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Abstract : The Lévy transform of a Brownian motion B is the Brownian motion $B'_t = \int_0^t sgn\,(B_s)\,dB_s$. Call T the corresponding transformation on the Wiener space W. We establish that a. s. the orbit of $w(\in W)$ under T is dense in W for the compact uniform convergence topology.

Introduction

Let $(B_t)_{t\geq 0}$ be a one-dimensional, issued from 0 Brownian motion, and $(L_t)_{t\geq 0}$ its local time at 0. The Lévy transform of B is the Brownian motion

$$B^1_t = \int_0^t sgn\,(B_s)\,dB_s = |B_t| - L_t.$$

Let T be the corresponding transformation in Ω, which maps B on B^1 : i.e. ω on $\mathbf{T}\omega$. w on Tw. At the end of chapter XII of [R,Y], Revuz and Yor asked for the ergodicity of T. From up to now, in [D,S], Dubins and Smorodinsky established the ergodicity of the discrete Lévy transform, an analog of T for random walks.
Our goal is to establish that the orbit under T of almost every path is dense in W equipped with the topology of uniform convergence on compact sets. This is a necessary but not sufficient condition for the possible ergodicity of Lévy transform, as it is the case for the density in $\mathbb{R}^+$ of the zeroes of the iterated Lévy transforms of Brownian motion established in [M].

Otherwise, the present paper is the natural consequence of [M]. As in [M], the strategy lies on the utilization of the *"Lévy's raises"* : after being

sufficiently down in the iterations, i.e. going from $T^n B$, for a big enough integer n, for at least one of $T^p B$, $p \leq n$, vanishes on given interval, we proceed for each raise from level n to level 0, by choosing a finite number of signs of selected excursions in such a way that the raised path takes the desired shape.

If the idea at the basis is simple, numerous difficulties arise when it is applied. So we present at paragraph 1, the **tools** we need, in a particularly simple situation : the **one time approximation**. Here, we introduce the technics of raisings which we illustrate with diagrams. Thus, we define the processes which we call B-raised Brownian motions. These processes enable us to easily manipulate the raisings, in particular on invariant events.

At paragraph 2, we come to a more elaborated situation, namely the d-time approximation. Here we need induction on the number of times. And we must act conditionnely on the hypothesis that none of the raised paths vanishes on (t_d, t_{d+1}).

Finally, at paragraph 3, we treat the continuous time approximation, when we read all the previous technics and a more acute analysis of the excursion straddling t.

In the end, I want to thank Marc Yor and Jean-Pierre Thouvenot for the good advices.

1 One-time approximation theorem.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where all random elements are defined, and $(\mathbf{W}, \mathcal{W}, \pi)$ the Wiener space. Any measurable map from Ω to any measure space, defined $\mathbf{P}-$a.e., will be called a random variable. If X is a r.v. with values in some measurable space A, the probability measure $\mathbf{P} \circ X^{-1}$ on A is called the law of X, and denoted by $\mathcal{L}(X)$. For instance, a $\mathbf{W}-$valued r.v. with law π is a Brownian motion.
The Lévy transform $\mathbf{T} : \mathbf{W} \longrightarrow \mathbf{W}$ is defined $\pi-$a.e. and preserves π. Given a Brownian motion B, we denote by B^n its $n-$Lévy iterate, that is, the Brownian motion $B^n = \mathbf{T}^n \circ B$. The one-time approximation theorem asserts that *for fixed $t > 0$ and for almost all ω, the sequence $B_t^n(\omega)$ is dense in $\mathbb{R}$.*

From now on, $t > 0$, $b \in \mathbb{R}$ and $\varepsilon > 0$ are fixed, and a Brownian motion B is given. The goal is to prove that the event $E = E^\varepsilon = \{\forall n \geq 0 | B_t^n - b| > \varepsilon\}$ is negligible. It suffices in fact to show $\mathbf{P}(E) < \varepsilon$, because $E^{\varepsilon_1} \subset E^{\varepsilon_2}$ when $\varepsilon_2 < \varepsilon_1$.

The idea is to construct from B another stochastic process $\Gamma : \Omega \longrightarrow \mathbf{W}$, which depends on t, b and ε, and has the following three properties :

(i) The law of the process Γ, i.e., the probability $F \longrightarrow \mathbf{P}(\Gamma^{-1}F)$ on $(\mathbf{W}, \mathcal{W})$, is absolutely continuous w.r.t. the law π of B.

(ii) For some deterministic $r \leq 0$, one has $\Gamma^r = B^r$, that is, $\mathbf{T}^r \circ \Gamma = \mathbf{T}^r \circ B$.

(iii) $\mathbf{P}\left(\forall n \geq 0 | \Gamma_t^n - b| > \varepsilon\right) < \varepsilon$.

Property (i) implies that $\mathbf{T} \circ \Gamma$ can be (almost everywhere) defined, in spite of $\mathbf{T}$ not being everywhere defined. Indeed, if $\mathbf{T}' : \mathbf{W} \longrightarrow \mathbf{W}$ is another version of $\mathbf{T}$, that is if $\mathbf{T}' = \mathbf{T}$ a.e., the set $\{\mathbf{T}' \neq \mathbf{W}\}$ is $\pi-$negligible; hence, by (i), $\Gamma^{-1}\{\mathbf{T}' \neq \mathbf{W}\}$ is $\mathbf{P}-$negligible, and $\mathbf{T}' \circ \Gamma = \mathbf{T} \circ \Gamma \mathbf{P}-$a.s.. Similarly, one can define the stochastic processes $\Gamma^n = \mathbf{T}^n \circ \Gamma$, which verify $\Gamma^0 = \Gamma$ and $\mathbf{T} \circ \Gamma^n = \Gamma^{n+1}$.

Proposition 1. *For fixed $G \in \mathcal{W}$ and $\varepsilon > 0$, let us suppose that there exists a stochastic process $\Gamma : \Omega \longrightarrow \mathbf{W}$ satisfying properties (i), (ii) and*

$$\mathbf{P}\left(\forall n \geq 0, \Gamma^n \in G\right) < \varepsilon. \tag{1}$$

Then :

$$\mathbf{P}\left(\forall n \geq 0, B^n \in G\right) < \varepsilon. \tag{2}$$

Proof Take $G \in \mathcal{W}$ and put $F = \bigcap_{n\geq 0} \mathbf{T}^{-n} G$. Then, for $r \geq 0$,

$$\mathbf{T}^{-r} F = \bigcap_{n\geq r} \mathbf{T}^{-n} G \supset \bigcap_{n\geq 0} \mathbf{T}^{-n} G = F. \tag{3}$$

But these two sets, F and $\mathbf{T}^{-r}F$, included in one another, have the same $\pi-$probability by $\mathbf{T}-$invariance; so equality $F = \mathbf{T}^{-r}F$ holds up to $\pi-$negligibility. As the laws of Γ and B are absolutely continuous w.r.t. π (this is where (i) is used), we have $\Gamma^{-1}(F) = \Gamma^{-1}(\mathbf{T}^{-r}F)$ and $B^{-1}F = B^{-1}(\mathbf{T}^{-r}F)$ up to $\mathbf{P}-$negligible events. In other words, almost surely, we have $\{\Gamma \in F\} = \{\Gamma^r \in F\}$ and $\{B \in F\} = \{B^r \in F\}$. Consequently, choosing r given by (ii) and using $\Gamma^r = B^r$, we have $\{\Gamma \in F\} = \{B \in F\}$ a.s.. That is to say :

$$\mathbf{P}\left(\forall n \geq 0, \Gamma^n \in G\right) = \mathbf{P}\left(\forall n \leq 0, B^n \in G\right) \tag{4}$$

■

Specializing $G = \{w \in \mathbf{W} \ ; \ |w(t) - b| > \varepsilon\}$, we obtain :

$$\mathbf{P}(E) < \varepsilon$$

Proposition 1 reduces the proof of the one-time approximation theorem to the construction of a process Γ verifying (i), (ii), and (iii). We shall first choose r in a suitable way, then work backwards, in r steps, from $\Gamma^r = B^r$ to $\Gamma = \Gamma^0$; each step (called a *Lévy raise*) will construct Γ^{n-1} from its Lévy transform $\Gamma^n = \mathbf{T} \circ \Gamma^{n-1}$. The sequence $(\Gamma^r, \Gamma^{r-1}, ..., \Gamma^0)$ is given a name :

Definition 1. *Given $r \in \mathbb{N}$, a sequence $(\Gamma^r, \Gamma^{r-1}, ..., \Gamma^0)$ is called a sequence of $B-$raised Brownian motions of index r if each Γ^n is a $\sigma(B^n)-$measurable, $\mathbf{W}-$valued r.v. with law absolutely continuous w.r.t. π, if $\Gamma^r = B^r$, and if we have $\Gamma^n = \mathbf{T} \circ \Gamma^{n-1}$ for $0 < n \leq r$.*

In fact, for convenience of exposition, let us enlarge the filtered probability space Ω, we suppose it contains the whole sequence $(B^n)_{n\in\mathbb{Z}}$ of the Brownian iterates.

So the Lévy transform $\mathbf{T}$ is defined at each level $n \in \mathbb{Z}$ and satisfies $\mathbf{T}B^n = B^{n+1}$, for all n in $\mathbb{Z}$. And we can generalize the preceding definition :

Definition 2. *Given $r \in \mathbb{N}$, a sequence $(\Gamma^n)_{n\geq r}$ is called a sequence of $B-$raised Brownian motions of index r if each Γ^n is a $\sigma(B^n)-$measurable, $\mathbf{W}-$valued r.v. with law absolutely continuous w.r.t. π, if $\Gamma^r = B^r$, and if we have $\Gamma^n = \mathbf{T} \circ \Gamma^{n-1}$ for all $n \leq r$.*

With this definition, we can rephrase Proposition 1 :

Corollary 1. *To prove the one-time approximation theorem, it suffices to exhibit a sequence* $(\Gamma^r, \Gamma^{r-1}, ..., \Gamma^0)$ *of* $B-$*raised Brownian motions of index* r *such that*

$$\mathbf{P}\left(|\Gamma_t^n - b| < \varepsilon \quad \textit{for some } n \ \in \{0, ..., r\}\right) > 1 - \varepsilon. \tag{5}$$

Proof Properties (i) and (ii) of Proposition 1 are granted by the definition of a sequence of $B-$raised Brownian motions, and (iii) is implied by (1).

■

A Lévy raise starts with a given $\mathbf{W}-$valued r.v. Γ^n, and yields some r.v. Γ^{n-1} with Lévy transform Γ^n. Given a $\mathbf{W}-$valued r.v. V, how can one find a r.v. U such that $V = \mathbf{T} \circ U$? Knowing V is equivalent to knowing $|U|$, so to define U one only needs to decide which sign is assigned to each excursion of $|U|$ away from zero. To make this rigorous, we need a formal definition of the excursions of a path and of their signs.

Notation 1. *For* $w \in \mathbf{W}$ *and* $q > 0$*, denote by* $Z(w) = \{s \geq 0 / w(s) = 0\}$ *the set of zeros of* w*, and define* $g_q(w) = \sup\left([0, q] \cap Z(w)\right) \geq 0$ *(last zero before* q*) and* $d_q(w) = \inf\left([q, \infty] \cap Z(w)\right) \neq \infty$ *(first zero after* q*).*

Fix a dense sequence (q_n) in $[0, \infty]$. To each $w \in \mathbf{W}$, we can attach the sequence (e_p) of disjoint, open intervals obtained from the sequence

$$\left((g_{q_1}, d_{q_1}), (g_{q_2}, d_{q_2}), ..., (g_{q_n}, d_{q_n}), \dots\right) \tag{6}$$

by deleting an interval whenever it already occurs earlier in the sequence. The e_p are the *excursion intervals* of w.$\pi-$almost surely, there are infinitely many of them, and they are the connected components of the open set $[0, \infty] \backslash Z(w)$. The interval $e_p(w)$ will be called the *p-th excursion interval* of w; e_p is an interval-valued measurable map, defined on $(\mathbf{W}, \mathcal{W})$ up to $\pi-$negligibility.
Since w does not vanish on $e_p(w)$, its sign is constant on this interval; this sign will be denoted by $\mathbf{S}_p(w)$, and the sequence $(\mathbf{S}_p)$ will be called $\mathbf{S}$. If B is a Brownian motion, the sequence of r.v. $\mathbf{S} \circ B = (\mathbf{S}_p \circ B)$ is a *coin-tossing*; this means, it is an i.i.d. sequence, with each r.v. $\mathbf{S}_p \circ B$ uniformly distributed on the set $\{-1, +1\}$. Moreover, $\mathbf{S}_p \circ B$ and $|B|$ are independent. (See Chap. XII of [R,Y]).

Lemma 1. *Define* $\mathbf{I} : \mathbf{W} \longrightarrow \mathbf{W}$ *by* $\mathbf{I}(w)(s) = \inf_{[0,s]} w$ *; that is,* $|w| = \mathbf{T}w - \mathbf{IT}w$ *for* $\pi-a.a.$ w.

Proof Fix $s \geq 0$. On $[0,s]$, $B^1 = |B| - L \geq -L_s = L_{g_s} = B^1_{g_s}$. So $B^1_{g_s} = \inf_{[0,s]} B^1$, and $|B_s| = B^1_s + L_s = B^1_s - \inf_{[0,s]} B^1$.

■

Lemma 2. *Let* A, A' *and* A'' *be three measure spaces; let* μ_1 *and* μ_2 *be two measures on* A, f *a measurable map from* A *to* A', *and* ν *a measure on* A''. *If* $\mu_1 \ll \mu_2$, *then*

(i) $\mu_1 \circ f^{-1} \ll \mu_2 \circ f^{-1}$;

(ii) $\mu_1 \otimes \nu \ll \mu_2 \otimes \nu$.

Proof (i) If $F \subset A'$ is measurable and if $(\mu_2 \circ f^{-1})(F) = 0$, then $\mu_2(f^{-1}F) = 0$, so $\mu_1 \circ f^{-1}(F) = \mu_1(f^{-1}F) = 0$.
(ii) If a measurable subset F of $A \times A''$ is negligible for $\mu_2 \otimes \nu$, then $\nu-$almost all its sections F_y verify $\mu_2(F_y) = 0$. Hence they also verify $\mu_1(F_y) = 0$, and consequently $(\mu_1 \otimes \nu)(F) = \int \mu_1(F_y)\nu(dy) = 0$.

■

Lemma 3. *Let* $\tau = (\tau_p)$ *be coin-tossing,* $\tau' = (\tau'_p)$ *a r.v. with values in* $\{-1,1\}^{\mathbb{N}}$ *such that* $\tau'_p = \tau_p$ *for all but a.s. finitely many* p, *and* X *a r.v. independent of* τ.
Then $\mathcal{L}(X,\tau') \ll \mathcal{L}(X,\tau)$.

This lemma says that changing finitely many values of τ does not perturb too much the joint law of X and τ. For instance, it implies that a process obtained from a Brownian motion by changing the signs of finitely many excursions has a law absolutely continuous w.r.t. π. This is called *'principe de retournement des excursions'* in [M].

Proof If $u = (u_1, u_2, ...)$ is an infinite sequence, denote by $u_{p]}$ the finite sequence $(u_1, ..., u_p)$ and by $u_{[p+1}$ the infinite sequence $(u_{p+1}, u_{p+2}, ...)$. We have $(x,u) = f_p(x, u_{p]}, u_{[p+1})$ for some function f_p.

We have to show that if F is measurable set such that $\mathbf{P}[(X,\tau) \in F] = 0$, then $\mathbf{P}[(X,\tau') \in F] = 0$. So assume $\mathbf{P}[(X,\tau) \in F] = 0$. For $p \in \mathbb{N}$, since $\tau_{p]}$ takes values in $\{-1,1\}^p$, we can write

$$(7) \qquad \sum_{\sigma \in \{-1,1\}^p} \mathbf{P}\left[f_p(X,\sigma,\tau_{[p+1}) \in F \text{ and } \tau_{p]} = \sigma\right] = \mathbf{P}\left[(X,\tau) \in F\right] = 0.$$

Using the independence of $\tau_{p]}$ and $(X,\tau_{[p+1})$, this becomes :

$$(8) \qquad \sum_{\sigma \in \{-1,1\}^p} 2^{-p}\mathbf{P}\left[f_p(X,\sigma,\tau_{[p+1}) \in F\right] = 0;$$

So for each $p \in \mathbb{N}$ and each $\sigma \in \{-1,1\}^p$, the event $\{f_p(X,\sigma,\tau_{[p+1} \in F\}$ is negligible. Since $\tau'_p(\omega) = \tau_p(\omega)$ for all p larger than some $N(\omega)$, one has

$$\begin{aligned}\mathbf{P}\left[(X,\tau') \in F\right] &= \lim_{p\to\infty} \mathbf{P}\left[(X,\tau') \in F \text{ and } \tau'_{[p+1} = \tau_{[p+1}\right] \\ &= \lim_{p\to\infty} \sum_{\sigma \in \{-1,1\}^p} \mathbf{P}\left[f_p(X,\sigma,\tau_{[p+1}) \in F \text{ and } \tau'_{p]} = \sigma \text{ and } \tau'_{[p+1} = \tau_{[p+1}\right].\end{aligned}$$

This is null because the event $f_p(X,\sigma,\tau_{[p+1}) \in F$ is negligible, as shown above.

