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Abstract

In this work, we give a description of all \( \sigma \)-finite measures on the space of rooted compact \( \mathbb{R} \)-trees which satisfy a certain regenerative property. We show that any infinite measure which satisfies the regenerative property is the "law" of a Lévy tree, that is, the "law" of a tree-valued random variable that describes the genealogy of a population evolving according to a continuous-state branching process. On the other hand, we prove that a probability measure with the regenerative property must be the law of the genealogical tree associated with a continuous-time discrete-state branching process.

1 Introduction

Galton-Watson trees are well known to be characterized among all random discrete trees by a regenerative property. More precisely, if \( \gamma \) is a probability measure on \( \mathbb{Z}_+ \), the law \( \Pi_\gamma \) of the Galton-Watson tree with offspring distribution \( \gamma \) is uniquely determined by the following two conditions: Under the probability measure \( \Pi_\gamma \),

(i) the ancestor has \( p \) children with probability \( \gamma(p) \),

(ii) if \( \gamma(p) > 0 \), then conditionally on the event that the ancestor has \( p \) children, the \( p \) subtrees which describe the genealogy of the descendants of these children, are independent and distributed according to \( \Pi_\gamma \).

The aim of this work is to study \( \sigma \)-finite measures satisfying an analogue of this property on the space of equivalence classes of rooted compact \( \mathbb{R} \)-trees.

An \( \mathbb{R} \)-tree is a metric space \((T, d)\) such that for any two points \( \sigma_1 \) and \( \sigma_2 \) in \( T \), there is a unique arc with endpoints \( \sigma_1 \) and \( \sigma_2 \), and furthermore this arc is isometric to a compact interval of the real line. In this work, all \( \mathbb{R} \)-trees are supposed to be compact. A rooted \( \mathbb{R} \)-tree is an \( \mathbb{R} \)-tree with a distinguished vertex called the root. Say that two rooted \( \mathbb{R} \)-trees
are equivalent if there is a root-preserving isometry that maps one onto the other. It was noted in [8] that the set $T$ of all equivalence classes of rooted compact $\mathbb{R}$-trees equipped with the pointed Gromov-Hausdorff distance $d_{GH}$ (see e.g. Chapter 7 in [4]), is a Polish space. Hence it is legitimate to consider random variables with values in $T$, that is, random $\mathbb{R}$-trees. A particularly important example is the CRT, which was introduced by Aldous [1], [2] with a different formalism. Striking applications of the concept of random $\mathbb{R}$-trees can be found in the recent papers [8] and [9].

Let $T$ be an $\mathbb{R}$-tree. We write $\mathcal{H}(T)$ for the height of the $\mathbb{R}$-tree $T$, that is the maximal distance from the root to a vertex of $T$. For every $t \geq 0$, we denote by $T_{\leq t}$ the set of all vertices of $T$ which are at distance at most $t$ from the root, and by $T_{> t}$ the set of all vertices which are at distance greater than $t$ from the root. To each connected component of $T_{> t}$ there corresponds a ”subtree” of $T$ above level $t$ (see section 2.2.3 for a more precise definition). For every $h > 0$, we define $Z(t, t+h)(T)$ as the number of subtrees of $T$ above level $t$ with height greater than $h$.

Let $\Theta$ be a $\sigma$-finite measure on $T$, such that $0 < \Theta(\mathcal{H}(T) > t) < \infty$ for every $t > 0$ and $\Theta(\mathcal{H}(T) = 0) = 0$. We say that $\Theta$ satisfies the regenerative property (R) if the following holds:

(R) For every $t, h > 0$ and $p \in \mathbb{N}$, under the probability measure $\Theta(\cdot | \mathcal{H}(T) > t)$ and conditionally on the event $\{Z(t, t+h) = p\}$, the $p$ subtrees of $T$ above level $t$ with height greater than $h$ are independent and distributed according to the probability measure $\Theta(\cdot | \mathcal{H}(T) > h)$.

This is a natural analogue of the regenerative property stated above for Galton-Watson trees. Beware that, unlike the discrete case, there is no natural order on the subtrees above a given level. So, the preceding property should be understood in the sense that the unordered collection of the $p$ subtrees in consideration is distributed as the unordered collection of $p$ independent copies of $\Theta(\cdot | \mathcal{H}(T) > h)$.

Property (R) is known to be satisfied by a wide class of infinite measures on $T$, namely the ”laws” of Lévy trees. Lévy trees have been introduced by T. Duquesne and J.F. Le Gall in [7]. Their precise definition is recalled in section 2.3, but let us immediately give an informal presentation.

Let $Y$ be a critical or subcritical continuous-state branching process. The distribution of $Y$ is characterized by its branching mechanism function $\psi$. Assume that $Y$ becomes extinct a.s., which is equivalent to the condition $\int_1^\infty \psi(u)^{-1}du < \infty$. The $\psi$-Lévy tree is a random variable taking values in $(T, d_{GH})$, which describes the genealogy of a population evolving according to $Y$ and starting with infinitesimally small mass. More precisely, the ”law” of the Lévy tree is defined in [7] as a $\sigma$-finite measure on the space $(T, d_{GH})$, such that $0 < \Theta_\psi(\mathcal{H}(T) > t) < \infty$ for every $t > 0$. As a consequence of Theorem 4.2 of [7], the measure $\Theta_\psi$ satisfies Property (R). In the special case $\psi(u) = u^\alpha$, $1 < \alpha \leq 2$ corresponding to the so-called stable trees, this was used by Miermont [12], [13] to introduce and to study certain fragmentation processes.
In the present work we describe all $\sigma$-finite measures on $\mathbb{T}$ that satisfy Property (R). We show that the only infinite measures satisfying Property (R) are the measures $\Theta_\psi$ associated with Lévy trees. On the other hand, if $\Theta$ is a finite measure satisfying Property (R), we can obviously restrict our attention to the case $\Theta(\mathbb{T}) = 1$ and we obtain that $\Theta$ must be the law of the genealogical tree associated with a continuous-time discrete-state branching process.

**Theorem 1.1** Let $\Theta$ be an infinite measure on $(\mathbb{T}, d_{GH})$ such that $\Theta(\mathcal{H}(T) = 0) = 0$ and $0 < \Theta(\mathcal{H}(T) > t) < +\infty$ for every $t > 0$. Assume that $\Theta$ satisfies Property (R). Then, there exists a continuous-state branching process, whose branching mechanism is denoted by $\psi$, which becomes extinct almost surely, such that $\Theta = \Theta_\psi$.

**Theorem 1.2** Let $\Theta$ be a probability measure on $(\mathbb{T}, d_{GH})$ such that $\Theta(\mathcal{H}(T) = 0) = 0$ and $0 < \Theta(\mathcal{H}(T) > t) < +\infty$ for every $t > 0$. Assume that $\Theta$ satisfies Property (R). Then there exists $a > 0$ and a critical or subcritical probability measure $\gamma$ on $\mathbb{Z}_+ \setminus \{1\}$ such that $\Theta$ is the law of the genealogical tree for a discrete-space continuous-time branching process with offspring distribution $\gamma$, where branchings occur at rate $a$.

In other words, $\Theta$ in Theorem 1.2 can be described in the following way: There exists a real random variable $J$ such that under $\Theta$:

(i) $J$ is distributed according to the exponential distribution with parameter $a$ and there exists $\sigma_J \in \mathbb{T}$ such that $T_{<J} = [\rho, \sigma_J]$,

(ii) the number of subtrees above level $J$ is distributed according to $\gamma$ and is independent of $J$,

(iii) for every $p \geq 2$, conditionally on $J$ and given the event that the number of subtrees above level $J$ is equal to $p$, these $p$ subtrees are independent and distributed according to $\Theta$.

Theorem 1.1 is proved in section 3, after some preliminary results have been established in section 2. A key idea of the proof is to use the regenerative property to embed discrete Galton-Watson trees in our random real trees (Lemma 3.3). A technical difficulty comes from the fact that real trees are not ordered whereas Galton-Watson trees are usually defined as random ordered discrete trees (cf subsection 2.2.4 below). To overcome this difficulty, we assign a random ordering to the discrete trees embedded in real trees. Another major ingredient of the proof of Theorem 1.1 is the construction of a "local time" $L_t$ at every level $t$ of a random real tree governed by $\Theta$. The local time process is then shown to be a continuous-state branching process with branching mechanism $\psi$, which makes it possible to identify $\Theta$ with $\Theta_\psi$. Theorem 1.2 is proved in section 4. Several arguments are similar to the proof of Theorem 1.1, so that we have skipped some details.
2 Preliminaries

In this section, we recall some basic facts about branching processes, $\mathbb{R}$-trees and Lévy trees.

2.1 Branching processes

2.1.1 Continuous-state branching processes

A (continuous-time) continuous-state branching process (in short a CSBP) is a Markov process $Y = (Y_t, t \geq 0)$ with values in the positive half-line $[0, +\infty)$, with a Feller semi-group $(Q_t, t \geq 0)$ satisfying the following branching property: For every $t \geq 0$ and $x, x' \geq 0$,

$$Q_t(x, \cdot) * Q_t(x', \cdot) = Q_t(x + x', \cdot).$$

Informally, this means that the union of two independent populations started respectively at $x$ and $x'$ will evolve like a single population started at $x + x'$.

We will consider only the critical or subcritical case, meaning that, for every $t \geq 0$ and $x \geq 0$,

$$\int_{[0, +\infty)} yQ_t(x, dy) \leq 1.$$

Then, if we exclude the trivial case where $Q_t(x, \cdot) = \delta_0$ for every $t > 0$ and $x \geq 0$, the Laplace functional of the semigroup can be written in the following form: For every $\lambda \geq 0$,

$$\int_{[0, +\infty)} e^{-\lambda y}Q_t(x, dy) = \exp(-xu(t, \lambda)),$$

where the function $(u(t, \lambda), t \geq 0, \lambda \geq 0)$ is determined by the differential equation

$$\frac{du(t, \lambda)}{dt} = -\psi(u(t, \lambda)), \; u(0, \lambda) = \lambda,$$

and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of the form

$$\psi(u) = \alpha u + \beta u^2 + \int_{(0, +\infty)} (e^{-ur} - 1 + ur)\pi(dr), \tag{1}$$

where $\alpha, \beta \geq 0$ and $\pi$ is a $\sigma$-finite measure on $(0, +\infty)$ such that $\int_{(0, +\infty)} (r \wedge r^2)\pi(dr) < \infty$. The process $Y$ is called the $\psi$-continuous-state branching process (in short the $\psi$-CSBP).

Continuous-state branching processes may also be obtained as weak limits of rescaled Galton-Watson processes. We recall that an offspring distribution is a probability measure on $\mathbb{Z}_+$. An offspring distribution $\mu$ is said to be critical if $\sum_{i \geq 0} i\mu(i) = 1$ and subcritical if $\sum_{i \geq 0} i\mu(i) < 1$. Let us state a result that can be derived from [10] and [11].
Theorem 2.1 Let \((\mu_n)_{n \geq 1}\) be a sequence of offspring distributions. For every \(n \geq 1\), denote by \(X^n\) a Galton-Watson process with offspring distribution \(\mu_n\), started at \(X^n_0 = n\). Let \((m_n)_{n \geq 1}\) be a nondecreasing sequence of positive integers converging to infinity. We define a sequence of processes \((Y^n)_{n \geq 1}\) by setting, for every \(t \geq 0\) and \(n \geq 1\),

\[
Y^n_t = n^{-1}X^n_{\lfloor mn_t \rfloor}.
\]

Assume that, for every \(t \geq 0\), the sequence \((Y^n_t)_{n \geq 1}\) converges in distribution to \(Y_t\) where \(Y = (Y_t, t \geq 0)\) is an almost surely finite process such that \(\mathbb{P}(Y_\delta > 0) > 0\) for some \(\delta > 0\). Then, \(Y\) is a continuous-state branching process and the sequence of processes \((Y^n)_{n \geq 1}\) converges to \(Y\) in distribution in the Skorokhod space \(\mathbb{D}(\mathbb{R}_+)\).

Proof: It follows from the proof of Theorem 1 of [11] that \(Y\) is a CSBP. Then, thanks to Theorem 2 of [11], there exists a sequence of offspring distributions \((\nu_n)_{n \geq 1}\) and a nondecreasing sequence of positive integers \((c_n)_{n \geq 1}\) such that we can construct for every \(n \geq 1\) a Galton-Watson process \(Z^n\) started at \(c_n\) and with offspring distribution \(\nu_n\) satisfying

\[
\left(c^{-1}_nZ^n_{\lfloor nt \rfloor}, t \geq 0\right) \xrightarrow{d_{n \to \infty}} (Y_t, t \geq 0),
\]

where the symbol \(\xrightarrow{d}\) indicates convergence in distribution in \(\mathbb{D}(\mathbb{R}_+)\).

