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► **To cite this version:**

Harald Luschgy, Gilles Pagès. High-resolution product quantization for Gaussian processes under sup-norm distortion. 2005. hal-00013489v1

HAL Id: hal-00013489

<https://hal.science/hal-00013489v1>

Preprint submitted on 8 Nov 2005 (v1), last revised 5 Sep 2007 (v2)

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High-resolution product quantization for Gaussian processes under sup-norm distortion

HARALD LUSCHGY* AND GILLES PAGÈS †

November 05

Abstract

We derive high-resolution upper bounds for optimal product quantization of pathwise continuous Gaussian processes respective to the supremum norm on $[0, T]^d$. Moreover, we describe a product quantization design which attains this bound. This is achieved under very general assumptions on random series expansions of the process. It turns out that product quantization is asymptotically only slightly worse than optimal functional quantization. The results are applied *e.g.* to fractional Brownian sheets and the Ornstein-Uhlenbeck process.

Key words: High-resolution quantization, product quantization, Gaussian process, series expansion.

2000 Mathematics Subject Classification: 60G15, 60G35, 41A25.

1 Introduction

We investigate the functional quantization problem for pathwise continuous Gaussian processes $X = (X_t)_{t \in I}$, $I = [0, T]^d$, where the distortion is based on the supremum norm in the path-space $E = \mathcal{C}(I)$. The general Banach space setting reads as follows.

Let $(E, \|\cdot\|)$ be a real separable Banach space and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow E$ be a centered Gaussian random vector taking its values in E with distribution \mathbb{P}_X . For $N \in \mathbb{N}$ and $r \in (0, \infty)$, the L^r -quantization problem for X of level N (or of nat-level $\log N$) consists in minimizing

$$(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r}$$

over all subsets $\alpha \subset E$ with $\text{card}(\alpha) \leq N$. Such a set is called N -codebook or N -quantizer. The minimal N th quantization error of X is then defined by

$$e_{N,r}(X, E) := \inf\{(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r} : \alpha \subset E, \text{card}(\alpha) \leq N\}. \quad (1.1)$$

It is still an open question whether L^r -optimal N -quantizers for Gaussian random vectors always exist (see [12]). For a given N -quantizer α one defines an associated closest neighbour projection

$$\pi_\alpha := \sum_{a \in \alpha} a 1_{C_a(\alpha)}$$

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and the Voronoi quantization of X induced by α by

$$\hat{X}^\alpha := \pi_\alpha(X), \quad (1.2)$$

where $\{C_a(\alpha) : a \in \alpha\}$ is a Voronoi partition induced by α , that is a Borel partition of E satisfying

$$C_a(\alpha) \subset \{x \in E : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}$$

for every $a \in \alpha$. Then one easily checks that

$$\mathbb{E}\|X - \hat{X}^\alpha\|^r = \mathbb{E} \min_{a \in \alpha} \|X - a\|^r. \quad (1.3)$$

In order to construct good N -quantizations for X we proceed by (scalar) product quantization. Let ξ_1, ξ_2, \dots be i.i.d. $\mathcal{N}(0, 1)$ -distributed random variables and let $(f_j)_{j \geq 1}$ be a sequence in E such that $\sum_{j=1}^{\infty} \xi_j f_j$ converges a.s. in E and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j. \quad (1.4)$$

For $m, N_1, \dots, N_m \in \mathbb{N}$ with $\prod_{j=1}^m N_j \leq N$, let $\alpha_j \subset \mathbb{R}$ be an L^r -optimal N_j -quantizer for ξ_j and let $\hat{\xi}_j = \hat{\xi}_j^{\alpha_j}$ be the (\mathbb{P}_{ξ_j} -a.s. uniquely defined) Voronoi quantization of ξ_j induced by α_j . Then, a L^r -product N -quantization of X with respect to (f_j) is defined by

$$\hat{X}^N := \sum_{j=1}^m \hat{\xi}_j f_j \quad (1.5)$$

and the quantization error induced by \hat{X}^N reads

$$(\mathbb{E} \|\sum_{j=1}^{\infty} \xi_j f_j - \hat{X}^N\|^r)^{1/r}.$$

The quantizer $\alpha = \hat{X}^N(\Omega)$ satisfies $\text{card}(\alpha) \leq N$ and

$$(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r} \leq (\mathbb{E} \|\sum_{j=1}^{\infty} \xi_j f_j - \hat{X}^N\|^r)^{1/r}.$$

Observe that \hat{X}^N is not a Voronoi quantization of X . The minimal N th product quantization error is then defined by

$$e_{N,r}(X, E)^{(pq)} := \inf \left\{ (\mathbb{E} \|\sum_{j=1}^{\infty} \xi_j f_j - \hat{X}\|^r)^{1/r} : X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j, \right. \\ \left. \hat{X} \text{ } L^r\text{-product } N\text{-quantization w.r.t. } (f_j) \right\} \quad (1.6)$$

Clearly, we have

$$e_{N,r}(X, E) \leq e_{N,r}(X, E)^{(pq)}. \quad (1.7)$$

We address the issue of high-resolution product quantization in $E = \mathcal{C}(I)$ under the sup-norm which concerns the performance of \hat{X}^N and the behaviour of $e_{N,r}(X, \mathcal{C}(I))^{(pq)}$ as $N \rightarrow \infty$. For a broad class of Gaussian processes we derive high-resolution upper estimates for $e_{N,r}(X, \mathcal{C}(I))^{(pq)}$. Furthermore, we describe a product quantization design \hat{X}^N which attains this bound. Combining

these estimates with precise high-resolution formulas for $e_{N,r}(X, \mathcal{C}(I))$ (see [5], [6], [11]) one may typically conclude

$$e_{N,r}(X, \mathcal{C}(I))^{(pq)} = O((\log \log N)^c e_{N,r}(X, \mathcal{C}(I)))$$

for some suitable constant $c > 0$. This suggests that the asymptotic quality of product quantization which is based on easy computations is only slightly worse than optimal quantization. However, it also suggests that the optimal rate, *i.e.* the rate of convergence to zero of $e_{N,r}(X, \mathcal{C}(I))$, cannot be achieved by product quantization.

It should be noticed that the $\mathcal{C}(I)$ -setting thus turns out to be different from the Hilbert space setting. For Hilbert spaces E (like $E = L^2(I, dt)$), sharp high-resolution formulas for $e_{N,r}(X, E)$ and L^2 -product quantization have been fully investigated in the past years (see [4], [16], [17], [18]). A typical result is then, at least for $r = 2$,

$$e_{N,r}(X, E)^{(pq)} = O(e_{N,r}(X, E))$$

so that optimal product quantization is in fact rate optimal.

The paper is organized as follows. Section 2 contains a brief survey about expansions of $\mathcal{C}(I)$ -valued Gaussian random vectors in the sense of (1.4). This is an important issue for product quantization since $e_{N,r}(X, \mathcal{C}(I))^{(pq)}$ comprises a minimization over such expansions (see (1.6)). In Section 3 we derive high-resolution upper estimates for $e_{N,r}(X, \mathcal{C}(I))^{(pq)}$ under very general assumptions on expansions. Section 4 contains a collection of examples including fractional Brownian sheets, Riemann-Liouville processes and the Ornstein-Uhlenbeck process.

It is convenient to use the symbols \sim and \approx , where $a_n \sim b_n$ means $a_n/b_n \rightarrow 1$ and $a_n \approx b_n$ means $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. Throughout all logarithms are natural logarithms and $[x]$ denotes the integer part of the real number x .

2 Expansions

We present the basic facts about expansions of Gaussian stochastic processes viewed as random vectors taking values in a Banach (function) space. They are essentially well known and contained more or less explicitly in [3], [14] and [20]. We give some proofs for the reader's convenience. Furthermore, we introduce examples which are investigated for quantization in Section 4.

2.1 The Banach space setting

Let $(E, \|\cdot\|)$ be a real separable Banach space. For $u \in E^*$ and $x \in E$, it is convenient to write

$$\langle u, x \rangle$$

in place of $u(x)$. Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow E$ be a centered Gaussian random vector with distribution \mathbb{P}_X . The covariance operator $C = C_X$ of X is defined by

$$C : E^* \rightarrow E, \quad Cu := \mathbb{E}\langle u, X \rangle X. \tag{2.1}$$

This operator is linear and (norm-)continuous. Let $H = H_X$ denote the reproducing kernel Hilbert space (Cameron Martin space) of the symmetric nonnegative definite kernel

$$E^* \times E^* \rightarrow \mathbb{R}, \quad (u, v) \mapsto \langle u, Cv \rangle.$$

Then H is a Hilbert subspace of E , that is $H \subset E$ and the inclusion map is continuous. The reproducing property reads

$$(h, Cu)_H = \langle u, h \rangle, \quad u \in E^*, \quad h \in H \tag{2.2}$$

where $(\cdot, \cdot)_H$ denotes the scalar product on H and the corresponding norm is given by

$$\|h\|_H = \sup\{|\langle u, h \rangle| : u \in E^*, \langle u, Cu \rangle \leq 1\}. \quad (2.3)$$

In particular, for $h \in H$,

$$\|h\| \leq \sup_{\|u\| \leq 1} \langle u, Cu \rangle^{1/2} \|h\|_H = \|C\|^{1/2} \|h\|_H. \quad (2.4)$$

The $\|\cdot\|_H$ -closure of $A \subset H$ is denoted by $\overline{A}^{(H)}$. Furthermore, H is separable, $C(E^*)$ is dense in $(H, \|\cdot\|_H)$, the unit ball

$$U_H := \{h \in H : \|h\|_H \leq 1\}$$

of H is a compact subset of E ,

$$\text{supp}(\mathbb{P}_X) = (\ker C)^\perp := \{x \in E : \langle u, x \rangle = 0 \text{ for every } u \in \ker C\} = \overline{H} \text{ in } E$$

and

$$H = \{x \in E : \|x\|_H < \infty\} \quad (2.5)$$

where

$$\|x\|_H = \sup\{|\langle u, x \rangle| : u \in E^*, \langle u, Cu \rangle \leq 1\}.$$

We are interested in expansions of X of the following type. Let ξ_1, ξ_2, \dots be i.i.d. $\mathcal{N}(0, 1)$ -distributed random variables.

Definition 1 A sequence $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$ is called *admissible for X* if

$$\sum_{j=1}^{\infty} \xi_j f_j \text{ converges a.s. in } E$$

and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j.$$

By adding zeros finite sequences in E may also serve as admissible sequences.

The key is the following characterization of admissibility. It relies on the Ito-Nisio theorem. Condition (v) is an abstract version of Mercer's theorem. Recall that a subset $G \subset E^*$ is said to be separating if for every $x, y \in E, x \neq y$ there exists $u \in G$ such that $\langle u, x \rangle \neq \langle u, y \rangle$.

Lemma 1 Let $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$. The following assertions are equivalent.

(i) The sequence $(f_j)_{j \geq 1}$ is admissible for X .

(ii) There is a separating linear subspace G of E^* such that for every $u \in G$,

$$(\langle u, f_j \rangle)_{j \geq 1} \text{ is admissible for } \langle u, X \rangle.$$

(iii) There is a separating linear subspace G of E^* such that for every $u \in G$,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \langle u, Cu \rangle.$$

(iv) For every $u \in E^*$,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu.$$

(v) For every $a > 0$,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle \langle v, f_j \rangle = \langle u, Cv \rangle$$

uniformly in $u, v \in \{y \in E^* : \|y\| \leq a\}$.

