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Abstract : The Lévy transform of a Brownian motion B is the Brownian motion $B'_t = \int_0^t sgn\,(B_s)\,dB_s$. Call T the corresponding transformation on the Wiener space W. We establish that a. s. the orbit of $w(\in W)$ under T is dense in W for the compact uniform convergence topology.

Introduction

Let $(B_t)_{t\geq 0}$ be a one-dimensional, issued from 0 Brownian motion, and $(L_t)_{t\geq 0}$ its local time at 0. The Lévy transform of B is the Brownian motion

$$B^1_t = \int_0^t sgn\,(B_s)\,dB_s = |B_t| - L_t.$$

Let T be the corresponding transformation in the Wiener space, which maps B on B^1, i.e. w on Tw. At the end of chapter XII of [R,Y], Revuz and Yor asked for the ergodicity of T. From up to now, in [D,S], Dubins and Smorodinsky established the ergodicity of the discrete Lévy transform, an analog of T for random walks.
Our goal is to establish that the orbit under T of almost every path is dense in W equipped with the topology of uniform convergence on compact sets. This is a necessary but not sufficient condition for the possible ergodicity of Lévy transform, as it is the case for the density in $\mathbb{R}^+$ of the zeroes of the iterated Lévy transforms of Brownian motion established in [M].

Otherwise, the present paper is the natural consequence of [M]. As in [M], the strategy lies on the utilization of the *"Lévy's raises"* : after being sufficiently down in the iterations, i.e. going from T^nB, for a big enough integer

n, for at least one of $T^p B$, $p \leq n$, vanishes on given interval, we proceed for each raise from level n to level 0, by choosing a finite number of signs of selected excursions in such a way that the raised path takes the desired shape.
If the idea at the basis is simple, numerous difficulties arise when it is applied. So we present at paragraph 1, the **tools** we need. In particular, we have chosen to construct processes rather than events. Because their manipulation is as simple as that of events in this way, and because they are more adequate in the inductive reasonings. In any way, the processes so called raised Brownian motions don't seem to bring more difficulty. We present also in this paragraph the technics of raising, which we illustrate with diagrams.
At paragraph 2, in order to avoid to handle too many difficulties at the same time, it seemed more appropriate to put the methods in action, first in an elementary situation, the **one-time approximation**. We treat it not for itself, but to prepare the ground for the future. So, the method used to prove the one-time approximation is not the fastest but it describes in simple situation what will be done afterwards. Thus, we introduce in this paragraph an algorithm which could be avoided, but it is the elementary form of the final one.
Then, at paragraph 3, we consider a more complex situation, the **d-time approximation**, where the induction is necessary. Here, it is impossible to use the excursions near from time 0. On the contrary, we must find zeroes between times t_d and t_{d+1}. In this context, a sophisticated algorithm is unavoidable.
Finally, at paragraph 4, when we come to our subject, the **uniform approximation**, the methods are in proper order, we still have to add the last difficulty which needs a deeper analysis of the path, precisely the notion of differential excursions.
In the end, I want to thank Marc Yor, Michel Emery, Jean-Pierre Thouvenot and Christophe Leuridan for the good advices.

1 Preliminaries

Let $(W, \mathcal{W}., P)$ be the Wiener's space of a Brownian motion B. As usual, for all $w \in W$, we call Lévy Transform of w, and we denote Tw, the map so defined :

$$\forall t \qquad (Tw)_t := |w_t| - L_t(w)$$

where :

$$L_t(w) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{|w_s|<\varepsilon} \, ds$$

is a possible definition of local time at 0 of w.
From Skorohod's lemma,

$$L_t(w) = \sup\{-(Tw)_s, 0 \leq s \leq t\} .$$

When w is a Brownian path, $L_t(w) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{|w_s|<\varepsilon} ds$, and when β is a Brownian motion, $T\beta$ is again a Brownian motion, the filtration of which being that of $|\beta|$, the absolute value of β.
We define inductively the iterated Lévy transforms of w :

$$T^0 w := w, \forall n \in \mathbb{N}, T^{n+1} w := T(T^n w).$$

We also set :

$$\forall n \in \mathbb{N}, B^n := T^n B,$$

and we will indifferently write $T^n w$ or w^n, $T^n \mathcal{W}.$ or $\mathcal{W}.^n$.
Because there is no loss of information, we consider the Lévy transform as a fall. So we will say that w is a Lévy *raise* of Tw. Of course, two raises of Tw have the same absolute value, and differ only by signs of excursions.
We say also that w is a Lévy raise of order n of w^n.
We claim to begin the elementary result, which is at the basis of the considerations to follow, as well as that of [M].
But let us call, as in [M], $\arg\min_{[u,v]} f$ resp. $\arg\max_{[u,v]} f$, the largest $t \in [u, v]$ such that $f(t) = \min_{[u,v]} f$, resp. $f(t) = \max_{[u,v]} f$, for all f continuous function on $[u, v]$.
Throughout this work, we are using the usual notations :
$g_t(w)$ is the first zero of w before t, and
$d_t(w)$ is the first zero of w after t.
We will write r.v. as shorthand for random variable.