■

Proposition 2. *(mechanism of a Lévy raise)*
Suppose given the following three r.v. :

(i) V, *a* $\mathbf{W}-$*valued r.v., such that* $\mathcal{L}(V) \ll \pi$;

(ii) $\tau = (\tau_p)_{p\in\mathbb{N}}$, *a coin-tossing independent of* V ;

(iii) $\tau' = (\tau'_p)_{p\in\mathbb{N}}$, *a r.v. valued in* $\{-1,1\}^{\mathbb{N}}$, *such that the random set* $p \in \mathbb{N} \quad \tau'_p(\omega) \neq \tau_p(\omega)$ *is a.s. finite.*

Then there exists a unique $\mathbf{w}-$*valued r.v.* U *such that*

$$(9) \qquad |U| = V - \mathbf{I} \circ V \text{ and } \mathbf{S}_p \circ U = \tau'_p \text{ for each } p.$$

It is measurable w.r.t. the $\sigma-$*field* $\sigma(V,\tau')$ *and we have* $\mathcal{L}(U) \ll \pi$ *and* $\mathbf{T}\circ U = V$. *For any* $n \geq 0$, *we have* $U = B^n$ *on the event* $\{V = B^{n+1} \text{ and } \tau' = \mathbf{S} \circ B^n\}$.

Proof We start from $\mathcal{L}(V) \ll \pi = \mathcal{L}(B) = \mathcal{L}(B^1)$. Using Lemma 2 (i) we write $\mathcal{L}(V, V - \mathbf{I} \circ V) \ll \mathcal{L}(B^1, B^1 - \mathbf{I} \circ B^1)$. By Lemma 2 (ii), the coin-tossing τ (resp. $\mathbf{S} \circ B$) which is independent of V (resp. B^1) can be added on the left (resp. right), and we obtain $\mathcal{L}(V, V - \mathbf{I} \circ V, \tau) \ll \mathcal{L}(B^1, B^1 - \mathbf{I} \circ B^1, \mathbf{S} \circ B)$; by Lemma 1, the right-hand side is $\mathcal{L}(B^1, |B|, \mathbf{S} \circ B)$. Lemma 3 allows us to replace τ by τ' in the left-hand side, so we finally have

$$\mathcal{L}(V, V - \mathbf{I} \circ V, \tau) \ll \mathcal{L}(B^1, |B|, \mathbf{S} \circ B) \tag{10}$$

Now, we call $\mathbf{W}^+$ the set of non-negative paths and $f : \mathbf{W}^+ \times \{-1, 1\}^{\mathbb{N}} \longrightarrow \mathbf{W}$ the measurable function such that $w = f(|w|, \mathbf{S}(w))$. We remark that $B = f(|B|, \mathbf{S} \circ B)$ and we define $U = f(V - \mathbf{I} \circ V, \tau')$; this is the unique r.v. U such that $|U| = V - \mathbf{I} \circ V$ and $\mathbf{S} \circ U = \tau'$. To verify that $\mathcal{L}(U) \ll \pi$ and $\mathbf{T} \circ U = V$, we apply Lemma 2 (i) to (2) with the functions $g(x, y, \sigma) = f(y, \sigma)$ and $h(x, y\sigma) = (f(y, \sigma), x)$. With g we obtain $\mathcal{L}(U) \ll \mathcal{L}(B)$, the first claim. With h we obtain $\mathcal{L}(U, V) \ll \mathcal{L}(B, B^1)$; this implies $\mathbf{T} \circ U = V$ since the joint law $\mathcal{L}(B)(B^1)$ is carried by the graph of $\mathbf{T}$.
Last, on the event $\{V = B^{n+1} and \tau' = \mathbf{s} \circ B^n\}$, using the definition of U and Lemma 1 we have

$$U = f(V - \mathbf{I} \circ V, \tau') = f(B^{n+1} - \mathbf{I} \circ B^{n+1}, \mathbf{S} \circ B^n) = f(|B^n|, \mathbf{S} \circ B^n) = B^n. \tag{11}$$

■

In view of Corollary 1, we will now describe the construction of a sequence of $B-$raised Brownian motions of index r, for some $r \in \mathbb{N}$ to be chosen later. The sequence of $\sigma-$fields $(\sigma(B^n))_{n \leq r}$ is decreasing; we may consider it as a reverse filtration, and define a r.v. J with values in $\mathbb{Z} \cap]-\infty, r]$ to be a *stopping time* if $\{J = n\}$ belongs to $\sigma(B^n)$ for each $n \in \mathbb{Z} \cap]-\infty, r]$ (or equivalently, $\{J \geq n\} \in \sigma(B^n)$, because the filtration is reverse). Naturally, B^J is defined by $B^J = B^n$ on $\{J = n\}$. If J is a stopping time and $J \leq n$, then $J - n$ is a stopping time too.
(Introducing such stopping times creates no ambiguity : we will never deal with the filtration generated by B, or B^n, or Γ^m, nor with stopping times for such filtrations.)

Lemma 4. *Let J be a stopping time such that $J \leq r$. The r.v. $\mathbf{S} \circ B^{J-1}$ is a coin-tossing independent of B^J.*

Proof Call ν the law of a coin-tossing (i.e., the law on $\{-1, 1\}^{\mathbb{N}}$ making the

coordinates independent and uniform on $\{-1,1\}$), and write, for arbitrary bounded functionals φ and ψ

$$\begin{aligned}\mathbf{E}\left[\varphi(B^J)\,\psi(\mathbf{S}\circ B^{J-1})\right] &= \sum_{n=1}^{r}\mathbf{E}\left[\mathbb{1}_{\{J=n\}}\varphi(B^n)\psi(\mathbf{S}\circ B^{n-1})\right]\\ &= \sum_{n=1}^{r}\mathbf{E}\left[\mathbb{1}_{\{J=n\}}\varphi(B^n)\right]\mathbf{E}\left[\psi(\mathbf{S}\circ B^{n-1})\right]\\ &= \sum_{n=1}^{r}\mathbf{E}\left[\mathbb{1}_{\{J=n\}}\varphi(B^n)\right] \quad \nu(\psi)=\mathbf{E}\left[\varphi(B^J)\right].\end{aligned}$$

■

Proposition 3. *Denote by $\mathcal{P}_f(\mathbb{N})$ the set of all finite subsets of $\mathbb{N}$. Fix r in $\mathbb{N}$, and let J be a stopping time with values $\leq r$. For each $n\geq 0$, let be given $\mathcal{N}^n$, a r.v. with values in $\mathcal{P}_f(\mathbb{N})$, and $\Sigma^n=(\Sigma^n_p, p\in\mathcal{N}^n)$, a r.v. taking its values in $\bigcup_{\mathcal{M}\in\mathcal{P}_f(\mathbb{N})}\{-1,1\}^{\mathcal{M}}$, such that $\Sigma^n(\omega)\in\{-1,1\}^{\mathcal{N}^n(\omega)}$.Suppose $\mathcal{N}^n$ and Σ^n are $\sigma(B^{J-n})-$measurable.*
Starting with $C^0=B^J$, we can define a sequence $(C^n)_{n\in\mathbb{N}}$ such that C^{n+1} is the $\mathbf{W}-$valued r.v. U obtained in Proposition 2 from

$$V=C^n,\qquad \tau=\mathbf{S}\circ B^{J-n-1},\qquad \tau'_p=\begin{cases}\Sigma^p(\omega) & \text{if } p\in\mathcal{N}^p(\omega)\\ \tau_p(\omega) & \text{else}\end{cases}$$

C^n is $\sigma(B^{J-n})-$measurable, and there exists a sequence $(\Gamma^n)_{n\geq r}$ of B-raised Brownian motions of index r such that $\Gamma^{J-n}=C^n$ for all $n\in\mathbb{N}$.

Proof First, we verify that the C^n can be constructed stepwise. Assuming C^n has already been constructed, has a law absolutely continuous w.r.t. π, and is $\sigma(B^{J-n})-$measurable, Proposition 2 applies to $V=C^n$ and $\tau=\mathbf{S}\circ B^{J-n-1}$ (they are independent by Lemma 4 applied to the stopping time $J-n$). The r.v. $C^{n+1}=U$ yielded by Proposition 2 also satisfies $\mathcal{L}(C^{n+1})\ll\pi$, and is measurable in $\sigma(C^n,\tau')$. But this $\sigma-$field is included in $\sigma(B^{J-n},\mathbf{S}\circ B^{J-n-1})$, and in $\sigma(B^{J-n-1})$ because $B^{J-n}=\mathbf{T}\circ B^{J-n-1}$; so the construction is possible.
The rest of the proof will exhibit a sequence $(\Gamma^n)_{n\geq r}$ of B-raised motions such that $\Gamma^{J-n}=C^n$. Starting with $\Gamma^r=B^r$, the other Γ^m will be inductively

defined : if $m < r$, suppose Γ^{m+1} has been defined, is $\sigma(B^{m+1})-$measurable, and verifies $\mathcal{L}(\Gamma^{m+1} \ll \pi$; define Γ^m as the r.v. U obtained in Proposition 2 from

(12)

$$V = \Gamma^{m+1}, \qquad \tau = \mathbf{S} \circ B^m, \qquad \tau'_p = \begin{cases} \Sigma^{J-m-1}(\omega) & \textit{if } \ p \in \mathcal{N}^{J-m-1}(\omega) \\ \tau_p(\omega) & \text{else} \end{cases}$$

This is possible since V and τ are independent and $\mathcal{N}^{J-m-1}$ is a.s. finte; the result Γ^m verifies $\mathcal{L}(\Gamma^n) \ll \pi$ and $\mathbf{T} \circ \Gamma^m = \Gamma^{m+1}$. To show that Γ^m is $\sigma(B^m)-$measurable, it suffices to show that so is τ'; this may be done separately on each of the events $\{J \leq m\}$, $\{J = m+1\}$, ..., $\{J = r\}$, because they form a $\sigma(B^m)-$partition of Ω. On $\{J \leq m\}$, we have $\tau' = \mathbf{S} \circ B^m$; this is $\sigma(B^m)-$measurable. To see what happens for other values of j, introduce φ^n and ψ^n such that $\mathcal{N}^n = \varphi^n(B^{J-n})$ and $\Sigma^n = \psi^n(B^{J-n})$ for $0 \leq n < k$. For $j \in \{m+1, ..., m+k\}$, we have on $\{J = j\}$

$$\tau' = \psi^n(B^{J-n}) \mathbb{1}_{\varphi^n(B^{J-n})} + \tau \left(\mathbb{1}_\Omega - \mathbb{1}_{\psi^n(B^{J-n})} \right) \tag{13}$$

This is $\sigma(B^m)-$measurable too. We have established that $\Gamma^r, ..., \Gamma^0$ exist and form a sequence of $B-$raised motions; it remains to see that $\Gamma^{J-n} = C^n$. This is done in two steps. Firstly, by induction on m, we have $\Gamma^m = B^m$ on $\{J \leq m\}$: this holds for $m = r$, and if it holds for $m+1$, it holds for m too, owing to the last statement in Proposition 2. Consequently, $\Gamma^m = B^m on \{J = m\}$, that is $\Gamma^J = B^J = C^0$. Secondly, to proceed by induction on n, we will assume that $\Gamma^{J-n} = C^n$ for some $n \geq 0$, and show $\Gamma^{J-n-1} = C^{n+1}$. It suffices to show this equality on the event $\{J = j\}$; on this event, using the definition of Γ^m with $m = j-n-1$ and the inequality $m = j-n-1 < j$, the r.v. Γ^{J-n-1} satisfies both $\mathbf{T}(\Gamma^{J-n-1}) = \mathbf{T}(\Gamma^{j-n-1}) = \Gamma^{j-n} = C^n = \mathbf{T}(C^{n+1})$ and

$$\mathbf{S} \circ \Gamma^{j-n-1} = \begin{cases} \Sigma^{n+1} & \text{on } \ \mathcal{N}^n \\ \mathbf{S} \circ B^{j-n-1} & \text{else} \end{cases} = \mathbf{S} \circ C^{n+1}. \tag{14}$$

These two equalities entail $\Gamma^{J-n-1} = C^{n+1}$ a.s. on $\{J = j\}$.

■

Corollary 2. *To prove the theorem, it suffices to find $r, J, \mathcal{N}^n$, and Σ^n as in Proposition 3, such that*

$$\mathbf{P}\left(|C_t^J - b| < \varepsilon \right) > 1 - \varepsilon. \tag{15}$$

Proof In that case, the sequence $(\Gamma^r, ..., \Gamma^0)$ of $B-$raised motions given by Proposition 3 verifies $\mathbf{P}\left(|\Gamma_t^0 - b| < \varepsilon\right) > 1 - \varepsilon$, so Corollary 1 applies.

■

Remark. *Corollary 2 improves on Corollary 1 by enabling us to work with Lévy raises from B^J up, with J random, instead of B^r in Corollary 1. The price to pay is that J must be a stopping time.*
Practically, the $\mathcal{N}^n$ and Σ_p^n used when performing the Lévy raise from C^n to C^{n+1} will be functionals of C^n; this automatically makes them $\sigma(B^{J-n})$-measurable, since so is C^n. so the step $C^n \longrightarrow C^{n+1}$ will consist in choosing finitely many excursions if C^n above its current minimum, and assigning them a sign; these choices depend only on the behavior of the path of C^n. The other excursions of C^n above its minimum are given random signs, using the signs of the excursions of B^{J-n-1} as a source of randomness. The path of C^{n+1} is obtained by putting all these signed excursions together.

Lemma 5. *Let $(j,k) \in \mathbb{N}^2$ and Q and R be two r.v. such that $k \leq j-1$ and $0 \leq Q \leq R$. On the event $\{\forall n \in \{k, ..., j-1\} Z \circ B^n \cap (Q,R) = \emptyset\}$ that the first iterates of B do not vanish between Q and R, there exists a (random) isometry $i : \mathbb{R} \longrightarrow \mathbb{R}$ such that $B^j = i \circ B^k$ on the interval (Q,R).*

Proof By induction, it suffices to show that if B^{j-1} does not vanish on the interval (Q,R), then $B^j = i \circ B^{j-1}$ on (Q,R), for some random isometry i. This is just Lemma 5 with $j = k+1$ and B^{j-1} instead of B^k, so we may suppose that $j = 1$.
On the event $\{Q = R\}$, the result is trivial. On $\{Q < R\} \cap \{Z \circ B \cap (Q,R) = \emptyset\}$, the local time L is constant on $[Q,R]$ because its support is $Z \circ B$, and the sign of B is constant on (Q,R); so $B^1 = |B| - L = i(B)$ on (Q,R), where i is the random isometry $x \mapsto xsgn\left(B_{(Q+R)/2}\right) - L_{(Q+R)/2}$.

■

It will be convenient to work with $\frac{\varepsilon}{3}$ instead of ε; for notational simplicity, we put $\varepsilon' = \frac{\varepsilon}{3}$.

Lemma 6. *There exists a number $j = j(t,\varepsilon) > 1$ (fixed in the sequel) such that the r.v. J_+ and $J_- = \sup\{n \in \{1,...,j-1\}; |B^n_t| < \varepsilon'\}$ (with $\sup\emptyset = 0$) and $J_+ = J_- + 1$, have the following four properties :*

(i) $1 \le J_+ \le j$; $0 \le J_- \le j-1$;

(ii) J_+ *and* J_- *are stopping times for the filtration* $(\sigma(B^j), \sigma(B^{j-1}), ..., \sigma(B^0))$;

(iii) $\mathbf{P}(J_+ = 1) < \varepsilon'$;

(iv) on the event $\{J_+ > 1\} = \{J_- > 0\}$, *we have* $|B^{J_-}_t| < \varepsilon'$.

Moreover, the law of the process B^{J_-} depends only upon j, t and ε, and is absolutely continuous w.r.t. the Wiener measure π.

Proof We shall first exhibit j and a r.v. K, with values in $\{1,...,j-1\}$, such that

- $\mathbf{P}(K = 1) < \varepsilon'$
- on the event $\{K > 1\}$, we have $|B^K_t| < \varepsilon'$.