Proof. Set $X_n := \sum_{j=1}^n \xi_j f_j$. (i) \Rightarrow (v). X_n converges a.s. in E to some E -valued random vector Y , say, with $X \stackrel{d}{=} Y$. It is well known that this implies $X_n \rightarrow Y$ in L_E^2 . Therefore,

$$\begin{aligned} \left| \sum_{j=1}^n \langle u, f_j \rangle \langle v, f_j \rangle - \langle u, Cv \rangle \right| &= |\mathbb{E} \langle u, X_n \rangle \langle v, X_n \rangle - \mathbb{E} \langle u, Y \rangle \langle v, Y \rangle| \\ &= |\mathbb{E} \langle u, Y - X_n \rangle \langle v, Y - X_n \rangle| \leq a^2 \mathbb{E} \|Y - X_n\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

uniformly in $u, v \in \{y \in E^* : \|y\| \leq a\}$. (v) \Rightarrow (iv) \Rightarrow (iii) is obvious. (iii) \Rightarrow (i). For every $u \in G$,

$$\mathbb{E} \exp(i \langle u, X_n \rangle) = \exp\left(-\sum_{j=1}^n \langle u, f_j \rangle^2 / 2\right) \rightarrow \exp(-\langle u, Cu \rangle / 2) = \mathbb{E} \exp(i \langle u, X \rangle).$$

The assertion (i) follows from the Ito-Nisio theorem (cf. [20], p. 271). (i) \Rightarrow (ii) \Rightarrow (iii) is obvious. \diamond

Note that the preceding lemma shows in particular that $(f_j)_{j \geq 1}$ is admissible for X if and only if $(f_{\sigma(j)})_{j \geq 1}$ is admissible for X for (some) every permutation σ of \mathbb{N} so that $\sum_j \xi_j f_j$ converges unconditionally a.s. in E for such sequences and all the a.s. limits under permutations of \mathbb{N} have distribution \mathbb{P}_X .

It is also an immediate consequence of Lemma 1(v) that admissible sequences (f_j) satisfy $\|f_j\| \rightarrow 0$.

The covariance operator admits factorizations $C = SS^*$, where $S : K \rightarrow E$ is a linear continuous operator and $(K, (\cdot, \cdot)_K)$ a real separable Hilbert space, which provide an useful tool for expansions. It is convenient to allow that S is not injective. One gets

$$\begin{aligned} S(K) &= H, & (2.6) \\ (Sk_1, Sk_2)_H &= (k_1, k_2)_K, k_1 \in K, k_2 \in (\ker S)^\perp, \\ \|S\| &= \|S^*\| = \|C\|^{1/2}, \\ \overline{S^*(E^*)} &= (\ker S)^\perp \text{ in } K, \\ (\ker S^*)^\perp &:= \{x \in E : \langle u, x \rangle = 0 \forall u \in \ker S^*\} = \overline{H} \text{ in } E. \end{aligned}$$

Proposition 1 Let $C = SS^*$, $S : K \rightarrow E$ be a factorization of C and let (e_j) be an orthonormal system in K satisfying $(\ker S)^\perp \subset \overline{\text{span}}\{e_j : j = 1, 2, \dots\}$. Then $(S(e_j))$ is admissible for X .

Proof. Clearly (e_j) is an orthonormal basis of $K_0 := \overline{\text{span}}\{e_j : j = 1, 2, \dots\}$. Since by (2.6)

$$S^*(E^*) \subset (\ker S)^\perp \subset K_0,$$

one obtains for every $u \in E^*$, by the Parseval identity,

$$\sum_j \langle u, Se_j \rangle^2 = \sum_j (S^*u, e_j)_K^2 = \|S^*u\|_K^2 = \langle u, Cu \rangle.$$

The assertion follows from Lemma 1. \diamond

EXAMPLES • Let $S : H \rightarrow E$ be the inclusion map. Then $C = SS^*$. Consequently, every orthonormal basis (f_j) of H is admissible for X .

• Let K be the closure of E^* in $L^2(\mathbb{P}_X)$ and $S : K \rightarrow E, Sk = \mathbb{E}k(X)X$. Then S is injective and $S^* : E^* \rightarrow K$ is the natural embedding. Thus $C = SS^*$. (K is sometimes called the energy space of X .) One obtains

$$H = S(K) = \{\mathbb{E}k(X)X : k \in K\}$$

and

$$(\mathbb{E}k_1(X)X, \mathbb{E}k_2(X)X)_H = \int k_1 k_2 d\mathbb{P}_X.$$

• Let E be a Hilbert space, $K = E$ and $S = C^{1/2}$. Then $C = SS^* = S^2$ and $(\ker S)^\perp = \overline{H}$. Consequently, if (e_j) is an orthonormal basis of the Hilbert subspace \overline{H} of E consisting of eigenvectors of C and (λ_j) the corresponding nonzero eigenvalues, then $(\sqrt{\lambda_j}e_j)$ is admissible for X and an orthonormal basis of $(H, (\cdot, \cdot)_H)$ (Karhunen-Loève basis).

In the following proposition it is demonstrated that any admissible set for X arises in the manner of Proposition 1.

Proposition 2 *Assume that $(f_j)_{j \geq 1}$ is admissible for X . Let K be an infinite dimensional real separable Hilbert space and $(e_j)_{j \geq 1}$ an orthonormal basis of K . Then there is a factorization $C = SS^*, S : K \rightarrow E$ such that $S(e_j) = f_j$ for every j .*

Proof. First, observe that $\sum_{j=1}^{\infty} c_j f_j$ converges in E for every $(c_j)_j \in l_2(\mathbb{N})$. In fact, using Lemma 1,

$$\begin{aligned} \left\| \sum_{j=n}^{n+m} c_j f_j \right\|^2 &= \sup_{\|u\| \leq 1} \langle u, \sum_{j=n}^{n+m} c_j f_j \rangle^2 \\ &\leq \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \sum_{j=1}^{\infty} \langle u, f_j \rangle^2 \\ &= \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \langle u, Cu \rangle \\ &= \sum_{j=n}^{n+m} c_j^2 \|C\| \rightarrow 0, \quad n, m \rightarrow \infty \end{aligned}$$

and thus the sequence is Cauchy in E . Now define $S : K \rightarrow E$ by

$$S(k) := \sum_{j=1}^{\infty} (k, e_j)_K f_j$$

where $\sum (k, e_j)_K f_j$ converges in E since $((k, e_j)_K)_j \in l_2(\mathbb{N})$. S is obviously linear. Moreover, for $k \in K$, using again Lemma 1,

$$\begin{aligned} \|Sk\|^2 &= \sup_{\|u\| \leq 1} \langle u, Sk \rangle^2 \\ &= \sup_{\|u\| \leq 1} \left(\sum_{j=1}^{\infty} (k, e_j)_K \langle u, f_j \rangle \right)^2 \\ &\leq \|k\|_K^2 \|C\|. \end{aligned}$$

Consequently, S is continuous and $S(e_j) = f_j$ for every j . Finally, $S^*(u) = \sum_{j=1}^{\infty} \langle u, f_j \rangle e_j$ and hence

$$SS^*u = \sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu$$

for every $u \in E^*$ by Lemma 1. \diamond

Corollary 1 Let $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$ be admissible for X . Then $\{f_j : j \geq 1\} \subset U_H, \overline{\text{span}}^{(H)}\{f_j : j \geq 1\} = H, \sum (f_j, h)_H f_j$ converges in H and

$$\sum_{j=1}^{\infty} (f_j, h)_H f_j = h$$

for every $h \in H$. Furthermore, (f_j) is an orthonormal basis of H if and only if $\|f_j\|_H = 1$ for every j .

Proof. Choose a factorization $C = SS^*, S : K \rightarrow H$ and an orthonormal basis $(e_j)_{j \geq 1}$ of K as in Proposition 2. The inclusion $\{f_j : j \geq 1\} \subset U_H$ and the assertion concerning the $\|\cdot\|_H$ -closed span follow immediately from (2.6). Since S is continuous as operator from K onto H , $\sum (e_j, k)_K S e_j$ converges in H ,

$$\sum_{j=1}^{\infty} (e_j, k)_K f_j = Sk$$

and since $(e_j, k)_K = (f_j, Sk)_H$ for every $k \in (\ker S)^\perp$, we obtain

$$\sum_{j=1}^{\infty} (f_j, h)_H f_j = h$$

for every $h \in H$. Write $e_j = e_j^{(1)} + e_j^{(2)}$ with $e_j^{(1)} \in (\ker S)^\perp$ and $e_j^{(2)} \in \ker S$. If $\|f_j\|_H = 1$ for every j , then using (2.6),

$$\|e_j^{(1)}\|_K = \|S e_j^{(1)}\|_H = \|f_j\|_H = 1 = \|e_j\|_K$$

and thus $e_j^{(2)} = 0$ for every j . This implies that S is injective so that for $i \neq j$,

$$(f_i, f_j)_H = (e_i, e_j)_K = 0. \quad \diamond$$

Let F be a further separable Banach space and $V : E \rightarrow F$ a linear continuous operator. Then $V(X)$ is centered Gaussian with covariance operator

$$C_{V(X)} = V C_X V^*$$

and

$$H_{V(X)} = V(H_X), \quad (2.7)$$

$$(Vh_1, Vh_2)_{H_{V(X)}} = (h_1, h_2)_{H_X}, h_1 \in H_X, h_2 \in (\ker(V|_{H_X}))^\perp.$$

If $(f_j)_{j \geq 1}$ is admissible for X in E , then clearly $(V(f_j))_{j \geq 1}$ is admissible for $V(X)$ in F . The following proposition contains the converse for injective operators V .

Proposition 3 Assume that $V : E \rightarrow F$ is injective. Let $(g_j)_{j \geq 1}$ be an admissible sequence for $V(X)$ in F . Then there exists a sequence $(f_j)_{j \geq 1}$ in E which is admissible for X such that $V(f_j) = g_j$ for every j .

Proof. By Corollary 1 and (2.7), we have $\{g_j : j \geq 1\} \subset V(H_X)$ so that $g_j = V(f_j)$ with $f_j \in H_X \subset E$. Using Lemma 1, we obtain for every $y \in F^*$,

$$\sum_{j=1}^{\infty} \langle V^* y, f_j \rangle^2 = \sum_{j=1}^{\infty} \langle y, V(f_j) \rangle^2 = \langle y, C_{V(X)} y \rangle = \langle V^* y, C_X V^* y \rangle$$

and hence

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \langle u, C_X u \rangle$$

for every $u \in V^*(F^*)$. Since $V^*(F^*) \subset E^*$ is separating if (and only if) V is injective it follows from Lemma 1 that (f_j) is admissible for X . \diamond

Almost sure representations of X can be obtained under a suitable restriction on the admissible sequence.