Lemma 1.
Let e be a positive excursion of Tw, $w \in W$, g its beginning and d its end. Then there exists a unique excursion e' of w whose support contains this of e. e' begins at $g' := \arg\min_{[0;g]} Tw$ and terminates at $d' := \inf\{t > g, Tw_t = \min\{Tw_s, s \in [0, g]\}\}$.
The height (in absolute value) h' of e' (i.e. $\sup\{|w_s|, s \in [g', d']\}$) is equal to the sum of the height h of e and the height of the excursion of Tw where it reaches its minimum on $[0, g]$.

Proof of lemma 1.
It is an immediate consequence of Tanaka's formula and Skorohod's lemma.

■

The next result, as in [M], is essential because it allows us to "give" the good signs to selected excursions.

Lemma 2. *(issued from lemma 2 in [M], p.502)*
Let be given an integer n^0, *a family* $(e_i^+)_{1\leq i\leq n^0}$ *of excursions of* B *with disjoint supports and beginnings* $(g_i)_{1\leq i\leq n^0}$ TB*-measurable, and a family* $(\varepsilon_i)_{1\leq i\leq n^0}$ *of* $\{-1,1\}$*-valued,* TB*-measurable r.v.*
Let $E_i := \lim_{t\downarrow\downarrow g_i} B_{g_i+t}$ *the sign of the corresponding excursion. Then*

- *The event* $A := \{\forall i \leq n^0,\ E_i = \varepsilon_i\}$ *verifies :*

$$P(A\,|\,|B|) = 2^{-n^0}$$

- $\sigma(|B|)$ *and* $\sigma(E_i, i \leq n^0)$ *are independent.*

- *Furthermore, if we number the other excursions of* B *(for instance by first arranging the excursions by decreasing heights* $\in [0,1[$, *then the excursions* $\in [1,2[$ *and so on, ..., and finally by choosing a bijection from* $\mathbb{N}^2$ *to* $\mathbb{N}$*) and if we introduce the r.v.* Z *corresponding to the sequence of their signs, then* Z *is an independent complement of* $\sigma(|B|) \vee \sigma(E_i, i \leq n^0)$ *in* $\sigma(B)$ *(i.e.* Z *is independent of* $\sigma(|B|) \vee \sigma(E_i, i \leq n^0)$ *and* $\sigma(|B|) \vee \sigma(E_i, i \leq n^0) \vee \sigma(Z) = \sigma(B)$*).*

Proof of lemma 2.
The theory of excursions ([R.Y], chapter XII) entails that the r.v. E_i are uniform on $\{-1,1\}$, independent among themselves, of $\sigma(|B|)$ and of the signs of the other excursions : Z.
In particular, we have :

$$P(A\,|\,|B|) = P(E_1 = \varepsilon_1, ..., E_{n^0} = \varepsilon_{n^0}) = 2^{-n^0}$$

Finally, the formulas of reconstitution of Brownian motion from the excursions allow to claim that Z is an independent complement of $\sigma(|B|) \vee \sigma(E_i, i \leq n^0)$ in $\sigma(B)$.

■

This lemma is the key to the :

Proposition 1. and definition
Being given an integer $r > 0$ and three finite sequences of r.v. on $(W, \mathcal{W}., P)$: $(Y_s)_{0\leq s\leq r-1}$, $(Y_{<s>})_{0\leq s\leq r-1}$ and $(Z^s)_{0\leq s\leq r-1}$ such that :

- *Y_s is the sequence of signs of a finite family of disjointed excursions $(e_i^s)_{1\leq i\leq n_s^0}$ whose beginnings, $(g_i^s)_{1\leq i\leq n_s^0}$ are a family of $\mathcal{W}_\infty^{s+1}$-measurable r.v.*
- *$Y_{<s>}$ is a sequence of n_s^0 $\mathcal{W}_\infty^{s+1}$-measurable r.v. with values in $\{-1, 1\}$*
- *Z^s is the r.v.constituted by the signs of the other excursions of B^s conveniently numbered.*

Then, for all s from 0 to $r-1$, the map :

$$\begin{array}{rcl} f_s : (W, \mathcal{W}_\infty^s, P) & \rightarrow & (W, \mathcal{W}_\infty^{s+1}, P) \times (Y_s(W), \sigma(Y_s), P_{Y_s}) \times (Z^s(W), \sigma(Z^s), P_{Zs}) \\ w^s & \rightarrow & (w^{s+1}, \widetilde{Y}_s(w^s), \widetilde{Z}^s(w^s)) \end{array}$$

$$\text{where for all } w \in W,\ Y_s(w) = \widetilde{Y}_s(w^s) \text{ and } Z^s(w) = \widetilde{Z}^s(w^s).$$

is an isomorphism of measured spaces.
We will call sequence of B-raised motions of index r *the sequence of processes so defined :*

