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► **To cite this version:**

Gwenaël Richomme. Conjugacy of morphisms and Lyndon decomposition of standard Sturmian words. 2005, pp.341-351. hal-00013457

**HAL Id: hal-00013457**

**<https://hal.science/hal-00013457v1>**

Submitted on 8 Nov 2005

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**LaRIA RESEARCH REPORT : LRR 2005-09**  
(October 2005)

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# Conjugacy of morphisms and Lyndon decomposition of standard Sturmian words\*

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October 28, 2005

## Abstract

Using the notions of conjugacy of morphisms, we answer a question of G. Melançon concerning the decomposition in Lyndon words of standard Sturmian words. We show some connections with morphisms preserving Lyndon words

## 1 Introduction

Finite (or infinite) Lyndon words can be encountered in many studies (see for instance [8, 9, 10]). They are the nonempty words which are smaller in lexicographic order than all their proper suffixes. The Lyndon factorization theorem [4] states that any finite word can be decomposed uniquely in a product of nonincreasing (in lexicographic order) Lyndon words. This result was extended to infinite words [19] (In such a case, the decomposition can end with an infinite Lyndon word). Thus some works concern the decomposition in Lyndon words of some infinite words (see for instance [3, 5, 11, 12, 18] for such results).

In [12], G. Melançon gives a decomposition in Lyndon words of standard Sturmian words. He asks the following question: in which cases, the sequence of nonincreasing Lyndon words appearing in the decomposition of a standard Sturmian word can be written  $(g^n(\ell_0))_{n \geq 0}$  with  $\ell_0$  a Lyndon word and  $g$  a morphism. In Section 5, we answer this question.

For this, we use results about morphisms preserving Lyndon words [14] and about conjugacy of morphisms [13]. In particular, we show that when a positive answer exists to the previous question,  $g$  preserves Lyndon words and is the conjugate of a morphism  $f$  that generates the decomposed standard Sturmian word.

In Section 2, we recall notions on Sturmian words and morphisms. Section 3 recalls both the decomposition in Lyndon words of standard Sturmian words obtained by G. Melançon, and his question. This section also recall notions on morphisms preserving Lyndon words. Section 4 presents notions on conjugacy of morphisms and introduces a new particular case, namely the strong

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\*This paper was presented at the 5th International Conference on Words which held in Montréal on september 2005 (Publications du LaCIM numéro 36, page 341-351 (S. Brlek, C. Reutenauer eds.).)

conjugacy. Using it in conjunction with morphisms preserving Lyndon words, we give a new proof that for any standard Sturmian words  $w$  over  $\{a < b\}$ ,  $aw$  is an infinite Lyndon word [3]. Finally, in Section 5, we answer G. Melançon. Note that at a first step, we express the decomposition of a standard Sturmian word using only morphisms.

## 2 Sturmian words and morphisms

We recall here notions on words (see for instance [8, 9] for more details).

An *alphabet*  $A$  is a set of symbols called *letters*. Here we consider only finite alphabets. A *word over*  $A$  is a sequence of letters from  $A$ . The *empty word*  $\varepsilon$  is the empty sequence. Equipped with the concatenation operation, the set  $A^*$  of finite words over  $A$  is a free monoid with neutral element  $\varepsilon$  and set of generators  $A$ . We denote by  $A^\omega$  the set of infinite words over  $A$ . As usually, for a finite word  $u$  and an integer  $n$ , the  $n^{\text{th}}$  power of  $u$ , denoted  $u^n$ , is the word  $\varepsilon$  if  $n = 0$  and the word  $u^{n-1}u$  otherwise. If  $u$  is not the empty word,  $u^\omega$  denotes the infinite word obtained by infinitely repeating  $u$ . A finite word  $w$  is said *primitive* if for any word  $u$ , the equality  $w = u^n$  (with  $n$  an integer) implies  $n = 1$ . Any word is the power of a unique primitive word called the *primitive root* of  $w$ .

Given a nonempty word  $u = u_1 \dots u_n$  with  $u_i \in A$ , the *length*  $|u|$  of  $u$  is the integer  $n$ . One has  $|\varepsilon| = 0$ . If for some words  $u, v, p, s$  (possibly empty),  $u = pvs$ , then  $v$  is a *factor* of  $u$ ,  $p$  is a *prefix* of  $u$  and  $s$  is a *suffix* of  $u$ . When  $p \neq u$  (resp.  $s \neq u$ ), we say that  $p$  is a *proper prefix* (resp.  $s$  is a *proper suffix*) of  $u$ . By  $|u|_a$  we denote the number of occurrences of the letter  $a$  in the word  $u$ .