Set $P = \sup\{s \in [0,t]; |B^2_s - B^2_t| > \varepsilon'\}$, with $\sup\emptyset = 0$. We have $|B^2 - B^2_t| \le \varepsilon'$ on the (random) non-empty interval (P,t). Put $g^n_t = g_t \circ B^n = \sup(Z \circ B^n \cap [0,t])$, and call H the first $n \ge 2$ such that $g^n_t > P$. This H is a.s. finite because $\sup_{m\ge 2} g^m_t = t$ a.s. (see [M]). On the event $\{H = n\}$, there are no zeroes of $B^2, ..., B^{n-1}$ in the interval (g^n_t, t). On this event, we have $g^n_t \in (P,t)$ and

$$\begin{aligned} |B^n_t| &= |B^n_t - B^n_{g^n_t}| \qquad \textit{by definition of } g^n_t \\ &= |B^2_t - B^2_{g^n_t}| \qquad \textit{by Lemma 5 applied to } B^1 \textit{ and the interval } (g^n_t, t) \\ &= \le \varepsilon' \qquad \textit{because } |B^2 - B^2_t| \le \varepsilon' \textit{ on } (P,t). \end{aligned}$$

As H is a.s. finite, there exists $j > 1$, depending only upon t and ε, such that $\mathbf{P}(H \ge j) < \varepsilon'$; so the r.v. $K = H$ verfies

$$1 \le K \le j, \qquad \mathbf{P}(K = 1) < \varepsilon', \quad \textit{and} \quad |B^K_t| < \varepsilon' \textit{ on } \{K > 1\};$$

in other words, (iii') and (iv') hold for j and K.
The mere existence of such a K immediately implies that the random variable $J_+ = 1 + \sup\{n \in \{1,...,j-1\}; |b^n_t| < \varepsilon'\}$ verifies (iii) and $|B^{J_+-1}_t| < \varepsilon'$ on $\{J_+ > 1\}$, whence (iv). Moreover, J_+ is a stopping time because, for $n \ge 1$, one has $J_+ \ge n$ if and only if $|B^{m-1}_t| < \varepsilon'$ for some $m \in \{n,...,j\}$, and because

$\sigma(|B^{m-1}|) = \sigma(B^m)$. And J_- is a stopping time too since $J_- = J_+ - 1$.
The law of B^{J_-} depends only on j, t, and ε because B^{J_-} is constructed using nothing but B, j, t, and ε. This law is absolutely continuous w.r.t. π, because $B^{J_-} = \sum_{n=0}^{j-1} B^n \mathbb{1}_{[J_-=n]}$ implies $\mathcal{L}(B^{J_-}) \leq \sum_{n=0}^{j-1} \mathcal{L}(B^n) = j\pi$.

■

Notation 2. *For $w \in \mathbf{W}$, the p-th excursion interval $e_p(w)$ was defined earlier; the number $\mathbf{h}_p(w) = \max_{s \in e_p(w)} |w(s)|$ will be called the height of the corresponding excursion.*

Lemma 7. *Let X be a process whose law is absolutely continuous w.r.t. Wiener measure. Almost surely,*

- $\lim_{p\to\infty} \mathbf{h}_p(X)\mathbb{1}_{\{e_p(X)\subset[0,t]\}} = 0$;
- $\sum_{p\in\mathbb{N}} \mathbf{h}_p(X)\mathbb{1}_{\{e_p(X)\subset[0,t]\}} = \infty$;
- *the set* $\left\{ \sum_{p\in\mathcal{M}} \mathbf{h}_p(X)\mathbb{1}_{\{e_p(X)\subset[0,t]\}}, \mathcal{M} \in \mathcal{P}_f(\mathbb{N}) \right\}$ *is dense in* $[0,\infty)$;
- *between any two different excursions of X, there exists a third one, with height smaller than any given random variable $\eta > 0$.*

Proof By a change of probability, we may suppose that X is a Brownian motion. It is known (see Exercise (VI.1.19) of [RY]) that when $\eta \to 0^+$, the number $\sum_p \mathbb{1}_{\{e_p(X)\subset[0,t]\}}\mathbb{1}_{\{\mathbf{h}_p(X)>\eta\}}$ of downcrossings of the interval $[0,\eta]$ by $|X|$ before t is a.s. equivalent to $\eta^{-1}L_t$, where L_t is the local time of X at 0. This easily implies (i) and (ii), wherefrom (iii) follows.
Last, between any two excursions of X there are infinitely many other ones (because X has no isolated zeroes) and, by (i), only finitely many with heights above η, whence (iv).

■

Notation 3. *An excursion whose interval is included in $[0,t]$ will be called a $t-$excursion.*

Lemma 8. *There exists a $\sigma(B^{J_+})-$measurable, $\mathbb{N}^*-$valued r.v. K such that for each r.v. K' $\sigma(B^{J_+})-$measurable and $\mathbb{N}^*-$valued r.v., there exists, on the event $[K' \geq K]$, $K'-1$ r.v. $P_1, ..., P_{K'-1}$ with values in $\mathbb{N}$ and $\sigma(B^{J_+})-$measurable, such that :*

(i) the $K'-1$ excursion intervals $e_{P_1}(B^{J_-}), \ldots, e_{P_{K'-1}}(B^{J_-})$ are disjoint and included in $[0,t]$;

(ii) the heights $\mathbf{h}_{P_1}(B^{J_-}), \ldots, \mathbf{h}_{P_{K'-1}}(B^{J_-})$ of the $K'-1$ excursions of B^{J_-} verify

$$|b| - \varepsilon' < \mathbf{h}_{P_1}(B^{J_-}) + \cdots + \mathbf{h}_{P_{K'-1}}(B^{J_-}) < |b| + \varepsilon' \qquad a.s. \tag{16}$$

Proof Lemma 7 (iii) applies to B^{J_-} (whose law is absolutely continuous by Lemma 6) implies that we can find random finite sets of $t-$excursions of B^{J_-} the sum of whose heights is $\frac{\varepsilon'}{2}$ close to $|b|$. Among them there is a set of minimal cardinality :
denote by K the (random) number of its elements. And the same Lemma 7 (iii) implies that, for every $\sigma(B^{J_-})-$measurable, $\mathbb{N}^*$ r.v. verifying $K' \geq K$ a.s., we can find a random finite set of $t-$excursions of B^{J_-} of cardinality K' satisfying (1).
All these choices can be made $\sigma(B^{J_+})-$measurably, because they depend only on $|B^{J_-}|$, which is $\sigma(B^{J_+})-$measurable since it equals $B^{J_+} - \mathbf{I} \circ B^{J_+}$ by Lemma 1.

■

These $K-1$ excursions with heights adding up to $|b| \pm \varepsilon'$ are called the building excursions; we will see in the proof of Proposition 5 how a Lévy raise can be performed so as to add height of a building excursion to the value at t. So, starting with an almost zero value at t (this is granted by Lemma 6 (iv)) and using Lemma 8, we can end up close to b after $K'-1$ raises, in view of Corollary 2. In fact, for a technical reason (Lemma 10), we will need an extra raise; all in all, there will be K' of them, as in Proposition 3 and its corollary. In virtue of the enlargement of the probability space, it is now possible that $K'(\omega) > J(\omega)$ with strictly positive probability. But our goal is to obtain the final result for B itself. So a simple way to achieve that is to apply Lemma 6, not to the initial Brownian motion B, but to its k-th Lévy iterate B^k for k to be specified later. The filtration in Lemma 6 (ii) becomes $\big(\sigma(B^{k+j}), \ldots, \sigma(B^k)\big)$ and the time-scale is shifted by k, yielding stopping

times with values in $\{k, ;k+j\}$. Observe that there is no vicious circle here : Lemma 6 gives a j which depends upon the constants t and ε only, then Lemma 8 refers to the stopping times from Lemma 6 but the constant k there depends only upon the other fixed constants, so we may licitly first fix all the constants ε_n, j, k, and only then apply Lemma 6 to the Brownian motion B^k which depends on k.

Corollary 3. *Put $r = k+j$. The r.v. $J = 1+\sup\{n; k<n<r$ and $|B^n_t| < \varepsilon'\}$ (with $\sup\emptyset = k$) has the following properties :*

(i) $k+1 \leq J \leq r$;

(ii) *J is a stopping time for the filtration $(\sigma(B^r), \ldots, \sigma(B^0))$;*

(iii) *the event $\Omega_2 = \{|B^{J-1}| < \varepsilon'\}$ belongs to $\sigma(B^J)$ and verifies $\mathbf{P}(\Omega_2) > 1-\varepsilon'$;*

(iv) *there exists an integer r large enough and depending only on π, b, t, ε, such that on some event $\Omega_1 \in \sigma(B^J)$ having probability $\mathbf{P}(\Omega_1) > 1-\varepsilon'$, B^{J-1} has $J-1$ different $t-$excursions which are measurable in $\sigma(B^J)$, and whose heights $H_1, \ldots, H_{J-1}$ (numbered in chronological order of the excursions) verify*

$$|H_1 + \cdots + H_{J-1} - |b|| < \varepsilon'. \tag{17}$$

Proof (i), (ii) and (iii) are immediate by calling J the stopping time J_+ from Lemma 6 applied to B^k instead of B, rephrasing accordingly Lemma 6.
For (iv), with the preceding choice of J, there exists an integer k such that $\mathbf{P}([K \leq k]) > 1-\varepsilon'$. Call Ω_1 the event $[K \leq k]$ for such a k. There we choose $K' = J$, we rephrase accordingly Lemma 6 and 8, and order $J-1$ excursions in chronological order (H_1 is the height of the first of these excursions, H_2 the height of the second one and so on).

■

Corollary 2 will be applied to these k, r and J. It remains to describe the $\mathcal{N}^n$ and Σ^n, i.e., to choose the signs of finitely many excursions when Lévy-raising from C^n to C^{n+1}. This will be done soon; we first need some notation and a lemma.

Notation 4. *If e' and e'' are two excursions of a path (or of a process), $e' \prec e''$ means that e' is anterior to e'' : $s' < s''$ for all $s' \in e'$ and $s'' \in e''$. For an excursion e of w, we denote by $i_w e := \inf\{w_s; s \in [0, d_e]\}$.*

Definition 3. *An excursion e of a path $w \in \mathbf{W}$ is said to be tall if it is positive (this implies that the process $\mathbf{I}w$ remains constant during e); and if for any excursion e' of w such that $i_w e' = i_w e$ and higher than e, then $e' = e$. Formally, e is tall if it is positive and if*

$$\max\left(w(s); s \geq 0, (\mathbf{I}w)(s) = i_w e\right) = \max\left(w(s); s \in e\right). \tag{18}$$

Lemma 9. *Let η be a positive number, $m \geq 1$ be an integer and $w \in \mathbf{W}$ a path. Let $e_1, \ldots, e_{m+1}$ be $m+1$ different $t-$excursions of w, numbered in chronological order : $e_1 \prec \cdots \prec e_{m+1}$; call $h_1, \ldots, h_{m+1}$ their respective heights. Let $f_1, \ldots, f_p$ denote all excursions of w which are anterior to e_{m+1} and whose heights are $\geq \min(\eta, h_1, \ldots, h_{m+1})$, numbered in reverse chronological order : let $g_1, \ldots, g_p$ be p excursions of w verifying $f_p \prec g_p \prec \cdots \prec f_1 \prec g_1 \prec e_{m+1}$.*
Suppose that

- *the excursion e_{m+1} is negative, and all $t-$excursions higher than e_{m+1} are positive;*
- *the excursions $f_1, \ldots, f_p$ are positive;*
- *the excursions $g_1, \ldots, g_p$ are negative; and every negative excursion anterior to g_q is smaller than g_q.*

Then $e_1, \ldots, e_m$ are tall, and $|i_w e_1| < |i_w e_2| < \cdots < |i_w e_m| < \eta$.

Proof Firstly, $|i_w f_1| < \eta$ because $f_1 \prec e_{m+1}$ and any excursion anterior to e_{m+1} and having height $\geq \eta$ is one of the f_q, hence positive.
Secondly, for $1 \leq q \leq p$, the excursion g_q is negative and higher than any negative excursion, anterior to it; so $\mathbf{I}w$ is not constant during g_q, and consequently we have

$$|i_w f_p| < |i_w f_{p-1}| < \cdots < |i_w f_1| < \textit{ height of } g_1, \tag{19}$$

where each $<$ sign is due to $\mathbf{I}w$ varying on the corresponding g_q.
thirdly, combining (20) with $|i_w f_1| < \eta$ (first step), and noticing that, by definition of the f_q, $(e_1, \ldots, e_m)$ is a sub-sequence of $(f_p, \ldots, d_1)$, we obtain

$$|i_w e_1| < \cdots < |i_w e_m| < \eta. \tag{20}$$

Last, it remains to establish that e_l is tall for $1 \leq l \leq m$. Let e' denote a positive excursion of w with height $h' \geq h_l$ and such that $i_w e' = i_w e_l$. From (20), we have $|i_w e'| = |i_w e_l| < \textit{height of } g_1$; so e' is anterior to g_1 and a fortiori anterior to e_{m+1}. As $h' \geq h_l$, e' must be one of the f_q (see their definition). But e_l is also one of the f_q and, due to (20), all $i_w f_q$ are different; so $e' = e_l$. This means that e_l is tall.

■

In the proof of Lemma 9, the negative excursions g_q are used to separate the f_q from each other. Yet, in the end, we are not interested in the behavior of all f_q but only in the e_l. It is possible to replace this lemma with a variant, where $2m$ excursions (instead of p ones, the g_q) are made negative, each e_l being flanked by two of them.

We will now perform the k Lévy raises $C^0 \to C^1 \to \cdots \to C^k$ described in Corollary 2. The first one is the construction of C^1 from $C^0 = B^J$.

Notation 5. *We put* $\varepsilon'' = \frac{\varepsilon'}{r^2} = \frac{\varepsilon}{3k^2}$.

Proposition 4. *With the notation of Corollary 3,* $\mathcal{N}^0$ *and* Σ^0 *can be chosen so that, on the event* Ω_1*,*

(i) $0 \leq C^1_t < \varepsilon'$*;*

(ii) $-(\mathbf{I} \circ C^1)(y) = H_{J-1}$*;*

(iii) C^1 *has* $J-2$ *tall* $t-$*excursions* $E_1 \prec E_2 \prec \cdots \prec E_{J-2}$*, with respective heights* $H_1, \ldots, H_{J-2}$*, and verifying*

$$|i_{C^1} E_1| < \cdots < |i_{C^1} E_{J-2}| < \varepsilon'' \tag{21}$$

Proof Since $\mathbf{T} \circ C^1 = C^0 = B^J = \mathbf{T} \circ B^{J-1}$, we have $|C^1| = |B^{J-1}|$ regardlessly of how $\mathcal{N}^0$ and Σ^0 are chosen. This choice will be described now; it will be $\sigma(B^J)-$measurable because it depends only on $|B^{J-1}|$. (In the rest of the proof, "we choose" is to be understood as "we choose $\sigma(B^J)-$measurably"). By Proposition 3, choosing $\mathcal{N}^0$ and Σ^0 means choosing the signs of finitely many excursions of C^1, whose absolute value is observed.
Outside the event Ω_1 we take $\mathcal{N}^0 = \emptyset$, that is, we do not fix the sign of any

particular excursion. The proof will describe the choices to be made on the $\sigma(B^J)-$event Ω_1, in order to ensure (i), (ii) and (iii); so we henceforth focus on this event only.
By Corollary 3 (iv), B^{J-1} has $J-1$ different $t-$excursion $E_1 \prec \cdots \prec E_{J-1}$ with respective heights $H_1, \ldots, H_{J-1}$. We choose to make excursion E_{J-1} negative in C^1, and to make positive all $t-$excursions higher than H_{J-1}; this ensures (ii).

We also choose to make positive the excursion which straddles t; owing to Corollary 3 (iii), this yields (i).

Now, consider all excursions which chronologically occur before E_{J-1}, and whose heights are $\geq \min(\varepsilon'', H_1, H_2, \ldots, H_{J-1})$; call $F_1, \ldots, F_p$ these excursions, in reverse chronological order : $F_p \prec \cdots \prec F_1 \prec E_{J-1}$ (p is random and $\sigma(B^J)-$measurable). We choose to make $F_1, \ldots F_p$ positive. Using Lemma 7 (iv), we choose between F_1 and E_{J-1} an excursion G_1 smaller than $\min(\varepsilon'', H_1, H_2, \ldots, H_{J-1})$, its sign has not yet been assigned, so we choose to make it negative, and to make positive all excursions higher than G_1 and anterior to it. Then, we make negative an excursion G_2 chosen between F_2 and F_1 and smaller than G_1 (such an excursion exists by Lemma 7 (iv) and is too small to have previously been chosen). We also make positive all excursions occurring before G_2 and higher than G_2. And so on : we keep doing this up to G_p.
Finally, we apply Lemma 9 with $\eta = \varepsilon''$ and $m = J-2$ to the E_1, F_q and G_q. The hypotheses of this lemma are satisfied due to the properties of $(E_1, \ldots, E_{J-1})$, to the definition of the F_q and G_q, and to the signs we have chosen. We obtain that, with these choices of signs, (iii) is satisfied for C^1 whatever the signs of the other excursions of C^1.

■

We have gone first step, from C^0 to C^1. We will soon perform the next $J-2$ steps, from C^1 to C^{J-1}; we first prove a lemma.