Let \((m_{nk})_{k \geq 1}\) be a strictly increasing subsequence of \((m_n)_{n \geq 1}\). For \(n \geq 1\), we set \(B^n = X^{nk}\) and \(b_n = n_k\) if \(n = m_{nk}\) for some \(k \geq 1\), and we set \(B^n = Z^n\) and \(b_n = c_n\) if there is no \(k \geq 1\) such that \(n = m_{nk}\). Then, for every \(t \geq 0\), \((b^{-1}_nB^{n}_{\lfloor nt \rfloor})_{n \geq 1}\) converges in distribution to \(Y_t\). Applying Theorem 4.1 of [10], we obtain that

\[
\left(b^{-1}_nB^{n}_{\lfloor nt \rfloor}, t \geq 0\right) \xrightarrow{d_{n \to \infty}} (Y_t, t \geq 0).
\]

In particular, we have,

\[
\left(Y^{nk}_t, t \geq 0\right) \xrightarrow{k \to \infty} (Y_t, t \geq 0).
\]

(2)

As (2) holds for every strictly increasing subsequence of \((m_n)_{n \geq 1}\), we obtain the desired result.

2.1.2 Discrete-state branching processes

A (continuous-time) discrete-state branching process (in short DSBP) is a continuous-time Markov chain \(Y = (Y_t, t \geq 0)\) with values in \(\mathbb{Z}_+\) whose transition probabilities \((P_t(i, j), t \geq 0)_{i,j \geq 0,j \geq 0}\) satisfy the following branching property: For every \(i \in \mathbb{Z}_+, t \geq 0\) and \(|s| \leq 1\),

\[
\sum_{j=0}^{\infty} P_t(i, j)s^j = \left(\sum_{j=0}^{\infty} P_t(1, j)s^j\right)^i.
\]
We exclude the trivial case where $P_t(i, i) = 1$ for every $t \geq 0$ and $i \in \mathbb{Z}_+$. Then, there exist $a > 0$ and an offspring distribution $\gamma$ with $\gamma(1) = 0$ such that the generator of $Y$ can be written of the form

$$Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots \\
-a & a\gamma(2) & a\gamma(3) & a\gamma(4) & \ldots \\
0 & -2a & 2a\gamma(2) & a\gamma(3) & \ldots \\
0 & 3a\gamma(0) & -3a & 3a\gamma(2) & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}.$$  

Furthermore, it is well known that $Y$ becomes extinct almost surely if and only if $\gamma$ is critical or subcritical. We refer the reader to [3] and [14] for more details.

### 2.2 Deterministic trees

#### 2.2.1 The space $(\mathbb{T}, d_{GH})$ of rooted compact $\mathbb{R}$-trees

We start with a basic definition.

**Definition 2.1** A metric space $(\mathcal{T}, d)$ is an $\mathbb{R}$-tree if the following two properties hold for every $\sigma_1, \sigma_2 \in \mathcal{T}$.

(i) There is a unique isometric map $f_{\sigma_1, \sigma_2}$ from $[0, d(\sigma_1, \sigma_2)]$ into $\mathcal{T}$ such that

$$f_{\sigma_1, \sigma_2}(0) = \sigma_1 \text{ and } f_{\sigma_1, \sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2.$$  

(ii) If $q$ is a continuous injective map from $[0, 1]$ into $\mathcal{T}$ such that $q(0) = \sigma_1$ and $q(1) = \sigma_2$, we have

$$q([0, 1]) = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)]).$$

A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root.

In what follows, $\mathbb{R}$-trees will always be rooted.

Let $(\mathcal{T}, d)$ be an $\mathbb{R}$-tree with root $\rho$, and $\sigma, \sigma_1, \sigma_2 \in \mathcal{T}$. We write $[[\sigma_1, \sigma_2]]$ for the range of the map $f_{\sigma_1, \sigma_2}$. In particular, $[[\rho, \sigma]]$ is the path going from the root to $\sigma$ and can be interpreted as the ancestral line of $\sigma$.

The height $\mathcal{H}(\mathcal{T})$ of the $\mathbb{R}$-tree $\mathcal{T}$ is defined by $\mathcal{H}(\mathcal{T}) = \sup\{d(\rho, \sigma) : \sigma \in \mathcal{T}\}$. In particular, if $\mathcal{T}$ is compact, its height $\mathcal{H}(\mathcal{T})$ is finite.

Two rooted $\mathbb{R}$-trees $\mathcal{T}$ and $\mathcal{T}'$ are called equivalent if there is a root-preserving isometry that maps $\mathcal{T}$ onto $\mathcal{T}'$. We denote by $\mathbb{T}$ the set of all equivalence classes of rooted compact $\mathbb{R}$-trees. We often abuse notation and identify a rooted compact $\mathbb{R}$-tree with its equivalence class.
The set \( \mathcal{T} \) can be equipped with the pointed Gromov-Hausdorff distance, which is defined as follows. If \( (E, \delta) \) is a metric space, we use the notation \( \delta_{\text{Haus}} \) for the usual Hausdorff metric between compact subsets of \( E \). Then, if \( T \) and \( T' \) are two rooted compact \( \mathbb{R} \)-trees with respective roots \( \rho \) and \( \rho' \), we define the distance \( d_{GH}(T, T') \) as

\[
d_{GH}(T, T') = \inf \{ \delta_{\text{Haus}}(\phi(T), \phi'(T')) \lor \delta(\phi(\rho), \phi'(\rho')) \},
\]

where the infimum is over all isometric embeddings \( \phi : T \to E \) and \( \phi' : T' \to E \) into a common metric space \( (E, \delta) \). We see that \( d_{GH}(T, T') \) only depends on the equivalence classes of \( T \) and \( T' \). Furthermore, according to Theorem 2 in [8], \( d_{GH} \) defines a metric on \( \mathcal{T} \) that makes it complete and separable. Notice that \( d_{GH}(T, T') \) makes sense more generally if \( T \) and \( T' \) are pointed compact metric spaces (see e.g. Chapter 7 in [4]). We will use this in the proof of Lemma 2.2 below.

We equip \( \mathcal{T} \) with its Borel \( \sigma \)-field. If \( T \in \mathcal{T} \), we set \( T \leq t = \{ \sigma \in T : d(\rho, \sigma) \leq t \} \) for every \( t \geq 0 \). Plainly, \( T \leq t \) is a real tree whose root is the same as the root of \( T \). Note that the mapping \( T \mapsto T \leq t \) from \( \mathcal{T} \) into \( \mathcal{T} \) is Lipschitz for the Gromov-Hausdorff metric.

### 2.2.2 The \( \mathbb{R} \)-tree coded by a function

We now recall a construction of rooted compact \( \mathbb{R} \)-trees which is described in [7]. Let \( g : [0, +\infty) \to [0, +\infty) \) be a continuous function with compact support satisfying \( g(0) = 0 \). We exclude the trivial case where \( g \) is identically zero. For every \( s, t \geq 0 \), we set

\[
m_g(s, t) = \inf_{r \in s \wedge t, s \vee t} g(r)
\]

and

\[
d_g(s, t) = g(s) + g(t) - 2m_g(s, t).
\]

We define an equivalence relation \( \sim \) on \( [0, +\infty) \) by declaring that \( s \sim t \) if and only if \( d_g(s, t) = 0 \) (or equivalently if and only if \( g(s) = g(t) = m_g(s, t) \)). Let \( \mathcal{T}_g \) be the quotient space

\[
\mathcal{T}_g = [0, +\infty)/\sim.
\]

Then, \( d_g \) induces a metric on \( \mathcal{T}_g \) and we keep the notation \( d_g \) for this metric. According to Theorem 2.1 of [7], the metric space \( (\mathcal{T}_g, d_g) \) is a compact \( \mathbb{R} \)-tree. By convention, its root is the equivalence class of 0 for \( \sim \) and is denoted by \( \rho_g \).

### 2.2.3 Subtrees of a tree above a fixed level

Let \( (\mathcal{T}, d) \in \mathcal{T} \) and \( t > 0 \). Denote by \( \mathcal{T}^{i:0} \), \( i \in I \) the connected components of the open set \( \mathcal{T}_{>t} = \{ \sigma \in \mathcal{T} : d(\rho(T), \sigma) > t \} \). Let \( i \in I \). Then the ancestor of \( \sigma \) at level \( t \), that is, the unique vertex on the line segment \([\rho, \sigma]\) at distance \( t \) from \( \rho \), must be the same for all \( \sigma \in \mathcal{T}^{i:0} \). We denote by \( \sigma_i \) this common ancestor and set \( \mathcal{T}^i = \mathcal{T}^{i:0} \cup \{ \sigma_i \} \). Then \( \mathcal{T}^i \)
is a compact rooted $\mathbb{R}$-tree with root $\sigma_i$. The trees $T^i, i \in I$ are called the subtrees of $T$ above level $t$. We now consider, for every $h > 0$,
\[ Z(t, t + h)(T) = \# \{ i \in I : H(T^i) > h \}. \]
By a compactness argument, we can easily verify that $Z(t, t + h)(T) < \infty$.

### 2.2.4 Discrete trees and real trees

We start with some formalism for discrete trees. We first introduce the set of labels
\[ U = \bigcup_{n \geq 0} \mathbb{N}^n, \]
where by convention $\mathbb{N}^0 = \{ \emptyset \}$. An element of $U$ is a sequence $u = u^1 \ldots u^n$, and we set $|u| = n$ so that $|u|$ represents the generation of $u$. In particular, $|\emptyset| = 0$. If $u = u_1 \ldots u_n$ and $v = v^1 \ldots v^m$ belong to $U$, we write $uv = u^1 \ldots u^n v^1 \ldots v^m$ for the concatenation of $u$ and $v$. In particular, $\emptyset u = u \emptyset = u$. The mapping $\pi : U \setminus \{ \emptyset \} \rightarrow U$ is defined by $\pi(u^1 \ldots u^n) = u^1 \ldots u_n^{-1}$ ($\pi(u)$ is the father of $u$). Note that $\pi^k(u) = \emptyset$ if $k = |u|$.

A rooted ordered tree $\theta$ is a finite subset of $U$ such that

(i) $\emptyset \in \theta$,

(ii) $u \in \theta \setminus \{ \emptyset \} \Rightarrow \pi(u) \in \theta$,

(iii) for every $u \in \theta$, there exists a number $k_u(\theta) \geq 0$ such that $u_j \in \theta$ if and only if $1 \leq j \leq k_u(\theta)$.

We denote by $A$ the set of all rooted ordered trees. If $\theta \in A$, we write $H(\theta)$ for the height of $\theta$, that is $H(\theta) = \max \{|u| : u \in \theta \}$. And for every $u \in \theta$, we define $\tau_u \theta \in A$ by $\tau_u \theta = \{ v \in U : uv \in \theta \}$. This is the tree $\theta$ shifted at $u$.

Let us define an equivalence relation on $A$ by setting $\theta \sim \theta'$ if and only if we can find a permutation $\varphi_u$ of the set $\{1, \ldots, k_u(\theta)\}$ for every $u \in \theta$ such that $k_u(\theta) \geq 1$, in such a way that
\[ \theta' = \{ \emptyset \} \cup \{ \varphi_{u}^{v_1}(u_1^{v_2}) \ldots \varphi_{u}^{v_{n-1}}(u_n^n) : u_1 \ldots u_n \in \theta, n \geq 1 \}. \]

In other words $\theta \sim \theta'$ if they correspond to the same unordered tree. Let $\mathbb{A} = A/\sim$ be the associated quotient space and let $p : A \rightarrow \mathbb{A}$ be the canonical projection. It is immediate that if $\theta \sim \theta'$, then $k_\emptyset(\theta) = k_\emptyset(\theta')$. So, for every $\xi \in \mathbb{A}$, we may define $k_\emptyset(\xi) = k_\emptyset(\theta)$ where $\theta$ is any representative of $\xi$. Let us fix $\xi \in \mathbb{A}$ such that $k_\emptyset(\xi) = k > 0$ and choose a representative $\theta$ of $\xi$. We can define $\{ \xi_1, \ldots, \xi^k \} = \{ p(\tau_1 \theta), \ldots, p(\tau_k \theta) \}$ as
the unordered family of subtrees of \( \xi \) above the first generation. Then, if \( F : \mathcal{A}^k \rightarrow \mathbb{R}_+ \) is any symmetric measurable function, we have

\[
\left( \#p^{-1}(\xi) \right)^{-1} \sum_{\theta \in p^{-1}(\xi)} F(\tau_1 \theta, \ldots, \tau_k \theta) = \left( \#p^{-1}(\xi^1) \right)^{-1} \ldots \left( \#p^{-1}(\xi^k) \right)^{-1} \sum_{\theta_1 \in p^{-1}(\xi^1)} \ldots \sum_{\theta_k \in p^{-1}(\xi^k)} F(\theta_1, \ldots, \theta_k). \tag{3}
\]

Note that the right-hand side of (3) is well defined since it is symmetric in \( \{\xi^1, \ldots, \xi^k\} \). The identity (3) is a simple combinatorial fact, whose proof is left to the reader.