Sturmian words may be defined in many equivalent ways (see [1] for instance). They are infinite binary words. Here we will consider them as the infinite balanced non ultimately periodic words. We recall that a (finite or infinite) word  $w$  over  $\{a, b\}$  is *balanced* if for any factors  $u$  and  $v$  of same length  $||u|_a - |v|_a| \leq 1$ , and that an infinite word  $w$  is *ultimately periodic* if  $w = uv^\omega$  for some finite words  $u$  and  $v$ .

Many studies of Sturmian words use Sturmian morphisms. Let  $A, B$  be two alphabets. A *morphism* (*endomorphism* if  $A = B$ )  $f$  from  $A^*$  to  $B^*$  is a mapping from  $A^*$  to  $B^*$  such that for all words  $u, v$  over  $A$ ,  $f(uv) = f(u)f(v)$ . We also say that  $f$  is a morphism on  $A$  or that  $f$  is defined on  $A$  (without any other precision when  $B$  has no importance). A morphism on  $A$  is entirely known by the images of the letters of  $A$ . A morphism extends naturally on infinite words. We denote just by juxtaposition the composition of morphisms. Given an endomorphism  $f$ , if  $\lim_{n \rightarrow \infty} f^n(a)$  exists, then this limit is denoted  $f^\omega(a)$  and is a fixed point of  $f$ : the word  $f^\omega(a)$  is said generated by  $f$ .

Sturmian morphisms are the morphisms in  $\{E, L_a, L_b, R_a, R_b\}^*$  where  $E, L_a, L_b, R_a, R_b$  are the endomorphisms defined on  $\{a, b\}$  by  $E(a) = b, E(b) = a, L_a(a) = a, L_a(b) = ab, L_b(a) = ba, L_b(b) = b, R_a(a) = a, R_a(b) = ba, R_b(a) = ab, R_b(b) = b$ . Many relations exists between Sturmian words and Sturmian morphisms. For instance, it is known [2, 6] that any Sturmian word can be defined as an infinite product of Sturmian morphisms.

A particular case of Sturmian words is the standard (or characteristic) one. For any standard Sturmian words, there exists a sequence  $(d_n)_{n \geq 0}$  of integers, called *the directive sequence* verifying  $d_1 \geq 0$  and  $d_k \geq 1$  for all  $k \geq 2$ , such that

$$w = \lim_{n \rightarrow \infty} s_n$$

where the sequence  $(s_n)_{n \geq -1}$  of words is defined by :  $s_{-1} = b$ ,  $s_0 = a$  and  $s_n = s_{n-1}^{d_n} s_{n-2}$  for  $n \geq 1$ . Let us observe that for every  $n \geq 0$ ,  $s_{2n}$  ends with  $a$ . Moreover [1],

$$\begin{aligned} s_{2n} &= L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}}(a) \\ &= L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}} L_a^{d_{2n+1}}(a) \\ s_{2n+1} &= L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}} L_a^{d_{2n+1}}(b) \\ &= L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}} L_a^{d_{2n+1}} L_b^{d_{2n+2}}(b) \end{aligned}$$

### 3 Lyndon words and morphisms

From now on we consider ordered alphabets. We denote  $\{\alpha_1 < \dots < \alpha_n\}$  the  $n$ -letter alphabet  $\{\alpha_1, \dots, \alpha_n\}$  with order  $\alpha_1 < \dots < \alpha_n$ . Given an ordered alphabet  $A$ , we denote by  $\preceq$  the lexicographic order whenever used on  $A^*$  or on  $A^\omega$ . Let recall that for two different (finite or infinite) words  $u$  and  $v$ ,  $u \prec v$  if and only if  $u = xay$ ,  $v = xbz$  with  $a, b \in A$ ,  $a < b$ ,  $x \in A^*$ ,  $y, z \in A^* \cup A^\omega$ , or if (when  $u$  is finite)  $u$  is a prefix of  $v$ .

A nonempty finite word  $w$  is a *Lyndon word* if for all nonempty words  $u$  and  $v$ ,  $w = uv$  implies  $w \prec vu$ . Equivalently [4, 8], a nonempty word  $w$  is a Lyndon word if all its nonempty proper suffixes are greater than it for the lexicographic order. For instance, on the one-letter alphabet  $\{a\}$ , only  $a$  is a Lyndon word. On  $\{a < b\}$  the Lyndon words of length at most 5 are  $a$ ,  $b$ ,  $ab$ ,  $aab$ ,  $abb$ ,  $aaaab$ ,  $aabb$ ,  $abbb$ ,  $aaaab$ ,  $aaabb$ ,  $aabab$ ,  $aabbb$ ,  $abbbb$ . Lyndon words are primitive.