Proposition 4 *Assume that the admissible sequence $(f_j)_{j \geq 1}$ for X in E is an orthonormal basis of H (cf. Corollary 1). Then there exist i.i.d. $\mathcal{N}(0, 1)$ -distributed random variables η_1, η_2, \dots such that*

$$X = \sum_{j=1}^{\infty} \eta_j f_j \quad a.s.$$

Proof. Choose the energy factorization $C = SS^*$ with K the closure of E^* in $L^2(\mathbb{P}_X)$ and $Sk = \mathbb{E}k(X)X$. Let $e_j := S^{-1}(f_j)$ and $\eta_j := e_j(X)$. Then $(e_j)_{j \geq 1}$ is an orthonormal basis of K and thus $\eta_j, j \geq 1$ are i.i.d. $\mathcal{N}(0, 1)$ -distributed. Consequently,

$$\sum_{j=1}^{\infty} \eta_j f_j \text{ converges a.s. in } E.$$

For every $u \in E^*$, we have

$$\langle u, \sum_{j=1}^{\infty} \eta_j f_j \rangle = \sum_{j=1}^{\infty} \eta_j (S^*u, e_j)_K = S^*u(X) = \langle u, X \rangle \text{ a.s.}$$

Since there is a countable separating subset of E^* this yields

$$X = \sum_{j=1}^{\infty} \eta_j f_j \quad a.s. \quad \diamond$$

2.2 Continuous Gaussian processes

Now let I be a compact metric space and $X = (X_t)_{t \in I}$ be a real pathwise continuous centered Gaussian process. Let $E := \mathcal{C}(I)$ be equipped with the sup-norm $\|x\| = \sup_{t \in I} |x(t)|$. Then X can be seen as $\mathcal{C}(I)$ -valued Gaussian random vector and the covariance operator $C : \mathcal{C}(I)^* \rightarrow \mathcal{C}(I)$ takes the form

$$\begin{aligned} Cu(t) &= \langle \delta_t, Cu \rangle = \langle C\delta_t, u \rangle \\ &= \langle \mathbb{E}X_t X, u \rangle = \int_I \mathbb{E}X_t X_s du(s). \end{aligned} \quad (2.8)$$

Factorizations of C can be obtained as follows. For Hilbert spaces K_i , let $\oplus_{i=1}^m K_i$ denote the Hilbertian (or l_2 -)direct sum.

Lemma 2 *For $i \in \{1, \dots, m\}$, let K_i be a real separable Hilbert space. Assume the representation*

$$\mathbb{E}X_s X_t = \sum_{i=1}^m (g_s^i, g_t^i)_{K_i}, \quad s, t \in I,$$

for vectors $g_t^i \in K_i$. Then

$$S : \bigoplus_{i=1}^m K_i \rightarrow \mathcal{C}(I), Sk(t) := \sum_{i=1}^m (g_t^i, k_i)_{K_i}$$

is a linear continuous operator, $(\ker S)^\perp = \overline{\text{span}}\{(g_t^1, \dots, g_t^m) : t \in I\}$ and $C = SS^*$.

Proof. Let $K := \bigoplus_{i=1}^m K_i$ and $g_t := (g_t^1, \dots, g_t^m)$. Then $\mathbb{E} X_s X_t = (g_s, g_t)_K$ and $Sk(t) = (g_t, k)_K$. First, observe that

$$\sup_{t \in I} \|g_t\|_K \leq \|C\|^{1/2} < \infty.$$

Indeed, for every $t \in I$, by (2.8),

$$\|g_t\|_K^2 = \mathbb{E} X_t^2 = \langle \delta_t, C \delta_t \rangle \leq \|C\|.$$

The function Sk is continuous for $k \in \text{span}\{g_s : s \in I\}$. This easily implies that Sk is continuous for every $k \in \overline{\text{span}}\{g_s : s \in I\}$ and thus for every $k \in K$. S is obviously linear and

$$\|Sk\| = \sup_{t \in I} |(g_t, k)_K| \leq \|C\|^{1/2} \|k\|_K.$$

Finally, $S^*(\delta_t) = g_t$ so that

$$SS^* \delta_t(s) = Sg_t(s) = \mathbb{E} X_s X_t = C \delta_t(s)$$

for every $s, t \in I$. Consequently, for every $u \in \mathcal{C}(I)^*, t \in I$,

$$\begin{aligned} SS^* u(t) &= \langle SS^* u, \delta_t \rangle = \langle u, SS^* \delta_t \rangle \\ &= \langle u, C \delta_t \rangle = \langle Cu, \delta_t \rangle = Cu(t) \end{aligned}$$

and hence $C = SS^*$. \diamond

EXAMPLE Let K be the first Wiener chaos, that is $K = \overline{\text{span}}\{X_t : t \in I\}$ in $L^2(\mathbb{P})$ and $g_t = X_t$. Then $Sk = \mathbb{E} kX$ and S is injective. If for instance $X = W$ (Brownian motion) and $I = [0, T]$, then

$$K = \left\{ \int_0^T f(s) dW_s : f \in L^2([0, T], dt) \right\}.$$

We derive from the preceding lemma and Proposition 1 the following corollary.

Corollary 2 *Assume the situation of Lemma 2. Let $(e_j^i)_j$ be an orthonormal system in K_i satisfying $\{g_t^i : t \in I\} \subset \overline{\text{span}}\{e_j^i : j = 1, 2, \dots\}$. Then, $(S_i(e_j^i))_{1 \leq i \leq m, j}$ is admissible for X , where $S_i k(t) = (g_t^i, k)_{K_i}$.*

The next corollary shows that the Karhunen-Loève expansion of X in some Hilbert space $L^2(I, \mu)$ already converges uniformly in $t \in I$. It appears as special case of Proposition 3.

Corollary 3 *Let μ be a finite Borel measure on I with $\text{supp}(\mu) = I$ and let $V : \mathcal{C}(I) \rightarrow L^2(I, \mu)$ denote the natural (injective) embedding. Let $(g_j)_{j \geq 1}$ be admissible for $V(X)$ in $L^2(I, \mu)$. Then there exists a sequence $(f_j)_{j \geq 1}$ in $\mathcal{C}(I)$ which is admissible for X such that $V(f_j) = g_j$ for every j .*

A final corollary is as follows.

Corollary 4 Let $(f_j)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ such that

$$\sum_{j=1}^{\infty} f_j(t)^2 < \infty \text{ for every } t \in I.$$

If the process Y with $Y_t = \sum_{j=1}^{\infty} \xi_j f_j(t)$ has a pathwise continuous modification X , then (f_j) is admissible for X and $X = \sum_{j=1}^{\infty} \xi_j f_j$ a.s..

Proof. Notice that $\sum \xi_j f_j(t)$ converges a.s. in \mathbb{R} and Y is a centered Gaussian process. Hence X is centered Gaussian. We have

$$\langle u, \sum_{j=1}^n \xi_j f_j \rangle \rightarrow \langle u, X \rangle \text{ a.s.}$$

for every $u \in G := \text{span}\{\delta_t : t \in I\}$. Since G is a separating subspace of $\mathcal{C}(I)^*$ the assertion follows from Lemma 1. \diamond

The admissibility feature is stable under tensor products. For $i \in \{1, \dots, d\}$, let I_i be a compact metric space and $X^i = (X_t^i)_{t \in I_i}$ a continuous centered Gaussian process. Set $I := \prod_{i=1}^d I_i$ and let $X = (X_t)_{t \in I}$ be a continuous centered Gaussian process with covariance function

$$\mathbb{E} X_s X_t = \prod_{i=1}^d \mathbb{E} X_{s_i}^i X_{t_i}^i, s, t \in I. \quad (2.9)$$

For instance, $X := \widehat{\otimes_{i=1}^d X^i}$ satisfies (2.9) provided X^1, \dots, X^d are independent. For real separable Hilbert spaces K_i , let $\widehat{\otimes_{i=1}^d K_i}$ denote the d -fold Hilbertian tensor product.

Proposition 5 For $i \in \{1, \dots, d\}$, let $(f_j^i)_{j \geq 1}$ be an admissible sequence for X^i in $\mathcal{C}(I_i)$. Then

$$(\widehat{\otimes_{i=1}^d f_{j_i}^i})_{\underline{j}=(j_1, \dots, j_d) \in \mathbb{N}^d}$$

is admissible for X with covariance (2.9) in $\mathcal{C}(I)$. Furthermore, if $C_{X^i} = S_i S_i^*$, $S_i : K_i \rightarrow \mathcal{C}(I_i)$ is a factorization of C_{X^i} , then $\widehat{\otimes_{i=1}^d S_i} : \widehat{\otimes_{i=1}^d K_i} \rightarrow \mathcal{C}(I)$ provides a factorization of C_X .

Proof. For $i \in \{1, \dots, d\}$, let K_i be a real separable Hilbert space and $(e_{j_i}^i)_{j_i}$ an orthonormal basis of K_i . Then $(\widehat{\otimes_{i=1}^d e_{j_i}^i})_{\underline{j}}$ is an orthonormal basis of $K := \widehat{\otimes_{i=1}^d K_i}$.

If $C_{X^i} = S_i S_i^*$, $S_i : K_i \rightarrow \mathcal{C}(I_i)$ is a factorization of C_{X^i} , set $g_t^i := S_i^* \delta_t$, $t \in I_i$. Then $\mathbb{E} X_s^i X_t^i = (g_s^i, g_t^i)_{K_i}$ and hence, by (2.9)

$$\mathbb{E} X_s X_t = \prod_{i=1}^d (g_{s_i}^i, g_{t_i}^i)_{K_i} = (\widehat{\otimes_{i=1}^d g_{s_i}^i}, \widehat{\otimes_{i=1}^d g_{t_i}^i})_K, s, t \in I.$$

Consequently, by Lemma 2

$$U : K \rightarrow \mathcal{C}(I), U k(t) = (\widehat{\otimes_{i=1}^d g_{t_i}^i}, k)_K$$

provides a factorization of C_X . Since

$$\begin{aligned} U(\widehat{\otimes_{i=1}^d e_{j_i}^i})(t) &= \prod_{i=1}^d (g_{t_i}^i, e_{j_i}^i)_{K_i} \\ &= \prod_{i=1}^d S_i e_{j_i}^i(t_i) = \widehat{\otimes_{i=1}^d (S_i e_{j_i}^i)}(t) \\ &= (\widehat{\otimes_{i=1}^d S_i})(\widehat{\otimes_{i=1}^d e_{j_i}^i})(t), t \in I, \end{aligned}$$

we obtain $U = \widehat{\otimes_{i=1}^d S_i}$ and thus $\widehat{\otimes_{i=1}^d S_i}$ provides a factorization of C_X .

If $(f_j^i)_{j \geq 1}$ is admissible for X^i , then by Proposition 2 assuming now that K_i is infinite dimensional, there is a factorization $C_{X^i} = T_i T_i^*$, $T_i : K_i \rightarrow \mathcal{C}(I_i)$ such that $T_i(e_j^i) = f_j^i$ for every j . Since $\otimes_{i=1}^d T_i : K \rightarrow \mathcal{C}(I)$ provides a factorization of C_X as shown above and $(\otimes_{i=1}^d T_i)(\otimes_{i=1}^d e_j^i) = \otimes_{i=1}^d f_j^i$, it follows from Proposition 1 that $(\otimes_{i=1}^d f_j^i)_{j \in \mathbb{N}^d}$ is admissible for X . \diamond

The following examples are further investigated in Section 4.