Lemma 10. *Let X be a process with law absolutely continuous w.r.t. π, and E a tall excursion of $\mathbf{T} \circ X$ with height H. There exists an excursion of X, with interval $\{s; (\mathbf{I} \circ \mathbf{T} \circ X)(s) = i_{\mathbf{T} \circ X} E\}$, and with height $H + |i_{\mathbf{T} \circ X} E|$.*

Proof First, recall a.s., Brownian motion B does not reach its current minimum $\mathbf{I} \circ B$ in the interior of a time-interval where $\mathbf{I} \circ B$ is constant. (This

is a consequence of $(\mathbf{I} \circ B)(s) < 0$ for $s > 0$ and of the Markov property at the first time that $B = \mathbf{I} \circ B$ after some rational).
Put $Y = \mathbf{T} \circ X$ and call F the interval $\{s \geq 0; (\mathbf{I} \circ Y)(s) = i_X E\}$; Y reaches its current minimum $\mathbf{I} \circ Y$ at both endpoints of F but not in the interior of F (see above). Since $|X| = Y - \mathbf{I} \circ Y$ by Lemma 1, we have that F is the support of some excursion of X. The height of that excursion is

$$\begin{aligned} \max(|X_s|; s \geq 0, \quad & and \quad (\mathbf{I} \circ Y)(s) = i_Y E) \\ & = \max\left(Y_s - (\mathbf{I} \circ Y)(s); s \geq 0, (\mathbf{I} \circ Y)(s) = i_Y E\right) \\ & = \max\left(Y_s; s \geq 0, (\mathbf{I} \circ Y)(s) = i_Y E\right) - i_Y E \\ & = \max\left(Y_s; s \in E\right) - i_Y E \qquad because\ E\ is\ tall \\ & = H + |i_Y E|. \end{aligned}$$

■

Proposition 5. *It is possible to define (partially) the sequences* $(\mathcal{N}^n)$ *and* (Σ^n) *on the event* $[n < J]$ *in such a manner that, on the event* $\Omega_1 \cap \Omega_2 \cap [J > n]$*, the process* C^n *satisfies :*

(i) $H_{J-n} \leq -(\mathbf{I} \circ C^n)_t < H_{j-n} + n\varepsilon''$;

(ii) *On* $[J > n+1]$, C^n *has* $J-n-1$ *tall* t*-excursions* $E_1^n \prec E_2^n \prec \cdots \prec E_{J-n-1}^n$ *such that*

$$|i_{C^n} E_1^n| < \cdots < |i_{C^n} E_{J-n-1}| < \varepsilon'',$$

and whose heights $H_1^n, \ldots, H_{J-n-1}^n$ *satisfy*

$$H_l \leq H_l^n < H_l + n\varepsilon'';$$

(iii) $H_{J-n} + \cdots + H_{J-1} < C_t^n < H_{J-n} + \cdots + H_{J-1} + \varepsilon' + n^2\varepsilon''$.

Proof Accordingly to Corollary 3, we consider the family of tall t-excursions of B^J satisfying relation (18), the building excursions. It is this family we have already chosen in Proposition 4 to define $\mathcal{N}^0$, Σ^0 *and* C^1.
Then Lemma 10 ensures the "heredity" of this family during the raisings. Hence we obtain the first part of (ii). It remains to notice that, while the building excursions are protected, each of them receives a "small" excursion. So its height add to the height of the building excursion, which entails (i)

and (ii).
For (iii), it suffices to remark that, each building excursion, before contributing to the height of the excursion of C^n straddling t, had received a number of "small" excursions lower than n.

■

Then we have to proceed to the last raising and verify the sequences $(\mathcal{N}^n)$, (Σ^n) have the good mesurability properties. It is the aim of :

Proposition 6. *We can define (entirely) the sequences* $(\mathcal{N}^n)$, (Σ^n), $n \leq r$, *with the adequate mesurability properties in such a way that :*

$$\mathbf{P}\left(|C_t^J - b| < \varepsilon\right) > 1 - \varepsilon$$

Proof Firstly, we put $\mathcal{N}^n = \emptyset$ *on* $[J < n]$, and Σ^n is the set reduced to the index of the excursion straddling t on $[J = n]$.
We begin by noticing that $\mathcal{N}^n \in \sigma(B^{J-n})$. The events $[J > n]$, $[J = n]$ *and* $[J < n]$ belong to $\sigma(B^{j-n})$.
Then, by defining Σ^{J-1} such that the excursion of C^J straddling t has the sign of b, we obtain, applying Proposition 5, that in $\Omega_1 \cap \Omega_2$, $\left|C_t^J - b\right| < \varepsilon$, with the preceding choices of ε' and ε''.
Finally,

$$\mathbf{P}(\Omega_1 \cap \Omega_2) > 1 - 2\varepsilon' = 1 - \varepsilon.$$

■

Thus we deduce that

$$\mathbf{P}\left(|\Gamma_t^0 - b| < \varepsilon\right) > \varepsilon.$$

And, from Corollary 2, the theorem on one-time approximation is obtained.

2 d-times approximation

The raises we have realized on the Brownian path, in paragraph 1, can be called "elementary" raises : we prepare, at level r, the excursions which enable the path to approach $\varphi(t)$ at time t, then we put them in action successively while holding up the remaining excursions among the selected ones. Now we must proceed by induction. But a new difficulty appears : the necessity of protecting, which has been obtained at the first d-times. This time, we can't anymore be satisfied with elementary raises. First, only excursions with support in $[t_d, t_{d+1}]$ can be used, so it is necessary to rely on the density of zeroes [M]. Then, going from an iteration which vanishes somewhere between t_d and t_{d+1}, we achieve *"horizontal"* raises to correctly configure the path at time t_{d+1}, at the considered level, in the manner of lemmas 4 and 5 : during these raises, we maintain the main excursions in $[0, t_d]$ hold up to preserve the path up to time t_d, while we correctly configure the path at time t_{d+1}, obtaining thus a path of *"essential"* level, this of the beginning of the configuration process. Then we come back to usual raises which we call *"vertical"* raises. We give to the main excursions the sign they had before the horizontal raises (this compels us to replace the $B-$raised Brownian motion. by a disjoint sum of such processes to keep these signs in memory). But the estimated value of the path at time t_{d+1} is based upon the hypothesis that no later vertical raise will vanish on $[t_d, t_{d+1}]$. In the opposite case, the configuration process must return to its beginning.
It's that we are doing in this paragraph :

Definition 4. *We call the sequence $(\Gamma^n)_{n\leq r}$ a disjointed sum of B-raised Brownian motions of index r if there exists*

- *a denumerable measurable partition of Ω, $(H_\lambda)_{\lambda\in\Lambda}$,*
- *for each $\lambda \in \Lambda$, a sequence $(\Gamma^n_\lambda)_{n\leq r}$ of $B-$raised Brownian motions of index r*

such that

$$\forall n \leq r, \Gamma^n = \sum_{\lambda\in\Lambda} \Gamma^n_\lambda \mathbb{1}_{H_\lambda} \tag{22}$$

We notice that : $\mathcal{L}(\Gamma^n) \ll \pi$. For, for all $F \in \mathcal{W}$ such that $\pi(F) = 0$, we have :

$$\begin{aligned} \mathbf{P}(\Gamma^n \in F) &= \sum_{\lambda\in\Lambda} \mathbf{P}\left(H_\lambda \cap [\Gamma^n_\lambda \in F]\right) \\ &\leq \sum_{\lambda\in\Lambda} \mathbf{P}\left(\Gamma^n_\lambda \in F\right) = 0 \end{aligned}$$

Now the paragraph is devoted to proving the following :

Let $\mathcal{P}_d$ be the property :
" Let be given : $\varepsilon > 0, (t_1, ..., t_d) \in \mathbb{R}^d$ such that $0 < t_1 < ... < t_d$ and φ a map from $\mathbb{R}^+ \to \mathbb{R}$. Then there exists an integer r and a disjointed sum $(\Gamma^i)_{0 \leq i \leq r}$ of $B-$raised Brownian motions of index r_d such that

$$\mathbf{P}\left(||\Gamma^0(t_1, ..., t_d) - \varphi(t_1, ..., t_d)||_\infty > \varepsilon\right) < \varepsilon$$

(with the convention that $f(t_1, ..., t_d)$ designates the vector of coordinates $(f(t_1), ..., f(t_d))$, for all map f from $\mathbb{R} \to \mathbb{R}$.) "
We proceed by induction on d.
At the first rank, the result yields from inequality in Corollary 4. In this case, the disjointed sum is nothing but a single sequence of $B-$raised Brownian motions.
Suppose $\mathcal{P}_d$ true.
We will apply this hypothesis to the B.M. B^{s_0} for an integer s_0 to be determined later. We will denote $\widetilde{w}$ the generic path of B^{s_0} : $\widetilde{w} := w^{s_0}$.
Thus there exists a disjointed sum of $B^{s_0}-$raised Brownian motions of index r_d, $\widetilde{\Gamma}$, such that :

$$\mathbf{P}(A_0^\varepsilon) > 1 - \frac{\varepsilon}{4},$$

where

$$A_0^\varepsilon \quad := \quad [\|\widetilde{\Gamma}(t_1, ..., t_d) - \varphi(t_1, ..., t_d)\|_\infty < \varepsilon]$$

As usual, we will denote : $\forall i \in \mathbb{N}$, $\widetilde{\Gamma}^i = \widetilde{w}^i$.
By definition, $\forall i > r_d$, $\tilde{w}^i = w^{s_0+i}$.

From the theorem on density of zeroes, ([M]), there exists a.s. an integer ℓ such that $\widetilde{\Gamma}^\ell(\omega)$ vanishes at least one time on $[t_d, t_{d+1}]$.
Let $L(w)$ be the smallest of these integers ℓ. L is a r.v. almost surely finite.
So there exists an integer ℓ_0 which we will choose $> r_d$ such that :

$$\mathbf{P}(A_1^\varepsilon) > 1 - 2\frac{\varepsilon}{4},$$

where

$$A_1^\varepsilon \quad := \quad A_0^\varepsilon \cap [L \leq \ell_0].$$

Our aim is to raise the path $\widetilde{w}^{\ell_0}$, which is nothing but $w^{s_0+\ell_0}$, from level $s_0+\ell_0$ to level 0, in such a way that the raised path approaches φ on $[t_d, t_{d+1}]$, while

remaining at the proximity of φ on $[0, t_d]$ acquired at level s_0, with the usual measurability conditions.
For this to be realized without damage, we will introduce the family of protecting excursions of $\widetilde{w}^k$: $(\widetilde{ep}_j^{(k)})_{1\leq j\leq p_k}$. This family is constituted by the excursions of $\widetilde{w}^k$ with beginning before t_d and height greater than ε'', and arranged in the reverse chronological order, for all k from 0 to ℓ_0.
The p_k, $0 \leq k \leq \ell_0$, are finite r.v. Let $(p_k^0)_{0\leq k\leq \ell_0}$ be a sequence of deterministic integers verifying :

$$\mathbf{P}(A_2^\varepsilon) > 1 - 3\frac{\varepsilon}{4},$$

where

$$A_2^\varepsilon := A_1^\varepsilon \cap \left(\bigcap_{k=0}^{\ell_0} [p_k \leq p_k^0] \right).$$

Now, we can modify the protecting excursions : $(\widetilde{ep}_j^{(k)})_{0\leq j\leq p_k^0}$ is constituted of the excursion straddling t_d and of the p_k^0 highest excursions before t_d, arranged in the reverse chronological order. We denote by t_j^{P,s_o+k} the beginning of $\widetilde{ep}_j^{(k)}$, for all j from 0 to p_0^k
We set :

$$\begin{aligned} \Lambda &:= \left\{(\lambda_k)_{0\leq k\leq \ell_0}\right\},\ \lambda_k \in \{-1,+1\}^{\{0,\dots,p_k^0\}} = \prod_{k=0}^{\ell_0} \{-1,+1\}^{\{0,\dots,p_k^0\}} \\ \Delta &:= \left\{[n\beta,\ (n+1)\beta[;\ n \in \mathbb{Z}\right\}^{\{0,\dots,\ell_0\}} \end{aligned}$$

The partition (H_ν^{d+1}) from which we are going to construct Γ is so defined :
$N_{d+1} := \Lambda \times \Delta$ is denumerable ;
$\forall(\lambda,\ \delta) \in \Lambda \times \Delta$,
$H_{\lambda,\,\delta}^{d+1} := \bigcap_{k=0}^{\ell_0} \left([(sgn(\widetilde{ep}_j^k))_{0\leq j\leq p_k^0} = \lambda_k] \ \cap [\tilde{w}_{t_d}^k \in \delta(k)] \right)$

To simplify the notations, we will omit the indices λ, δ when there is no ambiguity, or else replace them by ν.

Set : $b\delta(k)$, the beginning of the interval $\delta(k)$ and $\varepsilon(\lambda, k) := \prod_{i=0}^{k-1} \lambda_i(0)$.
First let us begin by an improvement of Lemma 5 :

Lemma 11. *Let $(0, \overrightarrow{i}, \overrightarrow{j})$ be an orthonormal basis of the plan in which we represent paths. Let τ_b^{a+} be the vertical translation of vector $(b-a)\overrightarrow{j}$ and τ_b^{a-} the reflection along the horizontal axis of equation : $y = \dfrac{a+b}{2}$.*
Consider $(t, k, p) \in \mathbb{Z} \times \mathbb{N}^2$ such that $w_t^k = a$ and $w_t^{k+p} = b$ and denote γ_t the first time posterior to t when at least one of the iterated Lévy transforms w^s,

$k \leq s \leq k+p-1$, *vanishes. Then we have :*

$$w^{k+p}_{|[t,\gamma_t]} = \begin{cases} \tau_b^{a+} \, o \, w^k_{|[t,\gamma_t]} & \text{if} \quad \prod_{i=k}^{k+p-1} w_t^i > 0 \\ \tau_b^{a-} \, o \, w^{-k}_{|[t,\gamma_t]} & \text{else} \end{cases}$$

We will denote $\tau^k_{k+p}(w)$ *the plan transformation, which transforms* $w^k_{|[t,\gamma_t]}$ *in* $w^{k+p}_{|[t,\gamma_t]}$.

Proof It is an immediate consequence of Tanaka's Lemma, when $p = 1$.
In general case, we break up the displacement τ which transforms $w^k_{/[t,\gamma_t]}$ in $w^{k+p}_{/[t,\gamma_t]}$ under the form $\tau = \tau_p \circ \tau_{p-1} \circ ... \circ \tau_1$ where τ_i transforms $w^{k+i-1}_{/[t,\gamma_t]}$ in $w^{k+i}_{/[t,\gamma_t]}$. From the preceding remark, each τ_i is a vertical translation or a reflection along an horizontal axis, according to the sign of w_t^{k+i-1}. Then we deduce the claim.

■

Lemma 12. *On* $H^{d+1}_{\lambda,\delta}$ *the displacements* $\tau^0_k(\widetilde{w})$ *and* $\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(k)}$ *differ from at most* 2β.

Proof From lemma 11, these displacements are of the same nature , vertical translations or reflections along the horizontal axis.

- In the case of a translation, $\varepsilon(\lambda,k) = +1$, we have :

$$\begin{aligned} \forall y \in \mathbb{R} \, , \, |\tau^0_k(\tilde{w})(y) - \tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(k)}(y)| &= |\widetilde{w}^k_{t_d} - \widetilde{w}^0_{t_d} - (b\delta(k) - b\delta(0))| \\ &= |\widetilde{w}^k_{t_d} - b\delta(k) - (\widetilde{w}_{t_d} - b\delta(0))| \\ &\leq \beta \end{aligned}$$

- In the case of a reflection, $\varepsilon(\lambda,k) = -1$, we have :

$$\begin{aligned} \forall y \in \mathbb{R} \, , \, |\tau^0_k(\widetilde{w})(y) - \tau^{b\delta(0)-}_{b\delta(k)}(y)| &= |(\widetilde{w}^k_{t_d} + \widetilde{w}^0_{t_d} - y) - (b\delta(k) + b\delta(0) - y)| \\ &= |(\widetilde{w}^k_{t_d} - b\delta(k)) + (\widetilde{w}_{t_d} - b\delta(0))| \\ &\leq 2\beta \end{aligned}$$

■

Lemma 13. *Let be given a stopping time J of the reverse filtration $(\sigma(B^n))_{n\leq r}$ (while values in $\mathbb{Z}\cap(-\infty;r]$), and K a $\sigma(B^J)-$measurable $\mathbb{N}-$valued r.v. . Then the r.v. $J-K$ is a stopping time of $(\sigma(B^n))_{n\leq r}$.*

Proof For every $p\in\mathbb{Z}\cap(-\infty;r]$, we have :

$$[J-K=p]=\bigcup_{j\in\mathbb{Z},\ j\geq p}[J=j]\cap[K=j-p].$$

As $[K=j-p]\in\sigma(B^J)$, $[J=j]\cap[K=j-p]\in\sigma(B^j)\subset\sigma(B^p)$.
Hence $[J-K=p]\in\sigma(B^p)$.