The second definition of Lyndon words extends to infinite words: An infinite word is an *infinite Lyndon word* if all its proper suffixes are greater than it for the lexicographic order. A useful result of G. Melançon [12] states that an infinite word is a Lyndon word if and only if it has an infinity of prefixes that are Lyndon words. See for instance [7] for a recent example of infinite Lyndon word.

Any nonempty finite or infinite Lyndon words can be decomposed as a nonincreasing product of Lyndon words. First, R.C. Lyndon proved (see [8] for instance):

*Any word  $w \in A^+$  may be written uniquely as a nonincreasing product of Lyndon words:  $w = \ell_1 \ell_2 \dots \ell_n$  where for each  $i$ ,  $\ell_i$  is a Lyndon word and  $\ell_n \preceq \ell_{n-1} \preceq \dots \preceq \ell_1$ .*

This result was generalized to infinite words [19]:

*Any right infinite word  $w$  may be uniquely expressed as a nonincreasing product of Lyndon words, finite or infinite, in one of the two following forms: either there exists an infinite nonincreasing sequence of finite Lyndon words  $(\ell_k)_{k \geq 0}$  such that*

$$w = \prod_{n \geq 0} \ell_n = \ell_0 \ell_1 \dots$$

*or there exist finite Lyndon words  $\ell_0, \dots, \ell_{m-1}$  ( $m \geq 0$ ) and an infinite word  $\ell_m$  such that  $\ell_m \prec \ell_{m-1} \preceq \ell_{m-2} \preceq \dots \preceq \ell_0$  and*

$$w = \ell_0 \dots \ell_{m-1} \ell_m.$$

As already said in the introduction, many works concern the decomposition in Lyndon words of some infinite words. In [12], G. Melançon obtains the decomposition of standard Sturmian words. We consider these words here on the alphabet  $\{a < b\}$ . For any word  $w$  ending with the letter  $a$ , let us denote  $\bar{w}$  the word such that  $w = \bar{w}a$ .

**Theorem 3.1** [12] *Let  $s$  be a standard Sturmian word with directive sequence  $(d_n)_{n \geq 1}$ . Let  $\ell_n = as_{2n}^{d_{2n+1}-1} s_{2n-1} \bar{s}_{2n}$ . (if  $d_1 = 0$  then  $\ell_0 = b$ ).*

*The words  $(\ell_n)_{n \geq 0}$  form a strictly decreasing sequence of Lyndon words and the unique factorization of  $s$  as a nonincreasing product of Lyndon words is*

$$s = \prod_{n \geq 0} \ell_n^{d_{2n+1}}.$$

G. Melançon wrote [12, Remark 3.7] :

*When is the sequence  $(\ell_n)_{n \geq 0}$  morphic ? More precisely, is it possible to give a morphism  $\varphi : \{a, b\}^* \rightarrow \{a, b\}^*$  and a Lyndon word  $\ell_0$  such that  $\ell_{n+1} = \varphi(\ell_n)$ ? This question has a positive answer in the case where the directive sequence is constant. For instance, if  $d_n = 2$  for all  $n \geq 0$ , then we may set  $\ell_0 = aab$  and use the morphism mapping  $a \mapsto aaabaab$  and  $b \mapsto aab$ .*

*A characteristic Sturmian word may be itself morphic. That is, may be the limit  $\lim_n \varphi^n(a)$  of a (nonerasing) morphism (satisfying  $\varphi(a) \in aA^*$ ). It is known that this is essentially equivalent to the fact that its directive sequence is periodic. Unfortunately, even when a characteristic Sturmian word  $s$  has a periodic directive sequence, it seems that the sequence  $(\ell_n)_{n \geq 0}$  is not always morphic, although it is possible to describe patterns in the factorization.*

The aim of this paper is to answer this question. The main ideas of our proof are generalizations of the following remarks: the morphism  $a \mapsto aaabaab$  and  $b \mapsto aab$  is the Sturmian morphism  $L_a^2 R_b^2$  and preserves Lyndon words. Moreover  $L_a^2 L_b^2$  is a conjugate of  $L_a^2 R_b^2$ ,  $L_a^2 L_b^2(a) = (l_0)^2 a$  and  $L_a^2 R_b^2(a) = a(l_0)^2$ . Let us note that, in [14], similar remarks are made about the decomposition of the Fibonacci word (the standard Sturmian word of directive sequence  $(1)_{n \geq 0}$ ). In Section 4, we recall notions on conjugacy of morphisms.

