jeulin's convergence lemma.

Keywords: Brownian motion, random time change, exit boundary, local time, additive functional, stochastic differential equation.

AMS Classification: 60J65, 60J60.

1. Consider a diffusion  $Y$  on an open interval  $I = (l, r)$  determined by the SDE

$$dY_t = \sigma(Y_t) dW_t + b(Y_t) dt,$$

where  $W$  is a standard Wiener process defined in a complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ . It is assumed that  $\sigma$  and  $b$  are continuous and  $\sigma(x) > 0$  for all  $x \in I$ . We assume also that  $Y$  is transient and

$$\lim_{t \rightarrow \zeta} Y_t = r \quad \text{a.s.}, \tag{1}$$

where  $\zeta$  is the life time of  $Y$ . Hence, if  $\zeta < \infty$  then

$$\zeta = H_r(Y) := \inf\{t : Y_t = r\}.$$

For the speed and the scale measure of  $Y$  we use

$$m^Y(dx) = 2\sigma^2(x)e^{B^Y(x)}dx \quad \text{and} \quad S^Y(dx) = e^{-B^Y(x)}dx, \quad (2)$$

respectively, where

$$B^Y(x) = 2 \int^x \frac{b(z)}{\sigma^2(z)} dz. \quad (3)$$

Let  $f$  be a positive and continuous function defined on  $I$ , and consider the perpetual integral functional

$$A_\zeta(f) := \int_0^\zeta f(Y_s) ds.$$

We are interested in finding necessary and sufficient conditions for a.s. finiteness of  $A_\zeta(f)$ . When  $Y$  is a Brownian motion with drift  $\mu > 0$  such a condition is that the function  $f$  is integrable at  $+\infty$  (see Engelbert and Senf [4] and Salminen and Yor [9]). This condition is derived in [9] via Ray-Knight theorems and the stationarity property of the local time processes (which makes Jeulin's lemma applicable). In this note a condition (see Theorem 2) valid for general  $Y$  is deduced by exploiting the fact that  $A_{H_x}(f)$  for  $x < r$  can, via random time change, be seen as the first hitting time of a point for another diffusion.

**2.** The next proposition presents the key result connecting perpetual integral functionals to first hitting times. The result is a generalization of Proposition 2.1 in [8] discussed in Propositions 2.1 and 2.3 in [2].

**Proposition 1.** *Let  $Y$  and  $f$  be as above, and assume that there exists a two times continuously differentiable function  $g$  such that*

$$f(x) = (g'(x)\sigma(x))^2, \quad x \in I. \quad (4)$$

Set for  $t > 0$

$$A_t := \int_0^t f(Y_s) ds. \quad (5)$$

and let  $\{a_t : 0 \leq t < A_\zeta\}$  denote the inverse of  $A$ , that is,

$$a_t := \min\{s : A_s > t\}, \quad t \in [0, A_\zeta).$$

Then the process  $Z$  given by

$$Z_t := g(Y_{a_t}), \quad t \in [0, A_\zeta),$$

is a diffusion satisfying the SDE

$$dZ_t = d\widetilde{W}_t + G(g^{-1}(Z_t)) dt, \quad t \in [0, A_\zeta).$$

where  $\widetilde{W}_t$  is a Brownian motion and

$$G(x) = \frac{1}{f(x)} \left( \frac{1}{2} \sigma(x)^2 g''(x) + b(x) g'(x) \right).$$

Moreover, for  $l < x < y < r$

$$A_{H_y(Y)} = \inf\{t : Z_t = g(y)\} =: H_{g(y)}(Z) \quad a.s. \quad (6)$$

with  $Y_0 = x$  and  $Z_0 = g(x)$ .