EXAMPLES • Let $W = (W_t)_{t \in [0, T]}$ be a *standard Brownian motion*. Since $\mathbb{E}W_s W_t = s \wedge t = \int_0^T 1_{[0, s]}(u) 1_{[0, t]}(u) du$, the (injective) operator

$$S : L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), Sk(t) = \int_0^t k(s) ds$$

provides a factorization of C_W so that we can apply Corollary 2. The orthonormal basis $e_j(t) = \sqrt{2/T} \cos(\pi(j - 1/2)t/T)$, $j \geq 1$ of $L^2([0, T], dt)$ yields the admissible sequence

$$f_j(t) = S e_j(t) = \frac{\sqrt{2T}}{\pi(j - 1/2)} \sin\left(\frac{\pi(j - 1/2)t}{T}\right), j \geq 1 \quad (2.10)$$

for W (Karhunen-Loève basis of H_W) and $e_j(t) = \sqrt{2/T} \sin(\pi jt/T)$ yields

$$f_j(t) = \frac{\sqrt{2T}}{\pi j} (1 - \cos(\frac{\pi jt}{T})), j \geq 1. \quad (2.11)$$

Then

$$\begin{aligned} f_j^1(t) &= \frac{\sqrt{T}}{\pi(j - 1/2)} \sin\left(\frac{\pi(j - 1/2)t}{T}\right), j \geq 1 \\ f_j^2(t) &= \frac{\sqrt{T}}{\pi j} (1 - \cos(\frac{\pi jt}{T})), j \geq 1 \end{aligned} \quad (2.12)$$

is also admissible for W . The trigonometric basis $e_0(t) = 1/\sqrt{T}$, $e_{2j}(t) = \sqrt{2/T} \cos(2\pi jt/T)$, $e_{2j-1}(t) = \sqrt{2/T} \sin(2\pi jt/T)$ yields the admissible sequence

$$\begin{aligned} f_0(t) &= \frac{t}{\sqrt{T}}, f_{2j}(t) = \frac{\sqrt{T}}{\sqrt{2\pi j}} \sin\left(\frac{2\pi jt}{T}\right), \\ f_{2j-1}(t) &= \frac{\sqrt{T}}{\sqrt{2\pi j}} (1 - \cos(\frac{2\pi jt}{T})), j \geq 1 \end{aligned} \quad (2.13)$$

(Paley-Wiener basis of H_W).

• We consider the Dzaparidze-van Zanten expansion of the *fractional Brownian motion* $X = (X_t)_{t \in [0, T]}$ with Hurst index $\rho \in (0, 1)$ and covariance function

$$\mathbb{E} X_s X_t = \frac{1}{2} (s^{2\rho} + t^{2\rho} - |s - t|^{2\rho}).$$

These authors discovered in [8] for $T = 1$ a time domain representation

$$\mathbb{E} X_s X_t = (g_s^1, g_t^1)_K + (g_s^2, g_t^2)_K$$

with $K = L^2([0, 1], dt)$ and kernels $g_t^i \in L^2([0, 1], dt)$. Hence by Lemma 2, the operator

$$S : L^2([0, 1], dt) \oplus L^2([0, 1], dt) \rightarrow \mathcal{C}([0, 1]), S(k_1, k_2)(t) = \int_0^1 g_t^1(s) k_1(s) ds + \int_0^1 g_t^2(s) k_2(s) ds$$

provides a factorization of C_X so that for every pair of orthonormal bases $(e_j^1)_{j \geq 1}$ and $(e_j^2)_{j \geq 1}$ of $L^2([0, 1], dt)$,

$$f_j^i(t) = \int_0^1 g_t^i(s) e_j^i(s) ds, \quad j \geq 1, i = 1, 2$$

is admissible in $\mathcal{C}([0, 1])$ for X . By Corollary 2, this is a consequence of the above representation of the covariance function (and needs no extra work). Then Dzaparidze and van Zanten [8] could calculate f_j^i explicitly for the Fourier-Bessel basis of order $-\rho$ and $1 - \rho$, respectively and arrived at the admissible family in $\mathcal{C}([0, 1])$

$$\begin{aligned} f_j^1(t) &= \frac{c_\rho \sqrt{2}}{|J_{1-\rho}(x_j)| x_j^{\rho+1}} \sin(x_j t), j \geq 1 \\ f_j^2(t) &= \frac{c_\rho \sqrt{2}}{|J_{-\rho}(y_j)| y_j^{\rho+1}} (1 - \cos(y_j t)), j \geq 1 \end{aligned}$$

where J_ν denotes the Bessel function of the first kind of order ν , $0 < x_1 < x_2 < \dots$ are the positive zeros of $J_{-\rho}$, $0 < y_1 < y_2 < \dots$ the positive zeros of $J_{1-\rho}$ and $c_\rho^2 = \Gamma(1 + 2\rho) \sin(\pi\rho)/\pi$. Consequently, by self-similarity of X , the sequence

$$\begin{aligned} f_j^1(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{1-\rho}(x_j)| x_j^{\rho+1}} \sin\left(\frac{x_j t}{T}\right), j \geq 1 \\ f_j^2(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{-\rho}(y_j)| y_j^{\rho+1}} (1 - \cos\left(\frac{y_j t}{T}\right)), j \geq 1 \end{aligned} \tag{2.14}$$

in $\mathcal{C}([0, T])$ is admissible for X . Using Lemma 1, one can deduce (also without extra work)

$$\mathbb{E} X_s X_t = \sum_{j=1}^{\infty} f_j^1(s) f_j^1(t) + \sum_{j=1}^{\infty} f_j^2(s) f_j^2(t)$$

uniformly in $(s, t) \in [0, T]^2$.

In the ordinary Brownian motion case $\rho = 1/2$, (2.14) coincides with (2.12). The interesting extension of (2.13) to fractional Brownian motions is discussed in [9].

- Let $X = (X_t)_{t \in [0, T]}$ be a *Brownian bridge* with covariance

$$\mathbb{E} X_s X_t = s \wedge t - \frac{st}{T} = \int_0^T (1_{[0, s]}(u) - \frac{s}{T})(1_{[0, t]}(u) - \frac{t}{T}) du.$$

By Lemma 2, the operator

$$S : L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), Sk(t) = \int_0^t k(s) ds - \frac{t}{T} \int_0^T k(s) ds$$

provides a factorization of C_X and $\ker S = \text{span}\{1_{[0, T]}\}$. The choice $e_j(t) = \sqrt{2/T} \cos(\pi j t / T)$, $j \geq 1$, of an orthonormal basis of $(\ker S)^\perp$ yields admissibility of

$$f_j(t) = S e_j(t) = \frac{\sqrt{2T}}{\pi j} \sin\left(\frac{\pi j t}{T}\right), \quad j \geq 1, \tag{2.15}$$

for X (Karhunen-Loève basis of H_X).

- One considers the *stationary Ornstein-Uhlenbeck process* as the solution of the Langevin equation

$$dX_t = -\beta X_t dt + \sigma W_t, t \in [0, T]$$

with X_0 independent of W and $N(0, \frac{\sigma^2}{2\beta})$ -distributed, $\sigma > 0, \beta > 0$. It admits the explicit representation

$$X_t = e^{-\beta t} X_0 + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s$$

and

$$\mathbb{E} X_s X_t = \frac{\sigma^2}{2\beta} e^{-\beta|s-t|} = \frac{\sigma^2}{2\beta} e^{-\beta(s+t)} + \sigma^2 e^{-\beta(s+t)} \int_0^{s \wedge t} e^{2\beta u} du.$$

Thus the (injective) operator

$$S : \mathbb{R} \oplus L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), S(c, k)(t) = \frac{c\sigma}{\sqrt{2\beta}} e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-s)} k(s) ds$$

provides a factorization of C_X so that for every orthonormal basis $(e_j)_{j \geq 1}$ of $L^2([0, T], dt)$, the functions

$$f_0(t) = \frac{\sigma}{\sqrt{2\beta}} e^{-\beta t}, f_j(t) = \sigma \int_0^t e^{-\beta(t-s)} e_j(s) ds, j \geq 1, \quad (2.16)$$

provide an admissible sequence for X .

Another representation is given by the Lamperti transformation $X = V(W)$ for the linear continuous operator

$$V : \mathcal{C}([0, e^{2\beta T}]) \rightarrow \mathcal{C}([0, T]), Vx(t) = \frac{\sigma}{\sqrt{2\beta}} e^{-\beta t} x(e^{2\beta t}).$$

The admissible sequence (f_j) in $\mathcal{C}([0, e^{2\beta T}])$ for $(W_t)_{t \in [0, e^{2\beta T}]}$ from (2.10) yields the admissible sequence

$$\tilde{f}_j(t) = V f_j(t) = \frac{\sigma}{\sqrt{\beta} \pi(j-1/2)} e^{\beta(T-t)} \sin(\pi(j-1/2)e^{-2\beta(T-t)}), j \geq 1 \quad (2.17)$$

for X .

- Sheet versions can be deduced from Proposition 5.

3 High-resolution product quantization

We investigate product functional quantization of centered continuous Gaussian processes $X = (X_t)_{t \in I}$ on $I = [0, T]^d$ in the space $E = \mathcal{C}(I)$ equipped with the sup-norm $\|x\| = \sup_{t \in I} |x(t)|$. Recall that the minimal N th product quantization error is given by

$$e_{N,r}(X)^{(pq)} := e_{N,r}(X, \mathcal{C}(I))^{(pq)} = \inf \left\{ \left(\mathbb{E} \left\| \sum_{j=1}^{\infty} \xi_j f_j - \hat{X} \right\|^r \right)^{1/r} : (f_j)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}} \text{ admissible for } X, \right. \\ \left. \hat{X} \text{ } L^r\text{-product } N\text{-quantization w.r.t. } (f_j) \right\}$$

where ξ_1, ξ_2, \dots are i.i.d. $\mathcal{N}(0, 1)$ -distributed random variables. The subsequent setting comprises a broad class of processes (see Section 4).

Let $(f_j)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ satisfy the following assumptions.

(A1) $\|f_j\| \leq C_1 j^{-\vartheta} \log(1+j)^\gamma$ for every $j \geq 1$ with $\vartheta > 1/2, \gamma \geq 0$ and $C_1 < \infty$.

(A2) f_j is a -Hölder-continuous and $[f_j]_a \leq C_2 j^b$ for every $j \geq 1$ with $a \in (0, 1], b \in \mathbb{R}$ and $C_2 < \infty$, where

$$[f]_a = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^a}$$

(and $|t|$ denotes the l_2 -norm of $t \in \mathbb{R}^d$).

In the sequel finite constants depending only on the parameters $T, \vartheta, \gamma, a, b, C_1, C_2, d$ and r are denoted by C and may be different from one formula to another one. Other dependencies are explicitly indicated.

First observe that Corollary 4 applies. By (A1), $\sum_{j=1}^{\infty} f_j(t)^2 \leq \sum_{j=1}^{\infty} \|f_j\|^2 < \infty$ for every $t \in I$, so that we can define a centered Gaussian process Y by $Y_t := \sum_{j=1}^{\infty} \xi_j f_j(t)$. Using (A1) and (A2) we have for $\rho \in (0, 1]$

$$\begin{aligned} |f_j(s) - f_j(t)| &= |f_j(s) - f_j(t)|^\rho |f_j(s) - f_j(t)|^{1-\rho} \\ &\leq ([f_j]_a |s - t|^a)^\rho (2\|f_j\|)^{1-\rho} \\ &\leq C_\rho j^{\rho(b+\vartheta)-\vartheta} \log(1+j)^{\gamma(1-\rho)} |s - t|^{a\rho} \end{aligned}$$

and hence

$$\sum_{j=1}^{\infty} [f_j]_{a\rho}^2 < \infty \text{ for every } \rho < \frac{\vartheta - 1/2}{(b + \vartheta)_+}.$$

This yields

$$\mathbb{E}|Y_s - Y_t|^2 = \sum_{j=1}^{\infty} |f_j(s) - f_j(t)|^2 \leq \left(\sum_{j=1}^{\infty} [f_j]_{a\rho}^2 \right) |s - t|^{2a\rho}$$

and using the Gaussian feature of Y , we obtain from the Kolmogorov-criterion that Y has a continuous modification X . Consequently by Corollary 4, (f_j) is admissible for X and

$$X = \sum_{j=1}^{\infty} \xi_j f_j \quad \text{a.s.}, \quad (3.1)$$

$(\mathbb{E}|X_s - X_t|^2)^{1/2} \leq C_\rho |s - t|^{a\rho}$ for every $s, t \in I$ and every $\rho \in (0, 1]$ with $\rho < \frac{\vartheta - 1/2}{(b + \vartheta)_+}$. An immediate consequence is the continuous embedding of the Cameron-Martin space H_X into the Hölder-space $\mathcal{C}^{a\rho}(I)$.