Let us now recall some results on morphisms preserving (finite) Lyndon words. These morphisms are studied in [14]. By definition, a morphism  $f$  preserves Lyndon word if for each Lyndon word  $w$ ,  $f(w)$  is a Lyndon word. Effective characterizations of such morphisms are given in [14]. Consequently Sturmian words preserving Lyndon words are known:

**Proposition 3.2** [14] *A Sturmian morphism on  $\{a < b\}$  is a Lyndon morphism if and only if it belongs to  $\{L_a, R_b\}^*$ .*

To end this section, let us observe that a study of morphisms preserving infinite Lyndon words is given in [15].

## 4 Strong Conjugacy

In this section, we recall the notion of conjugacy (see, e.g., [9, 13]). We also introduce the particular case of strong conjugacy which will be useful to answer G. Melançon.

Let  $A$  and  $B$  be two alphabets and let  $f$  and  $g$  be two morphisms from  $A^*$  to  $B^*$ . The morphism  $g$  is a (right) *conjugate* of  $f$  if there exists a word  $u$  such that for any word  $x$  over  $A$ ,  $f(x)u = ug(x)$ . We will also say that  $f$  and  $g$  are  *$u$ -conjugated*, and we will denote  $f \triangleleft_u g$ . Moreover if  $f(a) = ua$  and  $g(a) = au$  for a letter  $a$ ,  $f$  and  $g$  will be called *strongly (on  $a$ )  $u$ -conjugated*.

Let us recall that any morphism  $f$  has at least one conjugate: itself ( $f \triangleleft_\epsilon f$ ). The Fibonacci morphism  $\varphi = L_a E$  defined by  $\varphi(a) = ab$  and  $\varphi(b) = a$  has exactly two conjugates, itself and the morphism  $\tilde{\varphi} = R_a E$  ( $\tilde{\varphi}(a) = ba$ ,  $\tilde{\varphi}(b) = a$ ). A lot of relations between conjugacy of morphisms and Sturmian morphisms were given by P. Séébold [17] and generalized to a larger family of morphisms in [13].

Since  $\varphi(a)$  does not end with the letter  $a$ , no morphism is strongly conjugate (on  $a$ ) to the Fibonacci morphism. Nevertheless we can observe that  $\varphi^2$  ( $a \mapsto aba$ ,  $b \mapsto ab$ ) is strongly  $ab$ -conjugated to  $\varphi\tilde{\varphi}$  ( $a \mapsto aab$ ,  $b \mapsto ab$ ). More generally, for all integers  $x$  and  $y$  ( $y \neq 0$ ), the morphism  $L_a^x L_b^y$  is strongly conjugated to the morphism  $L_a^x R_b^y$ . This follows immediately the formulas:  $L_a^x L_b^y(a) = (a^x b)^y a$ ,  $L_a^x L_b^y(b) = a^x b$ ,  $L_a^x R_b^y(a) = a(a^x b)^y$ ,  $L_a^x R_b^y(b) = a^x b$  ( $L_a^x L_b^y \triangleleft_{(a^x b)^y} L_a^x R_b^y$ ).

A basic property of conjugacy is [9, 13]: for morphisms  $f, f', g, g'$ , and words  $u, u'$ , if  $f \triangleleft_u g$  and  $f' \triangleleft_{u'} g'$  then  $ff' \triangleleft_{f(u')u} gg'$  (of course  $f(u')u = ug(u')$ ). This property extends to strong conjugacy:

**Lemma 4.1** *Let  $f, f', g, g'$ , ( $a$  a letter) and  $u, u'$  words such that  $f$  is strongly (on  $a$ )  $u$ -conjugated to  $g$  and  $f'$  is strongly (on  $a$ )  $u'$ -conjugated to  $g'$ . Then  $ff'$  is strongly (on  $a$ )  $[f(u')u]$ -conjugated to  $gg'$ .*

*Proof.* We already know  $ff' \triangleleft_{f(u')u} gg'$ . By hypothesis,  $f(a) = ua$ ,  $g(a) = au$ ,  $f'(a) = u'a$  et  $g'(a) = au'$ . Thus  $ff'(a) = f(u'a) = f(u')ua$  and  $gg'(a) = g(au') = aug(u') = af(u')u$ . So  $ff'$  is strongly  $[f(u')u]$ -conjugated to  $gg'$ .  $\square$