■

Lemma 14. *Planing*
Let w belong to $\mathbf{W}$, and $t,\varepsilon'\in\mathbb{R}_*^+$. We suppose there is no interval on which w is constant.
Denote : $i_0^t(w)=\sup\{|w_s|;s\in[g_t(w),t]\}$, $t_0(w)=\sup\{s\in[g_t,t],|w_s|=i_0^t(w)\}$.

$$\begin{array}{rrcl}\forall n\in\mathbb{N}, & i_{n+1}^t(w) & = & \sup\left\{i_n^t(w)-|w_s|;s\in[t_n(w),t]\right\}\\ & t_{n+1}(w) & = & \sup\left\{s\in[t_n(w),t];i_n^t(w)-|w_s|=i_{n+1}^t(w)\right\}\end{array}$$

Then $(i_n^t(w))$ strictly decreases towards 0 or strictly decreases before reaching 0, and $(t_n(w))$ strictly increases towards t or strictly increases before reaching t.
Furthermore $K(w)=\inf\{n\in\mathbb{N};i_n^t(w)<\varepsilon'\}$ is a $\mathcal{W}-$measurable $\mathbb{N}-$valued r.v. .

Proof By construction, the sequences (i_n^t) and (t_n) are respectively positive decreasing and increasing bounded from above by t. So they converge.
Set i^t the limit of (i_n^t) and τ that of (t_n).
Suppose $i^t>0$. Then the oscillation of w is infinite at the neighbor of τ, which is in contradiction with the continuity of w. So $i^t=0$.

Suppose $\tau < t$. Then w is constant in $[\tau, t]$, which contradicts our hypothesis. So $\tau = t$.
Now, if $i^t_{n+1} = i^t_n$ for some n in $\mathbb{N}$, then, by construction, for all $p \geq n$, $i^t_p = i^t_n$, and it is the same thing for (t_n).
Finally the measurability of K is immediate.

■

Now, the following proposition plays the part of Lemma 6 and proposition 4 (i) :

Proposition 7. *(i) The r.v.* $J_0 = 1 + \sup\{n \leq s_0 + l_0,\ B^n \text{ vanishes on } (t_d, t_{d+1})\}$ *with* $\sup \emptyset = -\infty$, *and* $J_0 - 1$ *are stopping times of* $(\sigma(B^n))_{n \leq s_0 + l_0}$ *such that* $J_0 > s_0 + r_d$ *on* A^ε_1.
For all $(\omega, n) \in \Omega \times \mathbb{Z} \cap (-\infty, s_0 + l_0]$ *verifying :* $J_0(\omega) \leq n$, *we define :* $\Gamma^n(\omega) = B^n(\omega)$. *So,* $\Gamma^{J_0} = B^{J_0} = C^0$.
The r.v. $K(B^{J_0 - 1})$ *(notation of Lemma 14) is* $\sigma(B^{J_0})-$*measurable. So :* $J'_0 = J_0 - 1 - K(B^{J_0 - 1})$ *is a stopping time.*

(ii) We can define $\mathcal{N}^n(\omega)$, $\Sigma^n(\omega)$, $C^n(\omega)$ *and* $|C^{n+1}(\omega)|$, *for all* $n < J_0 - J'_0$ *on the event* $D^\varepsilon_1 = [B^{J_0 - 1} \text{ vanishes on } (t_d, t_{d+1})]$, *in such a manner that* $\left|\Gamma^{J'_0}_{t_{d+1}}\right| = \left|C^{J_0 - J'_0}_{t_{d+1}}\right| < \varepsilon'$ *on this event which is of probability 1.*

$$\left\| \Gamma^{J'_0}_{|[0,t_d]} - \Gamma^{J_0}_{|[0,t_d]} \right\|_\infty < K\varepsilon''$$

On the complementary event of D^ε_1, *we naturally set* $\mathcal{N}^n = \emptyset$.

Proof The first assertions are nothing but immediate consequences of definitions and Lemmas 13 and 14. Now we are going to show that a.s. J_0 has values in $\mathbb{Z} \cap (-\infty, r]$.
For all integer $n \geq r$ and all a, b reals such that $a < b$, let $A_n = [\exists i \in [r, n], B^i \text{ vanishes in } (a, b)]$. Then the sequence $(A_n)_{n \leq r}$ is decreasing (for the inclusion).
Let $A'_{r-n} = [\exists i \in [0, r - n], B^i \text{ vanishes in } (a, b)]$. The sequence $(A'_{r-n})_{n \leq r}$ is itself decreasing.
Since $(B^i)_{r \geq i \geq n}$ and $(B^j)_{0 \leq j \leq r-n}$ have the same law,

$$\forall n \leq r, \quad \mathbf{P}(A'_{r-n}) = \mathbf{P}(A_n)$$

So,

$$\begin{aligned} \mathbf{P}\left(\exists i \in (-\infty, r],\ B^i \text{ vanishes in } (a, b)\right) &= \lim_{n \to -\infty} \mathbf{P}(A_n) = \lim_{n \to \infty} \mathbf{P}(A'_{r-n}) \\ &= \mathbf{P}\left(\exists i \in [0, +\infty),\ B^i \text{ vanishes in } (a, b)\right) \\ &= 1 \qquad \text{from [M]} \end{aligned}$$

In order to define properly $\mathcal{N}^0$ and Σ^0 on D_1^ε, we apply Lemma 9, with $e_1, \ldots, e_m$ denoting the t_d-excursions of B^{J_0-1} of height greater than ε'', the excursion straddling t_{d+1}, and $t = t_{d+1}$, $\eta = \varepsilon'$. This step defines $\mathcal{N}^0$, Σ^0, C^1 and $|C^2|$.
Then it remains to apply this method $K(B^{J_0-1})-$times more in exactly the same way, to define $\mathcal{N}^n$, Σ^n and C^{n+1} for all n, on the event $[n \leq J_0 - J_0']$.
Then it is easily verified that, on D_1^ε, we have :

$$\left|\Gamma^{J_0'}_{t_{f+1}}\right| < \varepsilon'.$$

■

Then we are coming to the analog of Lemma 8 and Proposition 4 (ii) :

Proposition 8. *There exists a $\sigma(B^{J_0'+1})-$measurable, $\mathbb{N}-$valued r.v. K' such that there exists $K'-1$ r.v. $P_1, \ldots, P_{K'-1}$ themselves with values in $\mathbb{N}$ and $\sigma(B^{J_0'+1})-$measurable such that :*

(i) the $K'-1$ excursion intervals $e_{P_1}(\Gamma^{J_0'}), \ldots, e_{P_{K'-1}}(\Gamma^{J_0'})$ are disjoint and included in (t_d, t_{d+1}).

(ii) the heights $H_1, \ldots, H_{K'-1}$ of these $K'-1$ excursions of $\Gamma^{J_0'}$ verify on D_1^ε :

$$\left|\tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(J_0-s_0-1)}(\varphi(t_{d+1}))\right| - \varepsilon' < H_1 + \cdots + H_{K'-1} < \left|\tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(J_0-1-s_0)}(\varphi(t_{d+1}))\right| + \varepsilon'.$$

Proof The transposition of Lemma 8 and Proposition 4 (ii) to this new context is immediate because J_0' is a stopping time.

■

So we have prepared the following raises. The following proposition indicates, as Proposition 4 and 5, the mean by which the raised paths are realized, namely the corresponding values of $\mathcal{N}^n$ and Σ^n.

Proposition 9. *For all (n, ω) such that $J_0(\omega) - J_0'(\omega) \leq n < J_0(\omega) - J_0'(\omega) + K'(\omega)$, $\mathcal{N}^n(\omega)$ and $\Sigma^n(\omega)$ can be chosen so that :*

(i) $0 \leq C^{J_0-J_0'}_{t_{d+1}} < \varepsilon'$ and $\|\widetilde{\Gamma}^{J_0-1-s_0}_{|[0,t_d]} - C^{J_0'-J_0}_{|[0,t_d]}\|_\infty < (J_0 - J_0')\varepsilon''$.

(ii) $-(\mathbf{I} \circ C^{J_0-J_0'})(t_{d+1}) = H_{K'-1}$

(iii) $C^{J_0-J_0'}$ *has* $K'-2$ *tall excursions included in* (t_d,t_{d+1}) *:* $E_1<E_2<\cdots<E_{K'-2}$ *with respective heights* $H_1,\ldots,H_{K'-2}$ *verifying*

$$\left|i_{C^{J_0-J_0'}}E_1\right|<\left|i_{C^{J_0-J_0'}}E_2\right|<\cdots<\left|i_{C^{J_0-J_0'}}E_{K'-2}\right|.$$

(iv) $H_{K'-n+1}+\cdots+H_{K'-1}\leq C^n_{t_{d+1}}<H_{K'-n+1}+\cdots+H_{K'-1}+\varepsilon'+nK'\varepsilon''$.

(v) $H_{K'-n}<-(\mathbf{I}\circ C^n)(t_{d+1})<H_{K'-n}+K'\varepsilon''$.

(vi) C^n *has* $K'-n-2$ *tall excursions included in* $(t_d,t_{d+1})E^n_1<\cdots<E^n_{K'-n-2}$ *such that :* $|i_{C^n}E^n_1|<\cdots<|i_{C^n}E_{K'-n-2}|<\varepsilon''$*, and whose heights* $H^n_1,\ldots,H^n_{K'-n-1}$ *satisfy :*

$$H_l\leq H^n_l<H_l+n\varepsilon''\ for\ n+1\leq l<K'-1.$$

Proof Here again, the proof is identical with that of Propositions 4 and 5.

■

Otherwise, at each of the J_0-J_0' raises, each protected t_d-excursion of the raised path (i.e. the f_i's) receives a small excursion (one of the g_i's) of height lower than ε'' and during this procedure its support increases. So we can say that the $\widetilde{ep}_k^{J_0-s_0}$ of $\widetilde{\Gamma}^{J_0-s_0-1}$ are "conserved" with the meaning that for each of them we can find an excursion of $\Gamma^{J_0'}$ which contains its support and whose height is that of the excursion of $\widetilde{\Gamma}^{J_0-s_0-1}$ increased of at most $K(B^{1+J_0})$ times ε''.
For the non protected t_d-excursions of $\widetilde{\Gamma}^{J_0-s_0}=B^{J_0}$, they are not conserved in general, but it is easily seen that, on the union of their supports, the height of $\Gamma^{j_0'}$ never goes beyond $K(B^{1+J_0})$ times ε'', in the same manner.

At this point, it remains to give to the protected t_d-excursions of the next raise the sign, which is in memory, they had at level J_0, and to the excursion straddling t_{d+1} the sign of $\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(J_0-s_0-1)}(\varphi(t_{d+1}))$. It is the role of the

Proposition 10. *(analog of Proposition 6)*
It is possible to choose $\mathcal{N}^n(\omega)$ *and* $\Sigma^n(\omega)$*, for all* (n,ω) *such that* $n=J_0(\omega)-J_0'(\omega)+K'(\omega)$*, in order to have*

$$\left|C^{J_0-J_0'+K'}_{t_{d+1}}-\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(J_0-s_0-1)}(\varphi(t_{d+1}))\right|<\varepsilon'+\frac{1}{2}(K'-1)(K'-2)\varepsilon'' \tag{23}$$

$$\left\|C^{J_0-J_0'+K'}_{|[0,t_d]}-\widetilde{\Gamma}^{J_0'-K'-s_0}_{|[0,t_d]}\right\|_\infty<(J_0-J_0'+K')\varepsilon'' \tag{24}$$

Proof In this last raise, the procedure differs only by the fact that we give to the protected excursions the sign they had at level $J_0 - 1$, and the excursion straddling t_{d+1} the convenient sign. so we deduce the claimed results in the same manner.

■

Following Lemma 13, we remark once again that the r.v. $J_0' - K'$ is a stopping time. We call it J_0''.
So we have achieved the last "horizontal" raise, whose aim was only to correctly configure the path at time t_{d+1} (without damage in $[0, t_d]$). Let us consider that $\Gamma^{J_0''}(\omega)$ (or $C^{J_0 - J_0''}(\omega)$) as being really at level $J_0 - 1$.
To formalize this idea, we introduce the partially (for the time...) defined r.v. RL_n :

$$\begin{array}{rrcll} & RL_n(\omega) & = & J_0(\omega) - 1 & iff \quad J_0(\omega) - 1 \geq n \geq J_0''(\omega) \\ and & RL_n(\omega) & = & n & iff \quad n \geq J_0(\omega). \end{array}$$

The following raises will be "vertical", i.e. will verify : $RL_{n-1}(\omega) = RL_n(\omega) - 1$ while the path $\Gamma^n(\omega)$ will not vanish in (t_d, t_{d+1}) and $RL_n(\omega) < s_0$. The corresponding procedure is expressed in the :

Proposition 11. *It is possible to define $\mathcal{N}^n(\omega)$, $\Sigma^n(\omega)$ and RL_{J_0-n-1} inductively on the event $[n \geq J_0 - J_0'']$ if $C^n(\omega)$ doesn't vanish in (t_d, t_{d+1}) and $RL_{J_0-n}(\omega) > s_0$, in such a way that :*

(i) $\left\| C^n_{|[0,t_d]} - \widetilde{\Gamma}^{RL_{J_0-n} - s_0}_{|[0,t_d]} \right\|_\infty < 2^{n - J_0 + J_0''} \varepsilon''(J_0 - J_0'')$.

(ii) $\left| C^n_{t_{d+1}}(\omega) - \tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(RL_{J_0-n} - s_0)}(\varphi(t_{d+1})) \right| < \varepsilon'' + 2^{n - J_0 + J_0''} \varepsilon''$.

Proof We first recall that when we know $C^{n-1}(\omega)$, we know also wether C^n vanishes in (t_d, t_{d+1}). If it isn't the case, we put :

$$RL_{J_0-n-1}(\omega) = RL_{J_0-n}(\omega) - 1,$$

and define $\mathcal{N}^n(\omega)$ and $\Sigma^n(\omega)$ as we do in Proposition 9; but this time the counting of "errors" is radically different. It can happen between 0 and t_d that an excursion which was protected before become negative and, in the following raise, is going to add to another protected excursion. So, at each vertical raise, the "errors" are double of those of the preceding raise. Hence

the result in (ii).
In the same manner the excursion straddling t_{d+1} receives an excursion with beginning in $[0, t_d]$. So, to the errors soon acquainted at level $J_0^{''}$, we must add the error between 0 and t_d of the preceding level which entails (i).

■

So we have defined RL_n on the event

$$[J_0^{''} \geq n] \cap \left(\bigcap_{i=J_0^{''}-1}^{n} [\Gamma^n \; doesn't \; vanish \; in \; (t_d, t_{d+1})] \right) \cap \left[J_0^{''} - n \leq l_0 \right],$$

particularly on the event :

$$[J_0^{''} - 1 \geq n] \cap \left(\bigcap_{i=J_0^{''}-1}^{n} [\Gamma^n \; doesn't \; vanish \; in \; (t_d, t_{d+1})] \right) \cap \left[J_0^{''} - n = l_0 \right],$$

RL_n reaches s_0. We set :

$$J_1 \;\; = \;\; 1 + \sup \left\{ k < J_0^{''},\; \Gamma^k \; vanishes \; in \; (t_d, t_{d+1}) \; or \; RL_k = s_0 \right\}.$$

Let us notice that the procedure used for the raises from level J_0 to J_1 can be repeated as many times as $RL_n(\omega) > s_0$. So, we set inductively : on $\left[RL_{J_{n-1}} = s_0 \right], J_{n-1}^{''} = J_{n-1}$
else $J_{n-1}^{''}$ is constructed from J_{n-1}, as $J_0^{''}$ was from J_0 in Propositions 6 and 7, and :

$$J_n = 1 + \sup \left\{ k < J_{n-1}^{''},\; \Gamma^k \; vanishes \; in \; (t_d, t_{d+1}) \; or \; RL_k = s_0 \right\}$$

Let us denote $hz_n(\omega)$ (resp. $vt_n(\omega)$) the number of horizontal (resp. vertical) raises from level J_0 to level $J_0 - n$ having affected the path $C^n(\omega)$.
So $hz_n(\omega) + vt_n(\omega) = n$.
Then we can state :

Proposition 12. *(i) The r.v. RL_n are $\sigma(B^n)-$measurable, while the hz_n and vt_n one's are $\sigma(B^{J_0-n})-$measurable, and the J_n, $n \in \mathbb{N}$, are stopping times.*

(ii) It is possible to define $\mathcal{N}^n$ and Σ^n on the event $\left[J_{k-1}^{''} \geq n \geq J_k \right]$ in such a manner that :

(a) $\left\| C^n_{|[0,t_d]} - \widetilde{\Gamma}^{RL_{J_0-n}-s_0}_{|[0,t_d]} \right\|_\infty < hz_n \varepsilon'' 2^{vt_n}$

(b) $\left| C^n_{t_{d+1}} - \tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(RL_{J_0-n})}(\varphi(t_{d+1}))\right| < \varepsilon' + hz_n\varepsilon''2^{vt_n}$

Proof

(i) From their very definition, the r.v. RL_n, hz_n and vt_n are respectively $\sigma(B^n)$ and $\sigma(B^{J_0-n})-$measurable. For the nature of the J_n, we proceed by induction. The result is known for $n = 0$; suppose it is true for $n-1$. Then, for all $p \in \mathbb{N}$, we have :

$$[J_n = p] = \left[J''_{n-1} > p-1\right] \cap \left[\Gamma^{p-1}\ vanishes\ in\ (t_d, t_{d+1})\right] \cup [RL_{p-1} \geq s_0]$$

From the hypothesis and Lemma 13, J''_{n-1} is a stopping time, which entails :

$$\left[J''_{n-1} > p-1\right] \in \sigma(B^p)$$

and from the fact that RL_{p-1} is $\sigma(B^p)-$measurable,

$$[RL_{p-1} \geq s_0] \in \sigma(B^p)\ too.$$

Finally we know that :

$$\left[\Gamma^{p-1}\ vanishes\ in\ (t_d, t_{d+1})\right] \in \sigma(B^p).$$

Hence $[J_n = p] \in \sigma(B^p)$, and J_n is a stopping time.