Proposition 6 *We have $H_X \subset \mathcal{C}^{a\rho}(I)$ and the inclusion map is continuous for every $\rho \in (0, 1]$ with $\rho < \frac{\vartheta - 1/2}{(b + \vartheta)_+}$, where $\mathcal{C}^{a\rho}(I)$ is equipped with the norm $\| \|f\| \| = \|f\| + [f]_{a\rho}$.*

Proof. Let $h \in H = H_X$. By (2.2) and (2.8), we have

$$h(t) = \langle \delta_t, h \rangle = (h, C_X \delta_t)_H, C_X \delta_t = \mathbb{E} X_t X$$

and hence for every $u \in \mathcal{C}(I)^*$,

$$|\langle u, C_X(\delta_s - \delta_t) \rangle| = |\mathbb{E}(X_s - X_t) \langle u, X \rangle| \leq (\mathbb{E}(X_s - X_t)^2)^{1/2} \langle u, C_X u \rangle^{1/2}.$$

Consequently, using (2.3) and (3.1)

$$\begin{aligned} |h(s) - h(t)| &= |(h, C_X(\delta_s - \delta_t))_H| \\ &\leq \|h\|_H \|C_X(\delta_s - \delta_t)\|_H \\ &= \|h\|_H \sup\{|\langle u, C_X(\delta_s - \delta_t) \rangle| : \langle u, C_X u \rangle \leq 1\} \\ &\leq \|h\|_H (\mathbb{E}|X_s - X_t|^2)^{1/2} \\ &\leq C_\rho \|h\|_H |s - t|^{a\rho}. \end{aligned}$$

Thus in view of (2.4),

$$\|h\| \leq \|C_X\|^{1/2} \|h\|_H + C_\rho \|h\|_H. \quad \diamond$$

The design of a (scalar) product quantization for X is based on optimal quantizing the coefficients ξ_j in the expansion (3.1) of X . More precisely this yields product quantization with respect to the admissible sequence (f_j) . Let $N \in \mathbb{N}$, $r \in (0, \infty)$ and fix $m, N_1, \dots, N_m \in \mathbb{N}$ with $\prod_{j=1}^m N_j \leq N$. Let $\alpha_j \subset \mathbb{R}$ be the L^r -optimal N_j -quantizer for ξ_j and let $\hat{\xi}_j := \hat{\xi}_j^{\alpha_j}$ be the Voronoi quantization of ξ_j induced by α_j . Then define a L^r -product quantization of X by

$$\hat{X}^N := \sum_{j=1}^m \hat{\xi}_j f_j. \quad (3.2)$$

It is clear that $\text{card}(\hat{X}(\Omega)) \leq N$ so that \hat{X}^N is an N -quantization. If

$$\hat{\xi}_j = \sum_{b \in \alpha_j} b 1_{C_b(\alpha_j)}(\xi_j),$$

then

$$\hat{X}^N = \sum_{a \in \prod_{j=1}^m \alpha_j} \left(\sum_{j=1}^m a_j f_j \right) \prod_{j=1}^m 1_{C_{a_j}(\alpha_j)}(\xi_j) \quad (3.3)$$

where $a = (a_1, \dots, a_m) \in \prod_{j=1}^m \alpha_j$. Observe that \hat{X}^N is not a Voronoi quantization since it is based on the (much less complicated) Voronoi partitions for ξ_j , $j \leq m$.

For $r \in [1, \infty)$, the quantization error induced by \hat{X}^N satisfies

$$\begin{aligned} (\mathbb{E} \|X - \hat{X}^N\|^r)^{1/r} &= \|X - \hat{X}^N\|_{L_E^r} \\ &\leq \left\| \sum_{j=1}^m (\xi_j - \hat{\xi}_j) f_j \right\|_{L_E^r} + \left\| \sum_{j \geq m+1} \xi_j f_j \right\|_{L_E^r} \\ &\leq \sum_{j=1}^m \|\xi_j - \hat{\xi}_j\|_{L^r} \|f_j\| + \left\| \sum_{j \geq m+1} \xi_j f_j \right\|_{L_E^r} \end{aligned}$$

so that

$$(\mathbb{E} \|X - \hat{X}^N\|^r)^{1/r} \leq \sum_{j=1}^m \|f_j\| e_{N_j, r}(\mathcal{N}(0, 1)) + (\mathbb{E} \left\| \sum_{j \geq m+1} \xi_j f_j \right\|^r)^{1/r}. \quad (3.4)$$

For $r \in (0, 1]$, we have

$$\mathbb{E} \|X - \hat{X}^N\|^r \leq \sum_{j=1}^m \|f_j\|^r e_{N_j, r}(\mathcal{N}(0, 1))^r + \mathbb{E} \left\| \sum_{j \geq m+1} \xi_j f_j \right\|^r.$$

Now let us pass to the truncation error.

Theorem 1 *Assume that $(f_j)_{j \geq 1} \in \mathcal{C}(I)^\mathbb{N}$ satisfies (A1) and (A2). Then we have for every $n \geq 2$ and $r \in (0, \infty)$,*

$$(\mathbb{E} \left\| \sum_{j \geq n} \xi_j f_j \right\|^r)^{1/r} \leq \frac{C(\log n)^{\gamma+1/2}}{n^{\vartheta-1/2}}$$

and

$$(\mathbb{E} \left\| \sum_{j \geq n} \xi_j f_j \right\|^r)^{1/r} \leq \frac{C(\log n)^\gamma}{n^{\vartheta-1/2}} \text{ if } b + \vartheta \leq 0.$$

Proof. By equivalence of Gaussian moments

$$(\mathbb{E} \|\sum_{j \geq n} \xi_j f_j\|^r)^{1/r} \leq D \mathbb{E} \|\sum_{j \geq n} \xi_j f_j\| \quad (3.5)$$

for some universal constant D (cf. [14], Corollary 3.2). The upper estimate for $\mathbb{E} \|\sum_{j \geq n} \xi_j f_j\|$ is based on corresponding estimates for finite blocks of exponentially increasing length. For $m \geq 1$, set

$$Z = Z^{(m)} := \sum_{j=2^{m-1}+1}^{2^m} \xi_j f_j.$$

For a given $N \geq 1$ consider the grid $G_N = \{\frac{(2i-1)T}{2N} : i = 1, \dots, N\}^d$. Then

$$\|Z\| \leq \sup_{t \in G_N} |Z_t| + \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t|.$$

It follows from the Gaussian maximal inequality, that

$$\mathbb{E} \sup_{t \in G_N} |Z_t| \leq C \sqrt{\log(1 + N^d)} \sup_{t \in G_N} \sqrt{\mathbb{E} Z_t^2}.$$

Using (A1), we have for every $t \in I$,

$$\mathbb{E} Z_t^2 \leq \sum_{j=2^{m-1}+1}^{2^m} \|f_j\|^2 \leq C \sum_{j=2^{m-1}+1}^{2^m} j^{-2\vartheta} \log(1+j)^{2\gamma} \leq C 2^{m(1-2\vartheta)} m^{2\gamma}$$

so that

$$\mathbb{E} \sup_{t \in G_N} |Z_t| \leq C \sqrt{\log(1 + N)} 2^{-m(\vartheta-1/2)} m^\gamma.$$

Moreover, using (A2), we have for $|s - t| \leq CN^{-1}$

$$\begin{aligned} |Z_s - Z_t| &\leq \sum_{j=2^{m-1}+1}^{2^m} |\xi_j| |f_j(s) - f_j(t)| \\ &\leq C |s - t|^a \sum_{j=2^{m-1}+1}^{2^m} |\xi_j| [f_j]_a \\ &\leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} |\xi_j| j^b \end{aligned}$$

and hence

$$\mathbb{E} \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t| \leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} j^b \leq CN^{-a} 2^{m(1+b)}.$$

Thus we have established the estimate

$$\mathbb{E} \|Z^{(m)}\| \leq C (\sqrt{\log(1 + N)} 2^{-m(\vartheta-1/2)} m^\gamma + N^{-a} 2^{m(1+b)}). \quad (3.6)$$

As concerns the choice of N , set $N := [2^{um}] + 1$ with $u \in (0, \infty)$ satisfying $1 + b - au \leq \frac{1}{2} - \vartheta$. Then (3.6) reads

$$\mathbb{E} \|Z^{(m)}\| \leq C 2^{-m(\vartheta-1/2)} m^{\gamma+1/2}. \quad (3.7)$$

One notices that in case $b + \vartheta \leq -1/2$ one may choose $N = 1$ and obtains a power reduction from $m^{\gamma+1/2}$ to m^γ . This can be improved. In fact, we have

$$\begin{aligned}\mathbb{E}|Z_s - Z_t|^2 &= \sum_{j=2^{m-1}+1}^{2^m} |f_j(s) - f_j(t)|^2 \\ &\leq C|s - t|^{2a} \sum_{j=2^{m-1}+1}^{2^m} j^{2b} \leq C|s - t|^{2a} 2^{m(1+2b)}\end{aligned}$$

so that

$$d_Z(s, t) := (\mathbb{E}|Z_s - Z_t|^2)^{1/2} \leq C|s - t|^a 2^{m(b+1/2)}.$$

If $N(\varepsilon, d_Z)$ denote the covering numbers of I with respect to the intrinsic semimetric d_Z , then by chaining

$$\mathbb{E} \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t| \leq \mathbb{E} \sup_{d_Z(s,t) \leq \delta} |Z_s - Z_t| \leq C \int_0^\delta \sqrt{\log N(\varepsilon, d_Z)} d\varepsilon,$$

where $\delta := CN^{-a} 2^{m(b+1/2)}$ (cf. [21], p. 101). Since

$$N(\varepsilon, d_Z) \leq C \left(\frac{2^{m(b+1/2)}}{\varepsilon} \right)^{d/a}, \quad \varepsilon \leq \varepsilon_0$$

and $\int_0^1 \sqrt{\log(1/x)} dx < \infty$ one gets for large enough N

$$\int_0^\delta \sqrt{\log N(\varepsilon, d_Z)} d\varepsilon \leq C 2^{m(b+1/2)} \int_0^1 \sqrt{\log(1/x)} dx \leq C 2^{m(b+1/2)}.$$

Consequently,

$$\begin{aligned}\mathbb{E}\|Z^{(m)}\| &\leq C(\sqrt{\log(1+N)} 2^{-m(\vartheta-1/2)} m^\gamma + 2^{m(b+1/2)}) \\ &\leq C 2^{-m(\vartheta-1/2)} m^\gamma \quad \text{if } b + \vartheta \leq 0.\end{aligned}\tag{3.8}$$

Now we complete the proof. For $n \geq 2$, choose $m = m(n) \geq 1$ such that $2^{m-1} < n \leq 2^m$. Then

$$\left\| \sum_{j \geq n} \xi_j f_j \right\| \leq \sum_{j \geq m+1} \|Z^{(j)}\| + \left\| \sum_{j=n}^{2^m} \xi_j f_j \right\|.$$

Since $\mathbb{E}\left\| \sum_{j=n}^{2^m} \xi_j f_j \right\| \leq \mathbb{E}\|Z^{(m)}\|$ by the Anderson inequality (cf. [3], Cor. 3.3.7), we deduce from Equation (3.7)

$$\mathbb{E}\left\| \sum_{j \geq n} \xi_j f_j \right\| \leq C \sum_{j \geq m} \frac{j^{\gamma+1/2}}{2^{j(\vartheta-1/2)}} \leq \frac{C m^{\gamma+1/2}}{2^{m(\vartheta-1/2)}} \leq \frac{C(\log n)^{\gamma+1/2}}{n^{\vartheta-1/2}}.$$

If $b + \vartheta \leq 0$ then it follows from (3.8) that

$$\mathbb{E}\left\| \sum_{j \geq n} \xi_j f_j \right\| \leq \frac{C(\log n)^\gamma}{n^{\vartheta-1/2}}.$$

Combining these estimates with (3.5) yields the assertion. \diamond

Remarks • The rate for the truncation error only depends on ϑ and γ , that is on the decay of the length of functions f_j (as long as $b + \vartheta > 0$). The occurrence of expansions with $b + \vartheta \leq 0$ seems

to be a rare event and otherwise b plays no rôle (see the subsequent example). The case $\gamma = 0$ typically corresponds to 1-parameter processes with $I = [0, T]$.