**3.** To fix ideas, assume that the function  $g$  as introduced in Proposition 1 is increasing. We define  $g(r) := \lim_{x \rightarrow r} g(x)$ , and use the same convention for any increasing function defined on  $(l, r)$ . The state space of the diffusion  $Z$  is the interval  $(g(l), g(r))$  and a.s.  $\lim_{t \rightarrow \zeta(Z)} Z_t = g(r)$ . Clearly, letting  $y \rightarrow r$  in (6) it follows that

$$A_{H_r(Y)} = \inf\{t : Z_t = g(r)\} \quad a.s., \quad (7)$$

where both sides in (7) are either finite or infinite. Now we have

**Theorem 2.** For  $Y, A, f$  and  $g$  as above it holds that  $A_\zeta$  is a.s. finite if and only if for the diffusion  $Z$  the boundary point  $g(r)$  is an exit boundary, i.e.,

$$\int^{g(r)} S^Z(d\alpha) \int^\alpha m^Z(d\beta) < \infty, \quad (8)$$

where the scale  $S^Z$  and the speed  $m^Z$  of the diffusion  $Z$  are given by

$$S^Z(d\alpha) = e^{-B^Z(\alpha)} d\alpha \quad \text{and} \quad m^Z(d\beta) = 2 e^{B^Z(\beta)} d\beta$$

with

$$B^Z(\beta) = 2 \int^\beta G \circ g^{-1}(z) dz.$$

The condition (8) is equivalent with the condition

$$\int^r (S^Y(r) - S^Y(v)) f(v) m^Y(dv) < \infty. \quad (9)$$

*Proof.* As is well known from the standard diffusion theory, a diffusion hits its exit boundary with positive probability and an exit boundary cannot be unattainable (see [5] or [1]). This combined with (7) and the characterization of an exit boundary (see [1] No. II.6 p.14) proves the first claim. It remains to show that (8) and (9) are equivalent. We have

$$\begin{aligned} B^Z(\alpha) &= 2 \int^{g^{-1}(\alpha)} G(u) g'(u) du \\ &= 2 \int^{g^{-1}(\alpha)} \left( \frac{1}{2} \frac{g''(u)}{g'(u)} + \frac{b(u)}{\sigma^2(u)} \right) du \\ &= \log(g'(g^{-1}(\alpha))) + B^Y(g^{-1}(\alpha)). \end{aligned}$$

Consequently,

$$S^Z(d\alpha) = e^{-B^Z(\alpha)} d\alpha = \frac{1}{g'(g^{-1}(\alpha))} \exp(-B^Y(g^{-1}(\alpha))) d\alpha$$

and

$$m^Z(d\alpha) = 2 e^{B^Z(\alpha)} d\alpha = 2 g'(g^{-1}(\alpha)) \exp(B^Y(g^{-1}(\alpha))) d\alpha.$$

Substituting first  $\alpha = g(u)$  in the outer integral in (8) and after this  $\beta = g(v)$  in the inner integral yield

$$\begin{aligned} \int^{g(r)} S^Z(d\alpha) \int^\alpha m^Z(d\beta) &= 2 \int^r du e^{-B^Y(u)} \int^u dv (g'(v))^2 e^{B^Y(v)} \\ &= 2 \int^r dv (g'(v))^2 e^{B^Y(v)} \int_v^r du e^{-B^Y(u)} \end{aligned}$$

by Fubini's theorem. Using the expressions given in (2) for the speed and the scale of  $Y$  and the relation (4) between  $f$  and  $g$  complete the proof.  $\square$

**4.** It is easy to derive a condition that the mean of  $A_\zeta(f)$  is finite. Indeed,

$$\begin{aligned} \mathbf{E}_x(A_\zeta(f)) &= \int_0^\infty \mathbf{E}_x(f(Y_s)) ds \\ &= \int_l^r G_0^Y(x, y) f(y) m^Y(dy) < \infty, \end{aligned} \quad (10)$$

where  $G_0^Y$  is the Green kernel of  $Y$  w. r. t.  $m^Y$ . Under the assumption (1) we may take for  $x \geq y$

$$G_0^Y(x, y) = S^Y(r) - S^Y(x).$$

Consequently, the condition (9) may be viewed as a part of the condition (10).

**5.** Since the exit condition (8) plays a crucial rôle in our approach we discuss here shortly two proofs of this condition, thus making the paper as self-contained as possible.