A marked tree is a pair \( T = (\theta, \{h_u\}_{u \in \theta}) \) where \( \theta \in \mathcal{A} \) and \( h_u \geq 0 \) for every \( u \in \theta \). We denote by \( \mathcal{M} \) the set of all marked trees. We can associate with every marked tree \( T = (\theta, \{h_u\}_{u \in \theta}) \in \mathcal{M} \), an \( \mathbb{R} \)-tree \( T^T \) in the following way. Let \( \mathbb{R}^\theta \) be the vector space of all mappings from \( \theta \) into \( \mathbb{R} \). Write \( (e_u, u \in \theta) \) for the canonical basis of \( \mathbb{R}^\theta \). We define \( l_\emptyset = 0 \) and 

\[
l_u = \sum_{k=1}^{[u]} h_{x^k(u)} e_{x^k(u)} \quad \text{for } u \in \theta.
\]

Let us set

\[
T^T = \bigcup_{u \in \theta} [l_u, l_u + h_u e_u].
\]

\( T^T \) is a connected union of line segments in \( \mathbb{R}^\theta \). It is equipped with the distance \( d_T \) such that \( d_T(a, b) \) is the length of the shortest path in \( T^T \) between \( a \) and \( b \), and can be rooted in \( \rho(T^T) = 0 \) so that it becomes a rooted compact \( \mathbb{R} \)-tree.

If \( \theta \in \mathcal{A} \), we write \( T^\theta \) for the \( \mathbb{R} \)-tree \( T^T \) where \( T = (\theta, \{h_u\}_{u \in \theta}) \) with \( h_\emptyset = 0 \) and \( h_u = 1 \) for every \( u \in \theta \setminus \{\emptyset\} \), and we write \( d_\theta \) for the associated distance. We then set \( m_\emptyset = 0 \) and \( m_u = \sum_{k=0}^{[u]-1} e_{x^k(u)} = l_u + e_u \) for every \( u \in \theta \setminus \{\emptyset\} \).

It is easily checked that \( T^\theta = T^{\theta'} \) if \( \theta \sim \theta' \). Thus for every \( \xi \in \mathcal{A} \), we may write \( T^{\xi} \) for the tree \( T^\theta \) where \( \theta \) is any representative of \( \xi \).

We will now explain how to approximate a general tree \( T \) in \( \mathbb{T} \) by a discrete type tree. Let \( \varepsilon > 0 \) and set \( \mathbb{T}^{(\varepsilon)} = \{ T \in \mathbb{T} : \mathcal{H}(T) > \varepsilon \} \). For every \( T \in \mathbb{T}^{(\varepsilon)} \), we can construct by induction an element \( \xi^{\varepsilon}(T) \) of \( \mathcal{A} \) in the following way:

- If \( T \in \mathbb{T}^{(\varepsilon)} \) satisfies \( \mathcal{H}(T) \leq 2\varepsilon \), we set \( \xi^{\varepsilon}(T) = p(\emptyset) \).
- Let \( n \) be a positive integer. Assume that we have defined \( \xi^{\varepsilon}(T) \) for every \( T \in \mathbb{T}^{(\varepsilon)} \) such that \( \mathcal{H}(T) \leq (n + 1)\varepsilon \). Let \( T \) be an \( \mathbb{R} \)-tree such that \((n + 1)\varepsilon < \mathcal{H}(T) \leq (n + 2)\varepsilon \). We set \( k = \lceil \varepsilon, 2\varepsilon \rceil(T) \) and we denote by \( T^1, \ldots, T^k \) the \( k \) subtrees of \( T \) above level \( \varepsilon \) with height greater than \( \varepsilon \). Then \( \varepsilon < \mathcal{H}(T^i) \leq (n + 1)\varepsilon \) for every \( i \in \{1, \ldots, k\} \), so we can define \( \xi^{\varepsilon}(T^i) \). Let us choose a representative \( \theta^i \) of \( \xi^{\varepsilon}(T^i) \) for every \( i \in \{1, \ldots, k\} \). We set

\[
\xi^{\varepsilon}(T) = p \left( \{\emptyset\} \cup 1\theta^1 \cup \ldots \cup k\theta^k \right),
\]

where \( i\theta^i = \{iu : u \in \theta^i\} \). Clearly this does not depend on the choice of the representatives \( \theta^i \).
If \( r > 0 \) and \( T \) is a compact rooted \( \mathbb{R} \)-tree with metric \( d \), we write \( rT \) for the same tree equipped with the metric \( rd \).

**Lemma 2.2** For every \( \varepsilon > 0 \) and every \( T \in \mathbb{T}^{(\varepsilon)} \), we have

\[
    d_{GH}(\varepsilon T^{\varepsilon(T)}, T) \leq 4\varepsilon. \tag{4}
\]

**Proof:** Let \( \varepsilon > 0 \) and \( T \in \mathbb{T} \). Let \( \theta \) be any representative of \( \xi^\varepsilon(T) \). Recall the notation \((m_u, u \in \theta)\). We can construct a mapping \( \varphi : \theta \to T \) such that:

(i) For every \( \sigma \in T \), there exists \( u \in \theta \) such that \( d(\sigma, \varphi(u)) \leq 2\varepsilon \),

(ii) for every \( u \in \theta \), \( d(\varphi(u), \rho) = \varepsilon|u| \) where \( \rho \) denotes the root of \( T \),

(iii) for every \( u, u' \in \theta \), \( 0 \leq \varepsilon d_\theta(m_u, m_{u'}) - d(\varphi(u), \varphi(u')) \leq 2\varepsilon \).

To be specific, we always take \( \varphi(\emptyset) = \rho \), which suffices for the construction if \( \mathcal{H}(T) \leq 2\varepsilon \). If \((n + 1)\varepsilon < \mathcal{H}(T) \leq (n + 2)\varepsilon \) for some \( n \geq 1 \), we have as above

\[
    \theta = \{\emptyset\} \cup 1\theta^1 \cup \ldots \cup k\theta^k,
\]

where \( \theta^1, \ldots, \theta^k \) are representatives of respectively \( \xi^\varepsilon(T^1), \ldots, \xi^\varepsilon(T^k) \), if \( T^1, \ldots, T^k \) are the subtrees of \( T \) above level \( \varepsilon \) with height greater than \( \varepsilon \). With an obvious notation we define \( \varphi(ju) = \varphi_j(u) \) for every \( j \in \{1, \ldots, k\} \) and \( u \in \theta^j \). Properties (i)-(iii) are then easily checked by induction.

Let us now set \( \Phi = \{\varphi(u), u \in \theta\} \) and \( M = \{m_u, u \in \theta\} \). We equip \( \Phi \) with the metric induced by \( d \) and \( M \) with the metric induced by \( \varepsilon d_\theta \). Then, \( \Phi \) and \( M \) can be viewed as pointed compact metric spaces with respective roots \( \rho \) and \( 0 \). It is immediate that \( d_{GH}(\varepsilon T^\theta, M) \leq \varepsilon \). Furthermore, Property (i) above implies that \( d_{GH}(T, \Phi) \leq 2\varepsilon \). At last, according to Lemma 2.3 in \[8\] and Property (iii) above, we have \( d_{GH}(\Phi, M) \leq \varepsilon \). Lemma 2.2 then follows from the triangle inequality for \( d_{GH} \). \( \square \)

### 2.3 Lévy trees

Roughly speaking, a Lévy tree is a \( \mathbb{T} \)-valued random variable which is associated with a CSBP in such a way that it describes the genealogy of a population evolving according to this CSBP.

#### 2.3.1 The measure \( \Theta_\psi \)

We consider on a probability space \((\Omega, \mathbb{P})\) a \( \psi \)-CSBP \( Y = (Y_t, t \geq 0) \), where the function \( \psi \) is of the form \([8]\), and we suppose that \( Y \) becomes extinct almost surely. This condition is equivalent to

\[
    \int_1^\infty \frac{du}{\psi(u)} < \infty. \tag{5}
\]
This implies that at least one of the following two conditions holds:

$$\beta > 0 \text{ or } \int_{(0,1)} r \pi(dr) = \infty.$$  \hfill (6)

The Lévy tree associated to $Y$ will be defined as the tree coded by the so-called height process, which is a functional of the Lévy process with Laplace exponent $\psi$. Let us denote by $X = (X_t, t \geq 0)$ a Lévy process on $(\Omega, P)$ with Laplace exponent $\psi$. This means that $X$ is a Lévy process with no negative jumps, and that for every $\lambda, t \geq 0$,

$$E(\exp(-(\lambda X_t))) = \exp(t \psi(\lambda)).$$

Then, $X$ does not drift to $+\infty$ and has paths of infinite variation (by (6)).

We can define the height process $H = (H_t, t \geq 0)$ by the following approximation:

$$H_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{X_s \leq I_t^s+\varepsilon\}} ds,$$

where $I_t^s = \inf\{X_r : s \leq r \leq t\}$ and the convergence holds in probability (see Chapter 1 in [3]). Informally, we can say that $H$ measures the size of the set $\{s \in [0,t] : X_s \leq I_t^s\}$. Thanks to condition (6), we know that the process $H$ has a continuous modification (see Theorem 4.7 in [6]). From now on, we consider only this modification.

Let us now set $I_t = \inf\{X_s : 0 \leq s \leq t\}$ for every $t \geq 0$, and consider the process $X - I = (X_t - I_t, t \geq 0)$. We recall that $X - I$ is a strong Markov process, for which the point 0 is regular. The process $-I$ is a local time for $X - I$ at level 0. We write $N$ for the associated excursion measure. We let $\Delta(de)$ be the "law" of $(H_s, s \geq 0)$ under $N$. This makes sense because the values of the height process in an excursion of $X - I$ away from 0 only depend on that excursion (see section 1.2 in [3]). Then, $\Delta(de)$ is a $\sigma$-finite measure on $C([0,\infty))$, and is supported on functions with compact support such that $e(0) = 0$.

The Lévy tree is the tree $(T_e, de)$ coded by the function $e$, in the sense of section 2.2.2 under the measure $\Delta(de)$. We denote by $\Theta_\psi$ the $\sigma$-finite measure on $\mathbb{T}$ which is the "law" of the Lévy tree, that is the image of $\Delta(de)$ under the mapping $e \mapsto T_e$.

### 2.3.2 A discrete approximation of the Lévy tree

Let us now recall that the Lévy tree is the limit in the Gromov-Hausdorff distance of suitably rescaled Galton-Watson trees.

We start by recalling the definition of Galton-Watson trees which was given informally in the introduction above. Let $\gamma$ be a critical or subcritical offspring distribution. We exclude the trivial case where $\gamma(1) = 1$. Then, there exists a unique probability measure $\Pi_\gamma$ on $\mathcal{A}$ such that:

(i) For every $p \geq 0$, $\Pi_\gamma(k_\theta = p) = \gamma(p)$,
(ii) for every \( p \geq 1 \) with \( \gamma(p) > 0 \), under the probability measure \( \Pi \gamma(\cdot \mid k_0 = p) \), the shifted trees \( \tau_1\theta, \ldots, \tau_p\theta \) are independent and distributed according to \( \Pi \gamma \).

Recall that if \( r > 0 \) and \( T \) is a compact rooted \( \mathbb{R} \)-tree with metric \( d \), we write \( rT \) for the same tree equipped with the metric \( rd \). The following result is Theorem 4.1 in [7].

**Theorem 2.3** Let \( (\gamma_n)_{n \geq 1} \) be a sequence of critical or subcritical offspring distributions. For every \( n \geq 1 \), let us denote by \( X^n \) a Galton-Watson process with offspring distribution \( \gamma_n \), started at \( X^0_n = n \). Let \( (m_n)_{n \geq 1} \) be a nondecreasing sequence of positive integers converging to infinity. We define a sequence of processes \( (Y^n)_{n \geq 1} \) by setting, for every \( t \geq 0 \) and \( n \geq 1 \),

\[
Y^n_t = n^{-1}X^n_{[m_nt]}.
\]

Assume that, for every \( t \geq 0 \), \( (Y^n_t)_{n \geq 1} \) converges in distribution to \( Y_t \) where \( Y = (Y_t, t \geq 0) \) is a \( \psi \)-CSBP which becomes extinct almost surely. Assume furthermore that for every \( \delta > 0 \),

\[
\liminf_{n \to \infty} \mathbb{P}(Y^n_\delta = 0) > 0.
\]

Then, for every \( a > 0 \), the law of the \( \mathbb{R} \)-tree \( m^{-1}_nT^\theta \) under \( \Pi \gamma_n(\cdot \mid \mathcal{H}(\theta) \geq [an]) \) converges as \( n \to \infty \) to the probability measure \( \Theta_\psi(\cdot \mid \mathcal{H}(T) > a) \) in the sense of weak convergence of measures in the space \( \mathbb{T} \).