(ii) We have already seen that, when a raise is horizontal, the error on the interval $[0, t_d]$ increases of at most ε'', but when a raise is vertical, we must multiply it by two.
As there are hz_n horizontal raises, and vt_n vertical one's, we majorize the total error on $[0, t_d]$ between levels J_0 and $J_0 - n$ by :

$$hz_n\varepsilon'' + \sum_{i=J''_k}^{J_0-n+1} 2^{vt_{J_0-i}}\left(hz_{J_0-i}\varepsilon''\right).$$

For the error at time t_{d+1}, it is lower than ε' at level J''_k, if $\left[J''_k \geq J_0 - n \geq J_{k+1}\right]$. Then, at each of the $n-(J_0-J''_k)$ following raises, the excursion straddling t_{d+1} receives an excursion from $[0, t_d]$, hence the total error at time t_{d+1} is majored by :

$$\begin{aligned}
\varepsilon' + \sum_{i=J''_k}^{J_0+(n-1)} 2^{vt_{J_0-i}}\left(hz_{J_0-i}\varepsilon''\right) &= \sum_{k=0}^{J_0+(n-1)-J''_k} 2^{vt_{J_0-J''_k+k}}\left(hz_{J_0-J''_k}\varepsilon''\right) + \varepsilon' \\
&= 2^{vt_{J_0-J''_k}}\left(2^{J_0+n-J''_k}-1\right)\left(hz_{J_0-J''_k}\varepsilon''\right) + \varepsilon' \\
&\leq 2^{vt_n}\left(hz_n\varepsilon''\right) + \varepsilon'.
\end{aligned}$$

■

Proposition 13. *The r.v.* $S = \sup\{n \in \mathbb{Z}; RL_n = s_0\}$, $\sup \emptyset = -\infty$, *is a stopping time.*
Furthermore, the law of $S - s_0$ *doesn't depend upon* s_0 *and* ε''.

Proof For all $p \in \mathbb{Z}$, $[S = p] = [RL_p = s_0] \cap [RL_{p+1} > s_0]$. As RL_p is $\sigma(B^{p+1})-$measurable, we conclude $[S = p] \in \sigma(B^p)$, hence S is a stopping time.
Furthermore the r.v. $S-s_0$ is one which counts the number of raises necessary to obtain exactly l_0 vertical raises. Clearly its law doesn't depend upon s_0 and ε'' (but it depends upon ε').
Otherwise at each level J_n (of return to its beginning of the raising process) we notice that the r.v. vt_k increases at least one unity between levels J_n and J_{n-1}. So S is finite. From this point, the last part of the statement is easily deduced.

■

From Propositions 11 and 12, we can claim that there exists a sequence $(C^n)_{n \leq J_0 - S}$ associated with $\mathcal{N}^n$, Σ^n as in Proposition 3, at which we associate the sequence $(\Gamma^n)_{n \leq S}$, satisfying on A_1^ε :

$$(25) \quad \left\| \Gamma^S_{|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]} \right\|_\infty \quad < \quad hz_{J_0-S}\varepsilon'' + 2^{l_0} hz_{J_0-S}\varepsilon'' = \left(1 + 2^{l_0}\right) \varepsilon'' hz_{J_0-S}$$

$$(26) \quad \left| \Gamma^S_{t_{d+1}} - \varphi(t_{d+1}) \right| \quad < \quad \varepsilon' + 2^{l_0} \left(\varepsilon'' hz_{J_0-S}\right).$$

Before completing the construction of the disjointed sum of Brownian motions it is now necessary to restrain ourselves at a finite number of Γ'_ν, in order to define a s_0 large enough (but not infinite...) to put the $S_\nu \geq 0$ with a big probability.
As the $(H_\nu)_{\nu \in N_{d+1}}$ constitute a denumerable partition of Ω, it suffices to choose a finite part Π of N_{d+1} such that :

$$\mathbf{P}(A_4^\varepsilon) > 1 - 4\frac{\varepsilon}{4}, \quad \textit{where } A_4^\varepsilon = \left(\bigcup_{\nu \in \Pi} H_\nu \right) \cap A_2^\varepsilon.$$

Proposition 14. *There exists an integer* s_0, *such that :*

$$\forall \nu \in \Pi, \quad S_\nu \geq 0 \quad \textit{on the event } A_4^\varepsilon$$

Proof It is immediate since the law of each $S_\nu - s_0$, $\nu \in \Pi$ finite, doesn't depend upon s_0.

■

From now on, we choose such a s_0 (the smallest...).
At this point, we pursue the construction of the (Γ^n_ν) for all $n \geq 0$ and $\nu \in \Pi$ as in Proposition 6.

Proposition 15. *On the event $[S \geq n \geq 0]$, it is possible to define $\mathcal{N}^{J_0-n}$, Σ^{J_0-n}, $C^{J_0-n}_\nu$ and Γ^n_ν in such a way that : on the event $H_\nu \cap [S \geq n \geq 0]$, $\nu \in \Pi$*

$$\|\Gamma^0_{\nu,|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]}\|_\infty < 2\left(S_\nu - 1 + hz_{J_0-S_\nu}(2^{l_0}+1)\right)\varepsilon'' \tag{27}$$

$$|\Gamma^0_{\nu,t_{d+1}} - \varphi(t_{d+1})| < 2\left(\beta + \varepsilon' + \varepsilon''(S + 2^{l_0} hz_{J_0-S_\nu})\right) \tag{28}$$

Proof At each raise from level S to level 1, it suffices to protect all the excursions of C^{J_0-n-1} of height greater than ε''. This condition, since S is a stopping time, defines entirely $\mathcal{N}^{J_0-n-1}$ and Σ^{J_0-n-1}, so C^{J_0-n} and Γ^n on the event $[S \geq n-1 \geq 0]$.
At each of these S raises, each protected excursion receives a small excursion of height lower than ε''. And this situation is the worse for a non protected excursion. So :

$$\|\Gamma^n_{|[0,t_{d+1}]} - \Gamma^{n-1}_{|[0,t_{d+1}]}\|_\infty < \varepsilon''$$

Hence

$$\begin{aligned} \||\Gamma^1_{|[0,t_d]}| - |\widetilde{\Gamma}^0_{|[0,t_d]}|\|_\infty &< \left(S - 1 + hz_{J_0-S}(2^{l_0}+1)\right)\varepsilon'' \\ |\Gamma^1_{t_{d+1}} - |\widetilde{\Gamma}^0_{t_{d+1}}|| &< \varepsilon' + \varepsilon''\left(S + 2^{l_0} hz_{J_0-S}\right) + \beta. \end{aligned}$$

Finally, for the last raise, we give to the protected excursions of Γ^1 the sign of the corresponding excursions of $\widetilde{\Gamma}^0$, which determines $\mathcal{N}^{J_0}$, Σ^{J_0}, C^{J_0} and Γ^0 in such a manner that :

$$\begin{aligned} \|\Gamma^0_{\nu,|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]}\|_\infty &< 2\left(S - 1 + hz_{J_0-S}(2^{l_0}+1)\right)\varepsilon'' \\ |\Gamma^0_{t_{d+1}} - \varphi(t_{d+1})| &< 2\left(\beta + \varepsilon' + \varepsilon''\left(S + 2^{l_0} hz_{J_0-S}\right)\right). \end{aligned}$$

because each positive excursion receives a negative one, so the errors add one by one, hence they multiply by two.

Now we introduce the disjointed sum $\Gamma = \sum_{\nu} \mathbb{1}_{H_\nu} \Gamma_\nu$. Then we can rephrase the preceding assertions :
on the event A_4^ε

$$\|\Gamma^0_{|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]}\|_\infty < 2s_0\left(1 + 2^{l_0} + 1\right)\varepsilon'' \tag{29}$$

$$|\Gamma^0_{t_{d+1}} - \varphi(t_{d+1})| < 2\left(\beta + \varepsilon' + s_0(1 + 2^{l_0})\right)\varepsilon''. \tag{30}$$

because on this event, $l_0 \leq S \leq s_0$.

■

Proposition 16. *The process Γ we have thus defined is a disjointed sum of $B-$raised Brownian motions of index $r = s_0 + l_0$.*

Proof It suffices to notice that the r.v. J_n and S are stopping times of the reverse filtration $(\sigma(B^n))_{n \leq r}$. Consequently the r.v. $\mathcal{N}^n$ and Σ^n are $\sigma(B^{J_0 - n})-$measurable, as required by Proposition 3.

■

Finally we choose $\varepsilon' = \beta = \frac{\varepsilon}{6}$ and $\varepsilon'' = \frac{\varepsilon}{6s_0(2+2^{l_0})}$.
The last choice is permitted because the law of S doesn't depend upon ε''.
So we have proved the

Proposition 17. *For every $\varepsilon > 0$, there exists a disjointed sum of $B-$raised Brownian motions verifying :*

$$\mathbf{P}\left(|\Gamma(t_1, \ldots, t_{d+1}) - \varphi(t_1, \ldots, t_{d+1})| < 2\varepsilon\right) > 1 - \varepsilon.$$

Proof It is immediate from the hypothesis of induction, inequalities (1) and (2) and Proposition 14.

■

We deduce immediately that $\mathcal{P}_{d+1}$ is true. Hence $\mathcal{P}_d$ is true for every $d \in \mathbb{N}$.
Now we can state :

Theorem 1. *($d-$density of paths)*
$\forall \varepsilon > 0 \forall (t_1, \ldots, t_{d+1}) \in (\mathbb{R}_*^+)^d$

$$\mathbf{P}\left(\forall n \geq 0, \exists i \ from\ 1\ to\ d,\ |B^n_{t_i} - \varphi(t_i)| > \varepsilon\right) < \varepsilon.$$

Proof At this point, it is an immediate consequence of Proposition 1 : it suffices to choose for G the complementary event of $\{w \in W; \|w(t_1, \ldots, t_d) - \varphi(t_1, \ldots, t_d)\|_\infty \leq \varepsilon\}$ and apply Proposition 15.

■

3 Density of orbits

In this paragraph, we want the raised path to approach the map φ uniformly on $[0,1]$. The additional difficulty is twofold : first, to protect what has already been obtained *on the whole* segment $[0,t_d]$ (which has already begun in the preceding paragraph, with the introduction of $\| \quad \|_\infty$); secondly, to correctly configure the path *on the whole* segment $[t_d,t_{d+1}]$. Only a more precise analysis of the Lévy Transform, obtained at lemma 15, enables a fruitful utilization of the methods settled in the preceding paragraphs. Through a best knowledge of raises, we can show that the disjointed sum of $B-$Brownian motions furnished by the algorithm come up to the requirements of the following property we propose to show in this paragraph :

Whatever ε strictly positive, and $\varphi \in W_{|_{[0,1]}}$, there exists a disjointed sum Γ of $B-$raised Brownian motions such that :

$$\mathbf{P}\left(\|\Gamma_{|_{[0,1]}} - \varphi_{|_{[0,1]}}\|_\infty < \varepsilon\right) > 1-\varepsilon$$

We consider a modulus of uniform continuity α_0 associated to $\left(\frac{\varepsilon}{4},\varphi,[0,1]\right)$ and a real number α_1 such that $P(A_{0\,\varepsilon}) > 1-\frac{\varepsilon}{2}$ where

$$A_{0\,\varepsilon} = \left[\sup\{|B_t - B_u|,\ (t,u)\in[0,1]^2 \text{ and } |t-u|<\alpha_1\} < \frac{\varepsilon}{2}\right] ,$$

then we set $\alpha := \min(\alpha_0,\alpha_1)$, $d_0 := \left[\frac{1}{\alpha}\right]+1$, and for all $d\in\mathbb{N}$, $t_d = (d\alpha)\wedge 1$. We set again, for all integer $d\in[1,...,d_0]$,

$$A_\varepsilon^d := \left[\sup\{|B_t - B_u|,\ (t,u)\in[t_d,1]^2,\ |t-u|<\alpha\} < \frac{\varepsilon}{2}\right] .$$

Our aim is to show, by induction on d, the following property $\mathcal{P}_d$: " For all $\varepsilon>0$, there exists an integer r_d and a disjointed sum Γ of $B-$raised Brownian motions of index r_d such that :

$$\mathbf{P}\left(\left[\|\Gamma_{|_{[0,t_d]}} - \varphi_{|_{[0,t_d]}}\|_\infty < \varepsilon\right] \cap [|\Gamma_{t_d} - \varphi(t_d)| < \varepsilon_1] \cap A_\varepsilon^d\right) > 1-\varepsilon\left(1+\frac{d}{d_0}\right) \text{ "}$$

Notice that $\mathcal{P}_0$ immediately yields from the choice of α_1. We suppose now $\mathcal{P}_d$ true. We are going to apply this hypothesis to the Brownian motion B^{s_0}, for an integer s_0 which, as the real number ε_1, will be later specified.
As $A_\varepsilon^d \subset \left[\sup\{|B_t - B_u|,\ (t,u)\in[t_d,t_{d+1}]^2\} < \frac{\varepsilon}{2}\right] \cap A_\varepsilon^{d+1}$, and from the independence of the increments of Brownian motion, we can deduce the existence

of a disjointed sum $\widetilde{\Gamma}$ of $B^{s_0}-$raised Brownian motions of index r_d such that :

$$\mathbf{P}(A_0^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) ,$$

where

$$\begin{aligned} A_0^\varepsilon \;:=\; & \left[\|\widetilde{\Gamma}_{|[0,t_d]} - \varphi_{|[0,t_d]}\|_\infty < \varepsilon\right] \cap \left[|\widetilde{\Gamma}_{t_d} - \varphi(t_d)| < \varepsilon_1\right] \quad ... \\ & ... \quad \cap \left[\sup\{|B_t^{s_0+r_d} - B_u^{s_0+r_d}|,\, (t,u) \in [t_d, t_{d+1}]^2\} < \frac{\varepsilon}{2}\right] \cap A_\varepsilon^{d+1} \end{aligned}$$

We will denote : $\forall i \in \mathbb{N}$, $\widetilde{\Gamma}^i = \widetilde{w}^i$. By definition, $\forall i > r_d$, $\tilde{w}^i = w^{s_0+i}$.
From the theorem of density of zeroes ([M]), there exists a.s. an integer ℓ such that $\widetilde{w}^\ell$ vanishes at least one time on $[t_d, t_{d+1}]$.
Let $L(w)$ be the smallest of these integers ℓ. L is a r.v. almost surely finite. Then there exists an integer ℓ_0 which we will choose $> r_d$ such that :

$$\mathbf{P}(A_1^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{\varepsilon}{3d_0} \quad \text{où} \quad A_1^\varepsilon := A_0^\varepsilon \cap [L \leq \ell_0] .$$

Our aim is to raise the path $\widetilde{w}^{\ell_0}$, which is nothing but $w^{s_0+\ell_0}$ from level $s_0+\ell_0$ to level 0, in such a way that the raised path approaches φ on $[t_d, t_{d+1}]$ while remaining at the proximity of φ on $[0, t_d]$ acquired at level s_0, with usual measurability condition. For this to be realized, without damage, we will introduce the family of protecting excursions of $\widetilde{w}^k$, $(\widetilde{ep}_j^{(k)})_{1\leq j\leq p_k}$, constituted by the excursions of $\widetilde{w}^k$ with beginning before t_d and height greater than ε'' and arranged in the reverse chronological order, for all k from 0 to ℓ_0.
The p_k, $0 \leq k \leq \ell_0$, are finite r.v. Let $(p_k^0)_{0\leq k\leq \ell_0}$ be a sequence of deterministic integers verifying :

$$\mathbf{P}(A_2^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{2\varepsilon}{3d_0}$$

where

$$A_2^\varepsilon := A_1^\varepsilon \cap \left(\bigcap_{k=0}^{\ell_0} [p_k \leq p_k^0]\right) .$$

The notions to follow are just necessary for fine analysis of the Lévy raising.
For our needs, we will call $m-$excursion each map $e : \mathbb{R}^+ \to \mathbb{R}$ whose support is a not empty segment and which doesn't vanish at any point of the interior of the support. In particular, for $w \in W$ and $t > 0$, we will call $m-$excursion straddling t, and denote it by : $e_t(w)$, the map so defined :

$$e_t(w) : \mathbb{R}^+ \to \mathbb{R},\ \forall u \in \mathbb{R}^+,\ e_t(w)(u) = \begin{cases} 0 & \text{if } u \in [0, g_t(w)] \cup [d_t(w), +\infty[\\ w_u & \text{else} \end{cases}$$

We will introduce the map $de_t(w) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\forall u \in \mathbb{R}^+,\ de_t(w)(u) = \begin{cases} 0 & \text{if } u \in [0,t] \cup [R_t(w), +\infty[\\ & \text{where } R_t(w) = \sup\{u > t, \forall s \in]t, u], w_s \neq w_t\} \\ w_u - w_t & \text{else} \end{cases}$$

and we call it a differential $m-$ excursion of w.
We denote them e_t and de_t when there is no ambiguity, and e_t^s and de_t^s in the case of excursions of w^s.