We end this section with a first use of strong conjugacy concerning Sturmian words. One particular property of any standard Sturmian word  $w$  over  $\{a < b\}$  is that both  $aw$  and  $bw$  are Sturmian words [16]. Words  $aw$  (with  $w$  standard Sturmian) are also known as Christoffel words. In [3], it is shown, that Christoffel words are infinite Lyndon words:

**Proposition 4.2** [3] *For any standard Sturmian word  $w$  over  $\{a < b\}$ ,  $aw$  is an infinite Lyndon word.*

*Proof.* Let  $w$  be a standard word with directive sequence  $(d_n)_{n \geq 1}$ . We have already said that a standard word can be viewed as  $w = \lim_{n \rightarrow \infty} s_n$  for some words  $s_n$  defined in Section 2. In fact, we can verify that then  $w = \lim_{n \rightarrow \infty} s_{2n}$ . Let  $n \geq 1$ . We know that  $s_{2n} = L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}}(a)$ . As a consequence of Lemma 4.1 and of the fact that for all integers  $x$  and  $y$ , the morphism  $L_a^x L_b^y$  is strongly conjugated to the morphism  $L_a^x R_b^y$ , we can verify that  $L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}}$  is strongly conjugated to  $L_a^{d_1} R_b^{d_2} \dots L_a^{d_{2n-1}} R_b^{d_{2n}}$ .

In particular,  $aL_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}}(a) = L_a^{d_1} R_b^{d_2} \dots L_a^{d_{2n-1}} R_b^{d_{2n}}(a)a$ . By Proposition 3.2, the morphism  $L_a^{d_1} R_b^{d_2} \dots L_a^{d_{2n-1}} R_b^{d_{2n}}$  preserves Lyndon words. Hence  $L_a^{d_1} R_b^{d_2} \dots L_a^{d_{2n-1}} R_b^{d_{2n}}(a)$  is a Lyndon word. Consequently the word  $w$  has an infinity of Lyndon words as prefixes. It is a Lyndon word.  $\square$

Let us note that the previous proof technique can be used to state other results. For instance, we let the reader prove:

**Proposition 4.3** *Let  $A$  be an alphabet and  $a$  a letter in  $A$ . Let  $f, g$  be two nonerasing endomorphisms on  $A$  and let  $u$  be a word over  $A$  such that  $f$  is  $u$ -strongly conjugate to  $g$ . Then  $f^\omega(a)$  and  $g^\omega(a)$  exist and  $af^\omega(a) = g^\omega(a)$ .*

Thus if  $g$  generates on  $a$  an infinite Lyndon word (which is the case if it preserves Lyndon words or if it preserves infinite Lyndon words (see [15])),  $af^\omega(a)$  is an infinite Lyndon word.

The situation of Proposition 4.3 can be met for morphisms that are not Sturmian. For instance, this is the case with the morphisms:

$$f : \begin{cases} a \mapsto aba \\ b \mapsto abb \end{cases} \qquad g : \begin{cases} a \mapsto aab \\ b \mapsto bab \end{cases}$$

Moreover one can see that  $g$  preserves infinite Lyndon words and generates an infinite Lyndon word.

## 5 An answer to G. Melançon

In this section, we consider a standard Sturmian word  $w$  over the ordered alphabet  $\{a < b\}$  with directive sequence  $(d_n)_{n \geq 1}$  (Let recall that  $d_1 \geq 0$  and  $d_n \geq 1$  for all  $n \geq 2$ ). The sequence of words  $(s_n)_{n \geq 0}$  and  $(\ell_n)_{n \geq 0}$  are those defined respectively at the end of Section 2 and in Theorem 3.1. In particular,  $w = \lim_{n \rightarrow \infty} \prod_{n \geq 0} \ell_n^{d_{2n}}$  is the decomposition in Lyndon words of  $w$  (for each  $n \geq 0$ ,  $\ell_n$  is a Lyndon word and  $\ell_{n+1} \preceq \ell_n$ ). Our result is:

**Theorem 5.1** *With the hypotheses of this section, there exists a morphism  $g$  such that for all  $n \geq 0$ ,  $\ell_{n+1} = g(\ell_n)$  if and only if one of the two following cases hold:*

- $1 \leq d_1 \leq d_3$ , and for all  $n \geq 1$ ,  $d_{2n} = d_2$  and  $d_{2n+1} = d_3$ . In this case,  $\ell_0 = a^{d_1}b$  and  $g = L_a^{d_1} R_b^{d_2} L_a^{d_3-d_1}$ .
- $d_1 = 0$ ,  $1 \leq d_2 \leq d_4$ , and for all  $n \geq 1$ ,  $d_{2n+2} = d_4$  and  $d_{2n+1} = d_3$ . In this case,  $\ell_0 = b$  and  $g = R_b^{d_2} L_a^{d_3} R_b^{d_4-d_2}$ .

