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# High-resolution product quantization for Gaussian processes under sup-norm distortion 

HARALD LUSCHGY ${ }^{1}$ and GILLES PAGÈS ${ }^{2}$<br>${ }^{1}$ Universität Trier, FB IV-Mathematik, D-54286 Trier, Germany. E-mail: luschgy@uni-trier.de<br>${ }^{2}$ Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5, France. E-mail: gpa@ccr.jussieu.fr

We derive high-resolution upper bounds for optimal product quantization of pathwise continuous Gaussian processes with respect to the supremum norm on $[0, T]^{d}$. Moreover, we describe a product quantization design which attains this bound. This is achieved under very general assumptions on random series expansions of the process. It turns out that product quantization is asymptotically only slightly worse than optimal functional quantization. The results are applied to fractional Brownian sheets and the Ornstein-Uhlenbeck process.

Keywords: Gaussian process; high-resolution quantization; product quantization; series expansion

## 1. Introduction

In this paper, we investigate the functional quantization problem for pathwise continuous Gaussian processes $X=\left(X_{t}\right)_{t \in I}, I=[0, T]^{d}$, where the path space $E=\mathcal{C}(I)$ is endowed with the supremum norm. For any real separable space $(E,\|\cdot\|)$ and $r \in(0, \infty)$, optimal quantization means the best approximation in $L_{E}^{r}(\mathbb{P})$ of a random vector $X:(\Omega, \mathcal{A}, \mathbb{P}) \rightarrow$ $E$ by random vectors $\widehat{X}:(\Omega, \mathcal{A}, \mathbb{P}) \rightarrow E$ taking finitely many values in $E$. If $N \in \mathbb{N}$, $\operatorname{card}(\widehat{X}(\Omega)) \leq N$, then $\widehat{X}$ is called $N$-quantization. This leads to the minimal level $N$ quantization error defined by

$$
\begin{equation*}
e_{N, r}(X, E):=\inf \left\{\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r}: \widehat{X} N \text {-quantization of } X\right\} \tag{1.1}
\end{equation*}
$$

provided $X \in L_{E}^{r}(\mathbb{P})$. When $E=\mathbb{R}^{d}$, this problem is known as optimal vector quantization and has been extensively investigated since the early 1950s, with some applications to signal processing and transmission (see Gersho and Gray [11]) and to model-based clustering in statistics (see e.g., Tarpey [25]). Beyond these classical applications, optimal quantization has been used as a space discretization device to solve nonlinear problems, such as those arising in optimal stopping theory (American-style option pricing, reflected

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BSDE, Bally and Pagès [2]), nonlinear filtering (Pagès and Pham [22]), forward-backward SDE (see Delarue and Menozzi [5]) and SPDE (see Gobet et al. [12]). The mathematical foundations are treated in Graf and Luschgy [13]. Much attention has been paid to the infinite-dimensional case. This is the so-called functional quantization of stochastic processes: the aim is to quantize some processes viewed as random vectors taking values in their path spaces. Recently, a first application of functional quantization to statistical clustering of functional data has been investigated (see Tarpey and Kinateder [26] and Tarpey et al. [27]). The simplest application of functional quantization as a numerical method is to use it as an alternative to Monte Carlo simulation, using the quadrature formula

$$
\mathbb{E}(F(X)) \approx \mathbb{E}(F(\widehat{X}))=\sum_{a \in \alpha} F(a) \mathbb{P}(\widehat{X}=a), \quad \text { where } \alpha=\widehat{X}(\Omega)
$$

for sufficiently regular functionals $F: E \rightarrow \mathbb{R}$. If $\widehat{X}$ is an $L^{r}$-optimal $N$-quantization and $F$ is Lipschitz continuous, then the induced error is bounded by $[F]_{\text {Lip }} e_{N, r}(X, E), r \geq 1$. Some numerical applications are being developed for the pricing of path-dependent options (such as regular Asian options) in various models using $E=L^{2}([0, T], d t)$ (Black and Scholes, Heston, see Pagès and Printems [23], Wilbertz [29]). However, many important functionals of processes, like those related to barrier options or to options on maximum, are only continuous with respect to the sup-norm on $E=\mathcal{C}([0, T])$.

Let us now describe what we will call the product quantization scheme. Let $X$ be a centered $E$-valued Gaussian random vector. Let $\xi_{1}, \xi_{2}, \ldots$ be i.i.d. $\mathcal{N}(0,1)$-distributed random variables and let $\left(f_{j}\right)_{j \geq 1}$ be a sequence in $E$ such that $\sum_{j=1}^{\infty} \xi_{j} f_{j}$ converges a.s. in $E$ and

$$
\begin{equation*}
X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_{j} f_{j} \tag{1.2}
\end{equation*}
$$

Let us call such a sequence admissible for $X$. For background on expansions for Gaussian random vectors, the reader is referred to Bogachev [4] and Ledoux and Talagrand [17]. One checks that $\left(f_{j}\right)_{j \geq 1}$ is admissible for $X$ if and only if $\left(f_{j}\right)_{j \geq 1}$ is a normalized tight frame in the reproducing kernel Hilbert space (Cameron-Martin space) $H=H_{X}$, that is, $\left\{f_{j}, j \geq 1\right\} \subset H$ and $\sum_{j \geq 1}\left(f_{j}, h\right)_{H}^{2}=\|h\|_{H}^{2}$ for all $h \in H$ (see Luschgy and Pagès [21]). Then a sufficient (but not necessary) condition is that $\left(f_{j}\right)_{j \geq 1}$ is an orthonormal basis of $H_{X}$.

For $m, N_{1}, \ldots, N_{m} \in \mathbb{N}$ with $\prod_{j=1}^{m} N_{j} \leq N$, let $\widehat{\xi}_{j}$ be an $L^{r}$-optimal $N_{j}$-quantization for $\xi_{j}$, that is, $\left(\mathbb{E}\left|\xi_{j}-\widehat{\xi}_{j}\right|^{r}\right)^{1 / r}=e_{N_{j}, r}\left(\xi_{j}, \mathbb{R}\right)$. An $L^{r}$-product $N$-quantization of $X$ with respect to $\left(f_{j}\right)_{j \geq 1}$ is then defined by

$$
\begin{equation*}
\widehat{X}:=\widehat{X}^{\left(N_{1}, \ldots, N_{m}\right)}:=\sum_{j=1}^{m} \hat{\xi}_{j} f_{j} \tag{1.3}
\end{equation*}
$$

and the quantization error induced by $\widehat{X}$ is

$$
\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r}
$$

Note that if $\alpha_{j}=\widehat{\xi}_{j}(\Omega)$, then the codebook $\alpha=\widehat{X}(\Omega)$ of $\widehat{X}$ satisfies $\alpha=\left\{\sum_{j=1}^{m} a_{j} f_{j}: a \in\right.$ $\left.\prod_{j=1}^{m} \alpha_{j}\right\}$ and

$$
\left(\mathbb{E} \min _{a \in \alpha}\|X-a\|^{r}\right)^{1 / r} \leq\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r}
$$

The minimal $N$ th product quantization error is then defined by

$$
\begin{array}{r}
e_{N, r}^{(\text {prod })}(X, E):=\inf \left\{\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r}:\left(f_{j}\right)_{j \geq 1} \in E^{\mathbb{N}} \text { admissible for } X\right. \\
\left.\widehat{X} L^{r} \text {-product } N \text {-quantization w.r.t. }\left(f_{j}\right)\right\} . \tag{1.4}
\end{array}
$$

Clearly, we have

$$
\begin{equation*}
e_{N, r}(X, E) \leq e_{N, r}^{(\mathrm{prod})}(X, E) \tag{1.5}
\end{equation*}
$$