### 3 Proof of Theorem 1.1

Let \( \Theta \) be an infinite measure on \( (\mathbb{T},d_{GH}) \) satisfying the assumptions of Theorem 1.1. Clearly \( \Theta \) is \( \sigma \)-finite.

We start with two important lemmas that will be used throughout this section. Let us first define \( v : (0, \infty) \to (0, \infty) \) by \( v(t) = \Theta(\mathcal{H}(T) > t) \) for every \( t > 0 \). For every \( t > 0 \), we denote by \( \Theta^t \) the probability measure \( \Theta(\cdot \mid \mathcal{H}(T) > t) \).

**Lemma 3.1** The function \( v \) is nonincreasing, continuous and verifies

\[
v(t) \underset{t \to 0}{\longrightarrow} \infty \quad \text{and} \quad v(t) \underset{t \to \infty}{\longrightarrow} 0.
\]

**Proof:** We only have to prove the continuity of \( v \). To this end, we argue by contradiction and assume that there exists \( t > 0 \) such that \( \Theta(\mathcal{H}(T) = t) > 0 \). Let \( s > 0 \) and \( u \in (0, t) \) such that \( v(u) > v(t) \).

From the regenerative property (R), we have

\[
\Theta^s(\mathcal{H}(T) = s + t) = \Theta^s(\Theta^s(\mathcal{H}(T) = s + t \mid Z(s, s + u)))
\]

\[
= \Theta^s\left(Z(s, s + u)\Theta^u(\mathcal{H}(T) = t) (\Theta^t(\mathcal{H}(T) \leq t))^{Z(s,s+u)-1}\right)
\]

\[
= \frac{\Theta(\mathcal{H}(T) = t)}{v(u)} \Theta^s\left(Z(s, s + u) \left(1 - \frac{v(t)}{v(u)}\right)^{Z(s,s+u)-1}\right)
\]

\[
> 0.
\]
We have shown that \( \Theta(\mathcal{H}(T) = t+s) > 0 \) for every \( s > 0 \). This is absurd since \( \Theta \) is \( \sigma \)-finite. \( \Box \)

**Lemma 3.2** For every \( t > 0 \) and \( 0 < a < b \), the conditional law of the random variable \( Z(t, t+b) \), under the probability measure \( \Theta^t \) and given \( Z(t, t+a) \), is a binomial distribution with parameters \( Z(t, t+a) \) and \( v(b)/v(a) \) (where we define the binomial distribution with parameters 0 and \( p \in [0,1] \) as the Dirac measure \( \delta_0 \)).

**Proof:** This is a straightforward consequence of the regenerative property. \( \Box \)

### 3.1 The CSBP derived from \( \Theta \)

In this section, we consider a random forest of trees derived from a Poisson point measure with intensity \( \Theta \). We associate with this forest a family of Galton-Watson processes. We then construct local times at every level \( a > 0 \) as limits of the rescaled Galton-Watson processes. Finally we show that the local time process is a CSBP.

Let us now fix the framework. We consider a probability space \((\Omega, \mathbb{P})\) and on this space a Poisson point measure \( \mathcal{N} = \sum_{i \in \mathcal{I}} \delta_{T_i} \) on \( \mathcal{T} \), whose intensity is the measure \( \Theta \).

#### 3.1.1 A family of Galton-Watson trees

We start with some notation that we need in the first lemma. We consider on another probability space \((\Omega', \mathbb{P}')\), a collection \((\theta_\xi, \xi \in \mathcal{A})\) of independent \( \mathcal{A} \)-valued random variables such that for every \( \xi \in \mathcal{A} \), \( \theta_\xi \) is distributed uniformly over \( p^{-1}(\xi) \). In what follows, to simplify notation, we identify an element \( \xi \) of the set \( \mathcal{A} \) with the subset \( p^{-1}(\xi) \) of \( \mathcal{A} \).

Recall the definition of \( \xi^\varepsilon(T) \) before Lemma 2.2.

**Lemma 3.3** Let us define for every \( \varepsilon > 0 \), a mapping \( \theta^{(\varepsilon)} \) from \( \mathbb{T}^{(\varepsilon)} \times \Omega' \) into \( \mathcal{A} \) by

\[
\theta^{(\varepsilon)}(T, \omega) = \theta_{\xi^\varepsilon(T)}(\omega).
\]

Then for every positive integer \( p \), the law of the random variable \( \theta^{(\varepsilon)} \) under the probability measure \( \Theta^{p\varepsilon} \otimes \mathbb{P}' \) is \( \Pi_{\mu_\varepsilon}(\cdot \mid \mathcal{H}(\theta) \geq p-1) \) where \( \mu_\varepsilon \) denotes the law of \( Z(\varepsilon, 2\varepsilon) \) under \( \Theta^\varepsilon \).

**Proof:** Since \( \{\mathcal{H}(T) > p\varepsilon\} \times \Omega' = \{\mathcal{H}(\theta^{(\varepsilon)}) \geq p-1\} \) for every \( p \geq 1 \), it suffices to show the result for \( p = 1 \). Let \( k \) be a nonnegative integer. According to the construction of \( \xi^\varepsilon(T) \), we have

\[
\Theta^{\varepsilon} \otimes \mathbb{P}' \left( k_\theta \left( \theta^{(\varepsilon)} \right) = k \right) = \Theta^{\varepsilon}(Z(\varepsilon, 2\varepsilon) = k) = \mu_\varepsilon(k).
\]
Let us fix \( k \geq 1 \) with \( \mu_\varepsilon(k) > 0 \). Let \( F : \mathcal{A}^k \rightarrow \mathbb{R}_+ \) be a symmetric measurable function. Then we have

\[
\Theta^\varepsilon \otimes \mathbb{P}' \left( F \left( \tau_1 \theta(\varepsilon), \ldots, \tau_k \theta(\varepsilon) \right) \bigg| k_\emptyset(\theta(\varepsilon)) = k \right) \\
= \Theta^\varepsilon \otimes \mathbb{P}' \left( \sum_{\theta \in \xi^\varepsilon(T)} F \left( \tau_1 \theta, \ldots, \tau_k \theta \right) \mathbb{1}_{\{\theta \in \xi^\varepsilon(T)\}} \bigg| Z(\varepsilon, 2\varepsilon) = k \right) \\
= \Theta^\varepsilon \left( (\#\xi^\varepsilon(T))^{-1} \sum_{\theta \in \xi^\varepsilon(T)} F \left( \tau_1 \theta, \ldots, \tau_k \theta \right) \bigg| Z(\varepsilon, 2\varepsilon) = k \right).
\] (7)

On the event \( \{Z(\varepsilon, 2\varepsilon) = k\} \), we write \( T^1, \ldots, T^k \) for the \( k \) subtrees of \( T \) above level \( \varepsilon \) with height greater than \( \varepsilon \). Then, Formula (7) and the regenerative property yield

\[
\Theta^\varepsilon \left( (\#\xi^\varepsilon(T))^{-1} \sum_{\theta \in \xi^\varepsilon(T)} F \left( \tau_1 \theta, \ldots, \tau_k \theta \right) \bigg| Z(\varepsilon, 2\varepsilon) = k \right) \\
= \Theta^\varepsilon \left( (\#\xi^\varepsilon(T^1))^{-1} \cdots (\#\xi^\varepsilon(T^k))^{-1} \sum_{\theta_1 \in \xi^\varepsilon(T^1)} \cdots \sum_{\theta_k \in \xi^\varepsilon(T^k)} F \left( \theta_1, \ldots, \theta_k \right) \bigg| Z(\varepsilon, 2\varepsilon) = k \right) \\
= \int \Theta^\varepsilon(dT_1) \cdots \Theta^\varepsilon(dT_k) \left( (\#\xi^\varepsilon(T^1))^{-1} \cdots (\#\xi^\varepsilon(T^k))^{-1} \sum_{\theta_1 \in \xi^\varepsilon(T^1)} \cdots \sum_{\theta_k \in \xi^\varepsilon(T^k)} F \left( \theta_1, \ldots, \theta_k \right) \right) \\
= \int \Theta^\varepsilon \otimes \mathbb{P}'(dT_1, d\omega^1) \ldots \Theta^\varepsilon \otimes \mathbb{P}'(dT_k, d\omega^k) F \left( \theta(\varepsilon)(T_1, \omega^1), \ldots, \theta(\varepsilon)(T_k, \omega^k) \right),
\]
as in (7). We have thus proved that

\[
\Theta^\varepsilon \otimes \mathbb{P}' \left( F \left( \tau_1 \theta(\varepsilon), \ldots, \tau_k \theta(\varepsilon) \right) \bigg| k_\emptyset(\theta(\varepsilon)) = k \right) \\
= \int \Theta^\varepsilon \otimes \mathbb{P}'(dT_1, d\omega^1) \ldots \Theta^\varepsilon \otimes \mathbb{P}'(dT_k, d\omega^k) F \left( \theta(\varepsilon)(T_1, \omega^1), \ldots, \theta(\varepsilon)(T_k, \omega^k) \right).
\] (8)

Note that for every permutation \( \varphi \) of the set \( \{1, \ldots, k\} \), \( (\tau_1(\varphi(1)) \theta(\varepsilon), \ldots, \tau_k(\varphi(k)) \theta(\varepsilon)) \) and \( (\tau_1 \theta(\varepsilon), \ldots, \tau_k \theta(\varepsilon)) \) have the same distribution under \( \Theta^\varepsilon \otimes \mathbb{P}' \). Then, (8) means that the law of \( \theta(\varepsilon) \) under \( \Theta^\varepsilon \otimes \mathbb{P}' \) satisfies the branching property of the Galton-Watson trees. This completes the proof of the desired result. \( \square \)

Recall that \( \sum_{i \in I} \delta_{T_i} \) is a Poisson point measure on \( T \) with intensity \( \Theta \). Let us now set, for every \( t, h > 0 \),

\[
Z(t, t + h) = \sum_{i \in I} Z(t, t + h)(T_i)
\]

For every \( \varepsilon > 0 \), we define a process \( \mathcal{X}^\varepsilon = (\mathcal{X}^\varepsilon_k, k \geq 0) \) on \( (\Omega, \mathbb{P}) \) by the formula

\[
\mathcal{X}^\varepsilon_k = Z(k \varepsilon, (k + 1)\varepsilon), \ k \geq 0.
\]

**Proposition 3.4** For every \( \varepsilon > 0 \), the process \( \mathcal{X}^\varepsilon \) is a Galton-Watson process whose initial distribution is the Poisson distribution with parameter \( v(\varepsilon) \) and whose offspring distribution is \( \mu_\varepsilon \).
Proof: We first observe that $\mathcal{X}^\varepsilon_0 = \mathcal{N}(\mathcal{H}(T) > \varepsilon)$ is Poisson with parameter $\Theta(\mathcal{H}(T) > \varepsilon) = v(\varepsilon)$. Then let $p$ be a positive integer. We know from a classical property of Poisson measures that, under the probability measure $\mathbb{P}$ and conditionally on the event $\{\mathcal{X}^\varepsilon_0 = p\}$, the atoms of $\mathcal{N}$ that belong to the set $\mathbb{T}^\varepsilon$ are distributed as $p$ i.i.d. variables with distribution $\Theta^\varepsilon$. Furthermore, it follows from Lemma 3.3 that under $\Theta^\varepsilon$, the process $(Z(k\varepsilon, (k+1)\varepsilon))_{k \geq 0}$ is a Galton-Watson process started at one with offspring distribution $\mu^\varepsilon$. This completes the proof. 

As a consequence, we get the next proposition, which we will use throughout this work.

Proposition 3.5 For every $t > 0$ and $h > 0$, we have $\Theta(Z(t, t+h)) \leq v(h)$.