Lemma 15. *Let $w \in W$ and e a negative $m-$excursion of $\mathbf{T}w$, lower than all preceding it. Let γ be its beginning and δ its end. We set :*

$$\gamma_1 = \arg\min_{[0,\gamma]} \mathbf{T}w,\ \gamma_2 = \inf\{t \in supp(e); e(t) = \mathbf{T}w_{\gamma_1}\},\ \gamma_3 = \arg\min_{[\gamma,\delta]} e$$

Then :

$$\begin{cases} de_{\gamma_1} \text{ coincides with an excursion of } |w|, \text{ and its support is } [\gamma_1, \gamma_2] \\ de_{\gamma_3} \text{ coincides with an excursion of } |w|, \text{ which begins at } \gamma_3 \text{ and whose} \\ \text{ support contains } [\gamma_3, \delta] \end{cases}$$

Furthermore, $\forall u \in [\gamma_2, \gamma_3]$, de_u coincides with an excursion of $|w|$, if, and only if :

$$\begin{cases} de_u \text{ is a positive excursion} \\ e(u) = \inf\{e(t),\ t \in [\gamma, u]\} \end{cases}$$

It is the case in particular when de_u is the first positive excursion of the form de_v, $v \in [\gamma_2, \gamma_3]$ to overflow a given value.

Proof
From Tanaka's formula :

$$|w_t| = \mathbf{T}w_t + \sup\{-\mathbf{T}w_u, u \in [0,t]\}$$

Therefore,

$$|w_{\gamma_1}| = \mathbf{T}w_{\gamma_1} - \mathbf{T}w_{\gamma_1} = 0,$$

while, for all $t > \gamma_1$, sufficiently small :

$$\mathbf{T}w_t > \mathbf{T}w_{\gamma_1}.$$

So, de_{γ_1} is a positive excursion of $|w|$ which ends at γ_2.
In the same way, $\mathbf{T}w_t > \mathbf{T}w_{\gamma_3}$, for all $t \in [\gamma_3, \delta]$, therefore de_{γ_3} is an excursion of $|w|$ beginning at γ_3 whose support contains $[\gamma_3, \delta]$.
Let e' be an excursion of w with support included in $[\gamma_2, \gamma_3]$. Its beginning u, and its end v verify :

$$u = \underset{[0,v[}{\arg\min}\ \mathbf{T}w \qquad \text{and} \qquad v = \underset{[0,v]}{\arg\min}\ \mathbf{T}w.$$

So we deduce : $de_u = |e'|$.
Reciprocally, let $u \in [\gamma_2, \gamma_3]$ such that de_u is a positive excursion and $u = \underset{[0,u]}{\arg\min}\ \mathbf{T}w$.
Then, $u = \underset{[0,v[}{\arg\min}\ \mathbf{T}w$, where w is the end of de_u, because de_u is positive.
Thus, $w_u = w_v = 0$, and for all $t \in]u, v[$, $w_t \neq 0$.
Consequently, de_u is an excursion of $|w|$.
Let $h > 0$ be such that there exists $u \in [\gamma_2, \gamma_3]$ verifying de_u is the first positive excursion of the form de_v, $v \in [\gamma_2, \gamma_3]$, whose height overflows h. Then, for all $v < u$, the support of de_v can't contain this of de_u without denying the minimality of u.

■

We set :

$$\begin{aligned} \Lambda &:= \left\{(\lambda_k)_{0\le k\le \ell_0}\right\},\ \lambda_k \in \{-1,+1\}^{\{0,\dots,p_k^0\}} = \prod_{k=0}^{\ell_0} \{-1,+1\}^{\{0,\dots,p_k^0\}} \\ \Delta &:= \left\{[n\beta,\ (n+1)\beta[;\ n \in \mathbb{Z}\right\}^{\{0,\dots,\ell_0\}}. \end{aligned}$$

The partition (H_ν^{d+1}) from which we are going to construct Γ is so defined :
$\nu \in N_{d+1}$
$N_{d+1} := \Lambda \times \Delta$ is denumerable;
$\forall (\lambda,\ \delta) \in \Lambda \times \Delta$,
$H_{\lambda,\delta}^{d+1} := \bigcap_{k=0}^{\ell_0} \left([(sgn(\widetilde{ep}_j^k))_{0\le j\le p_k^0} = \lambda_k] \ \cap\ [\tilde{w}_{t_d}^k \in \delta(k)] \right)$

Set : $b\delta(k)$, the beginning of the interval $\delta(k)$ and $\varepsilon(\lambda, k) := \prod_{i=0}^{k-1} \lambda_i(0)$.

Lemma 16. *(analog of lemma 12)*
On $H_{\lambda,\delta}^{d+1}$ the displacements $\tau_k^0(\widetilde{w})$ and $\tau_{b\delta(k)}^{b\delta(0)\ \varepsilon(\lambda,k)}$ differ from at most 2β.

Proof See the proof of lemma 10.

■

This lemma allows us to replace the value to be anticipated $\tau_k^0(\widetilde{w})(\varphi(t_{d+1}))$, which doesn't have the good measurability by $\tau_{b\delta(k)}^{b\delta(0)\ \varepsilon(\lambda,k)}(\varphi(t_{d+1}))$ on $H_{\lambda,\delta}$.
The purpose of the following lemma is to prepare, at level s, when the iterated Brownian motion vanishes on $]t_d, t_{d+1}[$, the excursions which will allow the correctly raised path to approach φ at level 0 on $]t_d, t_{d+1}[$. It is the analogous of lemma 14.

Lemma 17. *Full planing.*
Let w belong to W, and $t,\ t',\ \varepsilon' \in \mathbb{R}_*^+$ be such that $t < t'$. We suppose there is no interval in which w is constant, and w vanishes in (t, t').
The following r.v. are functionals of $|w|$:

- $t_0(w) = g_{\inf\{s > d_t; |w_s| \geq \varepsilon'\}}$.
- $\forall n \in \mathbb{N}$, *while $t_n < t'$, we set :*

$$t_{n+1} := \begin{cases} \inf\{u \in [t_n; \arg\max |de_{t_n}|[, h(de_u) > \varepsilon' \text{ and } sgn(de_u) = -sgn(de_{t_n})\} \\ \text{if this set is not empty,} \\ \text{else :} \\ \arg\max |de_{t_n}| \wedge t \end{cases}$$

The sequence $(t_n)_{n\in\mathbb{N}}$ is strictly increasing and finite. Let $1 + K(w)$ be its cardinality.

Proof By construction, the sequence (t_n) is strictly increasing and lower than t'. Suppose the number of its terms is infinite. In this case, it would admit a limit $t_* \leq t'$, and the oscillation of w at t_* would be infinite, so contradicting the continuity of w. Then $(t_n)_n$ is finite.
The measurability and the finiteness of K_s are immediate.
The remaining of the proof follows the same way as in the proof of Lemma 14.

■

Let us remark that this Lemma gives us the possibility of planing the path after d_t in K raises.
For, during the first raise, we put negative the excursion straddling t and positive all the other excursions in $(0, t')$ of height greater than ε'. Then during the second raise, we put negative the excursion whose support contains

this of de_{t_1}, and so on. At the end of such K raises, the path on $[t_0, t']$ has an absolute value which doesn't exceed $\varepsilon' + K\varepsilon''$, and ε' on the excursion straddling t'.

Now Proposition 7 applies with very few changes :

Proposition 18. *(i) The r.v.* $J_0 = 1+\sup\{n \leq s_0+l_0; B^n \text{ vanishes on } (t_d, t_{d+1})\}$ *with* $\sup \emptyset = -\infty$, *and* $J_0 - 1$ *are stopping times of* $(\sigma(B^n))_{n \leq s_0+l_0}$ *such that* $J_0 > s_0 + r_d$ *on* A_1^ε, *and* $\mathbf{P}(J_0 = -\infty) = 0$.
For all $(\omega, n) \in \Omega \times (\mathbb{Z} \cap (-\infty, s_0 + l_0])$ *verifying* $J_0(\omega) \leq n$, *we define* $\Gamma^n(\omega) = B^n(\omega)$. *So,* $\Gamma^{J_0} = B^{J_0} = C^0$.
The r.v. $K(B^{J_0-1})$ *(notation of Lemma 17) is* $\sigma(B^{J_0})-$*measurable. So* $J_0^{'} = J_0 - 1 - K(B^{J_0-1})$ *is a stopping time.*

(ii) We can define $\mathcal{N}^n(\omega)$, $\Sigma^n(\omega)$, $C^n(\omega)$ *and* $|C^{n+1}(\omega)|$ *for all* $n < J_0 - J_0^{'}$ *on the event* $D_1^\varepsilon = [B^{J_0-1} \text{ vanishes on } (t_d, t_{d+1})]$, *which is of probability* 1, *in such a manner that :* $\left|\Gamma^{J_0^{'}}_{t_{d+1}}\right| = \left|C^{J_0-J_0^{'}}_{t_{d+1}}\right| < \varepsilon'$ *on this event, and we have*

$$\begin{aligned} \left\|\Gamma^{J_0^{'}}_{|[0,t_d]} - \Gamma^{J_0}_{|[0,t_d]}\right\|_\infty &< K(B^{J_0-1})\varepsilon'' \\ \text{and} \qquad \left\|\Gamma^{J_0^{'}}_{|[t_0(B^{J_0-1}),t_{d+1}]}\right\|_\infty &= \left\|C^{J_0-J_0^{'}}_{|[t_0(B^{J_0-1}),t_{d+1}]}\right\|_\infty < \varepsilon' \\ \left|\Gamma^{J_0^{'}}_{t_{d+1}}\right| &< \varepsilon'. \end{aligned}$$

Proof It is exactly the same as in Proposition 7, except the last point which is an immediate consequence of Lemma 17.

■

Then we are coming to the analog of Proposition 8 :

Proposition 19. *There exists a* $\sigma(B^{J_0^{'}+1})-$*measurable,* $\mathbb{N}-$*valued r.v.,* $K^{'}_{J_0}$ *such that there exists* $K^{'}_{J_0} - 1$ *r.v.* $P_1, \ldots, P_{K^{'}_{J_0}-1}$ *themselves with values in* $\mathbb{N}$ *and* $\sigma(B^{J_0^{'}+1})-$*measurable, such that :*

(i) the $K^{'}_{J_0} - 1$ *excursion intervals* $e_{P_1}(\Gamma^{J_0^{'}}), \ldots, e_{P_{K^{'}_{J_0}-1}}(\Gamma^{J_0^{'}})$ *are disjoint and included in* (t_d, t_{d+1})

(ii) *the heights* $H_1, \dots, H_{K'-1}$ *of these* $K'-1$ *excursions of* $\Gamma^{J'_0}$ *satisfy on* D^1_ε *:*

$$\left|\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(J_0^{-s_0-1})}(\varphi(t_{d+1}))\right| - \varepsilon' < H_1 + \cdots + H_{K'_{J_0}-1} < \left|\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(J_0^{-1-s_0})}(\varphi(t_{d+1}))\right| + \varepsilon'.$$

Proof (See Proposition 8)

■

Often in the sequel, we will denote K'_{J_0} simply by K', if there is no ambiguity. And likely we repeat word by word Proposition 9 :

Proposition 20. *For all* (n,ω) *such that* $J_0(\omega) - J'_0(\omega) \le n < J_0(\omega) - J'_0(\omega) + K'(\omega)$, $\mathcal{N}^n(\omega)$ *and* $\Sigma^n(\omega)$ *can be chosen so that :*

(i) $0 \le C^{J_0-J'_0}_{t_{d+1}} < \varepsilon'$ *and* $\|\widetilde{\Gamma}^{J_0-1-s_0}_{|[0,t_d]} - C^{J'_0-J_0}_{|[0,t_d]}\|_\infty < (J_0 - J'_0)\varepsilon''$.

(ii) $-(\mathbf{I} \circ C^{J_0-J'_0})(t_{d+1}) = H_{K'_{J_0}-1}$

(iii) $C^{J_0-J'_0}$ *has* $K'-2$ *tall excursions included in* (t_d, t_{d+1}) *:* $E_1 < E_2 < \cdots < E_{K'-2}$ *with respective heights* $H_1, \dots, H_{K'-2}$ *verifying* $\left|i_{C^{J_0-J'_0}} E_1\right| < \left|i_{C^{J_0-J'_0}} E_2\right| < \cdots < \left|i_{C^{J_0-J'_0}} E_{K'-2}\right|$.

(iv) $H_{K'_{J_0}-n+1} + \cdots + H_{K'_{J_0}-1} \le C^{n+J_0-J'_0}_{t_{d+1}} < H_{K'_{J_0}-n+1} + \cdots + H_{K'_{J_0}-1} + \varepsilon' + nK'\varepsilon''$.

(v) $H_{K'_{J_0}-n} < -(\mathbf{I} \circ C^{n+J_0-J'_0})(t_{d+1}) < H_{K'_{J_0}-n} + K'_{J_0}\varepsilon''$.

(vi) C^n *has* $K'_{J_0} - n - 2$ *tall excursions included in* $(t_d, t_{d+1}) E^n_1 < \cdots <$ $E^n_{K'_{J_0}-n-2}$ *such that :* $|i_{C^n} E_1| < \cdots < \left|i_{C^n E_{K'_{J_0}-n-2}}\right| < \varepsilon''$, *and whose heights* $H^n_1, \dots, H^n_{K'_{J_0}-n-2}$ *satisfy :*

$$H_l \le H^n_l < H_l + n\varepsilon'' \text{ for } n+1 \le l < K'_{J_0} - 1.$$

Proof The same Proposition, the same proof.

■

In our pursuit of the procedure in paragraph 2, we are now rephrasing Proposition 10 :

Proposition 21. *It is possible to choose $\mathcal{N}^n(\omega)$ and $\Sigma^n(\omega)$, for all (n,ω) such that $n = J_0(\omega) - J'_0(\omega) + K'_{J_0}(\omega)$, in order to have :*

(i)

$$\left\| C^{J_0 - J'_0 + K'_{J_0}}_{|[0,t_d]} - \widetilde{\Gamma}^{J'_0 - K'_{J_0} - s_0}_{|[0,t_d]} \right\|_\infty < \left(J_0 - J'_0 + K'_{J_0}\right) \varepsilon''$$

(ii)

$$\left| C^{J_0 - J'_0 + K'_{J_0}}_{t_{d+1}} - \tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(J_0 - s_0 - 1)}(\varphi(t_{d+1})) \right| < \varepsilon' + \frac{1}{2}(K'_{J_0} - 1)(K'_{J_0} - 2)\varepsilon''$$

and

$$\left\| C^{J_0 - J'_0 + K'_{J_0}}_{(t_{d+1})} - \widetilde{\Gamma}^{J_0 - J'_0 + K'_{J_0}}_{|[t_0(B^{J_0}-1),t_{d+1}]} \right\|_\infty < \varepsilon' + \frac{1}{2}(K'_{J_0} - 1)(K'_{J_0} - 2)\varepsilon''$$

Proof For the additional part in (ii), we notice that the building excursions which appear in Proposition 18, the $e_{P_0}(\Gamma^{J'_0})$'s, are successively protected. Once protected, each of them receives a small excursion of height lower than ε'' at each raise. So we deduce (ii).