We address the issue of high-resolution product quantization in $E=\mathcal{C}(I)$ under the sup-norm, which concerns the performance of $\widehat{X}=\widehat{X}^{\left(N_{1}, \ldots, N_{m}\right)}$ under a suitable choice of the marginal quantization levels $N_{j}$ and the behaviour of $e_{N, r}^{(\operatorname{prod})}(X, \mathcal{C}(I))$ as $N \rightarrow$ $\infty$. For a broad class of Gaussian processes, we derive high-resolution upper estimates for $e_{N, r}^{(\text {prod })}(X, \mathcal{C}(I))$. Furthermore, we describe a product quantization design $\widehat{X}$ which attains this bound. Combining these estimates with precise high-resolution formulas for $e_{N, r}(X, \mathcal{C}(I))$ (see Dereich et al. [6], Dereich and Scheutzow [7], Graf et al. [14]), one may typically conclude that

$$
e_{N, r}^{(\mathrm{prod})}(X, \mathcal{C}(I))=O\left((\log \log N)^{c} e_{N, r}(X, \mathcal{C}(I))\right)
$$

for some suitable constant $c>0$. This suggests that the asymptotic quality of product quantization, which is based on easy computations, is only slightly worse than optimal quantization. The optimality of this rate for product quantization rate remains open, although one may reasonably guess that it is optimal.

The paper is organized as follows. In Section 2, we derive high-resolution upper estimates for $e_{N, r}^{(\text {prod })}(X, \mathcal{C}(I))$ under very general assumptions on expansions. Section 3 contains a collection of examples, including fractional Brownian sheets, Riemann-Liouville processes and the Ornstein-Uhlenbeck process.

It is convenient to use the symbols $\sim$ and $\approx$, where $a_{n} \sim b_{n}$ means $a_{n} / b_{n} \rightarrow 1$ and $a_{n} \approx$ $b_{n}$ means $a_{n}=O\left(b_{n}\right)$ and $a_{n}=\Omega\left(b_{n}\right)$. Throughout, all logarithms are natural logarithms and $[x]$ denotes the integer part of the real number $x$.

## 2. High-resolution product quantization

We investigate high-resolution product functional quantization of centered continuous Gaussian processes $X=\left(X_{t}\right)_{t \in I}$ on $I=[0, T]^{d}$ in the space $E=\mathcal{C}(I)$ equipped with the sup-norm $\|x\|=\sup _{t \in I}|x(t)|$. Let

$$
e_{N, r}^{(\text {prod })}(X):=e_{N, r}^{(\text {prod })}(X, \mathcal{C}(I))
$$

The subsequent setting comprises a broad class of processes.
Let $\left(f_{j}\right)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ satisfy the following assumptions:
(A1) $\left\|f_{j}\right\| \leq C_{1} j^{-\vartheta} \log (1+j)^{\gamma}$ for every $j \geq 1$ with $\vartheta>1 / 2, \gamma \geq 0$ and $C_{1}<\infty$;
(A2) $f_{j}$ is $a$-Hölder-continuous and $\left[f_{j}\right]_{a} \leq C_{2} j^{b}$ for every $j \geq 1$ with $a \in(0,1], b \in \mathbb{R}$ and $C_{2}<\infty$, where

$$
[f]_{a}=\sup _{s \neq t} \frac{|f(s)-f(t)|}{|s-t|^{a}}
$$

(and $|t|$ denotes the $l_{2}$-norm of $t \in \mathbb{R}^{d}$ ).
In the sequel, finite constants depending only on the parameters $T, \vartheta, \gamma, a, b, C_{1}, C_{2}, d$ and $r$ are denoted by $C$ and may differ from one formula to another one. Other dependencies are explicitly indicated.

First, observe that by (A1), $\sum_{j=1}^{\infty} f_{j}(t)^{2} \leq \sum_{j=1}^{\infty}\left\|f_{j}\right\|^{2}<\infty$ for every $t \in I$, so we can define a centered Gaussian process $Y$ by $Y_{t}:=\sum_{j=1}^{\infty} \xi_{j} f_{j}(t)$. Using (A1) and (A2), we have, for $\rho \in(0,1]$,

$$
\begin{aligned}
\left|f_{j}(s)-f_{j}(t)\right| & =\left|f_{j}(s)-f_{j}(t)\right|^{\rho}\left|f_{j}(s)-f_{j}(t)\right|^{1-\rho} \\
& \leq\left(\left[f_{j}\right]_{a}|s-t|^{a}\right)^{\rho}\left(2\left\|f_{j}\right\|\right)^{1-\rho} \\
& \leq C_{\rho} j^{\rho(b+\vartheta)-\vartheta} \log (1+j)^{\gamma(1-\rho)}|s-t|^{a \rho}
\end{aligned}
$$

and hence

$$
\sum_{j=1}^{\infty}\left[f_{j}\right]_{a \rho}^{2}<\infty \quad \text { for every } \rho<\frac{\vartheta-1 / 2}{(b+\vartheta)_{+}}
$$

This yields

$$
\begin{equation*}
\mathbb{E}\left|Y_{s}-Y_{t}\right|^{2}=\sum_{j=1}^{\infty}\left|f_{j}(s)-f_{j}(t)\right|^{2} \leq\left(\sum_{j=1}^{\infty}\left[f_{j}\right]_{a \rho}^{2}\right)|s-t|^{2 a \rho} \tag{2.1}
\end{equation*}
$$

and using the Gaussian feature of $Y$, we obtain from the Kolmogorov criterion that $Y$ has a continuous modification $X$. Consequently, $\left(f_{j}\right)$ is admissible for $X$ and

$$
\begin{equation*}
X=\sum_{j=1}^{\infty} \xi_{j} f_{j} \quad \text { a.s. } \tag{2.2}
\end{equation*}
$$

For $r \in[1, \infty)$, the quantization error induced by the $L^{r}$-product $N$-quantization $\widehat{X}:=$ $\widehat{X}^{\left(N_{1}, \ldots, N_{m}\right)}($ see $(1.3))$ satisfies

$$
\begin{aligned}
\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r} & =\|X-\widehat{X}\|_{L_{E}^{r}(\mathbb{P})} \\
& \leq\left\|\sum_{j=1}^{m}\left(\xi_{j}-\hat{\xi}_{j}\right) f_{j}\right\|_{L_{E}^{r}(\mathbb{P})}+\left\|\sum_{j \geq m+1} \xi_{j} f_{j}\right\|_{L_{E}^{r}(\mathbb{P})} \\
& \leq \sum_{j=1}^{m}\left\|\xi_{j}-\hat{\xi}_{j}\right\|_{L^{r}(\mathbb{P})}\left\|f_{j}\right\|+\left\|\sum_{j \geq m+1} \xi_{j} f_{j}\right\|_{L_{E}^{r}(\mathbb{P})}
\end{aligned}
$$

so that

$$
\begin{equation*}
\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r} \leq \sum_{j=1}^{m}\left\|f_{j}\right\| e_{N_{j}, r}(\mathcal{N}(0,1))+\left(\mathbb{E}\left\|\sum_{j \geq m+1} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \tag{2.3}
\end{equation*}
$$

For $r \in(0,1)$, we have

$$
\begin{equation*}
\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r} \leq \mathbb{E}\|X-\widehat{X}\| \leq \sum_{j=1}^{m}\left\|f_{j}\right\| \mathbb{E}\left|\xi_{j}-\widehat{\xi}_{j}\right|+\mathbb{E}\left\|\sum_{j \geq m+1} \xi_{j} f_{j}\right\| \tag{2.4}
\end{equation*}
$$

Let us now consider the truncation error.
Theorem 1. Assume that $\left(f_{j}\right)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ satisfies (A1)-(A2). Then, for every $n \geq 2$ and $r \in(0, \infty)$,

$$
\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \leq \frac{C(\log n)^{\gamma+1 / 2}}{n^{\vartheta-1 / 2}}
$$

and

$$
\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \leq \frac{C(\log n)^{\gamma}}{n^{\vartheta-1 / 2}}, \quad \text { if } b+\vartheta \leq 0
$$