Let  $Y$  be an arbitrary regular diffusion living on the interval  $I$  with the end points  $l$  and  $r$ . The scale function of  $Y$  is denoted by  $S$  and the speed measure by  $m$ . It is also assumed that the killing measure of  $Y$  is identically zero. Recall the definition due to Feller

$$r \text{ is exit} \quad \Leftrightarrow \quad \int^r S(d\alpha) \int^\alpha m(d\beta) < \infty. \quad (11)$$

Note that by Fubini's theorem

$$\int^r S(d\alpha) \int^\alpha m(d\beta) = \int^r m(d\beta)(S(r) - S(\beta)),$$

and, hence,  $S(r) < \infty$  if  $r$  is exit. Moreover, if  $r$  is exit then  $H_r < \infty$  with positive probability.

**5.1.** We give now some details of the proof of (11) following closely Kallenberg [7] (see also Breiman [3]). For  $l < a < b < r$  let  $H_{ab} := \inf\{t : Y_t = a \text{ or } b\}$ . Then for  $a < x < b$

$$\mathbf{E}_x(H_{ab}) = \int_a^b \widehat{G}_0^Y(x, z) m(dz), \quad (12)$$

where  $\widehat{G}_0^Y$  is the (symmetric) Green kernel of  $Y$  killed when it exits  $(a, b)$ , i.e.,

$$\widehat{G}_0^Y(x, z) = \frac{(S(b) - S(x))(S(y) - S(a))}{S(b) - S(a)} \quad x \geq y.$$

If  $r$  is exit there exists  $h > 0$  such that  $\mathbf{P}_x(H_r < h) > 0$  for any fixed  $x \in (a, r)$ . Using this property it can be deduced (see [7] p. 377) that for any  $a \in (l, r)$

$$\mathbf{E}_x(H_{ar}) < \infty,$$

which, from (12), is seen to be equivalent with (11).

**5.2.** Another proof of (11) can be found in Itô and McKean [5] p. 130. To present also this briefly recall first the formula

$$\mathbf{E}_x(\exp(-\lambda H_b)) = \frac{\psi_\lambda(x)}{\psi_\lambda(b)}, \quad (13)$$

where  $\lambda > 0$  and  $\psi_\lambda$  is an increasing solution of the generalized differential equation

$$\frac{d}{dm} \frac{d}{dS} u = \lambda u. \quad (14)$$

Letting  $b \rightarrow r$  in (13) it is seen that

$$r \text{ is exit} \quad \Leftrightarrow \quad \lim_{b \rightarrow r} \psi_\lambda(b) < \infty.$$

Let  $\psi_\lambda^+$  denote the (right) derivative of  $\psi_\lambda$  with respect to  $S$ . Since  $\psi_\lambda$  is increasing it holds that  $\psi_\lambda^+ > 0$ . The fact that  $\psi_\lambda$  solves (14) yields for  $z < r$

$$\psi_\lambda^+(r) - \psi_\lambda^+(z) = \lambda \int_z^r \psi_\lambda(a) m(da).$$

In particular,  $\psi_\lambda^+$  is increasing and  $\psi_\lambda^+(r) > 0$ . Hence, assuming now that  $\psi_\lambda(r) < \infty$  we obtain  $S(r) < \infty$ , and, further,

$$\begin{aligned} \lambda \psi_\lambda(z) \int_z^r S(d\alpha) \int_z^\alpha m(d\beta) &\leq \lambda \int_z^r S(d\alpha) \int_z^\alpha \psi_\lambda(\beta) m(d\beta) \\ &= \int_z^r S(d\alpha) (\psi_\lambda^+(\alpha) - \psi_\lambda^+(z)) \\ &= \psi_\lambda(r) - \psi_\lambda(z) - \psi_\lambda^+(z) (S(r) - S(z)) < \infty, \end{aligned}$$

which yields the condition on the right hand side of (11). Assume next that the condition on the right hand side of (11) holds, and consider for  $z < \beta$

$$0 \leq (\psi_\lambda(\beta))^{-1} (\psi_\lambda^+(\beta) - \psi_\lambda^+(z)) = (\psi_\lambda(\beta))^{-1} \int_z^\beta \psi_\lambda(\alpha) m(d\alpha).$$

Integrating over  $\beta$  gives

$$\begin{aligned} \log(\psi_\lambda(r)) - \log(\psi_\lambda(z)) - \psi_\lambda^+(z) \int_z^r (\psi_\lambda(\beta))^{-1} S(d\beta) \\ = \int_z^r S(d\beta) (\psi_\lambda(\beta))^{-1} \int_z^\beta \psi_\lambda(\alpha) m(d\alpha) \\ \leq \int_z^r S(d\beta) \int_z^\beta m(d\alpha) < \infty, \end{aligned}$$

which implies that  $\psi_\lambda(r) < \infty$ , thus completing the proof.