• The $e_{N,r}^{(pq)}$ -problem comprises the optimization of admissible sequences and in view of (3.4) is thus related to the l -numbers of X defined by

$$l_{n,r}(X) = l_{n,r}(X, \mathcal{C}(I)) := \inf\{(\mathbb{E}\|\sum_{j \geq n} \xi_j g_j\|^r)^{1/r} : (g_j) \text{ admissible for } X \text{ in } \mathcal{C}(I)\}. \quad (3.9)$$

Rate optimal solutions of the $l_{n,r}$ -problem in the sense of $l_{n,r}(X) \approx (\mathbb{E}\|\sum_{j \geq n} \xi_j g_j\|^r)^{1/r}$ as $n \rightarrow \infty$ have recently been investigated (see [13], [7], [8], [9], [1]). Admissible sequences of the type (A1) and (A2) seem to be promising candidates. For instance, for the Brownian motion W on $I = [0, 1]$, it is known that

$$l_{n,r}(W) \approx \left(\frac{\log n}{n}\right)^{1/2}.$$

Hence, by Theorem 1, any admissible sequence in $\mathcal{C}([0, 1])$ for W satisfying (A1) and (A2) with $\vartheta = 1$ and $\gamma = 0$ (like the Karhunen-Loève basis of H_W) is a rate optimal solution of the $l_{n,r}(W)$ -problem.

EXAMPLE (*Weierstrass processes*): Let

$$f_j(t) = j^{-\vartheta} \sin(j^{b+\vartheta}t), \quad j \geq 1, \quad \vartheta > 1/2, \quad b \in \mathbb{R}, \quad t \in [0, T].$$

Then $\|f_j\| \leq j^{-\vartheta}$ and $[f_j]_1 = j^b$. Since $f_j(0) = 0$, we also have $\|f_j\| \leq Tj^b$ so that (A1) and (A2) are satisfied with $\tilde{\vartheta} = \max\{\vartheta, -b\}$ and $a = 1$. The covariance function of $X = \sum_{j=1}^{\infty} \xi_j f_j$ is given by

$$\mathbb{E} X_s X_t = \sum_{j=1}^{\infty} j^{-2\vartheta} \sin(j^{b+\vartheta}s) \sin(j^{b+\vartheta}t).$$

Now in the "Weierstrass case" $b + \vartheta > 0$, we get from Theorem 1

$$(\mathbb{E}\|\sum_{j \geq n} \xi_j f_j\|^r)^{1/r} \leq \frac{C\sqrt{\log n}}{n^{\vartheta-1/2}}$$

while in the "non-Weierstrass case" $b + \vartheta \leq 0$ appears the better rate

$$(\mathbb{E}\|\sum_{j \geq n} \xi_j f_j\|^r)^{1/r} \leq \frac{C}{n^{-b-1/2}}.$$

We come to the minimal product quantization error $e_{N,r}(X)^{(pq)}$. Let $r \in [1, \infty)$ and set $\nu_j := j_0^{-\vartheta} \log(1 + j_0)^\gamma$ if $j < j_0 := \lceil e^{\gamma/\vartheta} \rceil$ and $\nu_j := j^{-\vartheta} \log(1 + j)^\gamma$ if $j \geq j_0$. Then the sequence $(\nu_j)_j$ is decreasing. Since

$$\lim_{k \rightarrow \infty} k e_{k,r}(\mathcal{N}(0, 1)) \text{ exists in } (0, \infty)$$

(cf. [10]), we deduce from (3.4), (A1) and Theorem 1 the estimate

$$(\mathbb{E}\|X - \hat{X}^N\|^r)^{1/r} \leq C\left(\sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1 + m)^{\gamma+1/2}}{m^{\vartheta-1/2}}\right)$$

for every $m, N_1, \dots, N_m \in \mathbb{N}$ with $\prod_{j=1}^m N_j \leq N$. (The case $b + \vartheta \leq 0$ is treated analogously.) Consequently,

$$e_{N,r}(X)^{(pq)} \leq C \inf \left\{ \sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1+m)^{\gamma+1/2}}{m^{\vartheta-1/2}} : m, N_1, \dots, N_m \in \mathbb{N}, \prod_{j=1}^m N_j \leq N \right\}. \quad (3.10)$$

For a given $N \in \mathbb{N}$, we may first optimize the integer bit allocation given by the N_j 's for fixed m and then optimize m . To this end, note that the continuous allocation problem reads

$$\inf \left\{ \sum_{j=1}^m \nu_j y_j^{-1} : y_j > 0, \prod_{j=1}^m y_j \leq N \right\} = \sum_{j=1}^m \nu_j z_j^{-1} = N^{-1/m} m (\prod_{j=1}^m \nu_j)^{1/m}$$

where

$$z_j = N^{1/m} \nu_j (\prod_{j=1}^m \nu_j)^{-1/m}$$

and $z_1 \geq \dots \geq z_m$. One can produce an (approximate) integer solution by setting

$$N_j = [z_j] = \lfloor N^{1/m} \nu_j (\prod_{j=1}^m \nu_j)^{-1/m} \rfloor, j \in \{1, \dots, m\} \quad (3.11)$$

provided $z_m \geq 1$. Then

$$\sum_{j=1}^m \nu_j N_j^{-1} \leq 2m N^{-1/m} (\prod_{j=1}^m \nu_j)^{1/m} \leq C m N^{-1/m} \nu_m.$$

Since the constraint on m reads $m \in I(N)$ with

$$I(N) := \{m \in \mathbb{N} : N^{1/m} \nu_m (\prod_{j=1}^m \nu_j)^{-1/m} \geq 1\}, \quad (3.12)$$

we arrive at

$$e_{N,r}(X)^{(pq)} \leq C \inf_{m \in I(N)} \left(\frac{N^{-1/m} \log(1+m)^\gamma}{m^{\vartheta-1}} + \frac{\log(1+m)^{\gamma+1/2}}{m^{\vartheta-1/2}} \right) \quad (3.13)$$

for every $N \in \mathbb{N}$. One checks that $I(N)$ is finite, $I(N) = \{1, \dots, m^*(N)\}$, $m^*(N)$ increases to infinity and

$$m^*(N) \sim \frac{\log N}{\vartheta} \text{ as } N \rightarrow \infty. \quad (3.14)$$

Finally, after a bit reflection let

$$m = m(N) \in I(N) \text{ with } m(N) \leq \frac{2 \log N}{\log \log N} \text{ for } N \geq 3 \quad (3.15)$$

$$\text{such that } m(N) \sim \frac{2 \log N}{\log \log N} \text{ as } N \rightarrow \infty.$$

This is possible in view of (3.14). The case $r \in (0, 1]$ can be treated similarly. We thus obtain the following result.

Theorem 2 *Assume that X admits an admissible set $(f_j)_{j \geq 1}$ in $\mathcal{C}(I)$ satisfying (A1) and (A2). Then we have for every $N \geq 3$ and $r \in (0, \infty)$,*

$$e_{N,r}(X)^{(pq)} \leq \frac{C(\log \log N)^{\vartheta+\gamma}}{(\log N)^{\vartheta-1/2}} \quad (3.16)$$

and

$$e_{N,r}(X)^{(pq)} \leq \frac{C(\log \log N)^{\vartheta+\gamma-1/2}}{(\log N)^{\vartheta-1/2}} \text{ if } b + \vartheta \leq 0.$$

Furthermore, the L^r -product N -quantization \hat{X}^N with respect to (f_j) with tuning parameters defined in (3.11) and (3.15) achieves these rates.

We may reasonably conjecture that for many specific processes the above rate is the true one. This would imply that product quantization achieves the optimal rate for quantization, namely the rate of convergence to zero of $e_{N,r}(X) := e_{N,r}(X, \mathcal{C}(I))$, only up to a $\log \log N$ -term in the formula (3.16). This is in contrast to the Hilbert space setting where the optimal rate is attained by product quantization (cf. [17]). To be precise we summarize the results on $e_{N,r}(X)$ in the present setting.

Theorem 3 (a) *Assume that X admits an admissible sequence in $\mathcal{C}(I)$ satisfying (A1) and (A2). Then*

$$e_{N,r}(X) = O\left(\frac{(\log \log N)^{\gamma+1/2}}{(\log N)^{\vartheta-1/2}}\right) \quad (3.17)$$

and

$$e_{N,r}(X) = O\left(\frac{(\log \log N)^\gamma}{(\log N)^{\vartheta-1/2}}\right) \text{ if } b + \vartheta \leq 0. \quad (3.18)$$

(b) *Assume that X admits an admissible sequence satisfying (A1). Let μ be a finite Borel measure on I and let $V : \mathcal{C}(I) \rightarrow L^2(I, \mu)$ denote the natural embedding. Then*

$$e_{N,r}(V(X), L^2(\mu)) = O\left(\frac{(\log \log N)^\gamma}{(\log N)^{\vartheta-1/2}}\right)$$

and

$$e_{N,2}(V(X), L^2(\mu))^{(pq)} = O\left(\frac{(\log \log N)^\gamma}{(\log N)^{\vartheta-1/2}}\right).$$

Proof. (a) The assertions follow from Theorem 1, Proposition 4.1 in [15] which relates l -numbers (see (3.9)) and small ball probabilities (but this relation being not always sharp) and a precise link between these probabilities and $e_{N,r}(X)$ (cf. [5], [11]).