Proof. By equivalence of Gaussian moments,

$$
\begin{equation*}
\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \leq D \mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\| \tag{2.5}
\end{equation*}
$$

for some constant $D$ depending on $r$ (cf. Ledoux and Talagrand [17], Corollary 3.2). The upper estimate for $\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|$ is based on corresponding estimates for finite blocks
of exponentially increasing length. For $m \geq 1$, set

$$
Z=Z^{(m)}:=\sum_{j=2^{m-1}+1}^{2^{m}} \xi_{j} f_{j}
$$

For a given $N \geq 1$, consider the grid $G_{N}=\left\{\frac{(2 i-1) T}{2 N}: i=1, \ldots, N\right\}^{d}$. Then

$$
\|Z\| \leq \sup _{t \in G_{N}}\left|Z_{t}\right|+\sup _{|s-t| \leq C N^{-1}}\left|Z_{s}-Z_{t}\right|
$$

It follows from the Gaussian maximal inequality that

$$
\mathbb{E} \sup _{t \in G_{N}}\left|Z_{t}\right| \leq C \sqrt{\log \left(1+N^{d}\right)} \sup _{t \in G_{N}} \sqrt{\mathbb{E} Z_{t}^{2}}
$$

Using (A1), we have, for every $t \in I$,

$$
\mathbb{E} Z_{t}^{2} \leq \sum_{j=2^{m-1}+1}^{2^{m}}\left\|f_{j}\right\|^{2} \leq C \sum_{j=2^{m-1}+1}^{2^{m}} j^{-2 \vartheta} \log (1+j)^{2 \gamma} \leq C 2^{m(1-2 \vartheta)} m^{2 \gamma}
$$

so that

$$
\mathbb{E} \sup _{t \in G_{N}}\left|Z_{t}\right| \leq C \sqrt{\log (1+N)} 2^{-m(\vartheta-1 / 2)} m^{\gamma}
$$

Moreover, using (A2), we have, for $|s-t| \leq C N^{-1}$,

$$
\begin{aligned}
\left|Z_{s}-Z_{t}\right| & \leq \sum_{j=2^{m-1}+1}^{2^{m}}\left|\xi_{j}\right|\left|f_{j}(s)-f_{j}(t)\right| \\
& \leq C|s-t|^{a} \sum_{j=2^{m-1}+1}^{2^{m}}\left|\xi_{j}\right|\left[f_{j}\right]_{a} \\
& \leq C N^{-a} \sum_{j=2^{m-1}+1}^{2^{m}}\left|\xi_{j}\right| j^{b}
\end{aligned}
$$

and hence

$$
\mathbb{E} \sup _{|s-t| \leq C N^{-1}}\left|Z_{s}-Z_{t}\right| \leq C N^{-a} \sum_{j=2^{m-1}+1}^{2^{m}} j^{b} \leq C N^{-a} 2^{m(1+b)}
$$

Thus we have established the estimate

$$
\begin{equation*}
\mathbb{E}\left\|Z^{(m)}\right\| \leq C\left(\sqrt{\log (1+N)} 2^{-m(\vartheta-1 / 2)} m^{\gamma}+N^{-a} 2^{m(1+b)}\right) \tag{2.6}
\end{equation*}
$$

As concerns the choice of $N$, set $N:=\left[2^{u m}\right]+1$, with $u \in(0, \infty)$ satisfying $1+b-a u \leq$ $\frac{1}{2}-\vartheta$. Equation (2.6) then becomes

$$
\begin{equation*}
\mathbb{E}\left\|Z^{(m)}\right\| \leq C 2^{-m(\vartheta-1 / 2)} m^{\gamma+1 / 2} \tag{2.7}
\end{equation*}
$$

We note that in the case $b+\vartheta \leq-1 / 2$, we may choose $N=1$ and thereby obtain a power reduction from $m^{\gamma+1 / 2}$ to $m^{\gamma}$. This can be improved. In fact, we have

$$
\begin{aligned}
\mathbb{E}\left|Z_{s}-Z_{t}\right|^{2} & =\sum_{j=2^{m-1}+1}^{2^{m}}\left|f_{j}(s)-f_{j}(t)\right|^{2} \\
& \leq C|s-t|^{2 a} \sum_{j-2^{m-1}+1}^{2^{m}} j^{2 b} \leq C|s-t|^{2 a} 2^{m(1+2 b)}
\end{aligned}
$$

so that

$$
d_{Z}(s, t):=\left(\mathbb{E}\left|Z_{s}-Z_{t}\right|^{2}\right)^{1 / 2} \leq C|s-t|^{a} 2^{m(b+1 / 2)}
$$

If $N\left(\varepsilon, d_{Z}\right)$ denotes the covering numbers of $I$ with respect to the intrinsic semi-metric $d_{Z}$, then, by chaining,

$$
\mathbb{E} \sup _{|s-t| \leq C N^{-1}}\left|Z_{s}-Z_{t}\right| \leq \mathbb{E} \sup _{d_{Z}(s, t) \leq \delta}\left|Z_{s}-Z_{t}\right| \leq C \int_{0}^{\delta} \sqrt{\log N\left(\varepsilon, d_{Z}\right)} \mathrm{d} \varepsilon
$$

where $\delta:=C N^{-a} 2^{m(b+1 / 2)}$ (cf. Van der Waart and Wellner [28], page 101). Since

$$
N\left(\varepsilon, d_{Z}\right) \leq C\left(\frac{2^{m(b+1 / 2)}}{\varepsilon}\right)^{d / a}, \quad 0<\varepsilon \leq \varepsilon_{0}
$$

and $\int_{0}^{1} \sqrt{\log (1 / x)} \mathrm{d} x<+\infty$, we obtain, for sufficiently large $N$,

$$
\int_{0}^{\delta} \sqrt{\log N\left(\varepsilon, d_{Z}\right)} \mathrm{d} \varepsilon \leq C 2^{m(b+1 / 2)} \int_{0}^{1} \sqrt{\log (1 / x)} \mathrm{d} x \leq C 2^{m(b+1 / 2)}
$$

Consequently,

$$
\begin{align*}
\mathbb{E}\left\|Z^{(m)}\right\| & \leq C\left(\sqrt{\log (1+N)} 2^{-m(\vartheta-1 / 2)} m^{\gamma}+2^{m(b+1 / 2)}\right) \\
& \leq C 2^{-m(\vartheta-1 / 2)} m^{\gamma} \quad \text { if } b+\vartheta \leq 0 . \tag{2.8}
\end{align*}
$$

We now complete the proof. For $n \geq 2$, choose $m=m(n) \geq 1$ such that $2^{m-1}<n \leq 2^{m}$. Then

$$
\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\| \leq \sum_{j \geq m+1}\left\|Z^{(j)}\right\|+\left\|\sum_{j=n}^{2^{m}} \xi_{j} f_{j}\right\|
$$

Since $\mathbb{E}\left\|\sum_{n \leq j \leq 2^{m}} \xi_{j} f_{j}\right\| \leq \mathbb{E}\left\|Z^{(m)}\right\|$ by the Anderson inequality (cf. Bogachev [4], Corollary 3.3.7), we deduce from equation (2.7) that

$$
\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\| \leq C \sum_{j \geq m} \frac{j^{\gamma+1 / 2}}{2^{j(\vartheta-1 / 2)}} \leq \frac{C m^{\gamma+1 / 2}}{2^{m(\vartheta-1 / 2)}} \leq \frac{C(\log n)^{\gamma+1 / 2}}{n^{\vartheta-1 / 2}}
$$

If $b+\vartheta \leq 0$, then it follows from (2.8) that

$$
\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\| \leq \frac{C(\log n)^{\gamma}}{n^{\vartheta-1 / 2}}
$$

Combining these estimates with (2.5) yields the assertion.