We observe that in each case, the morphism  $g$  is a Sturmian morphism that preserves Lyndon words (see Proposition 3.2). Moreover the word  $w$  is generated by a Sturmian morphism  $(L_a^{d_1} L_b^{d_2} L_a^{d_3-d_1}$  or  $L_b^{d_2} L_a^{d_3} L_b^{d_4-d_2})$ .

In order to prove the previous theorem, using the strong conjugacy, we first express each Lyndon word  $\ell_n$  with morphisms. For  $n \geq 0$ , we denote:

$$f_n = (L_a^{d_1} L_b^{d_2}) \dots (L_a^{d_{2n-1}} L_b^{d_{2n}})$$

$$g_n = (L_a^{d_1} R_b^{d_2}) \dots (L_a^{d_{2n-1}} R_b^{d_{2n}})$$

The interest of the morphisms  $f_n$  is immediate since we have already seen relations between them and the words  $s_n$  ( $s_{2n} = f_n(a)$ ,  $s_{2n+1} = f_{n+1}(b)$ ). We also observe that each  $g_n$  is a morphism that preserves Lyndon words. As a consequence of Lemma 4.1 and of the fact that for all integers  $x$  and  $y$ , the morphism  $L_a^x L_b^y$  is strongly conjugated to the morphism  $L_a^x R_b^y$ , we have:

**Lemma 5.2** *For all  $n \geq 1$ ,  $f_n$  is strongly (on  $a$ ) conjugated to  $g_n$ .*



Now we give a new formula for the words  $(\ell_n)_{n \geq 0}$ :

**Lemma 5.3** For all  $n \geq 0$ ,  $\ell_n = g_n L_a^{d_{2n+1}}(b)$

*Proof.*

$$\begin{aligned} \ell_n a &= a s_{2n}^{d_{2n+1}-1} s_{2n-1} \overline{s_{2n}} a \\ &= a s_{2n}^{d_{2n+1}-1} s_{2n-1} s_{2n} \\ &= a f_n(a^{d_{2n+1}-1} b a). \end{aligned}$$

If  $n = 0$ ,  $\ell_n a = a^{d_1} b a = L_a^{d_1}(b) a = g_0 L_a^{d_1}(b) a$ .

When  $n \geq 1$ , let  $u_n$  be the word such that  $f \triangleleft_{u_n} g_n$ . By Lemma 5.2,  $f_n(a) = u_n a$ ,  $g_n(a) = a u_n$ . Thus

$$\begin{aligned} \ell_n a &= a f_n(a^{d_{2n+1}-1} b) u_n a \\ &= a u_n g_n(a^{d_{2n+1}-1} b) a \\ &= g_n(a^{d_{2n+1}} b) a \\ &= g_n L_a^{d_{2n+1}}(b) a \end{aligned}$$

Consequently for all  $n \geq 0$ ,  $\ell_n = g_n L_a^{d_{2n+1}}(b)$ .  $\square$

Let us observe that Lemma 5.2 allows to give a new proof of the fact that the words  $(\ell_n)_{n \geq 0}$  form a strictly decreasing sequence of Lyndon words. Indeed, by Proposition 3.2, each morphism  $g_n L_a^{d_{2n+1}}$  is a Lyndon morphism, hence  $g_n L_a^{d_{2n+1}}$  is a Lyndon word. Moreover  $R_b^{d_{2n}} L_a^{d_{2n+1}}(b)$  for each  $n \geq 1$ , then  $R_b^{d_{2n}} L_a^{d_{2n+1}}(b) \prec b$  which implies  $\ell_n = g_n L_a^{d_{2n+1}}(b) \prec g_{n-1} L_a^{d_{2n-1}}(b) = \ell_{n-1}$  (since any morphism preserving Lyndon words also strictly preserves the lexicographic order on finite words [14]).

**Proof of Theorem 5.1.** Note that the “if” part of the theorem is immediate. Assume the sequence  $(\ell_n)_{n \geq 0}$  is morphic. Let  $g$  be the morphism such that, for all  $n \geq 0$ ,  $g(\ell_n) = \ell_{n+1}$ . Observe that the morphism  $g$  cannot be erasing since otherwise this contradicts the fact that  $\ell_2$  is a primitive word (as a Lyndon word).