(b) Let $(f_j)_{j \geq 1}$ be an admissible sequence in $\mathcal{C}(I)$ for X satisfying (A1) and consider an L^2 -product N -quantization of $V(X)$ based on (Vf_j) ,

$$\widehat{V(X)}^N = \sum_{j=1}^m \hat{\xi}_j V(f_j)$$

(see (3.2)). Then using the independence of $\xi_j - \hat{\xi}_j, j \geq 1$ and the stationary property $\hat{\xi}_j = \mathbb{E}(\xi_j | \hat{\xi}_j)$ of the quantization $\hat{\xi}_j$,

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^{\infty} \xi_j V(f_j) - \widehat{V(X)}^N \right\|_{L^2(\mu)}^2 &= \sum_{j=1}^m \mathbb{E} |\xi_j - \hat{\xi}_j|^2 \|Vf_j\|_{L^2(\mu)}^2 + \sum_{j \geq m+1} \|Vf_j\|_{L^2(\mu)}^2 \\ &\leq C \left(\sum_{j=1}^m n_j^{-2} j^{-2\vartheta} \log(1+j)^{2\gamma} + \sum_{j \geq m+1} j^{-2\vartheta} \log(1+j)^{2\gamma} \right). \end{aligned}$$

Now argue as in [16] to conclude that

$$e_{N,2}(V(X), L^2(\mu))^{(pq)} = O\left(\frac{(\log \log N)^\gamma}{(\log N)^{\vartheta-1/2}}\right). \quad \diamond$$

Sometimes (3.17) provides the true rate for $e_{N,r}(X)$ (like for the 2-parameter Brownian sheet), sometimes it yields the best known upper bound (like for the d -parameter Brownian sheet with $d \geq 3$) and sometimes (3.18) provides the true rate (like for the Brownian motion). The latter fact typically occurs when the rate of $e_{N,r}(X)$ and the ‘‘Hilbert rate’’ of $e_{N,r}(V(X), L^2(dt))$ coincide (see Section 4). It remains an open question to find conditions for this to happen.

The following proposition ensures stability of conditions (A1) and (A2) under tensor products (see Proposition 5).

Proposition 7 For $i \in \{1, \dots, d\}$, let $(f_j^i)_{j \geq 1} \in \mathcal{C}([0, T])^{\mathbb{N}}$ satisfy (A1) and (A2) with parameters $\vartheta_i, \gamma_i, a_i, b_i$ such that $\gamma_i = 0$. Then a suitable arrangement of $(\otimes_{i=1}^d f_j^i)_{\underline{j} \in \mathbb{N}^d}$ satisfies (A1) and (A2) with parameters $\vartheta = \min_{1 \leq i \leq d} \vartheta_i, \gamma = \vartheta(m-1)$, where $m = \text{card}\{i \in \{1, \dots, d\} : \vartheta_i = \vartheta\}$, $a = \min_{1 \leq i \leq d} a_i$ and $b = (\max_{1 \leq i \leq d} b_i)_+$.

Proof. For $\underline{j} = (j_1, \dots, j_d) \in \mathbb{N}^d$, set $f_{\underline{j}} = \otimes_{i=1}^d f_{j_i}^i$ so that $f_{\underline{j}}(t) = \prod_{j=1}^d f_{j_i}^i(t_i), t \in [0, T]^d$. We have

$$\|f_{\underline{j}}\| \leq \prod_{i=1}^d \|f_{j_i}^i\| \leq C \prod_{i=1}^d j_i^{-\vartheta_i}$$

and

$$|f_{\underline{j}}(s) - f_{\underline{j}}(t)| \leq C \max_{1 \leq i \leq d} j_i^b |s - t|^a.$$

Let $u_{\underline{j}} := \prod_{i=1}^d j_i^{-\vartheta_i}$. Choose a bijective map $\psi : \mathbb{N} \rightarrow \mathbb{N}^d$ such that $u_k := u_{\psi(k)}$ is decreasing in $k \geq 1$. Set $f_k := f_{\psi(k)}$. Then

$$u_k \approx C k^{-\vartheta} (\log k)^{\vartheta(m-1)} \text{ as } k \rightarrow \infty$$

(cf. [19], Theorem 2.1). Consequently,

$$\|f_k\| \leq C k^{-\vartheta} (\log k)^{\vartheta(m-1)}$$

and for $\underline{j} = \psi(k)$,

$$j_i \leq \prod_{i=1}^d j_i \leq \prod_{i=1}^d j_i^{\vartheta_i/\vartheta} \leq C k (\log k)^{-(m-1)} \leq C k,$$

hence

$$|f_k(s) - f_k(t)| \leq C k^b |s - t|^a. \quad \diamond$$

4 Examples

4.1 Brownian motion and Brownian sheet

The admissible sequences (2.10)-(2.13) for the *standard Brownian motion* over $[0, T]$ all satisfy the conditions (A1) and (A2) with parameters $\vartheta = 1, \gamma = 0, a = 1$ and $b = 0$. Consequently, by Theorem 2, the performance of product quantization is ruled by

$$e_{N,r}(BM)^{(pq)} = O\left(\frac{\log \log N}{(\log N)^{1/2}}\right) \quad (4.1)$$

while (see [5], [11])

$$e_{N,r}(BM) \approx (\log N)^{-1/2}. \quad (4.2)$$

The same result holds for *Brownian bridge* over $[0, T]$ using e.g. the admissible sequence (2.15).

By Proposition 5 and 7, the tensor products of functions (2.10)-(2.13) are admissible for the *Brownian sheet* (or Chentsov-Wiener field) X over $[0, T]^d$ with covariance function

$$\mathbb{E} X_s X_t = \prod_{i=1}^d s_i \wedge t_i$$

and satisfy (A1) and (A2) with $\vartheta = 1, \gamma = d-1, a = 1$ and $b = 0$. Therefore, by Theorems 2 and 3,

$$e_{N,r}(BS)^{(pq)} = O\left(\frac{(\log \log N)^d}{(\log N)^{1/2}}\right) \quad (4.3)$$

and

$$e_{N,r}(BS) = O\left(\frac{(\log \log N)^{d-1/2}}{(\log N)^{1/2}}\right). \quad (4.4)$$

Using $e_{N,r}(BS) \geq T^{-d}e_{N,r}(BS, L^2([0, T]^d, dt))$ and employing the sharp rate for the Hilbert space case (see [17]), one obtains the lower estimate

$$e_{N,r}(BS) = \Omega\left(\frac{(\log \log N)^{d-1}}{(\log N)^{1/2}}\right). \quad (4.5)$$

For $d = 2$, the "non-Hilbert-rate" (4.4) is known to be the true one for $e_{N,r}(BS)$ (see [11]) and it is very likely also the true rate for $d \geq 3$. In any case we see no dramatic loss of performance by using product quantization.

4.2 Fractional Brownian motions and fractional Brownian sheets

Consider the admissible family (2.14) in $\mathcal{C}([0, T])$ for the *fractional Brownian motion* over $[0, T]$ with Hurst index $\rho \in (0, 1)$. Using the asymptotic properties

$$x_j \sim y_j \sim \pi j, J_{1-\rho}(x_j) \sim J_{-\rho}(y_j) \sim \frac{\sqrt{2}}{\pi} j^{-1/2} \text{ as } j \rightarrow \infty$$

(cf. [8]), one observes that a suitable arrangement of the functions (2.14) (like $f_{2j} = f_j^1, f_{2j-1} = f_j^2$) satisfies (A1) and (A2) with parameters $\vartheta = \rho + 1/2, \gamma = 0, a = 1$ and $b = 1/2 - \rho$. Consequently,

$$e_{N,r}(FBM)^{(pq)} = O\left(\frac{(\log \log N)^{\rho+1/2}}{(\log N)^\rho}\right) \quad (4.6)$$

while (see [6], [11])

$$e_{N,r}(FBM) \approx (\log N)^{-\rho}. \quad (4.7)$$

The tensor products of functions (2.14) are admissible for the *fractional Brownian sheet* X over $[0, T]^d$ with covariance function

$$\mathbb{E} X_s X_t = \left(\frac{1}{2}\right)^d \prod_{i=1}^d (s_i^{2\rho_i} + t_i^{2\rho_i} - |s_i - t_i|^{2\rho_i}),$$

$\rho_i \in (0, 1)$, and satisfy conditions (A1) and (A2) with $\vartheta = \rho + 1/2$, where $\rho = \min_{1 \leq i \leq d} \rho_i$, $\gamma = \vartheta(m - 1)$, where $m = \text{card}\{i \in \{1, \dots, d\} : \rho_i = \rho\}$, $a = 1$ and $b = \max_{1 \leq i \leq d} (1/2 - \rho_i)_+$. Therefore, by Theorems 2 and 3,

$$e_{N,r}(FBS)^{(pq)} = O\left(\frac{(\log \log N)^{m(\rho+1/2)}}{(\log N)^\rho}\right) \quad (4.8)$$

and

$$e_{N,r}(FBS) = O\left(\frac{(\log \log N)^{m(\rho+1/2)-\rho}}{(\log N)^\rho}\right). \quad (4.9)$$

The Hilbert space setting $E = L^2([0, T]^d, dt)$ provides the lower estimate

$$e_{N,r}(FBS) = \Omega\left(\frac{(\log \log N)^{(m-1)(\rho+1/2)}}{(\log N)^\rho}\right) \quad (4.10)$$

(see [16], [17]). The true rate of $e_{N,r}(FBS)$ is known only for the case $m = 1$ where the true rate is the "Hilbert rate" (4.10) (see [5]) and in case $m = 2$ where (4.9) is the true rate (see [2], [11]). A reasonable conjecture is that (4.9) is also the true rate for $m \geq 3$.

4.3 Riemann-Liouville and other moving average processes

For $\psi \in L^2([0, T], dt)$ and a standard Brownian motion W , let

$$X_t = \int_0^t \psi(t-s) dW_s, \quad t \in [0, T]$$

and assume that X has a pathwise continuous modification. Since

$$\mathbb{E} X_s X_t = \int_0^{s \wedge t} \psi(s-u) \psi(t-u) du,$$

the operator $S : L^2([0, T], dt) \rightarrow \mathcal{C}([0, T])$, $Sk(t) = \int_0^t \psi(t-s)k(s)ds$ provides a factorization of C_X so that

$$\begin{aligned} f_j(t) &= \sqrt{\frac{2}{T}} \int_0^t \psi(t-s) \cos\left(\frac{\pi(j-1/2)s}{T}\right) ds \\ &= \sqrt{\frac{2}{T}} \int_0^t \psi(s) \cos\left(\frac{\pi(j-1/2)(t-s)}{T}\right) ds, \quad j \geq 1, \end{aligned} \quad (4.11)$$

is an admissible sequence for X . Observe that (4.11) provides well defined continuous functions even for $\psi \in L^1([0, T], dt)$.

Lemma 3 *Let $\psi \in L^1([0, 1], dt)$.*

(a) *If $\varphi(t) = \int_0^t |\psi(s)| ds$ is β -Hölder continuous with $\beta \in (0, 1]$, then the sequence (f_j) from (4.11) satisfies (A2) with $a = \beta$ and $b = 1$. In particular, if $\psi \in L^2([0, T], dt)$, then (A2) is satisfied with $a = 1/2$ and $b = 1$.*

(b) *If ψ has finite variation over $[0, T]$, then (A1) is satisfied with $\vartheta = 1$ and $\gamma = 0$.*

Proof. Let $\lambda_j = (\pi(j-1/2)/T)^{-2}$. (a) For $s < t$, we have

$$f_j(s) - f_j(t) = \sqrt{\frac{2}{T}} \left\{ \int_0^s \psi(u) (\cos((s-u)/\sqrt{\lambda_j}) - \cos((t-u)/\sqrt{\lambda_j})) du - \int_s^t \psi(u) \cos((t-u)/\sqrt{\lambda_j}) du \right\}$$

so that

$$|f_j(s) - f_j(t)| \leq \sqrt{\frac{2}{T}} \left(\frac{|s-t|}{\sqrt{\lambda_j}} \|\psi\|_{L^1(dt)} + \int_s^t |\psi(u)| du \right).$$

(b) We have

$$\begin{aligned} f_j(t) &= -\sqrt{2\lambda_j/T} \int_0^t \psi(s) d(\sin((t-s)/\sqrt{\lambda_j})) \\ &= \sqrt{2\lambda_j/T} (\psi(0) \sin(t/\sqrt{\lambda_j}) + \int_0^t \sin((t-s)/\sqrt{\lambda_j}) d\psi(s)) \end{aligned}$$

so that

$$\|f_j\| \leq \sqrt{2\lambda_j/T} (|\psi(0)| + \text{Var}(\psi, [0, T])). \quad \diamond$$

This lemma yields an universal upper bound

$$e_{N,r}(X)^{(pq)} = O\left(\frac{\log \log N}{(\log N)^{1/2}}\right)$$

for functions ψ having finite variation.