## Remarks.

- The rate for the truncation error depends only on $\vartheta$ and $\gamma$, that is, on the decay of the size of functions $f_{j}$ (provided $b+\vartheta>0$ ). The occurrence of expansions with $b+\vartheta \leq 0$ seems to be a rare event and otherwise $b$ plays no role (see the subsequent example). The case $\gamma=0$ typically corresponds to one-parameter processes with $I=[0, T]$.
- The $e_{N, r}^{(\text {prod })}$-problem comprises the optimization of admissible sequences and, in view of (2.3) and (2.4), is thus related to the $l$-numbers of $X$ defined by

$$
\begin{equation*}
l_{n, r}(X)=l_{n, r}(X, \mathcal{C}(I)):=\inf \left\{\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} g_{j}\right\|^{r}\right)^{1 / r}:\left(g_{j}\right) \text { admissible for } X \text { in } \mathcal{C}(I)\right\} \tag{2.9}
\end{equation*}
$$

Rate-optimal solutions of the $l_{n, r}$-problem, in the sense of $l_{n, r}(X) \approx\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} g_{j}\right\|^{r}\right)^{1 / r}$ as $n \rightarrow \infty$, have recently been investigated (see Kühn and Linde [16], Dzhaparidze and van Zanten $[8,9,10]$, Ayache and Taqqu [1]). Admissible sequences of type (A1) and (A2) seem to be promising candidates.

Example 1 (Weierstrass processes). Let

$$
f_{j}(t)=j^{-\vartheta} \sin \left(j^{b+\vartheta} t\right), \quad j \geq 1, \vartheta>1 / 2, b \in \mathbb{R}, t \in[0, T] .
$$

Then $\left\|f_{j}\right\| \leq j^{-\vartheta}$ and $\left[f_{j}\right]_{\tilde{\sim}}=j^{b}$. Since $f_{j}(0)=0$, we also have $\left\|f_{j}\right\| \leq T j^{b}$, so (A1) and (A2) are satisfied, with $\tilde{\vartheta}=\max \{\vartheta,-b\}$ and $a=1$. The covariance function of $X=$ $\sum_{j=1}^{\infty} \xi_{j} f_{j}$ is given by

$$
\mathbb{E} X_{s} X_{t}=\sum_{j \geq 1} j^{-2 \vartheta} \sin \left(j^{b+\vartheta} s\right) \sin \left(j^{b+\vartheta} t\right)
$$

Now, in the "Weierstrass case" $b+\vartheta>0$, we obtain, from Theorem 1,

$$
\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \leq \frac{C \sqrt{\log n}}{n^{\vartheta-1 / 2}}
$$

while in the "non-Weierstrass case," $b+\vartheta \leq 0$ appears the better rate:

$$
\left(\mathbb{E}\left\|\sum_{j \geq n} \xi_{j} f_{j}\right\|^{r}\right)^{1 / r} \leq \frac{C}{n^{-b-1 / 2}}
$$

We pass to the minimal product quantization error $e_{N, r}^{(\mathrm{prod})}(X)$.
Theorem 2. Assume that $X$ admits an admissible set $\left(f_{j}\right)_{j \geq 1}$ in $\mathcal{C}(I)$ satisfying (A1) and (A2). We then have, for every $N \geq 3$ and $r \in(0, \infty)$,

$$
\begin{equation*}
e_{N, r}^{(\text {prod })}(X) \leq \frac{C(\log \log N)^{\vartheta+\gamma}}{(\log N)^{\vartheta-1 / 2}} \tag{2.10}
\end{equation*}
$$

and

$$
e_{N, r}^{(\mathrm{prod})}(X) \leq \frac{C(\log \log N)^{\vartheta+\gamma-1 / 2}}{(\log N)^{\vartheta-1 / 2}} \quad \text { if } b+\vartheta \leq 0
$$

Furthermore, the $L^{r}$-product $N$-quantization $\widehat{X}$ with respect to $\left(f_{j}\right)$, with tuning parameters defined in (2.11) and (2.15) below, achieves these rates.

Proof. Let $r \in[1, \infty)$ and set $\nu_{j}:=j_{0}^{-\vartheta} \log \left(1+j_{0}\right)^{\gamma}$ if $j<j_{0}:=\left[e^{\gamma / \vartheta}\right]$ and $\nu_{j}:=$ $j^{-\vartheta} \log (1+j)^{\gamma}$ if $j \geq j_{0}$. The sequence $\left(\nu_{j}\right)_{j}$ is then decreasing. Since

$$
\lim _{k \rightarrow \infty} k e_{k, r}(\mathcal{N}(0,1), \mathbb{R}) \text { exists in }(0, \infty)
$$

(cf. Graf and Luschgy [13]), we deduce from (2.3), (A1) and Theorem 1 the estimate

$$
\left(\mathbb{E}\|X-\widehat{X}\|^{r}\right)^{1 / r} \leq C\left(\sum_{j=1}^{m} \nu_{j} N_{j}^{-1}+\frac{\log (1+m)^{\gamma+1 / 2}}{m^{\vartheta-1 / 2}}\right)
$$

for every $m, N_{1}, \ldots, N_{m} \in \mathbb{N}$ with $\prod_{j=1}^{m} N_{j} \leq N$. (The case $b+\vartheta \leq 0$ is treated analogously.) Consequently,

$$
\begin{array}{r}
e_{N, r}^{(\mathrm{prod})}(X) \leq C \inf \left\{\sum_{j=1}^{m} \nu_{j} N_{j}^{-1}+\frac{\log (1+m)^{\gamma+1 / 2}}{m^{\vartheta-1 / 2}}: m, N_{1}, \ldots, N_{m} \in \mathbb{N}\right. \\
\left.\prod_{j=1}^{m} N_{j} \leq N\right\} \tag{2.11}
\end{array}
$$

For a given $N \in \mathbb{N}$, we may first optimize the integer bit allocation given by the $N_{j}$ 's for fixed $m$ and then optimize $m$. To this end, note that the continuous allocation problem reads

$$
\inf \left\{\sum_{j=1}^{m} \nu_{j} y_{j}^{-1}: y_{j}>0, \prod_{j=1}^{m} y_{j} \leq N\right\}=\sum_{j=1}^{m} \nu_{j} z_{j}^{-1}=N^{-1 / m} m\left(\prod_{j=1}^{m} \nu_{j}\right)^{1 / m}
$$

where

$$
z_{j}=N^{1 / m} \nu_{j}\left(\prod_{k=1}^{m} \nu_{k}\right)^{-1 / m}
$$

and $z_{1} \geq \cdots \geq z_{m}$. One can produce an (approximate) integer solution by setting

$$
\begin{equation*}
N_{j}=\left[z_{j}\right]=\left[N^{1 / m} \nu_{j}\left(\prod_{k=1}^{m} \nu_{k}\right)^{-1 / m}\right], \quad j \in\{1, \ldots, m\} \tag{2.12}
\end{equation*}
$$

provided $z_{m} \geq 1$. Then

$$
\sum_{j=1}^{m} \nu_{j} N_{j}^{-1} \leq 2 m N^{-1 / m}\left(\prod_{j=1}^{m} \nu_{j}\right)^{1 / m} \leq C m N^{-1 / m} \nu_{m}
$$