We first consider the case  $d_1 \geq 2$ . Observe  $\ell_0 = a^{d_1} b$  and

$$g(a^{d_1} b) = \ell_1 = [a(a^{d_1} b)^{d_2}]^{d_3} a^{d_1} b.$$

Assume  $g(a) = a$ , and so  $g(b) = ab(a^{d_1} b)^{d_2-1} [a(a^{d_1} b)^{d_2}]^{d_3-1} a^{d_1} b$ . The word  $\ell_2 = g(\ell_1)$  has  $g(a^{d_1+1} b)$  as prefix. Thus the words  $a^{d_1+2}$  and  $ba^{d_1} b$  are factors of  $\ell_2$ . This contradicts the fact that  $\ell_2$ , as a factor of a Sturmian word, is balanced. Hence  $g(a) \neq a$ .

Since  $d_1 \geq 2$  and  $g(a^{d_1} b)$  starts with  $a^{d_1+1} b$ , the word  $a^{d_1+1} b$  is a prefix of  $g(a)$ . More precisely,  $a(a^{d_1} b)^{d_2}$  must be a prefix of  $g(a)$ . Finally, we can verify that  $g(a) = (a(a^{d_1} b)^{d_2})^k$  for an integer  $k \geq 1$ . It follows  $g(b) = (a(a^{d_1} b)^{d_2})^{d_3-kd_1} a^{d_1} b$  which implies  $d_3 \geq kd_1$ .

Assume  $k \geq 2$ . The word  $\ell_2 = g(\ell_1)$  contains  $g(ba^{d_1} b)$  and  $g(a^{d_1+1} b)$  as factors. The word  $g(ba^{d_1} b)$  ends with  $bub$  where  $u = (a^{d_1} b)^{d_2} [a(a^{d_1} b)^{d_2}]^{d_3} a^{d_1}$ . Furthermore the word  $g(a^{d_1+1} b) = [a(a^{d_1} b)^{d_2}]^{d_3+k} a^{d_1} b$  starts with  $aua$ . This contradicts the fact that  $\ell_2$  is balanced.

Hence  $k = 1$ ,  $d_3 \geq d_1$ ,  $g(a) = a(a^{d_1}b)^{d_2}$ ,  $g(b) = [a(a^{d_1}b)^{d_2}]^{d_3-d_1}a^{d_1}b$ . We observe that  $g = L_a^{d_1}R_b^{d_2}L_a^{d_3-d_1}$  and that it is an injective morphism.

Now we can prove that, for all  $n \geq 1$ ,  $d_{2n} = d_2$  and  $d_{2n+1} = d_3$ . We act by induction on  $n$ . There is nothing to do for  $n = 1$ . Let  $n \geq 1$ . Assume that we have already proved  $d_{2p} = d_2$  and  $d_{2p+1} = d_3$  for all integers  $p$  with  $1 \leq p \leq n$ . We have  $\ell_{n+1} = g_{n+1}L_a^{d_{2n+3}}(b) = L_a^{d_1}(R_b^{d_2}L_a^{d_3})^n R_b^{d_{2n+2}}L_a^{d_{2n+3}}(b) = (L_a^{d_1}R_b^{d_2}L_a^{d_3-d_1})^n L_a^{d_1}R_b^{d_{2n+2}}L_b^{d_{2n+3}}(b) = g^n(L_a^{d_1}R_b^{d_{2n+2}}L_b^{d_{2n+3}}(b))$ . Moreover  $\ell_{n+1} = g^n(\ell_1)$ . Since  $g$  is injective,  $\ell_1 = L_a^{d_1}R_b^{d_{2n+2}}L_b^{d_{2n+3}}(b)$ . This implies  $d_{2n+2} = d_2$  and  $d_{2n+3} = d_3$ .