Proof: Since compact real trees have finite height, the Galton-Watson process $\mathcal{X}^\varepsilon$ dies out $\mathbb{P}$ a.s. This implies that $\mu^\varepsilon$ is critical or subcritical so that $(\mathcal{X}^\varepsilon_k, k \geq 0)$ is a super-martingale. Let $t, h > 0$. We can find $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $t = k\varepsilon$ and $\varepsilon \leq h$. Thus we have,

$$\Theta(Z(t, t+\varepsilon)) = \Theta(Z(k\varepsilon, (k+1)\varepsilon)) = E(\mathcal{X}^\varepsilon_k) \leq E(\mathcal{X}^\varepsilon_0) = v(\varepsilon).$$

(9)

Using Lemma 3.2 and (9), we get

$$\Theta(Z(t, t+h)) = \Theta \left( Z(t, t+\varepsilon) \frac{v(h)}{v(\varepsilon)} \right) \leq v(h).$$

(10)

3.1.2 A local time process

Proposition 3.6 For every $t \geq 0$, there exists a random variable $L_t$ on the space $\mathbb{T}$ such that $\Theta$ a.e.

$$\frac{Z(t, t+h)}{v(h)} \xrightarrow{h \to 0} L_t.$$

Proof: Let us start with the case $t = 0$. As $Z(0, h) = 1_{\{H(T) > h\}}$ for every $h > 0$, Lemma 3.1 gives $v(h)^{-1}Z(0, h) \longrightarrow 0$ $\Theta$ a.e. as $h \longrightarrow 0$, so we set $L_0 = 0$.

Let us now fix $t > 0$. Thanks to Lemma 3.1, we can define a decreasing sequence $(\varepsilon_n)_{n \geq 1}$ by the condition $v(\varepsilon_n) = n^4$ for every $n \geq 1$. We claim that there exists a random variable $L_t$ on the space $\mathbb{T}$ such that, $\Theta$ a.e.,

$$\frac{Z(t, t+\varepsilon_n)}{n^4} \xrightarrow{n \to \infty} L_t.$$

(10)

Indeed, using Lemma 3.2, we have, for every $n \geq 1$,

$$\Theta^\varepsilon \left( \frac{Z(t, t+\varepsilon_n)}{n^4} - \frac{Z(t, t+\varepsilon_{n+1})}{(n+1)^4} \right)^2$$

15
\[
\begin{align*}
\Theta^t \left( \frac{1}{n^8} \Theta^t \left( \left| Z(t, t + \varepsilon_n) - \frac{n^4}{(n + 1)^4} Z(t, t + \varepsilon_{n+1}) \right|^2 \right) \right) \\
\leq \Theta^t \left( \frac{Z(t, t + \varepsilon_{n+1})}{4n^8} \right) \\
\leq \frac{(n + 1)^4}{4v(t)n^8},
\end{align*}
\]
where the last bound follows from Proposition 3.3 and the definition of \( \varepsilon_{n+1} \). Thanks to the Cauchy-Schwarz inequality, we get
\[
\Theta^t \left( \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n + 1)^4} \right| \right) \leq \frac{(n + 1)^2}{2n^4 \sqrt{v(t)}} \leq \frac{2}{n^2 \sqrt{v(t)}}.
\]
The bound (12) implies
\[
\Theta \left( \sum_{n=1}^{\infty} \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n + 1)^4} \right| \right) < \infty.
\]
In particular, \( \Theta \) a.e.,
\[
\sum_{n=1}^{\infty} \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n + 1)^4} \right| < \infty.
\]
Our claim (10) follows.

For every \( h \in (0, \varepsilon_1] \), we can find \( n \geq 1 \) such that \( \varepsilon_{n+1} \leq h \leq \varepsilon_n \). Then, we have \( Z(t, t + \varepsilon_n) \leq Z(t, t + h) \leq Z(t, t + \varepsilon_{n+1}) \) \( \Theta \) a.e., and \( n^4 \leq v(h) \leq (n + 1)^4 \) so that
\[
\frac{Z(t, t + \varepsilon_n)}{(n + 1)^4} \leq \frac{Z(t, t + h)}{v(h)} \leq \frac{Z(t, t + \varepsilon_{n+1})}{n^4}.
\]
We then deduce from (11) that \( \Theta \) a.e.,
\[
\frac{Z(t, t + h)}{v(h)} \xrightarrow{h \to 0} L_t
\]
which completes the proof. \( \square \)

**Definition 3.1** We define a process \( \mathcal{L} = (\mathcal{L}_t, t \geq 0) \) on \( (\Omega, \mathbb{P}) \) by setting \( \mathcal{L}_0 = 1 \) and for every \( t > 0 \),
\[
\mathcal{L}_t = \sum_{i \in I} L_t(T_i).
\]
Notice that \( L_t(T) = 0 \) if \( \mathcal{H}(T) \leq t \) so that the above sum is finite a.s.
Corollary 3.7 For every $t \geq 0$, we have $\mathbb{P}$ a.s. 

$$\frac{Z(t, t + h)}{v(h)} \xrightarrow{h \to 0} \mathcal{L}_t.$$ 

Moreover, this convergence holds in $L^1(\mathbb{P})$ uniformly in $t \in [0, \infty)$.

Proof: The first assertion is an immediate consequence of Proposition 3.6. Let us focus on the second assertion. From Lemma 3.2, $\Theta(n^{-4} Z(t, t + \varepsilon_n) - (n + 1)^{-4} Z(t, t + \varepsilon_{n+1})) = 0$ for every $t \geq 0$ and $n \geq 1$. Thus, from the second moment formula for Poisson measures, we get, for every $t \geq 0$ and $n \geq 1$,

$$\mathbb{E} \left( \left( \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n+1)^4} \right)^2 \right) = \Theta \left( \left( \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n+1)^4} \right)^2 \right).$$

Now, we have

$$\Theta \left( \left( \frac{Z(0, \varepsilon_n)}{n^4} - \frac{Z(0, \varepsilon_{n+1})}{(n+1)^4} \right)^2 \right) = \Theta \left( \left( \frac{\mathbb{1}_{\{H(\mathcal{I}) \geq \varepsilon_n\}}}{n^4} - \frac{\mathbb{1}_{\{H(\mathcal{I}) \geq \varepsilon_{n+1}\}}}{(n+1)^4} \right)^2 \right) = \frac{1}{n^4} - \frac{1}{(n+1)^4}$$

and for every $t > 0$, thanks to the bound (11),

$$\Theta \left( \left( \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n+1)^4} \right)^2 \right) \leq \frac{(n+1)^4}{4n^8}.$$

So for every $t \geq 0$ and $n \geq 1$, we have from the Cauchy-Schwarz inequality

$$\mathbb{E} \left( \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + \varepsilon_{n+1})}{(n+1)^4} \right| \right) \leq \frac{(n+1)^2}{n^4}, \quad (13)$$

Then $n^{-4} Z(t, t + \varepsilon_n) \xrightarrow{n \to \infty} \mathcal{L}_t$ in $L^1$ as $n \to \infty$ and, for every $n \geq 2$,

$$\mathbb{E} \left( \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \mathcal{L}_t \right| \right) \leq \sum_{k=n}^{\infty} \frac{(k+1)^2}{k^4} \leq \sum_{k=n}^{\infty} \frac{4}{k^2} \leq \frac{8}{n}.$$ 

In the same way as in the proof of (11), we have the following inequality: If $h \in (0, \varepsilon_1], t \geq 0$ and $n$ is a positive integer such that $\varepsilon_{n+1} \leq h \leq \varepsilon_n$,

$$\mathbb{E} \left( \left| \frac{Z(t, t + \varepsilon_n)}{n^4} - \frac{Z(t, t + h)}{v(h)} \right| \right) \leq \frac{\sqrt{v(h)}}{n^4} \leq \frac{16}{\sqrt{v(h)}}.$$ 

Then, for every $h \in (0, \varepsilon_2]$ and $t \geq 0$, we get

$$\mathbb{E} \left( \left| \frac{Z(t, t + h)}{v(h)} - \mathcal{L}_t \right| \right) \leq 16 \left( v(h)^{-1/2} + v(h)^{-1/4} \right),$$

which completes the proof. \qed

We will now establish a regularity property of the process $(\mathcal{L}_t, t \geq 0)$. 

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Proposition 3.8 The process \((L_t, t \geq 0)\) admits a modification, denoted by \((\tilde{L}_t, t \geq 0)\), which is right-continuous with left-limits, and which has no fixed discontinuities.

Proof: We start with two lemmas.

Lemma 3.9 There exists \(\lambda \geq 0\) such that \(\mathbb{E}(L_t) = e^{-\lambda t}\) for every \(t \geq 0\).

Proof: We claim that the function \(t \in [0, +\infty) \mapsto \mathbb{E}(L_t)\) is multiplicative, meaning that for every \(t, s \geq 0\), \(\mathbb{E}(L_{t+s}) = \mathbb{E}(L_t)\mathbb{E}(L_s)\). As \(L_0 = 1\) by definition, \(\mathbb{E}(L_0) = 1\). Let \(t, s > 0\) and \(0 < h < s\). Let us denote by \(T^1, \ldots, T^{Z(t,t+h)}\) the subtrees of \(T\) above level \(t\) with height greater than \(h\). Then, using the regenerative property, we can write

\[
\Theta(Z(t+s,t+s+h)) = \Theta\left(\sum_{i=1}^{Z(t,t+h)} Z(s,s+h)(T^i)\right) = \Theta(Z(t,t+h))\Theta^h(Z(s,s+h)),
\]

which implies

\[
\mathbb{E}(Z(t+s,t+s+h)) = \mathbb{E}(Z(t,t+h))\mathbb{E}\left(\frac{Z(s,s+h)}{v(h)}\right). \tag{14}
\]

Thus, dividing by \(v(h)\) and letting \(h \to 0\) in (14), we get our claim from Corollary 3.7. Moreover, thanks to Proposition 3.5 and Corollary 3.7, we know that \(\mathbb{E}(L_t) \leq 1\) for every \(t \geq 0\). Then, we obtain in particular that the function \(t \in [0, \infty) \mapsto \mathbb{E}(L_t)\) is nonincreasing.

To complete the proof, we have to check that \(\mathbb{E}(L_t) > 0\) for every \(t > 0\). If we assume that \(\mathbb{E}(L_t) = 0\) for some \(t > 0\) then \(L_t = 0, \Theta\) a.e. Let \(s, h > 0\) such that \(0 < h < s\). With the same notation as in the beginning of the proof, we can write

\[
\Theta(H(T) > t+s) = \Theta\left(\exists i \in \{1, \ldots, Z(t,t+h)\} : H(T^i) > s\right)
\]

\[
= \Theta\left(1 - \left(1 - \frac{v(s)}{v(h)}\right)^{Z(t,t+h)}\right). \tag{15}
\]

Now, thanks to Proposition 3.6, \(\Theta\) a.e.,

\[
\left(1 - \frac{v(s)}{v(h)}\right)^{Z(t,t+h)} \xrightarrow[h \to 0]{} \exp(-L_tv(s)) = 1.
\]

Moreover, \(\Theta\) a.e.,

\[
1 - \left(1 - \frac{v(s)}{v(h)}\right)^{Z(t,t+h)} \leq \mathbb{1}_{\{H(T) > t\}}.
\]

Then, using dominated convergence in (15) as \(h \to 0\), we obtain \(\Theta(H(T) > t+s) = 0\) which contradicts the assumptions of Theorem 1.1. \(\square\)
Lemma 3.10 Let us denote by $D = \{k2^{-n}, k \geq 1, n \geq 0\}$ the set of positive dyadic numbers and define $G_t = \sigma(L_s, s \in D, s \leq t)$ for every $t \in D$. Then $(L_t, t \in D)$ is a nonnegative supermartingale with respect to the filtration $(G_t, t \in D)$.

Proof: Let $p$ be a positive integer, let $s_1, \ldots, s_p, s, t \in D$ such that $s_1 < \ldots < s_p \leq s < t$ and let $f: \mathbb{R}^p \to \mathbb{R}_+$ be a bounded continuous function. We can find a positive integer $n$ such that $2^n t, 2^n s,$ and $2^n s_i$ for $i \in \{1, \ldots, p\}$ are nonnegative integers. The process $\mathcal{X}_{2^{-n}}^s$ is a subcritical Galton-Watson process, so

$$
\mathbb{E}(\mathcal{X}_{2^{-n}}^s f(\mathcal{X}_{2^{-n}}^{s_1}, \ldots, \mathcal{X}_{2^{-n}}^{s_p})) \leq \mathbb{E}(\mathcal{X}_{2^{-n}}^s f(\mathcal{X}_{2^{-n}}^{s_1}, \ldots, \mathcal{X}_{2^{-n}}^{s_p})).
$$

Therefore we have also,

$$
\mathbb{E}
\left(
\frac{Z(t, t + 2^{-n})}{v(2^{-n})} f
\left(
\frac{Z(s_1, s_1 + 2^{-n})}{v(2^{-n})}, \ldots, \frac{Z(s_p, s_p + 2^{-n})}{v(2^{-n})}
\right)
\right)
\leq
\mathbb{E}
\left(
\frac{Z(s, s + 2^{-n})}{v(2^{-n})} f
\left(
\frac{Z(s_1, s_1 + 2^{-n})}{v(2^{-n})}, \ldots, \frac{Z(s_p, s_p + 2^{-n})}{v(2^{-n})}
\right)
\right).
$$

(16)

We can then use Corollary 3.7 to obtain $\mathbb{E}(L_t f(L_{s_1}, \ldots, L_{s_p})) \leq \mathbb{E}(L_s f(L_{s_1}, \ldots, L_{s_p})).$ \Box

We now complete the proof of Proposition 3.8. Let us set, for every $t \geq 0$,

$$
\tilde{G}_t = \bigcap_{s > t, s \in D} G_s.
$$

From Lemma 3.10 and classical results on supermartingales, we can define a right-continuous supermartingale $(\tilde{L}_t, t \geq 0)$ with respect to the filtration $(\tilde{G}_t, t \geq 0)$ by setting, for every $t \geq 0$,

$$
\tilde{L}_t = \lim_{s \uparrow t, s \in D} L_s.
$$

(17)

where the limit holds $\mathbb{P}$ a.s. and in $L^1$ (see e.g. Chapter VI in [5] for more details). We claim that $(\tilde{L}_t, t \geq 0)$ is a càdlàg modification of $(L_t, t \geq 0)$ with no fixed discontinuities.