Since the constraint on $m$ reads $m \in I(N)$ with

$$
\begin{equation*}
I(N):=\left\{m \in \mathbb{N}: N^{1 / m} \nu_{m}\left(\prod_{j=1}^{m} \nu_{j}\right)^{-1 / m} \geq 1\right\} \tag{2.13}
\end{equation*}
$$

we arrive at

$$
\begin{equation*}
e_{N, r}^{(\text {prod })}(X) \leq C \inf _{m \in I(N)}\left(\frac{N^{-1 / m} \log (1+m)^{\gamma}}{m^{\vartheta-1}}+\frac{\log (1+m)^{\gamma+1 / 2}}{m^{\vartheta-1 / 2}}\right) \tag{2.14}
\end{equation*}
$$

for every $N \in \mathbb{N}$. We check that $I(N)$ is finite, $I(N)=\left\{1, \ldots, m^{*}(N)\right\}, m^{*}(N)$ increases to infinity and

$$
\begin{equation*}
m^{*}(N) \sim \frac{\log N}{\vartheta} \quad \text { as } N \rightarrow \infty \tag{2.15}
\end{equation*}
$$

Finally, let

$$
\begin{equation*}
m=m(N) \in I(N), \quad \text { with } m(N) \leq \frac{2 \log N}{\log \log N} \quad \text { for } N \geq 3 \tag{2.16}
\end{equation*}
$$

such that

$$
m(N) \sim \frac{2 \log N}{\log \log N} \quad \text { as } N \rightarrow \infty
$$

This is possible in view of (2.14). Using (2.4), the case $r \in(0,1)$ follows from $r=1$ since the $L^{r}$-optimal $N_{j}$-quantizations $\widehat{\xi}_{j}$ satisfy $\mathbb{E}\left|\xi_{j}-\widehat{\xi}_{j}\right| \leq C N_{j}^{-1}, j \geq 1$; see Graf et al. [15].

We may reasonably conjecture that for many specific processes, the above rate is the true one. This would imply that product quantization achieves the optimal rate for quantization, namely the rate of convergence to zero of $e_{N, r}(X):=e_{N, r}(X, \mathcal{C}(I))$, only up to a $\log \log N$ term in formula (2.16). This is in contrast to the Hilbert space setting, where the optimal rate is attained by product quantization (cf. Luschgy and Pagès [20]). To be precise, we summarize the results on $e_{N, r}(X)$ in the present setting.

Proposition 1. (a) Assume that $X$ admits an admissible sequence in $\mathcal{C}(I)$ satisfying (A1) and (A2). Then

$$
\begin{equation*}
e_{N, r}(X)=O\left(\frac{(\log \log N)^{\gamma+1 / 2}}{(\log N)^{\vartheta-1 / 2}}\right) \tag{2.17}
\end{equation*}
$$

and

$$
\begin{equation*}
e_{N, r}(X)=O\left(\frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1 / 2}}\right), \quad \text { if } b+\vartheta \leq 0 \tag{2.18}
\end{equation*}
$$

(b) Assume that $X$ admits an admissible sequence satisfying (A1). Let $\mu$ be a finite Borel measure on $I$ and let $V: \mathcal{C}(I) \rightarrow L^{2}(I, \mu)$ denote the natural embedding. Then

$$
e_{N, r}\left(V(X), L^{2}(\mu)\right)=O\left(\frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1 / 2}}\right)
$$

and

$$
e_{N, 2}^{(\operatorname{prod})}\left(V(X), L^{2}(\mu)\right)=O\left(\frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1 / 2}}\right)
$$

Proof. (a) The proof is not constructive. We use Proposition 4.1 in Li and Linde [18], which relates $l$-numbers (see (2.9)) and small ball probabilities (but this relation is not always sharp). By combining this relation and Theorem 1, we obtain

$$
\begin{aligned}
& -\log (\mathbb{P}(\|X\| \leq \varepsilon))=O\left(\varepsilon^{-1 /(\vartheta-1 / 2)}\left(\log \left(\frac{1}{\varepsilon}\right)\right)^{(\gamma+1 / 2) /(\vartheta-1 / 2)}\right) \\
& -\log (\mathbb{P}(\|X\| \leq \varepsilon))=O\left(\varepsilon^{-1 /(\vartheta-1 / 2)}\left(\log \left(\frac{1}{\varepsilon}\right)\right)^{\gamma /(\vartheta-1 / 2)}\right), \quad \text { if } b+\vartheta \leq 0
\end{aligned}
$$

as $\varepsilon \rightarrow 0$. We may then apply a known, precise relationship between these probabilities and $e_{N, r}(X)$ (cf. Dereich et al. [6], Graf et al. [14]) and this leads to the desired estimate.
(b) Let $\left(f_{j}\right)_{j \geq 1}$ be an admissible sequence in $\mathcal{C}(I)$ for $X$ satisfying (A1) and consider an $L^{2}$-product $N$-quantization of $V(X)$ based on $\left(V f_{j}\right)_{j \geq 1}$,

$$
\widehat{V(X)}^{N}=\sum_{j=1}^{m} \hat{\xi}_{j} V\left(f_{j}\right)
$$

where $\widehat{\xi}_{j}$ are $L^{2}$-optimal Voronoi $N_{j}$-quantizers; see Luschgy and Pagès [19]. Then, using the independence of $\xi_{j}-\hat{\xi}_{j}, j \geq 1$, and the stationarity property $\hat{\xi}_{j}=\mathbb{E}\left(\xi_{j} \mid \hat{\xi}_{j}\right)$ of the quantization $\hat{\xi}_{j}$, we have

$$
\begin{aligned}
& \mathbb{E}\left\|\sum_{j=1}^{\infty} \xi_{j} V\left(f_{j}\right)-\widehat{V(X)}^{N}\right\|_{L^{2}(\mu)}^{2} \\
& \quad=\sum_{j=1}^{m} \mathbb{E}\left|\xi_{j}-\hat{\xi}_{j}\right|^{2}\left\|V f_{j}\right\|_{L^{2}(\mu)}^{2}+\sum_{j \geq m+1}\left\|V f_{j}\right\|_{L^{2}(\mu)}^{2} \\
& \quad \leq C\left(\sum_{j=1}^{m} N_{j}^{-2} j^{-2 \vartheta} \log (1+j)^{2 \gamma}+\sum_{j \geq m+1} j^{-2 \vartheta} \log (1+j)^{2 \gamma}\right)
\end{aligned}
$$

We then argue along the lines of Luschgy and Pagès [19] to conclude that

$$
e_{N, 2}^{(\operatorname{prod})}\left(V(X), L^{2}(\mu)\right)=O\left(\frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1 / 2}}\right)
$$

Sometimes, (2.17) provides the true rate for $e_{N, r}(X)$ (as for the two-parameter Brownian sheet), sometimes it yields the best known upper bound (as for the $d$-parameter Brownian sheet with $d \geq 3$ ) and sometimes (2.18) provides the true rate (as for Brownian motion). The latter typically occurs when the rate of $e_{N, r}(X)$ and the "Hilbert rate" of $e_{N, r}\left(V(X), L^{2}(\mathrm{~d} t)\right)$ coincide (see Section 3). It remains an open question to find conditions for this to happen.

## 3. Examples

### 3.1. Fractional Brownian motions and fractional Brownian sheets

We consider the Dzaparidze-van Zanten expansion of the fractional Brownian motion $X=\left(X_{t}\right)_{t \in[0, T]}$ with Hurst index $\rho \in(0,1)$ and covariance function

$$
\mathbb{E} X_{s} X_{t}=\frac{1}{2}\left(s^{2 \rho}+t^{2 \rho}-|s-t|^{2 \rho}\right)
$$

These authors discovered, in Dzhaparidze and van Zanten [9], that the sequence

$$
\begin{align*}
f_{j}^{1}(t)=\frac{T^{\rho} c_{\rho} \sqrt{2}}{\left|J_{1-\rho}\left(x_{j}\right)\right| x_{j}^{\rho+1}} \sin \left(\frac{x_{j} t}{T}\right), & j \geq 1 \\
f_{j}^{2}(t)=\frac{T^{\rho} c_{\rho} \sqrt{2}}{\left|J_{-\rho}\left(y_{j}\right)\right| y_{j}^{\rho+1}}\left(1-\cos \left(\frac{y_{j} t}{T}\right)\right), & j \geq 1 \tag{3.1}
\end{align*}
$$

in $\mathcal{C}([0, T])$ is admissible for $X$, where $J_{\nu}$ denotes the Bessel function of the first kind of order $\nu, 0<x_{1}<x_{2}<\cdots$ are the positive zeros of $J_{-\rho}, 0<y_{1}<y_{2}<\cdots$ the positive zeros of $J_{1-\rho}$ and $c_{\rho}^{2}=\Gamma(1+2 \rho) \sin (\pi \rho) / \pi$.