Now we consider the case  $d_1 = 1$ . We have  $\ell_0 = ab$  and  $\ell_1 = [a(ab)^{d_2}]^{d_3}ab$ . As in case  $d_1 \geq 2$ , we cannot have  $g(a) = a$ . Hence  $g(a)$  starts with  $aa$ . We observe that  $g(a)$  cannot ends with  $a$ , since otherwise the balanced word  $\ell_2 = g(\ell_1)$  contains  $aaa$  and  $bab$ . We observe also that  $g(a) \neq [a(ab)^{d_2}]^i a(ab)^k$  for any integer  $k, i$  such that  $1 \leq k < d_2$  and  $i \geq 0$ . Indeed otherwise the word  $\ell_2$  containing both  $g(aa)$  and  $g(ab)$  should contains the factors  $a(ab)^k aa$  and  $b(ab)^k ab$  (since  $(ab)^{d_2+1}$  ends  $g(ab)$ ): this contradicts the fact that  $\ell_2$  is balanced. It follows that  $g(a) = [a(ab)^{d_2}]^k$  with  $1 \leq k \leq d_3$  and  $g(b) = [a(a^{d_1}b)^{d_2}]^{d_3-k}a^{d_1}b$ . Exactly as in case  $d_1 \geq 2$ , we can then prove that  $k = 1$ ,  $g = L_a R_b^{d_2} L_a^{d_3-1}$  and for all integers  $n \geq 1$ ,  $d_{2n} = d_2$  and  $d_{2n+1} = d_3$ .

From now on, we consider the case  $d_1 = 0$ . we have  $\ell_0 = b$  and so  $g(b) = \ell_1 = (ab^{d_2})^{d_3}b$ . Moreover  $\ell_2 = R_b^{d_2}L_a^{d_3}R_b^{d_4}L_a^{d_5}(b)$ , that is

$$\ell_2 = [ab^{d_2}[(ab^{d_2})^{d_3}b]^{d_4}]^{d_5}(ab^{d_2})^{d_3}b.$$

Furthermore  $\ell_2 = g^2(b) = g((ab^{d_2})^{d_3}b)$ . It follows that

$$g((ab^{d_2})^{d_3}) = [ab^{d_2}g(b)^{d_4}]^{d_5}$$

Since the word  $ab^{d_2}g(b)^{d_4} = ab^{d_2}[(ab^{d_2})^{d_3}b]^{d_4}$  is a primitive word,  $g(ab^{d_2}) = [ab^{d_2}g(b)^{d_4}]^x$  and  $xd_3 = d_5$  for an integer  $x \geq 1$ . Since  $ab^{d_2}$  is not a suffix of  $g(b)$ ,  $d_2 \leq d_4$ .

Let us prove that  $x = 1$ , that is,  $d_3 = d_5$ . Assume by contradiction that  $x \geq 2$ . The word  $\ell_2$  has  $(ab^{d_2})^{d_3+1}$  as a prefix and  $[(ab^{d_2})^{d_3}b]^2$  as a suffix. Let  $u = ab^{d_2}g(b)^{d_4}$ :  $g(ab^{d_2}) = u^x$ . The word  $\ell_3 = g(\ell_2)$  contains the factor  $g((ab^{d_2})^{d_3+1}) = u^{(d_3+1)x} = uu^{d_5}uu^{x-2}$  which contains the factor  $ab^{d_2}g(b)^{d_4}u^{d_5}ab^{d_2}(ab^{d_2})^{d_3}b$  which starts with  $ab^{d_2}g(b)^{d_4}u^{d_5}(ab^{d_2})^{d_3}a$ . Observe now that  $g((ab^{d_2})^{d_3}b) = [ab^{d_2}g(b)^{d_4}]^{d_5}g(b)$  ends with  $b^{d_2+1}g(b)^{d_4}$ . Consequently the word  $\ell_3$  also contains the factor  $b^{d_2+1}g(b)^{d_4}g((ab^{d_2})^{d_3}b) = bb^{d_2}g(b)^{d_4}u^{d_5}(ab^{d_2})^{d_3}b$ . We have a contradiction with the fact that  $\ell_3$  is a balanced word.

From what precedes,  $g(ab^{d_2}) = ab^{d_2}g(b)^{d_4}$  and so  $g(a) = ab^{d_2}g(b)^{d_4-d_2} = ab^{d_2}((ab^{d_2})^{d_3}b)^{d_4-d_2}$ . Moreover  $g(b) = (ab^{d_2})^{d_3}b$ . We observe  $g = R_b^{d_2}L_a^{d_3}R_b^{d_4-d_2}$ . As in case  $d_1 \geq 2$ , we can state that, for all integers  $n \geq 2$ ,  $d_{2n} = d_4$  and  $d_{2n-1} = d_3$ .  $\square$

## 6 Conclusion

This paper shows the interest of conjugacy of morphisms and of morphisms preserving Lyndon words as tools to tackle problems concerning Sturmian words and/or Lyndon words. We are now working to find other situations where these tools can be useful. In particular, we are looking for the decomposition in Lyndon words of any Sturmian words.

## Acknowledgement

Many thanks to P. Séébold for useful discussions on strong conjugacy.

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