We first prove that $(\tilde{L}_t, t \geq 0)$ is a modification of $(L_t, t \geq 0)$. For every $t \geq 0$ and every sequence $(s_n)_{n \geq 0} \in D$ such that $s_n \downarrow t$ as $n \uparrow \infty$, we have thanks to (17) and Lemma 3.9,

$$
\mathbb{E}(\tilde{L}_t) = \lim_{n \to \infty} \mathbb{E}(L_{s_n}) = \mathbb{E}(L_t).
$$

Let us now show that for every $t \geq 0$, $L_t \leq \tilde{L}_t$ $\mathbb{P}$ a.s. Let $\alpha, \varepsilon > 0$ and $\delta \in (0, 1)$. Thanks to Corollary 3.7, we can find $h_0 > 0$ such that for every $h \in (0, h_0)$ and $n \geq 0$,

$$
\mathbb{E}
\left(
\left|\frac{Z(t, t + h)}{v(h)} - L_t\right|
\right) \leq \varepsilon \alpha \quad \text{and} \quad \mathbb{E}
\left(
\left|\frac{Z(s_n, s_n + h)}{v(h)} - L_{s_n}\right|
\right) \leq \varepsilon \alpha.
$$

We choose $h \in (0, h_0)$ and $n_0 \geq 0$ such that $s_n - t + h \leq h_0$ and $v(h) \leq (1 + \delta)v(s_n - t + h)$ for every $n \geq n_0$. We notice that $Z(t, s_n + h) \leq Z(s_n, s_n + h)$ so that, for every $n \geq n_0$,

$$
\mathbb{P}(L_t > (1 + \delta) L_{s_n} + \varepsilon)
$$
\[
\begin{align*}
\leq \mathbb{P} \left( \mathcal{L}_t - \frac{\mathcal{Z}(t, s_n + h)}{v(s_n - t + h)} > (1 + \delta) \mathcal{L}_{s_n} - (1 + \delta) \frac{\mathcal{Z}(s_n, s_n + h)}{v(h)} + \varepsilon \right) \\
\leq 2\varepsilon^{-1} \mathbb{E} \left( \frac{\mathcal{Z}(t, s_n + h)}{v(s_n - t + h)} - \mathcal{L}_t \right) + 2\varepsilon^{-1} (1 + \delta) \mathbb{E} \left( \frac{\mathcal{Z}(s_n, s_n + h)}{v(h)} - \mathcal{L}_{s_n} \right) \\
\leq 6\alpha.
\end{align*}
\]

We have thus shown that
\[
\mathbb{P}(\mathcal{L}_t > (1 + \delta) \mathcal{L}_{s_n} + \varepsilon) \longrightarrow 0. \tag{18}
\]

So, \(\mathbb{P}(\mathcal{L}_t - (1 + \delta) \tilde{\mathcal{L}}_t > \varepsilon) = 0\) for every \(\varepsilon > 0\), implying that \(\mathcal{L}_t \leq (1 + \delta) \tilde{\mathcal{L}}_t\), \(\mathbb{P}\) a.s. This leads us to the claim \(\mathcal{L}_t \leq \tilde{\mathcal{L}}_t\) a.s. Since we saw that \(\mathbb{E}(\mathcal{L}_t) = \mathbb{E}(\tilde{\mathcal{L}}_t)\), we have \(\mathcal{L}_t = \tilde{\mathcal{L}}_t\) \(\mathbb{P}\) a.s. for every \(t \geq 0\).

Now, \((\tilde{\mathcal{L}}_t, t \geq 0)\) is a right-continuous supermartingale. Thus, \((\tilde{\mathcal{L}}_t, t \geq 0)\) is also left-limited and we have \(\mathbb{E}(\tilde{\mathcal{L}}_t) \leq \mathbb{E}(\tilde{\mathcal{L}}_{t^-})\) for every \(t > 0\). Moreover, we can prove in the same way as we did for (18) that, for every \(t > 0\) and every sequence \((s_n, n \geq 0)\) in \(D\) such that \(s_n \uparrow t\) as \(n \uparrow \infty\),
\[
\mathbb{P}(\tilde{\mathcal{L}}_{s_n} > (1 + \delta) \tilde{\mathcal{L}}_t + \varepsilon) \longrightarrow 0,
\]

implying that \(\tilde{\mathcal{L}}_{t^-} \leq \tilde{\mathcal{L}}_t\), \(\mathbb{P}\) a.s. So, \(\mathcal{L}_t = \mathcal{L}_{t^-}\) \(\mathbb{P}\) a.s. for every \(t > 0\) meaning that \((\tilde{\mathcal{L}}_t, t \geq 0)\) has no fixed discontinuities. \(\square\)

From now on, to simplify notation, we replace \((\mathcal{L}_t, t \geq 0)\) by its càdlàg modification \((\tilde{\mathcal{L}}_t, t \geq 0)\).

### 3.1.3 The CSBP

We will prove that the suitably rescaled family of Galton-Watson processes \((\mathcal{X}^\varepsilon)_{\varepsilon > 0}\) converges to the local time \(\mathcal{L}\).

Thanks to Lemma 3.1, we can define a sequence \((\eta_n)_{n \geq 1}\) by the condition \(v(\eta_n) = n\) for every \(n \geq 1\). We set \(m_n = [\eta_n^{-1}]\) where \([x]\) denotes the integer part of \(x\). We recall from Proposition 3.4 that \(\mathcal{X}^n_{\eta_n}\) is a Galton-Watson process on \((\Omega, \mathbb{P})\) whose initial distribution is the Poisson distribution with parameter \(n\). For every \(n \geq 1\), we define a process \(\mathcal{Y}^n = (\mathcal{Y}^n_t, t \geq 0)\) on \((\Omega, \mathbb{P})\) by the following formula,
\[
\mathcal{Y}^n_t = n^{-1} \mathcal{X}^n_{[m_n t]}, \quad t \geq 0.
\]

**Proposition 3.11** For every \(t \geq 0\), \(\mathcal{Y}^n_t \longrightarrow \mathcal{L}_t\) in probability as \(n \longrightarrow \infty\).

**Proof:** The result for \(t = 0\) is a consequence of the definition of \(\mathcal{L}_0\) together with simple estimates for the Poisson distribution. Let \(t, \delta > 0\). We can write
\[
\begin{align*}
\mathbb{P}(|\mathcal{Y}^n_t - \mathcal{L}_t| > 2\delta) & \leq \mathbb{P}(|\mathcal{Y}^n_t - \mathcal{L}_{\eta_n[m_n t]}| > \delta) + \mathbb{P}(|\mathcal{L}_{\eta_n[m_n t]} - \mathcal{L}_t| > \delta) \\
& \leq \delta^{-1} \mathbb{E}(|\mathcal{Y}^n_t - \mathcal{L}_{\eta_n[m_n t]}|) + \mathbb{P}(|\mathcal{L}_{\eta_n[m_n t]} - \mathcal{L}_t| > \delta).
\end{align*}
\]

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Now, Corollary 3.7 and Proposition 3.8 imply respectively that

\[
\mathbb{E} \left( \left| Y^n_t - L_{\eta_n[m,t]} \right| \right) \rightarrow 0 \quad \text{and} \quad \mathbb{P} \left( |L_{\eta_n[m,t]} - L_t| > \delta \right) \rightarrow 0,
\]

which completes the proof. \qed

Corollary 3.12 For every \( t \geq 0 \), the law of \( Y^n_t \) under \( \mathbb{P}(\cdot \mid X_0^n = n) \) converges weakly to the law of \( L_t \) under \( \mathbb{P} \) as \( n \rightarrow \infty \).

Proof: For positive integers \( n \) and \( k \), we denote by \( \mu^n \) the offspring distribution of the Galton-Watson process \( X^n_\eta \), by \( f^n \) the generating function of \( \mu^n \) and by \( f^n_k \) the \( k \)-th iterate \( f^n_k = f^n \circ \ldots \circ f^n \) of \( f^n \). Let \( \lambda > 0 \) and \( t \geq 0 \). We have,

\[
\mathbb{E}(\exp(-\lambda Y^n_t)) = \sum_{p=0}^{\infty} e^{-n\frac{n^p}{p!}} (f^n_{[m,t]}(e^{-\lambda/n}))^p = \exp\left(-n \left(1 - f^n_{[m,t]}(e^{-\lambda/n})\right)\right).
\]

From Proposition 3.11, it holds that

\[
\exp\left(-n \left(1 - f^n_{[m,t]}(e^{-\lambda/n})\right)\right) \rightarrow \mathbb{E}(\exp(-\lambda L_t)).
\]

Let us set \( u(t,\lambda) = -\log(\mathbb{E}[\exp(-\lambda L_t)]) \). It follows that,

\[
n \left(1 - f^n_{[m,t]}(e^{-\lambda/n})\right) \rightarrow u(t,\lambda).
\]

Thus, we obtain,

\[
\mathbb{E} \left( \exp(-\lambda Y^n_t) \mid X_0^n = n \right) = (f^n_{[m,t]}(e^{-\lambda/n}))^n \rightarrow \exp(-u(t,\lambda)) = \mathbb{E}[\exp(-\lambda L_t)].
\]

At this point, we can use Theorem 2.1 to assert that \( (L_t, t \geq 0) \) is a CSBP and that the law of \( (Y^n_t, t \geq 0) \) under the probability measure \( \mathbb{P}(\cdot \mid X_0^n = n) \) converges to the law of \( (L_t, t \geq 0) \) as \( n \rightarrow \infty \) in the space of probability measures on the Skorokhod space \( \mathbb{D}(\mathbb{R}_+) \). To verify the assumptions of Theorem 2.1, we need to check that there exists \( \delta > 0 \) such that \( \mathbb{P}(L_\delta > 0) > 0 \). This is obvious from Lemma 3.9.

3.2 Identification of the measure \( \Theta \)

In the previous section, we have constructed from \( \Theta \) a CSBP \( L \), which becomes extinct almost surely. We denote by \( \psi \) the associated branching mechanism. We can consider the \( \sigma \)-finite measure \( \Theta_\psi \), which is the law of the Lévy tree associated with \( L \). Our goal is to show that the measures \( \Theta \) and \( \Theta_\psi \) coincide.

Recall that \( \mu^n \) denotes the offspring distribution of the Galton-Watson process \( X^n_\eta \).
Lemma 3.13 For every \( a > 0 \), the law of the \( \mathbb{R} \)-tree \( \eta_n T^0 \) under \( \Pi_{\mu^n} (\cdot \mid \mathcal{H} (\theta) \geq [am_n]) \) converges as \( n \rightarrow \infty \) to the probability measure \( \Theta_\psi (\cdot \mid \mathcal{H} (T) > a) \) in the sense of weak convergence of measures in the space \( T \).

Proof: We first check that, for every \( \delta > 0 \),

\[
\liminf_{n \to \infty} \mathbb{P} (\mathcal{Y}^n_\delta = 0) > 0. \tag{19}
\]

Indeed, we have

\[
\mathbb{P} (\mathcal{Y}^n_\delta = 0) = \mathbb{P} (\mathcal{N} (\mathcal{H} (T) > (m_n \delta) + 1) \eta_n) = \exp (-v (([m_n \delta] + 1) \eta_n)).
\]

As \( v \) is continuous, it follows that \( \mathbb{P} (\mathcal{Y}^n_\delta = 0) \to \exp (-v (\delta)) \) as \( n \to \infty \) implying (19).