Using the asymptotic properties

$$
x_{j} \sim y_{j} \sim \pi j, \quad J_{1-\rho}\left(x_{j}\right) \sim J_{-\rho}\left(y_{j}\right) \sim \frac{\sqrt{2}}{\pi} j^{-1 / 2} \quad \text { as } j \rightarrow \infty
$$

(cf. Dzhaparidze and van Zanten [9]), one observes that a suitable arrangement of the functions (3.1) (like $f_{2 j}=f_{j}^{1}, f_{2 j-1}=f_{j}^{2}$ ) satisfies (A1) and (A2) with parameters $\vartheta=$ $\rho+1 / 2, \gamma=0, a=1$ and $b=1 / 2-\rho$. Consequently,

$$
\begin{equation*}
e_{N, r}^{(\mathrm{prod})}(F B M)=O\left(\frac{(\log \log N)^{\rho+1 / 2}}{(\log N)^{\rho}}\right) \tag{3.2}
\end{equation*}
$$

while (see Dereich and Scheutzow [7], Graf et al. [14])

$$
\begin{equation*}
e_{N, r}(F B M) \approx(\log N)^{-\rho} \tag{3.3}
\end{equation*}
$$

The tensor products of functions (3.1) are admissible for the fractional Brownian sheet $X$ over $[0, T]^{d}$ with covariance function

$$
\mathbb{E} X_{s} X_{t}=\left(\frac{1}{2}\right)^{d} \prod_{i=1}^{d}\left(s_{i}^{2 \rho_{i}}+t_{i}^{2 \rho_{i}}-\left|s_{i}-t_{i}\right|^{2 \rho_{i}}\right)
$$

$\rho_{i} \in(0,1)$, and satisfy conditions (A1) and (A2) with $\vartheta=\rho+1 / 2, \rho=\min _{1 \leq i \leq d} \rho_{i}$, $\gamma=\vartheta(m-1)$, where $m=\operatorname{card}\left\{i \in\{1, \ldots, d\}: \rho_{i}=\rho\right\}, a=1$ and $b=\max _{1 \leq i \leq d}(1 / 2-$ $\left.\left.\rho_{i}\right)\right)_{+}$. This is a consequence of the following lemma which ensures stability of conditions (A1) and (A2) under tensor products.

Lemma 1. For $i \in\{1, \ldots, d\}$, let $\left(f_{j}^{i}\right)_{j \geq 1} \in \mathcal{C}([0, T])^{\mathbb{N}}$ satisfy (A1) and (A2) with parameters $\vartheta_{i}, \gamma_{i}, a_{i}, b_{i}$ such that $\gamma_{i}=0$. Then a decreasing arrangement of $\left(\otimes_{i=1}^{d} f_{j_{i}}^{i}\right)_{\underline{j} \in \mathbb{N}^{d}}$ satisfies (A1) and (A2) with parameters $\vartheta=\min _{1 \leq i \leq d} \vartheta_{i}, \gamma=\vartheta(m-1)$, where $\bar{m}=$ $\operatorname{card}\left\{i \in\{1, \ldots, d\}: \vartheta_{i}=\vartheta\right\}, a=\min _{1 \leq i \leq d} a_{i}$ and $b=\left(\max _{1 \leq i \leq d} b_{i}\right)_{+}$.

Proof. For $\underline{j}=\left(j_{1}, \ldots, j_{d}\right) \in \mathbb{N}^{d}$, set $f_{\underline{j}}=\bigotimes_{i=1}^{d} f_{j_{i}}^{i}$ so that $f_{\underline{j}}(t)=\prod_{j=1}^{d} f_{j_{i}}^{i}\left(t_{i}\right), t \in$ $[0, T]^{d}$. We have

$$
\left\|f_{\underline{j}}\right\| \leq \prod_{i=1}^{d}\left\|f_{j_{i}}^{i}\right\| \leq C \prod_{i=1}^{d} j_{i}^{-\vartheta_{i}} \quad \text { and } \quad\left|f_{\underline{j}}(s)-f_{\underline{j}}(t)\right| \leq C \max _{1 \leq i \leq d} j_{i}^{b}|s-t|^{a}
$$

Let $u_{j}:=\prod_{i=1}^{d} j_{i}^{-\vartheta_{i}}$. Choose a bijective map $\psi: \mathbb{N} \rightarrow \mathbb{N}^{d}$ such that $u_{k}:=u_{\psi(k)}$ is decreasing in $^{-} k \geq 1$. Set $f_{k}:=f_{\psi(k)}$. Then

$$
u_{k} \approx C k^{-\vartheta}(\log k)^{\vartheta(m-1)} \quad \text { as } k \rightarrow \infty
$$

(cf. Papageorgiou and Wasilkowski [24], Theorem 2.1). Consequently,

$$
\left\|f_{k}\right\| \leq C k^{-\vartheta}(\log k)^{\vartheta(m-1)}
$$

and, for $\underline{j}=\psi(k)$,

$$
j_{i} \leq \prod_{i=1}^{d} j_{i} \leq \prod_{i=1}^{d} j_{i}^{\vartheta_{i} / \vartheta} \leq C k(\log k)^{-(m-1)} \leq C k
$$

hence

$$
\left|f_{k}(s)-f_{k}(t)\right| \leq C k^{b}|s-t|^{a}
$$

Therefore, by Theorem 2 and Proposition 1,

$$
\begin{equation*}
e_{N, r}^{(\text {prod })}(F B S)=O\left(\frac{(\log \log N)^{m(\rho+1 / 2)}}{(\log N)^{\rho}}\right) \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
e_{N, r}(F B S)=O\left(\frac{(\log \log N)^{m(\rho+1 / 2)-\rho}}{(\log N)^{\rho}}\right) \tag{3.5}
\end{equation*}
$$

The Hilbert space setting $E=L^{2}\left([0, T]^{d}, \mathrm{~d} t\right)$ provides the lower estimate

$$
\begin{equation*}
e_{N, r}(F B S)=\Omega\left(\frac{(\log \log N)^{(m-1)(\rho+1 / 2)}}{(\log N)^{\rho}}\right) \tag{3.6}
\end{equation*}
$$

(see Luschgy and Pagès $[19,20]$ ). The true rate of $e_{N, r}(F B S)$ is known only for the case $m=1$, where the true rate is the "Hilbert rate" (2.6) (see Dereich et al. [6]), and for the case $m=2$, where (3.5) is the true rate (see Belinsky and Linde [3], Graf et al. [14]). A reasonable conjecture is that (3.5) is also the true rate for $m \geq 3$.