In the sequel we do not care about improvements of parameter b in (A2) since the condition $b + \vartheta \leq 0$ cannot be achieved in this setting (cf. Theorems 2 and 3).

Lemma 4 Let $\psi \in L^1([0, T], dt)$.

(a) If ψ is positive and decreasing on $(0, T]$ and $\varphi(t) = \int_0^t \psi(s) ds$ is β -Hölder continuous with $\beta \in (0, 1]$, then the sequence (f_j) from (4.11) satisfies $\|f_j\| \leq Cj^{-\beta}$. If $\beta > 1/2$, then (A1) is satisfied with $\vartheta = \beta$ and $\gamma = 0$.

(b) If $\psi(0) = 0$, ψ is β -Hölder continuous with $\beta \in (0, 1]$, ψ is differentiable on $(0, T]$ such that ψ' is positive and decreasing on $(0, T]$, then (A1) is satisfied with $\vartheta = 1 + \beta$ and $\gamma = 0$.

Proof. Let $\lambda_j = (\pi(j - 1/2)/T)^{-2}$. (a) For $t \leq \sqrt{\lambda_j}$, we have

$$|f_j(t)| \leq \sqrt{2/T} \varphi(\sqrt{\lambda_j}).$$

Using the second integral mean value formula we obtain for $t \in [\sqrt{\lambda_j}, T]$ and some $\delta_j \in [\sqrt{\lambda_j}, t]$

$$\begin{aligned} |f_j(t)| &\leq \sqrt{2/T} (|\int_0^{\sqrt{\lambda_j}} \psi(s) \cos((t-s)/\sqrt{\lambda_j}) ds| + |\int_{\sqrt{\lambda_j}}^t \psi(s) \cos((t-s)/\sqrt{\lambda_j}) ds|) \\ &= \sqrt{2/T} (|\int_0^{\sqrt{\lambda_j}} \psi(s) \cos((t-s)/\sqrt{\lambda_j}) ds| + \psi(\sqrt{\lambda_j}) |\int_{\sqrt{\lambda_j}}^{\delta_j} \cos((t-s)/\sqrt{\lambda_j}) ds|) \\ &\leq \sqrt{2/T} (\varphi(\sqrt{\lambda_j}) + 2\sqrt{\lambda_j} \psi(\sqrt{\lambda_j})) \\ &\leq 3\sqrt{2/T} \varphi(\sqrt{\lambda_j}). \end{aligned}$$

Consequently,

$$\|f_j\| \leq 3\sqrt{2/T} \varphi(\sqrt{\lambda_j}) \leq C\lambda_j^{\beta/2}.$$

(b) The function ψ is absolutely continuous on $[0, T]$ so that an integration by parts yields

$$f_j(t) = \sqrt{2\lambda_j/T} \int_0^t \psi'(s) \sin((t-s)/\sqrt{\lambda_j}) ds.$$

Arguing like in (a) (with ψ replaced by ψ') one derives that

$$\|f_j\| \leq 3\sqrt{2\lambda_j/T} \psi(\sqrt{\lambda_j}) \leq C\lambda_j^{(1+\beta)/2}. \quad \diamond$$

Now let $\psi(t) = t^{\rho-1/2}$ with $\rho \in (0, \infty)$. Then

$$X_t = X_t^\rho = \int_0^t (t-s)^{\rho-1/2} dW_s, t \in [0, T] \quad (4.12)$$

so that X^ρ is a *Riemann-Liouville* process of order ρ . Using the $(\rho \wedge \frac{1}{2})$ -Hölder continuity of the application $t \mapsto X_t^\rho$ from $[0, T]$ into $L^2(\mathbb{P})$ and the Kolmogorov criterion one checks that X^ρ has a pathwise continuous modification.

Lemma 5 Let $\psi(t) = t^{\rho-1/2}$, $\rho \in (0, \infty)$. Then the sequence (f_j) from (4.11) satisfies (A2) with $a = \min\{1, \rho + 1/2\}$, $b = 1$ and (A1) for $\rho \in (0, 3/2]$ with $\vartheta = \rho + 1/2$ and $\gamma = 0$.

Proof. Immediate consequence of Lemma 3 and Lemma 4. \diamond

We deduce for Riemann-Liouville processes of order $\rho \in (0, 3/2]$

$$e_{N,r}(RL)^{(p,q)} = O\left(\frac{(\log \log N)^{\rho+1/2}}{(\log N)^\rho}\right) \quad (4.13)$$

while for every $\rho \in (0, \infty)$ (see [15], [11])

$$e_{N,r}(RL) \approx (\log N)^{-\rho}. \quad (4.14)$$

For $\beta \in (0, \infty)$, define the *fractionally integrated Brownian motion* by

$$Y_t^\beta = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} W_s ds. \quad (4.15)$$

The Ito formula yields $Y_t^\beta = \Gamma(1+\beta)^{-1} X_t^{\beta+1/2}$ so that for $\beta \in (0, 1]$

$$e_{N,r}(FIBM)^{(pq)} = O\left(\frac{(\log \log N)^{\beta+1}}{(\log N)^{\beta+1/2}}\right)$$

and for $\beta \in (0, \infty)$

$$e_{N,r}(FIBM) \approx (\log N)^{-(\beta+1/2)}.$$

To go beyond $\rho = 3/2$ needs to slightly change the way we quantize. Let $\psi(t) = t^{\rho-1/2}$ with $\rho > 3/2$ and choose $k \in \mathbb{N}$ such that $k + 1/2 < \rho \leq k + 3/2$. Set $\lambda_j = (\pi(j - 1/2)/T)^{-2}$. For $k \in \{2n - 1, 2n\}, n \in \mathbb{N}$ integration by parts yields the expansion

$$\begin{aligned} f_j(t) &= \sum_{m=1}^n (-1)^{m-1} \lambda_j^m \sqrt{2/T} \psi^{(2m-1)}(t) + (-1)^n \lambda_j^n \sqrt{2/T} \int_0^t \psi^{(2n)}(s) \cos((t-s)/\sqrt{\lambda_j}) ds \\ &=: g_j(t) + h_j(t), t \in [0, T]. \end{aligned}$$

Since $\psi^{(2n)}(t) = Ct^{\beta-1}$ if $k = 2n - 1$ and $\psi^{(2n)}(t) = Ct^\beta$ if $k = 2n$ with $\beta = \rho - k - 1/2 \in (0, 1]$ we deduce from Lemma 3 and Lemma 4 that the sequence (h_j) in $C([0, T])$ satisfies (A1) with $\vartheta = \rho + 1/2, \gamma = 0$ and (A2) with $a = \rho - k - 1/2, b = -k$ if $k = 2n - 1$ and $a = 1, b = -k + 1$ if $k = 2n$. Clearly, the sequence (g_j) also satisfies the conditions (A1) and (A2) (with $\vartheta = 2, \gamma = 0, b = -2$ and $a = \rho - k - 1/2$ if $k = 2n - 1$ and $a = 1$ if $k = 2n$). Consequently, there exist centered continuous Gaussian processes $U = (U_t)_{t \in [0, T]}$ and Z such that $U = \sum_{j=1}^{\infty} \xi_j g_j$ a.s., $Z = \sum_{j=1}^{\infty} \xi_j h_j$ a.s.,

$$X = X^\rho \stackrel{d}{=} U + Z \quad (4.16)$$

and $U \in \text{span}\{\psi^{(2m-1)} : m = 1, \dots, n\}$ a.s. Observe that

$$U = \sum_{m=1}^n (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \eta_m$$

where $\eta_m = \sum_{j=1}^{\infty} \lambda_j^m \xi_j$ is $N(0, \sum_{j=1}^{\infty} \lambda_j^{2m})$ -distributed.

Now use e.g. $[N^{1/2n}]$ -quantizations of η_m and a $[\sqrt{N}]$ -product quantization of Z for the quantization of X (which is clearly not optimal in practice but remains rate optimal). Let $\alpha_m \subset \mathbb{R}$ be the L^r -optimal $[N^{1/2n}]$ -quantizer for η_m , $\hat{\eta}_m := \hat{\eta}_m^{\alpha_m}$,

$$\hat{U}^{\sqrt{N}} := \sum_{m=1}^n (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \hat{\eta}_m$$

and let $\hat{Z}^{\sqrt{N}}$ be the L^r -product $[\sqrt{N}]$ -quantization of Z from Theorem 2. Then, a (modified) L^r -product N -quantization of X with respect to (f_j) is defined by

$$\hat{X}^N := \hat{U}^{\sqrt{N}} + \hat{Z}^{\sqrt{N}}. \quad (4.17)$$

Using Theorem 2 one shows for the quantization error

$$\begin{aligned}
\|U + Z - \hat{X}^N\|_{L^r_E} &\leq C(\|U - \hat{U}^{\sqrt{N}}\|_{L^r_E} + \|Z - \hat{Z}^{\sqrt{N}}\|_{L^r_E}) \\
&\leq C\left(\sum_{m=1}^n \sqrt{2/T} \|\psi^{(2m-1)}\| \|\eta_m - \hat{\eta}_m\|_{L^r} + \|Z - \hat{Z}^{\sqrt{N}}\|_{L^r_E}\right) \\
&\leq \frac{C}{N^{1/2n}} + \frac{C(\log \log \sqrt{N})^{\rho+1/2}}{(\log \sqrt{N})^\rho} \\
&\leq \frac{C(\log \log N)^{\rho+1/2}}{(\log N)^\rho}
\end{aligned}$$

so that with the above modification (4.13) remains true for $\rho > 3/2$.

4.4 Ornstein-Uhlenbeck process

By Lemma 3, the admissible sequence

$$f_0(t) = \frac{\sigma}{\sqrt{2a}} e^{-\beta t}, f_j(t) = \sigma \sqrt{\frac{2}{T}} \int_0^t e^{-\beta(t-s)} \cos\left(\frac{\pi(j-1/2)s}{T}\right) ds, j \geq 1$$

(see (2.16)) for the stationary Ornstein-Uhlenbeck process $(X_t)_{t \in [0, T]}$ with covariance

$$\mathbb{E} X_s X_t = \frac{\sigma^2}{2\beta} e^{-\beta|s-t|}$$

satisfies conditions (A1) and (A2) with $\vartheta = 1, \gamma = 0, a = 1$ and $b = 1$. Consequently,

$$e_{N,r}(OU)^{(pq)} = O\left(\frac{\log \log N}{(\log N)^{1/2}}\right) \quad (4.18)$$

while (see [11])

$$e_{N,r}(OU) \approx (\log N)^{-1/2}. \quad (4.19)$$

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