We recall that the law of \( \mathcal{Y}^n \) under the probability measure \( \mathbb{P} (\cdot \mid X^0_n = n) \) converges to the law of \( (\mathcal{L}_t, t \geq 0) \). Then, thanks to (19), we can apply Theorem 2.3 to get that, for every \( a > 0 \), the law of the \( \mathbb{R} \)-tree \( m_n^{-1} T^0 \) under \( \Pi_{\mu^n} (\cdot \mid \mathcal{H} (\theta) \geq [am_n]) \) converges to the probability measure \( \Theta_\psi (\cdot \mid \mathcal{H} (T) > a) \) in the sense of weak convergence of measures in the space \( T \). As \( m_n^{-1} \eta_n \to 1 \) as \( n \to \infty \), we get the desired result. □

We can now complete the proof of Theorem 1.1. Indeed, thanks to Lemmas 2.2 and 3.3, we can construct on the same probability space \( (\Omega, \mathcal{P}) \), a sequence of \( T \)-valued random variables \( (T_n)_{n \geq 1} \) distributed according to \( \Theta (\cdot \mid \mathcal{H} (T) > ([am_n] + 1) \eta_n) \) and a sequence of \( A \)-valued random variables \( (\theta_n)_{n \geq 1} \) distributed according to \( \Pi_{\mu^n} (\cdot \mid \mathcal{H} (\theta) \geq [am_n]) \) such that for every \( n \geq 1 \), \( \mathcal{P} \) a.s.,

\[
d_{GH} (T_n, \eta_n T^{\theta_n}) \leq 4 \eta_n.
\]

Then, using Lemma 3.13, we have \( \Theta (\cdot \mid \mathcal{H} (T) > ([am_n] + 1) \eta_n) \to \Theta_\psi (\cdot \mid \mathcal{H} (T) > a) \) as \( n \to \infty \) in the sense of weak convergence of measures on the space \( T \). So we get

\[
\Theta (\cdot \mid \mathcal{H} (T) > a) = \Theta_\psi (\cdot \mid \mathcal{H} (T) > a)
\]

for every \( a > 0 \), and thus \( \Theta = \Theta_\psi \).

4 Proof of Theorem 1.2

Let \( \Theta \) be a probability measure on \((T, d_{GH})\) satisfying the assumptions of Theorem 1.2.

In this case, we define \( v : [0, \infty) \to (0, \infty) \) by \( v(t) = \Theta (\mathcal{H} (T) > t) \) for every \( t \geq 0 \). Note that \( v(0) = 1 \) is well defined here. For every \( t > 0 \), we denote by \( \Theta^t \) the probability measure \( \Theta (\cdot \mid \mathcal{H} (T) > t) \). The following two results are proved in a similar way to Lemmas 3.1 and 3.2.

Lemma 4.1 The function \( v \) is nonincreasing, continuous and goes to 0 as \( t \to \infty \).

Lemma 4.2 For every \( t > 0 \) and \( 0 < a < b \), the conditional law of the random variable \( Z(t, t + b) \), under the probability measure \( \Theta^t \) and given \( Z(t, t + a) \), is a binomial distribution with parameters \( Z(t, t + a) \) and \( v(b)/v(a) \).
4.1 The DSBP derived from $\Theta$

We will follow the same strategy as in section 3 but instead of a CSBP we will now construct an integer-valued branching process.

4.1.1 A family of Galton-Watson trees

We recall that $\mu_\varepsilon$ denotes the law of $Z(\varepsilon, 2\varepsilon)$ under the probability measure $\Theta^\varepsilon$, and that $(\theta_\xi, \xi \in A)$ is a sequence of independent $A$-valued random variables defined on a probability space $(\Omega', \mathbb{P}')$ such that for every $\xi \in A$, $\theta_\xi$ is distributed uniformly over $p^{-1}(\xi)$. The following lemma is proved in the same way as Lemma 3.3.

Lemma 4.3 Let us define for every $\varepsilon > 0$, a mapping $\theta^{(\varepsilon)}$ from $T(\varepsilon) \times \Omega'$ into $A$ by

$$\theta^{(\varepsilon)}(T, \omega) = \theta_{\xi^T}(\omega).$$

Then for every positive integer $p$, the law of the random variable $\theta^{(\varepsilon)}$ under the probability measure $\Theta^\varepsilon \otimes \mathbb{P}'$ is $\Pi_{\mu_\varepsilon}(\cdot \mid \mathcal{H}^{(\varepsilon)} \geq p - 1)$.

For every $\varepsilon > 0$, we define a process $X^{\varepsilon} = (X^{\varepsilon}_k, k \geq 0)$ on $T$ by the formula

$$X^{\varepsilon}_k = Z(k\varepsilon, (k+1)\varepsilon), \ k \geq 0.$$ 

We show in the same way as Proposition 3.4 and Proposition 3.5 the following two results.

**Proposition 4.4** For every $\varepsilon > 0$, the process $X^{\varepsilon}$ is under $\Theta$ a Galton-Watson process whose initial distribution is the Bernoulli distribution with parameter $v(\varepsilon)$ and whose offspring distribution is $\mu_\varepsilon$.

**Proposition 4.5** For every $t > 0$ and $h > 0$, we have $\Theta(Z(t, t+h)) \leq v(h) \leq 1$.

The next proposition however is particular to the finite case and will be useful in the rest of this section.

**Proposition 4.6** The family of probability measures $(\mu_\varepsilon)_{\varepsilon > 0}$ converges to the Dirac measure $\delta_1$ as $\varepsilon \to 0$. In other words,

$$\Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) = 1) \to 1.$$ 

**Proof:** We first note that

$$2\Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) \geq 1) - \Theta^\varepsilon(Z(\varepsilon, 2\varepsilon)) \leq \Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) = 1) \leq \Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) \geq 1).$$

Moreover, $\Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) \geq 1) = \Theta^\varepsilon(\mathcal{H}(T) > 2\varepsilon) = v(2\varepsilon)/v(\varepsilon)$ and $\Theta^\varepsilon(Z(\varepsilon, 2\varepsilon)) \leq 1$. So,

$$\frac{2v(2\varepsilon)}{v(\varepsilon)} - 1 \leq \Theta^\varepsilon(Z(\varepsilon, 2\varepsilon) = 1) \leq \frac{v(2\varepsilon)}{v(\varepsilon)}. \quad (20)$$

We let $\varepsilon \to 0$ in (20) and we use Lemma 4.1 to obtain the desired result. \hfill \square
4.1.2 Construction of the DSBP

**Proposition 4.7** For every $t \geq 0$, there exists an integer-valued random variable $L_t$ on the space $\mathbb{T}$ such that $\Theta(L_t) \leq 1$ and $\Theta$ a.s.,

\[ Z(t, t + h) \uparrow_{h \downarrow 0} L_t. \]

**Proof:** Let $t \geq 0$. The function $h \in (0, \infty) \mapsto Z(t, t + h) \in \mathbb{Z}_+$ is nonincreasing so that there exists a random variable $L_t$ with values in $\mathbb{Z}_+ \cup \{\infty\}$ such that, $\Theta$ a.s.,

\[ Z(t, t + h) \uparrow_{h \downarrow 0} L_t. \]

Thanks to the monotone convergence theorem, we have

\[ \Theta(Z(t, t + h)) \longrightarrow_{h \to 0} \Theta(L_t). \]

Now, by Proposition 4.5, $\Theta(Z(t, t + h)) \leq 1$ for every $h > 0$. Then, $\Theta(L_t) \leq 1$ which implies in particular that $L_t < \infty$ $\Theta$ a.s. $\square$

**Proposition 4.8** For every $t > 0$, the following two convergences hold $\Theta$ a.s.,

\[ Z(t - h, t) \uparrow_{h \downarrow 0} L_t, \tag{21} \]

\[ Z(t - h, t + h) \uparrow_{h \downarrow 0} L_t. \tag{22} \]

**Proof:** Let $t > 0$ be fixed throughout this proof. By the same arguments as in the proof of Proposition 4.7, we can find a $\mathbb{Z}_+$-valued random variable $L_t$ such that $\Theta(L_t) \leq 1$ and $Z(t - h, t) \uparrow L_t$ as $h \downarrow 0$, $\Theta$ a.s. If $h \in (0, t)$, we write $T^1, \ldots, T^{Z(t-h,t)}$ for the subtrees of $\mathcal{T}$ above level $t - h$ with height greater than $h$. Then, from the regenerative property,

\[ \Theta(|Z(t, t + h) - Z(t - h, t)| \geq 1) \]

\[ = \Theta\left( \Theta\left( \sum_{i=1}^{Z(t-h,t)} |Z(h, 2h)(T^i) - 1| \geq 1 \right| Z(t-h, t) \right) \]

\[ \leq \Theta\left( \Theta\left( |Z(h, 2h)(T^i) - 1| \geq 1 \text{ for some } i \in \{1, \ldots, Z(t-h, t)\} \right| Z(t-h, t) \right) \]

\[ \leq \Theta\left( Z(t-h, t)\Theta^h |Z(h, 2h) - 1| \geq 1 \right) \tag{23}. \]

Since $Z(t-h, t)\Theta^h |Z(h, 2h) - 1| \geq 1$ $\leq L_t$ $\Theta$ a.s., Proposition 4.6 and the dominated convergence theorem imply that the right-hand side of (23) goes to 0 as $h \to 0$. Thus $L_t = L_t$ $\Theta$ a.s.

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Likewise, there exists a random variable $\hat{L}_t$ with values in $\mathbb{Z}_+$ such that, $\Theta$ a.s., $Z(t-h, t+h) \uparrow \hat{L}_t$ as $h \downarrow 0$. Let us now notice that, for every $h > 0$, $\Theta$ a.s., $Z(t-h, t+h) \leq Z(t-h, t)$. Moreover, thanks to Lemma 4.2, we have

$$\Theta(Z(t-h, t) \geq Z(t-h, t+h) + 1) = 1 - \Theta \left( \left( \frac{v(2h)}{v(h)} \right)^{Z(t-h, t)} \right) \geq 1 - \Theta \left( \left( \frac{v(2h)}{v(h)} \right)^{\hat{L}_t} \right).$$

The right-hand side of (24) tends to 0 as $h \to 0$. So $\hat{L}_t \Theta$ a.s.

We will now establish a regularity property of the process $(L_t, t \geq 0)$.

**Proposition 4.9** The process $(L_t, t \geq 0)$ admits a modification which is right-continuous with left limits, and which has no fixed discontinuities.

**Proof:** We start the proof with three lemmas. The first one is proved in a similar but easier way as Lemma 3.9.

**Lemma 4.10** There exists $\lambda \geq 0$ such that $\Theta(L_t) = e^{-\lambda t}$ for every $t \geq 0$.

For every $n \geq 1$ and every $t \geq 0$ we set $Y^n_t = X_{[nt]}^{1/n}$.

**Lemma 4.11** For every $t \geq 0$, $Y^n_t \to L_t$ as $n \to \infty$, $\Theta$ a.s.

This lemma is an immediate consequence of Proposition 4.8.

**Lemma 4.12** Let us define $G_t = \sigma(L_s, s \leq t)$ for every $t \geq 0$. Then $(L_t, t \geq 0)$ is a nonnegative supermartingale with respect to the filtration $(G_t, t \geq 0)$.

**Proof:** Let $s, t, s_1, \ldots, s_p \geq 0$ such that $0 \leq s_1 \leq \ldots \leq s_p \leq s < t$ and let $f : \mathbb{R}^p \to \mathbb{R}_+$ be a bounded measurable function. For every $n \geq 1$, the offspring distribution $\mu_{1/n}$ is critical or subcritical so that $(X_{k/n}^{1/n}, k \geq 0)$ is a supermartingale. Thus we have

$$\Theta(X_{[nt]}^{1/n} f (X_{[ns_1]}^{1/n}, \ldots, X_{[ns_p]}^{1/n})) \leq \Theta(X_{[ns]}^{1/n} f (X_{[ns_1]}^{1/n}, \ldots, X_{[ns_p]}^{1/n})).$$

Lemma 4.11 yields $\Theta(L_t f (L_{s_1}, \ldots, L_{s_p})) \leq \Theta(L_s f (L_{s_1}, \ldots, L_{s_p}))$ since $f$ is bounded and $X_{k/n}^{1/n} \leq L_u \Theta$ a.s. for every $u \geq 0$.

Let us set, for every $t \geq 0$,

$$\widetilde{G}_t = \bigcap_{s > t} G_s.$$

Recall that $D$ denotes the set of positive dyadic numbers. From Lemma 4.12 and classical results on supermartingales, we can define a right-continuous supermartingale $(\widetilde{L}_t, t \geq 0)$ with respect to the filtration $(\widetilde{G}_t, t \geq 0)$ by setting, for every $t \geq 0$,

$$\widetilde{L}_t = \lim_{s \uparrow t, s \in D} L_s \quad (25)$$
where the limit holds $\Theta$ a.s. and in $L^1$. In a way similar to Section 3 we can prove that $(\tilde{L}_t, t \geq 0)$ is a càdlàg modification of $(L_t, t \geq 0)$ with no fixed discontinuities.