### 3.2. Riemann-Liouville and other moving average processes

For $\psi \in L^{2}([0, T], \mathrm{d} t)$ and a standard Brownian motion $W$, let

$$
X_{t}=\int_{0}^{t} \psi(t-s) \mathrm{d} W_{s}, \quad t \in[0, T]
$$

and assume that $X$ has a pathwise continuous modification. Since

$$
\begin{align*}
\mathbb{E} X_{s} X_{t} & =\int_{0}^{s \wedge t} \psi(s-u) \psi(t-u) \mathrm{d} u \\
f_{j}(t) & =\sqrt{\frac{2}{T}} \int_{0}^{t} \psi(t-s) \cos \left(\frac{\pi(j-1 / 2) s}{T}\right) \mathrm{d} s \\
& =\sqrt{\frac{2}{T}} \int_{0}^{t} \psi(s) \cos \left(\frac{\pi(j-1 / 2)(t-s)}{T}\right) \mathrm{d} s, \quad j \geq 1 \tag{3.7}
\end{align*}
$$

is an admissible sequence for $X$. Observe that (3.7) provides well-defined continuous functions, even for $\psi \in L^{1}([0, T], \mathrm{d} t)$.

Lemma 2. Let $\psi \in L^{1}([0,1], \mathrm{d} t)$.
(a) If $\varphi(t)=\int_{0}^{t}|\psi(s)| \mathrm{d}$ s is $\beta$-Hölder continuous with $\beta \in(0,1]$, then the sequence $\left(f_{j}\right)$ from (3.7) satisfies (A2) with $a=\beta$ and $b=1$. In particular, if $\psi \in L^{2}([0, T], \mathrm{d} t)$, then (A2) is satisfied with $a=1 / 2$ and $b=1$.
(b) If $\psi$ has finite variation over $[0, T]$, then (A1) is satisfied with $\vartheta=1$ and $\gamma=0$.

Proof. Let $\lambda_{j}=(\pi(j-1 / 2) / T)^{-2}$. (a) For $s<t$, we have

$$
\begin{array}{r}
f_{j}(s)-f_{j}(t)=\sqrt{\frac{2}{T}}\left\{\int_{0}^{s} \psi(u)\left(\cos \left((s-u) / \sqrt{\lambda_{j}}\right)-\cos ((t-u) / \sqrt{\lambda j})\right) \mathrm{d} u\right. \\
\left.-\int_{s}^{t} \psi(u) \cos \left((t-u) / \sqrt{\lambda_{j}}\right) \mathrm{d} u\right\}
\end{array}
$$

so that

$$
\left|f_{j}(s)-f_{j}(t)\right| \leq \sqrt{\frac{2}{T}}\left(\frac{|s-t|}{\sqrt{\lambda_{j}}}\|\psi\|_{L^{1}(\mathrm{~d} t)}+\int_{s}^{t}|\psi(u)| \mathrm{d} u\right)
$$

(b) We have

$$
\begin{aligned}
f_{j}(t) & =-\sqrt{2 \lambda_{j} / T} \int_{0}^{t} \psi(s) \mathrm{d}\left(\sin \left((t-s) / \sqrt{\lambda_{j}}\right)\right) \\
& =\sqrt{2 \lambda_{j} / T}\left(\psi(0) \sin \left(t / \sqrt{\lambda_{j}}\right)+\int_{0}^{t} \sin \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} \psi(s)\right)
\end{aligned}
$$

so that

$$
\left\|f_{j}\right\| \leq \sqrt{2 \lambda_{j} / T}(|\psi(0)|+\operatorname{Var}(\psi,[0, T]))
$$

This lemma yields a universal upper bound,

$$
e_{N, r}^{(\text {prod })}(X)=O\left(\frac{\log \log N}{(\log N)^{1 / 2}}\right)
$$

for functions $\psi$ having finite variation.
In the sequel, we do not concern ourselves with improvements of the parameter $b$ in (A2) since the condition $b+\vartheta \leq 0$ cannot be achieved in this setting.

Lemma 3. Let $\psi \in L^{1}([0, T], \mathrm{d} t)$.
(a) If $\psi$ is positive and decreasing on $(0, T]$ and $\varphi(t)=\int_{0}^{t} \psi(s) \mathrm{d} s$ is $\beta$-Hölder continuous with $\beta \in(0,1]$, then the sequence $\left(f_{j}\right)$ from (3.7) satisfies $\left\|f_{j}\right\| \leq C j^{-\beta}$. If $\beta>1 / 2$, then (A1) is satisfied with $\vartheta=\beta$ and $\gamma=0$.
(b) If $\psi(0)=0, \psi$ is $\beta$-Hölder continuous with $\beta \in(0,1]$ and $\psi$ is differentiable on $(0, T]$ such that $\psi^{\prime}$ is positive and decreasing on $(0, T]$, then (A1) is satisfied with $\vartheta=1+\beta$ and $\gamma=0$.

Proof. Let $\lambda_{j}=(\pi(j-1 / 2) / T)^{-2}$. (a) For $t \leq \sqrt{\lambda_{j}}$, we have

$$
\left|f_{j}(t)\right| \leq \sqrt{2 / T} \varphi\left(\sqrt{\lambda_{j}}\right)
$$

Using the second integral mean value formula, we obtain, for $t \in\left[\sqrt{\lambda_{j}}, T\right]$ and some $\delta_{j} \in\left[\sqrt{\lambda_{j}}, t\right]$,

$$
\begin{aligned}
\left|f_{j}(t)\right| & \leq \sqrt{2 / T}\left(\left|\int_{0}^{\sqrt{\lambda_{j}}} \psi(s) \cos \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s\right|+\left|\int_{\sqrt{\lambda_{j}}}^{t} \psi(s) \cos \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s\right|\right) \\
& =\sqrt{2 / T}\left(\left|\int_{0}^{\sqrt{\lambda_{j}}} \psi(s) \cos \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s\right|+\psi\left(\sqrt{\lambda_{j}}\right)\left|\int_{\sqrt{\lambda_{j}}}^{\delta_{j}} \cos \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s\right|\right) \\
& \leq \sqrt{2 / T}\left(\varphi\left(\sqrt{\lambda_{j}}\right)+2 \sqrt{\lambda_{j}} \psi\left(\sqrt{\lambda_{j}}\right)\right) \\
& \leq 3 \sqrt{2 / T} \varphi\left(\sqrt{\lambda_{j}}\right) .
\end{aligned}
$$

Consequently,

$$
\left\|f_{j}\right\| \leq 3 \sqrt{2 / T} \varphi\left(\sqrt{\lambda_{j}}\right) \leq C \lambda_{j}^{\beta / 2}
$$

(b) The function $\psi$ is absolutely continuous on $[0, T]$, so an integration by parts yields

$$
f_{j}(t)=\sqrt{2 \lambda_{j} / T} \int_{0}^{t} \psi^{\prime}(s) \sin \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s
$$

Arguing as in (a) (with $\psi$ replaced by $\psi^{\prime}$ ), we deduce that

$$
\left\|f_{j}\right\| \leq 3 \sqrt{2 \lambda_{j} / T} \psi\left(\sqrt{\lambda_{j}}\right) \leq C \lambda_{j}^{(1+\beta) / 2}
$$

Now, let $\psi(t)=t^{\rho-1 / 2}$ with $\rho \in(0, \infty)$. Then

$$
\begin{equation*}
X_{t}=X_{t}^{\rho}=\int_{0}^{t}(t-s)^{\rho-1 / 2} \mathrm{~d} W_{s}, \quad t \in[0, T] \tag{3.8}
\end{equation*}
$$

so that $X^{\rho}$ is a Riemann-Liouville process of order $\rho$. Using the ( $\rho \wedge \frac{1}{2}$ )-Hölder continuity of the application $t \mapsto X_{t}^{\rho}$ from $[0, T]$ into $L^{2}(\mathbb{P})$ and the Kolmogorov criterion, we can check that $X^{\rho}$ has a pathwise continuous modification.

Lemma 4. Let $\psi(t)=t^{\rho-1 / 2}, \rho \in(0, \infty)$. Then the sequence $\left(f_{j}\right)$ from (3.7) satisfies (A2) with $a=\min \{1, \rho+1 / 2\}, b=1$ and (A1) for $\rho \in(0,3 / 2]$ with $\vartheta=\rho+1 / 2$ and $\gamma=0$.