■

Let us denote $J''_0 := J'_0 - K'_{J_0}(w)$.
Similarly Proposition 11 becomes :

Proposition 22. *It is possible to define $\mathcal{N}^n(\omega)$, $\Sigma^n(\omega)$ and $RL_{J_0-n}(\omega)$ inductively on the event $[n \geq J_0 - J''_0]$ if $C^n(\omega)$ doesn't vanish in (t_d, t_{d+1}) and $RL_{J_0-n}(\omega) > s_0$, in such a way that :*

(i) $\left\| C^n_{|[0,t_d]} - \widetilde{\Gamma}^{RL_{J_0-n} - s_0}_{|[0,t_d]} \right\|_\infty < 2^{n - J_0 + J''_0} \varepsilon''(J_0 - J''_0).$

(ii) $\left\| C^n_{|[t_0(B^{J_0}-1),t_{d+1}]} - C^n_{t_{d+1}} \right\|_\infty < \varepsilon' + K(B^{J_0}-1)\varepsilon''.$

(iii) $\left| C^n_{t_{d+1}} - \tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(RL_{J_0-n} - s_0)}(\varphi(t_{d+1})) \right| < \varepsilon' + \frac{1}{2}(K'_{J_0} - 1)(K'_{J_0} - 2)\varepsilon'' + 2^{n - J_0 + J''_0}\varepsilon''.$

Proof We first recall that when we know $C^{n-1}(\omega)$, we know also wether C^n vanishes in (t_d, t_{d+1}). If it isn't the case, we put :

$$RL_{J_0-n-1}(\omega) = RL_{J_0-n}(\omega) - 1,$$

and define $\mathcal{N}^n(\omega)$ and $\Sigma^n(\omega)$ as we do in Proposition 9; but this time the counting of "errors" is radically different. It can happen between 0 and t_d that an excursion which was protected before become negative and, in the following raise, is going to add to another protected excursion. So, at each vertical raise, the "errors" are double of those of the preceding raise. Hence the result in (ii).
In the same manner the excursion straddling t_{d+1} receives an excursion with beginning in $[0, t_d]$. So, to the errors soon acquainted at level J_0'' we must add the error between 0 and t_d of the preceding level which entails (i).
For (iii) : here the raises which are involved, are the planing one's, i.e. the $K(B^{J_0-1})$ first raises. At most, at each instant of the interval $[t_0(B^{J_0-1}), t_{d+1}]$, the path C^n has received $K(B^{J_0-1})$ small excursions. Then, this part of the path is just successively translated, which entails (iii)

■

Then, without any change, we define $J_n,\ n \geq 0,\ hz_n$ *and* vt_n, and Proposition 12 takes place.

Proposition 23. *(i) For all $n \in \mathbb{N}$, the r.v. J_n are stopping times, while the RL_n are $\sigma(B^{n+1})-$measurable, the hz_n and vt_n, $\sigma(B^{J_0-n})-$measurable.*

(ii) It is possible to define $\mathcal{N}^n$ and Σ^n on the event $\left[J''_{k-1} \geq n \geq J_k\right]$ in such a manner that :

(a) $\left\| C^n_{|[0,t_d]} - \widetilde{\Gamma}^{RL_{J_0-n}-s_0}_{|[0,t_d]} \right\|_\infty < hz_n \varepsilon'' 2^{vt_n}$

(b) $\left\| C^n_{|[t_0(B^{J_{k-1}}),t_{d+1}]} - C^n_{t_{d+1}} \right\|_\infty < \varepsilon' + K(\Gamma^{J_{k-1}})\varepsilon''$

(c) $\left| C^n_{t_{d+1}} - \tau^{b\delta(0)\ \varepsilon(\lambda,k)}_{b\delta(RL_{J_0-n}-s_0)}(\varphi(t_{d+1})) \right| < \varepsilon' + \frac{1}{2}(K'_{J_0}-1)(K'_{J_0}-2)\varepsilon'' + hz_n\varepsilon''2^{vt_n}$

Proof It is the same as in Proposition 12.
For (b) we notice that, after the intervention of the planing excursions, this part of the path is merely translated, without being affected by any other modification.

■

As before, $S = \sup\{n \leq r; RL_n = s_0\}$.

Proposition 24. *(i) S is a stopping time such that $S - s_0$ doesn't depend upon s_0 and ε''.*

(ii) Let (Γ^n) be the sequence associated to the C^n of Proposition 22. It satisfies on A_1^ε :

$$\begin{aligned}
\left\|\Gamma^S_{|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]}\right\|_\infty &< hz_{J_0-S}\varepsilon'' + 2^{l_0} hz_{J_0-S}\varepsilon'' = \left(1+2^{l_0}\right)\varepsilon'' hz_{J_0-S} \\
\left\|\Gamma^S_{|[t_0(B^{J_S-1}),t_{d+1}]} - \Gamma^S_{t_{d+1}}\right\|_\infty &< \varepsilon' + K(B^{J_S-1})\varepsilon'' \\
\left|\Gamma^S_{t_{d+1}} - \varphi(t_{d+1})\right| &< 2\beta + \varepsilon' + \frac{1}{2}(hz_n - 1)(hz_n - 2)\varepsilon^{''} 2^{l_0}\left(\varepsilon'' hz_{J_0-S}\varepsilon^{''}\right).
\end{aligned}$$

Proof For (i), see Proposition 12.
Then (ii) follows immediately from preceding Propositions.

■

We notice that, up to now, we have correctly defined the sequences $(C^n)_{n\leq J_0-S}$ and $(\Gamma^n)_{n\geq S}$. *Now we restrain the partition $(H_\nu)_{\nu\in N_{d+1}}$: there exists a finite part Π of N_{d+1} such that* :

(31)
$$\mathbf{P}(A_3^\varepsilon) > 1-\varepsilon(1+\frac{d}{d_0}) - \frac{3\varepsilon}{3d_0} = 1-\varepsilon(1+\frac{d+1}{d_0}) where\ A_3^\varepsilon = A_2^\varepsilon \cap \left(\bigcup_{\nu\in\Pi} H_\nu\right)$$

And we set, as usual, $S = \sum_{\nu\in N_{d+1}} S_\nu \mathbb{1}_{H_\nu}$.
Then Proposition 13 remains unchanged : we choose such an integer s_0.
Now Proposition 14 becomes :

Proposition 25. *On the event $[S \geq n \geq 0]$, it is possible to define $\mathcal{N}^{J_0-n}$, Σ^{J_0-n}, $C_\nu^{J_0-n}$ and Γ_ν^n in such a way that : on the event $\left(\bigcup_{\nu\in\Pi} H_\nu\right) \cap [S \geq n \geq 0]$, we have :*

$$\begin{aligned}
(32)\quad \|\Gamma^0_{|[0,t_d]} - \widetilde{\Gamma}^0_{|[0,t_d]}\|_\infty &< 2\left(S - 1 + hz_{J_0-S}(2^{l_0}+1)\right)\varepsilon^{''} \\
(33)\quad \|\Gamma^0_{|[t_0(\Gamma^{J_0}),t_{d+1}]} - \Gamma^0_{t_{d+1}}\|_\infty &< \varepsilon' + K(\Gamma^{J_S})\varepsilon^{''} \\
(34)\quad |\Gamma^S_{t_{d+1}} - \varphi(t_{d+1})| &< \varepsilon' + \frac{1}{2}(hz_n - 1)(hz_n - 2)\varepsilon^{''} + 2\beta + 2^{l_0}\varepsilon^{''} hz_{J_0-S}.
\end{aligned}$$

Proof The same as in Proposition 14.
For (38) and (39), we use the arguments of Propositions 21 and 22.

■

Then Proposition 15 remains unchanged :
Γ *so defined is a disjointed sum of* $B-$*raised Brownian motions.*
So, let us choose : $\varepsilon' = \frac{\varepsilon}{16}$, $\beta = \frac{\varepsilon}{32}$, and $\varepsilon'' = \frac{\varepsilon}{16(\frac{1}{2}(s_0-1)(s_0-2)+2^{l_0}s_0}$.
From the independence of the stopping times $S_\nu - s_0$, and consequently to the r.v. S, upon s_0 and ε'', these choices don't create any vicious circle, and we can claim :

Proposition 26. *For all* $\varepsilon > 0$*, there exists a disjointed sum of* $B-$*raised Brownian motions verifying on the event* A_3^ε *:*

$$\left\|\Gamma^0_{|[0,t_d]} - \varphi_{|[0,t_d]}\right\|_\infty < \frac{3}{2}\varepsilon \tag{35}$$

$$\left\|\Gamma^0_{|[t_0(B^{J_0}),t_{d+1}]} - \varphi_{|[t_0(B^{J_0}),t_{d+1}]}\right\|_\infty < \frac{\varepsilon}{8} \tag{36}$$

$$\left|\Gamma^0_{t_{d+1}} - \varphi(t_{d+1})\right| < \frac{\varepsilon}{4}. \tag{37}$$

Proof We deduce immediately these increases from Proposition 24 since $S \leq s_0$, $hz_{J_0-S} \leq s_0$, $K(\Gamma^{J_S}) \leq s_0$.

■

At this point, the last task to achieve is to control Γ^0 between times $t_0(B^{J_0})$ and t_{d+1}.

So we are going now to analyze more in details its behavior on (t_d, t_{d+1}). Let us denote γ_{t_d} the first time after t_d at which one of the w^σ, $0 \leq \sigma \leq s_0+l_0$, vanishes on (t_d, t_{d+1}), and σ_0 the corresponding level.
Notice that there exists an integer k such that $\sigma_0 = J_k$. Set :

$$\sigma_1 = \sigma_0 - \sum_{k\in\mathbb{N}} \left(K(\Gamma_{J_k}) + K'_{J_k}\right) \mathbb{1}_{\sigma_0 = J_k}.$$

Let us introduce the rectangle $Rect_{\sigma_1}$ defined by the four straight lines with equations :
$x = t_d$, $x = t_{d+1}$, $y = \inf \tau^{b\delta(0)\,\varepsilon(\lambda, RL_{\sigma_0})}_{b\delta(RL_{\sigma_0})}(\varphi)|_{[t_d,t_{d+1}]}$ *and*
$y = \sup \tau^{b\delta(0)\,\varepsilon(\lambda, RL_{\sigma_0})}_{b\delta(RL_{\sigma_0})}(\varphi)|_{[t_d,t_{d+1}]}$
$Rect_{\sigma_1}$ contains by definition the path of $\tau^{b\delta(0)\,\varepsilon(\lambda, RL_{\sigma_0})}_{b\delta(RL_{\sigma_0})}(\varphi)|_{[t_d,t_{d+1}]}$ and, from the choice of α_0, its height is lower than $\dfrac{\varepsilon}{4}$.

Now consider the path of $w^{\sigma_1}|_{[t_d,t_{d+1}]}$, it takes one of the two forms given in the appendix.
In the two cases by hypothesis, the total variation of $w^{s_0+l_0}$ on $[t_d, t_{d+1}]$ is lower than $\frac{\varepsilon}{2}$. So, by lemma 5, and the definition of γ_{t_d}, it is equal to that of w^{σ_0}, on $[t_d, \gamma_{t_d}]$. Consequently the path w^{σ_1} can move again from $Rect_{\sigma_1}$ but at most from $\frac{\varepsilon}{4} + \frac{\varepsilon}{2}$ on the same interval.
And rapidly, it is bound to join φ in $Rect_{\sigma_1}$ by the building excursions, the flat part remaining flat.
Therefore, the rectangle RR_{σ_1} with the same center and vertical straight lines bordering it, and height that of $Rect_{\sigma_1} + \frac{3\varepsilon}{2}$, contains the path $w^{\sigma_1}|_{[t_d,t_{d+1}]}$.
During the following raises, the rectangle $Rect_{\sigma_1}$, according to lemma 5, moves by isometry. We call R_σ its new positions, and likewise RR_σ that of RR_{σ_1}.
We can easily check that, for all $\sigma > \sigma_1$ corresponding to a vertical raise, the path $w^\sigma|_{[t_d,t_{d+1}]}$ is contained in RR_σ.
Finally, for $w \in A_3^\varepsilon \cap H^\nu$, $\nu \in \Pi$, at level 0 we have the desired property :

$$\left\|\Gamma^0_\nu|_{[t_d,t_{d+1}]} - \varphi|_{[t_d,t_{d+1}]}\right\|_\infty < \varepsilon$$

So we have proved the following :

Proposition 27. *For all $\varepsilon > 0$, there exists a disjointed sum of $B-$raised Brownian motions such that : on A_3^ε,*

$$\left\|\Gamma^0_{[t_d,t_{d+1}]} - \varphi|_{[t_d,t_{d+1}]}\right\| < 3\frac{\varepsilon}{2} \tag{38}$$

$$\left|\Gamma^0_{t_{d+1}} - \varphi(t_{d+1})\right| < \frac{\varepsilon}{4}. \tag{39}$$

By replacing ε by $\frac{2}{3}\varepsilon$, we have establish that $\mathcal{P}_{d+1}$ is true. So, by induction, $\mathcal{P}_d$ is true for all $d \leq d_0$, and we can claim :

Proposition 28. *For all $\varepsilon > 0$, there exists a disjointed sum of $B-$raised Brownian motions such that :*

$$\mathbf{P}\left(\left[\|\Gamma^0_{|[0,1]} - \varphi_{|[0,1]}\|_\infty < \varepsilon\right] \cap \left[|\Gamma_1 - \varphi(1)| < \frac{\varepsilon}{4}\right]\right) > 1 - 2\varepsilon.$$

Then we can apply Proposition 1 to $G = \left[\|w_{|[0,1]} - \varphi_{|[0,1]}\|_\infty > \varepsilon\right]$.
So,

$$\mathbf{P}\left(\forall n \geq 0, \left\|B^n_{|[0,1]} - \varphi_{|[0,1]}\right\|_\infty > \varepsilon\right) < 2\varepsilon$$

We deduce immediately :

$$\mathbf{P}\left(\forall n \geq 0, \left\|B^n_{|[0,1]} - \varphi_{|[0,1]}\right\|_\infty > \varepsilon\right) = 0.$$

But this property is true again when we replace 1 by a, for all $a > 0$: This means :

Theorem 2.
For almost every $\omega \in \Omega$, the orbit of $B(\omega)$:

$$orb(B(\omega)) = \{B^n(\omega);\ , n \in \mathbb{N}\}$$

is dense in W, equipped with the topology of uniform convergence on compact sets.

Let us notice that if, in place of restrain ourselves with the open sets B, we have shown :
$\forall B$ closed set in W,

$$\mathbf{P}(B) > 0 \Rightarrow \mathbf{P}(orb(w) \cap B \neq \emptyset) = 1$$

Then every set A $\mathbf{T}$-invariant, measurable and not negligible, would contain the event $[orb(w) \cap B \neq \emptyset]$ and so, would be almost sure. Therefore, $\mathbf{T}$ would be ergodic.
To end, we are going to claim in an equivalent way, following thus an interesting suggestion of J.P Thouvenot :

$$\forall (\varphi, \varepsilon) \in W|_{[0,1]} \times \mathbb{R}^+_*\ ,$$

the reverse martingale $\mathbf{P}(w \in B(\varphi, \varepsilon)|\mathcal{W}^n_\infty)$ admits a regular conditional version $\mathbf{P}(w \in B(\varphi, \varepsilon)|w^n)$, and we have :

Theorem 3.

$$\mathbf{P}\ a.s.\ ,\ \lim_{n\to\infty} \mathbf{P}(w \in B(\varphi, \varepsilon)|w^n) > 0$$

Proof of theorem 3.
Suppose the contrary, and let :

$$A := \left[w \in W, \lim_{n\to\infty} \mathbf{P}(w \in B(\varphi, \varepsilon)|w^n) = 0\right]$$

As $\mathbf{P}(w \in B(\varphi,\varepsilon)|w^n) = \mathbf{P}(w \in B(\varphi,\varepsilon)|w^{n+1})$, because $\mathbf{T}$ is measure-preserving. So we have :

$$w \in A \Leftrightarrow \mathbf{T}w \in A$$

So A is $\mathbf{T}$-invariant. Consequently :

$$E\left(\mathbb{1}_A \mathbf{P}(w \in B(\varphi,\varepsilon)|w^n)\right) = \mathbf{P}\left(A \cap [w \in B(\varphi,\varepsilon)]\right) = \mathbf{P}\left(A \cap [w^n \in B(\varphi,\varepsilon)]\right)$$

But by hypothesis :

$$\lim_{n\to\infty} E\left(\mathbb{1}_A \mathbf{P}(w \in B(\varphi,\varepsilon)|w^n)\right) = 0$$

Therefore,

$$\mathbf{P}\left(A \cap [orb(w) \cap B(\varphi,\varepsilon) \neq \emptyset]\right) = 0$$

which, from theorem 1, entails that $\mathbf{P}(A) = 0$

■

Finally, let us remark that, if we could show :

$$\lim_{n\to\infty} \mathbf{P}\left(\left[w \in B(\varphi,\varepsilon)\right]|w^n\right) = \mathbf{P}\left(\left[w \in B(\varphi,\varepsilon)\right]\right) \ ,$$

Than, not only $\mathbf{T}$ would be ergodic but exact which means :

$\mathcal{W}_\infty^\infty := \bigcap_{n\in\mathbb{N}} \mathcal{W}_\infty^n$ would be trivial.

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