**6.** As an application of Theorem 2, we consider a Bessel process with dimension parameter  $\delta > 2$ . Let  $R$  denote this process. It is well known that  $\lim_{t \rightarrow \infty} R_t = +\infty$  and that  $R$  solves the SDE

$$dR_t = dW_t + \frac{\delta - 1}{2R_t} dt,$$

where  $W$  is a standard Brownian motion. Here the function  $B^R$  (cf. (3)) takes the form

$$B^R(v) = (\delta - 1) \log v,$$

and, consequently,

$$\begin{aligned} \int_0^\infty dv (g'(v))^2 e^{B^R(v)} \int_v^\infty du e^{-B^R(u)} \\ = \int_0^\infty dv (g'(v))^2 v^{\delta-1} \int_v^\infty du u^{-\delta+1} \\ = \int_0^\infty dv (g'(v))^2 v^{\delta-1} \frac{1}{\delta-2} v^{-\delta+2} \end{aligned}$$

leading to

$$\int_0^\infty f(R_t) dt < \infty \quad \Leftrightarrow \quad \int_0^\infty u f(u) du < +\infty.$$

Another way to derive this condition is via local times and Jeulin's lemma [6]. Indeed, by the occupation time formula and Ray-Knight theorem for the total local times of  $R$  (see, e.g. [10] Theorem 4.1 p. 52) we have

$$\begin{aligned} \int_0^\infty f(R_s) ds &\stackrel{(d)}{=} \int_0^\infty f(a) \frac{\rho_{a^\gamma}}{\gamma a^{\gamma-1}} da \\ &= \frac{1}{\gamma} \int_0^\infty a f(a) \frac{\rho_{a^\gamma}}{a^\gamma} da \end{aligned}$$

where  $\delta = 2 + \gamma$  and  $\rho$  is a squared 2-dimensional Bessel process. Using the scaling property, it is seen that the distribution of the random variable  $\rho_{a^\gamma}/a^\gamma$  does not depend on  $a$ . Hence, we obtain by Jeulin's lemma that if the function  $a \mapsto a f(a)$ ,  $a > 0$ , is locally integrable on  $[0, \infty)$  then

$$\int_0^\infty f(R_s) ds < \infty \quad \Leftrightarrow \quad \int_0^\infty a f(a) da < \infty. \quad (15)$$



The same argument allows us to recover the result in [9], that is,

$$\int_0^\infty g(W_s^{(\mu)}) ds < \infty \quad \Leftrightarrow \quad \int_0^\infty g(x) dx < \infty. \quad (16)$$

where  $g$  is any non-negative locally integrable function and  $W^{(\mu)}$  denotes a Brownian motion with drift  $\mu > 0$ . To see this, write  $g(x) = f(e^x)$  and use Lamperti's representation

$$\exp(W_s^{(\mu)}) = R_{A_s^{(\mu)}}^{(\mu)}, \quad s \geq 0,$$

where

$$A_s^{(\mu)} = \int_0^s du \exp(2W_u^{(\mu)})$$

and  $R^{(\mu)}$  is a Bessel process with dimension  $d = 2(1 + \mu)$  starting from 1, we obtain (cf. [8] Remark 3.3.(3))

$$\int_0^\infty f(\exp(W_s^{(\mu)})) ds = \int_0^\infty (R_u^{(\mu)})^{-2} f(R_u^{(\mu)}) du \quad \text{a.s.},$$

and, in order to get (16) it now only remains to use the equivalence (15).

We wish to underline the fact that in Theorem 2 it is assumed that the function  $f$  is continuous whereas the approach via Jeulin's lemma, which we developed above, demands only local integrability.

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