Proof. This is an immediate consequence of Lemmas 2 and 3.
We deduce, for Riemann-Liouville processes of order $\rho \in(0,3 / 2]$, that

$$
\begin{equation*}
e_{N, r}^{(\mathrm{prod})}(R L)=O\left(\frac{(\log \log N)^{\rho+1 / 2}}{(\log N)^{\rho}}\right) \tag{3.9}
\end{equation*}
$$

while for every $\rho \in(0, \infty)$ (see [18], Graf et al. [14]),

$$
\begin{equation*}
e_{N, r}(R L) \approx(\log N)^{-\rho} \tag{3.10}
\end{equation*}
$$

To go beyond $\rho=3 / 2$, we must slightly change the way we quantize. Let $\psi(t)=$ $t^{\rho-1 / 2}$, with $\rho>3 / 2$, and choose $k \in \mathbb{N}$ such that $k+1 / 2<\rho \leq k+3 / 2$. Set $\lambda_{j}=(\pi(j-$ $1 / 2) / T)^{-2}$. For $k \in\{2 n-1,2 n\} n \in \mathbb{N}$, integration by parts yields the expansion

$$
\begin{aligned}
f_{j}(t) & =\sum_{m=1}^{n}(-1)^{m-1} \lambda_{j}^{m} \sqrt{2 / T} \psi^{(2 m-1)}(t)+(-1)^{n} \lambda_{j}^{n} \sqrt{2 / T} \int_{0}^{t} \psi^{(2 n)}(s) \cos \left((t-s) / \sqrt{\lambda_{j}}\right) \mathrm{d} s \\
& =: g_{j}(t)+h_{j}(t), \quad t \in[0, T]
\end{aligned}
$$

Since $\psi^{(2 n)}(t)=C t^{\beta-1}$ if $k=2 n-1$ and $\psi^{(2 n)}(t)=C t^{\beta}$ if $k=2 n$ with $\beta=\rho-k-1 / 2 \in$ $(0,1]$, we deduce from Lemma 2 and Lemma 3 that the sequence $\left(h_{j}\right)$ in $C([0, T])$ satisfies (A1) with $\vartheta=\rho+1 / 2, \gamma=0$ and (A2) with $a=\rho-k-1 / 2, b=-k$ if $k=2 n-1$ and $a=1, b=-k+1$ if $k=2 n$. Clearly, the sequence $\left(g_{j}\right)$ also satisfies the conditions (A1) and (A2) (with $\vartheta=2, \gamma=0, b=-2$ and $a=\rho-k-1 / 2$ if $k=2 n-1$ and $a=1$ if $k=2 n$ ). Consequently, there exist centered continuous Gaussian processes $U=\left(U_{t}\right)_{t \in[0, T]}$ and $Z$ such that $U=\sum_{j=1}^{\infty} \xi_{j} g_{j}$ a.s., $Z=\sum_{j=1}^{\infty} \xi_{j} h_{j}$ a.s.,

$$
\begin{equation*}
X=X^{\rho} \stackrel{d}{=} U+Z \tag{3.11}
\end{equation*}
$$

and $U \in \operatorname{span}\left\{\psi^{(2 m-1)}: m=1, \ldots, n\right\}$ a.s. Observe that

$$
U=\sum_{m=1}^{n}(-1)^{m-1} \sqrt{2 / T} \psi^{(2 m-1)} \eta_{m}
$$

where $\eta_{m}=\sum_{j=1}^{\infty} \lambda_{j}^{m} \xi_{j}$ is $\mathcal{N}\left(0, \sum_{j=1}^{\infty} \lambda_{j}^{2 m}\right)$-distributed.
Now use, for example, $\left[N^{1 / 2 n}\right]$-quantizations of $\eta_{m}$ and a $[\sqrt{N}]$-product quantization of $Z$ for the quantization of $X$ (which is clearly not optimal in practise, but remains rate optimal). Let $\widehat{\eta}_{m}$ be an $L^{r}$-optimal $\left[N^{1 / 2 n}\right]$-quantization for $\eta_{m}$,

$$
\hat{U}^{\sqrt{N}}:=\sum_{m=1}^{n}(-1)^{m-1} \sqrt{2 / T} \psi^{(2 m-1)} \hat{\eta}_{m}
$$

and let $\hat{Z}^{\sqrt{N}}$ be the $L^{r}$-product $[\sqrt{N}]$-quantization of $Z$ from Theorem 2. A (modified) $L^{r}$-product $N$-quantization of $X$ with respect to $\left(f_{j}\right)$ is then defined by

$$
\begin{equation*}
\widehat{X}:=\hat{U}^{\sqrt{N}}+\hat{Z}^{\sqrt{N}} \tag{3.12}
\end{equation*}
$$

Using Theorem 2, we can show for the quantization error, that

$$
\begin{aligned}
\|U+Z-\widehat{X}\|_{L_{E}^{r}} & \leq C\left(\left\|U-\hat{U}^{\sqrt{N}}\right\|_{L_{E}^{r}}+\left\|Z-\hat{Z}^{\sqrt{N}}\right\|_{L_{E}^{r}}\right) \\
& \leq C\left(\sum_{m=1}^{n} \sqrt{2 / T}\left\|\psi^{(2 m-1)}\right\|\left\|\eta_{m}-\hat{\eta}_{m}\right\|_{L^{r}}+\left\|Z-\hat{Z}^{\sqrt{N}}\right\|_{L_{E}^{r}}\right) \\
& \leq \frac{C}{N^{1 / 2 n}}+\frac{C(\log \log \sqrt{N})^{\rho+1 / 2}}{(\log \sqrt{N})^{\rho}} \\
& \leq \frac{C(\log \log N)^{\rho+1 / 2}}{(\log N)^{\rho}}
\end{aligned}
$$

so that, with the above modification, (3.9) remains true for $\rho>3 / 2$.
Now, consider the stationary Ornstein-Uhlenbeck process as the solution of the Langevin equation

$$
\mathrm{d} X_{t}=-\beta X_{t} \mathrm{~d} t+\sigma \mathrm{d} W_{t}, \quad t \in[0, T]
$$

with $X_{0}$ independent of $W$ and $\mathcal{N}\left(0, \frac{\sigma^{2}}{2 \beta}\right)$-distributed, $\sigma>0, \beta>0$. It admits the explicit representation

$$
\begin{equation*}
X_{t}=e^{-\beta t} X_{0}+\sigma \mathrm{e}^{-\beta t} \int_{0}^{t} \mathrm{e}^{\beta s} \mathrm{~d} W_{s} \tag{3.13}
\end{equation*}
$$

and

$$
\mathbb{E} X_{s} X_{t}=\frac{\sigma^{2}}{2 \beta} \mathrm{e}^{-\beta|s-t|}
$$

By Lemma 2, the admissible sequence

$$
f_{0}(t)=\frac{\sigma}{\sqrt{2 a}} \mathrm{e}^{-\beta t}, \quad f_{j}(t)=\sigma \sqrt{\frac{2}{T}} \int_{0}^{t} \mathrm{e}^{-\beta(t-s)} \cos \left(\frac{\pi(j-1 / 2) s}{T}\right) \mathrm{d} s, \quad j \geq 1
$$

satisfies conditions (A1) and (A2) with $\vartheta=1, \gamma=0, a=1$ and $b=1$. Consequently,

$$
\begin{equation*}
e_{N, r}^{(\operatorname{prod})}(O U)=O\left(\frac{\log \log N}{(\log N)^{1 / 2}}\right) \tag{3.14}
\end{equation*}
$$

while (see Graf et al. [14])

$$
\begin{equation*}
e_{N, r}(O U) \approx(\log N)^{-1 / 2} \tag{3.15}
\end{equation*}
$$

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