From now on, to simplify notation we replace $(L_t, t \geq 0)$ by its càdlàg modification $(\tilde{L}_t, t \geq 0)$.

**Proposition 4.13** $(L_t, t \geq 0)$ is a DSBP which becomes extinct $\Theta$ a.s.

**Proof:** By the same arguments as in the proof of (21), we can prove that, for every $0 < s < t$, the following convergence holds in probability under $\Theta$,

$$Z\left(\frac{[nt] - [ns]}{n}, \frac{[nt] - [ns] + 1}{n}\right) \rightarrow_{n \to \infty} L_{t-s}. \quad (26)$$

Let $s, t, s_1, \ldots, s_p \geq 0$ such that $0 \leq s_1 \leq \ldots \leq s_p \leq s < t$, $\lambda > 0$ and let $f : \mathbb{R}^ p \to \mathbb{R}$ be a bounded measurable function. For every $n \geq 1$, under $\Theta^{1/n}$, $(X^{1/n}_k, k \geq 0)$ is a Galton-Watson process started at one so that

$$\Theta^{1/n}(f(Y^n_{s_1}, \ldots, Y^n_{s_p}) \exp(-\lambda Y^n_t)) = \Theta^{1/n}(f(Y^n_{s_1}, \ldots, Y^n_{s_p}) (\Theta(\exp(-\lambda X^{1/n}_{[nt]-[ns]}))) Y^n_t).$$

From Lemma 4.11, (26) and dominated convergence, we get

$$\Theta(f(L_{s_1}, \ldots, L_{s_p}) \exp(-\lambda L_t)) = \Theta(f(L_{s_1}, \ldots, L_{s_p}) (\Theta(\exp(-\lambda L_{t-s}))) L^*_t).$$

Then, $(L_t, t \geq 0)$ is a continuous-time Markov chain with values in $\mathbb{Z}_+$ satisfying the branching property. Furthermore, since $\mathcal{H}(\mathcal{T}) < \infty \Theta$ a.s., it is immediate that $(\tilde{L}_t, t \geq 0)$ becomes extinct $\Theta$ a.s. \hfill $\square$

### 4.2 Identification of the probability measure $\Theta$

Let us now define, for every $\mathcal{T} \in \mathbb{T}$ and $t \geq 0$, $N_t(\mathcal{T}) = \#\{\sigma \in \mathcal{T} : d(\rho, \sigma) = t\}$ where we recall that $\rho$ denotes the root of $\mathcal{T}$.

**Proposition 4.14** For every $t \geq 0$, $N_t = L_t \Theta$ a.s.

Note that for every $t \geq 0$, $L_t$ is the number of subtrees of $\mathcal{T}$ above level $t$.

**Proof:** Since $\Theta(\mathcal{H}(\mathcal{T}) = 0) = 0$, we have $L_0 = 1 = N_0 \Theta$ a.s. Thanks to Propositions 4.7, 4.8 and 4.9, for every $t > 0$, $\Theta$ a.s., there exists $h_0 > 0$ such that for every $h \in (0, h_0]$, $L_t = L_{t-h} = L_{t+h} = Z(t-h, t+h)$.

The remaining part of the argument is deterministic. We fix $t, h_0 > 0$ and a (deterministic) tree $\mathcal{T} \in \mathbb{T}$. We assume that there is a positive integer $p$ such that for every $h \in (0, h_0]$,

$$L_t = L_{t-h} = L_{t+h} = Z(t-h, t+h) = p.$$

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and we will verify that \( N_t(T) = p \). We denote by \( T^1, \ldots, T^p \) the \( p \) subtrees of \( T \) above level \( t - h_0 \) and we write \( \rho_i \) for the root of the subtree \( T^i \). For every \( i \in \{1, \ldots, p\} \), we have \( \mathcal{H}(T^i) > 2h_0 \) so that there exists \( x_i \in T^i \) such that \( d(\rho_i, x_i) = 2h_0 \). Let us prove that for every \( i \in \{1, \ldots, p\} \),

\[
\mathcal{T}_{\leq 2h_0} = \{ \sigma \in T^i : d(\rho_i, \sigma) \leq 2h_0 \} = [\rho_i, x_i].
\]  

(27)

To this end, we argue by contradiction and assume that we can find \( i \in \{1, \ldots, p\} \) and \( t_i \in T^i \) such that \( d(\rho_i, t_i) \leq 2h_0 \) and \( t_i \notin [\rho_i, x_i] \). Let \( z_i \) be the unique vertex of \( T^i \) satisfying \([\rho_i, z_i] = [\rho_i, x_i] \cap [\rho_i, t_i] \). We choose \( c > 0 \) such that \( d(\rho_i, z_i) < c < d(\rho_i, t_i) \).

Then it is not difficult to see that \( T \) has at least \( p + 1 \) subtrees above level \( t - h_0 + c \). This is a contradiction since \( L_{t-h_0+c} = p \). So \( N_t(T) = p \), which completes the proof. \( \square \)

Proposition 1.14 means that \((L_t, t \geq 0)\) is a modification of the process \((N_t, t \geq 0)\) which describes the evolution of the number of individuals in the tree. Let us denote by \( Q \) the generator of \((L_t, t \geq 0)\) which is of the form

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
a\gamma(0) & -a & a\gamma(2) & a\gamma(3) & a\gamma(4) & \cdots \\
0 & 2a\gamma(0) & -2a & 2a\gamma(2) & a\gamma(3) & \cdots \\
0 & 0 & 3a\gamma(0) & -3a & 3a\gamma(2) & \cdots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix},
\]

where \( a > 0 \) and \( \gamma \) is a critical or subcritical offspring distribution with \( \gamma(1) = 0 \).

For every \( t \geq 0 \) we let \( \mathcal{F}_t \) be the \( \sigma \)-field on \( \mathbb{T} \) generated by the mapping \( T \mapsto \mathcal{T}_{\leq t} \) and completed with respect to \( |\Theta| \). Thus \((\mathcal{F}_t, t \geq 0)\) is a filtration on \( \mathbb{T} \).

**Lemma 4.15** Let \( t > 0 \) and \( p \in \mathbb{N} \). Under \( \Theta \), conditionally on \( \mathcal{F}_t \) and given \( \{L_t = p\} \), the \( p \) subtrees of \( T \) above level \( t \) are independent and distributed according to \( \Theta \).

**Proof:** Thanks to Lemma 2.2 and Lemma 4.3, we can construct on the same probability space \((\Omega, \mathcal{P})\), a sequence of \( \mathbb{T} \)-valued random variables \((T_n)_{n \geq 1}\) distributed according to \( \Theta^{1/n} \) and a sequence of \( \mathcal{A} \)-valued random variables \((\theta_n)_{n \geq 1}\) distributed according to \( \Pi_{\mu_{1/n}} \) such that, for every \( n \geq 1 \),

\[
d_{GH}(T_n, n^{-1}T_{\theta_n}) \leq 4n^{-1}.
\]

(28)

For every \( n \geq 1 \) and \( k \geq 0 \), we define \( X^u_k = \# \{ u \in \theta_n : |u| = k \} \). Let \( t \geq 0 \) and \( p \geq 1 \), let \( g : \mathbb{T} \to \mathbb{R} \) be a bounded continuous function and let \( G : \mathbb{T}^p \to \mathbb{R} \) be a bounded continuous symmetric function. For \( n \geq 1 \), on the event \( \{X^u_{|nt|} = p\} \), we set \( \{u^1, \ldots, u^p\} = \{ u \in \theta_n : |u| = |nt| \} \) and \( \theta^i_n = \tau_{nt} \theta_n \) for every \( i \in \{1, \ldots, p\} \). Then we can write, thanks to the branching property of Galton-Watson trees,

\[
E \left( \prod \{X^u_{|nt|} = p\} g \left( n^{-1}T_{\leq |nt|}^\theta \right) G \left( n^{-1}T_{\leq |nt|}^\theta, \ldots, n^{-1}T_{\leq |nt|}^\theta \right) \right) = E \left( \prod \{X^u_{|nt|} = p\} g \left( n^{-1}T_{\leq |nt|}^\theta \right) \right) \left( \Pi_{\mu_{1/n}} \right)^{\otimes p} \left( G \left( n^{-1}T_{\theta_1}, \ldots, n^{-1}T_{\theta_p} \right) \right),
\]

(29)
where \( \theta_1, \ldots, \theta_p \) denote the coordinate variables under the product measure \((\Pi_{\mu_{J/n}})^{\otimes p}\). As a consequence of (28), we see that the law of \( n^{-1} T^\theta \) under \( \Pi_{\mu_{J/n}} \) converges to \( \Theta \) in the sense of weak convergence of measures on the space \( \mathbb{T} \). Then, thanks to Lemma 4.11, the right-hand side of (29) converges as \( n \to \infty \) to
\[
\Theta(\mathbb{1}_{\{L_t=p\}} g(T_{\leq \tau})) \Theta^{\otimes p}(G(T_1, \ldots, T_p)).
\]
Similarly, the left-hand side of (29) converges as \( n \to \infty \) to
\[
\Theta(\mathbb{1}_{\{L_t=p\}} g(T_{\leq \tau}) G(T^1, \ldots, T^p)),
\]
where \( T^1, \ldots, T^p \) are the \( p \) subtrees of \( T \) above level \( t \) on the event \( \{L_t = p\} \). This completes the proof.

Let us define \( J = \inf\{t \geq 0 : L_t \neq 1\} \). Then \( J \) is an \( (\mathcal{F}_t)_{t \geq 0} \)-stopping time.

Lemma 4.16 Let \( p \in \mathbb{N} \). Under \( \Theta \), given \( \{L_J = p\} \), the \( p \) subtrees of \( T \) above level \( J \) are independent and distributed according to \( \Theta \), and are independent of \( J \).

Proof: Let \( p \in \mathbb{N} \), let \( f : \mathbb{R}_+ \to \mathbb{R} \) be a bounded continuous function and let \( G : \mathbb{T}^p \to \mathbb{R} \) be a bounded continuous symmetric function. On the event \( \{L_J = p\} \), we denote by \( T^1, \ldots, T^p \) the \( p \) subtrees of \( T \) above level \( J \). Let \( n \geq 1 \) and \( k \geq 0 \). On the event \( \{L_{(k+1)/n} = p\} \), we denote by \( T^1_{(n,k)}, \ldots, T^p_{(n,k)} \) the \( p \) subtrees of \( T \) above level \( (k+1)/n \).

On the one hand, the right-continuity of the mapping \( t \mapsto L_t \) gives
\[
\Theta\left( \sum_{k=1}^{\infty} \mathbb{1}_{\{L_{(k+1)/n} = p\}} G(T^1_{(n,k)}, \ldots, T^p_{(n,k)}) f((k+1)/n) \mathbb{1}_{\{k/n < J \leq (k+1)/n\}} \right) \to_{n \to \infty} \Theta(\mathbb{1}_{\{L_J = p\}} G(T^1, \ldots, T^p) f(J)).
\]

On the other hand, thanks to Lemma 4.11, we can write for every \( n \geq 1 \) and \( k \geq 0 \),
\[
\Theta\left( \mathbb{1}_{\{L_{(k+1)/n} = p\}} G(T^1_{(n,k)}, \ldots, T^p_{(n,k)}) f((k+1)/n) \mathbb{1}_{\{k/n < J \leq (k+1)/n\}} \right) = \Theta(\mathbb{1}_{\{L_J = p\}} f(J)) \Theta^{\otimes p}(G(T_1, \ldots, T_p)).
\]

It follows that \( \Theta(\mathbb{1}_{\{L_J = p\}} G(T^1, \ldots, T^p) f(J)) = \Theta(\mathbb{1}_{\{L_J = p\}} f(J)) \Theta^{\otimes p}(G(T_1, \ldots, T_p)) \). □

We can now complete the proof of Theorem 1.2. The random variable \( J \) is the first jump time of the DSBP \( (L_t, t \geq 0) \) so that \( J \) is distributed according to the exponential distribution with parameter \( a \) and is independent of \( L_J \). Thanks to Proposition 4.14, there exists \( \sigma_J \in T \) such that \( T_{\leq J} = [\rho, \sigma_J] \). Lemma 4.16 gives the last part of the description of \( \Theta \).

Another way to describe \( \Theta \) is as follows: Assume that we are given on the same probability space \((\Omega, \mathcal{P})\) an \( A \)-valued random variable \( \theta \) distributed according to \( \Pi_{\theta} \) and an independent sequence of independent random variables \( (h_u, u \in U) \) with values in \([0, \infty)\), such that each variable \( h_u \) is distributed according to the exponential distribution with parameter \( a \). We set \( T = (\theta, \{h_u\}_{u \in \theta}) \) and \( T = \mathcal{T}^T \). Then the random variable \( T \) is distributed according to \( \Theta \).
References