- $\Gamma^r := B^r$
- *$\forall s$ from r to 1,* $\Gamma^{s-1}(w) = f_{s-1}^{-1}(\Gamma^s(w), \widetilde{Y}_{<s-1>}(\Gamma^s(w)), Z^{s-1}(w))$

We will say that (Γ^s) is a sequence of $B - m.r.(r)$ and Γ^0 is a $B - m.r.(r)$.
We have :

$$\forall s \in \mathbb{N}, T\Gamma^s = \Gamma^{s+1}.$$

Furthermore,

$$\forall s \in \{r-1, ..., 0\} \forall w \in \bigcap_{s=r-1}^{i} [Y_s = Y_{<s>}], \Gamma^s(w) = B^s(w) = w^s,$$

the event $\bigcap_{s=r-1}^{i} [Y_s = Y_{<s>}]$ being $\mathcal{W}_\infty^i$-measurable and not negligible (of probability $\prod_{s=r-1}^{i} 2^{-n_s^0}$).

Proof of Proposition 1.
It is an immediate consequence of a repeated use of the preceding lemma.

■

When we need to keep in memory some information not contained in $\mathcal{W}_\infty^r$, we will rely on the following notion :
We call *disjointed sum of* $B - m.r.(r)$ a family $(\Gamma^s)_{0\leq s\leq r}$ of processes, such that there exists :

- A denumerable $\mathcal{W}_\infty$-measurable partition $(H_\nu)_{\nu\in N}$ of W.
- For all ν in N, a sequence of $B - m.r.(r)$ $(\Gamma_\nu^s)_{0\leq s\leq r}$ verifying :

$$\forall s \in \{0, ..., r-1\}, \Gamma^s = \sum_{\nu\in N} 1_{H_\nu}\, \Gamma_\nu^s.$$

We will denote Y_s^ν, $Y_{<s>}^\nu$, Z_ν^s, the r.v. associated to Γ_ν^s.

Our strategy is to construct a $B - m.r.(r)$ or a disjointed sum of $B - m.r.(r)$ which fulfill at level 0 some conditions. That returns to prove that these conditions are realized with strictly positive probability by the Brownian path itself. We have chosen the presentation through $B - m.r.(r)$ because it seems attractive to us to have a family of processes ranging from level r to level 0 which realizes the desired conditions, a family which necessitates nothing but two r.v. at each level : Y_s and $Y_{<s>}$.
Our task will now be precisely to construct these r.v.; and to perform this task, we will base ourself on the possibility to have at our disposal reserves of excursions, property that we express in the following :

Lemma 3.
Let be given e an excursion of B, with beginning g, and g' a r.v. such that $0 < g' < g$.
Then, if we arrange the family of excursions of B whose support lies in $[g', g]$, $(e_i)_{i\in\mathbb{N}}$, and if we call h_i the height of e_i, then :

$$\sum_{n\in\mathbb{N}} h_n = \infty\ , \qquad \textit{and} \tag{1}$$

(2) $$\lim_{n\to\infty} h_n = 0\ .$$

In particular, the set of sums of heights of finite number of such excursions is dense in $\mathbb{R}^+$.

Proof of lemma 3.
We know that :

$$\lim_{\varepsilon\to 0} \varepsilon d_\varepsilon^c(t) = \frac{1}{2} L(t) \ \ a.s.$$

where d_ε^c designates the number of down crossings of $[0,\varepsilon]$ (see [R,Y], chapter VI).
As g is an increasing limit of d_r, $r \in \mathbb{Q} \cap [0,g]$, we deduce that $\varlimsup_{\varepsilon\to 0} \varepsilon d_\varepsilon^{g'}(g)$ is strictly positive, we call it $\ell_g^{g'}$ (here, $d_\varepsilon^{g'}(g)$ designates the number of done crossings of $[0,\varepsilon]$ by $w_{|[g',g]}$).
Hence $\varlimsup_{\varepsilon\to 0} \varepsilon N_g^{g'}(\varepsilon) \geq \ell_g^{g'}$, where $N_g^{g'}(\varepsilon)$ designates the number of excursions greater than ε, whose support lies in $[g',g]$.
But :

$$\sum_{n\in\mathbb{N}} h_n = \int_0^\infty N_g^{g'}(\varepsilon) d\varepsilon.$$

Therefore,

$$\sum_{n\in\mathbb{N}} h_n = +\infty.$$

The continuity of paths ensures the null limit at infinity.

■

We need the notion of plug excursion :
Being given an excursion e, that we want to preserve during the raising, we put e positive and border it with e' a negative excursion lower than all the preceding ones, and anterior to e, and e'' a negative excursion, lower than all the preceding ones and posterior to e : e' and e'' are plug excursions for e.
This is the aim of :

Lemma 4.
Let $(e_i^{*s})_{1\leq i\leq q_s}$ *be a sequence of excursions of* w^s *arranged in the reverse*

chronological order, such that :
q_s and all the beginnings $(g_i^{*s})_{1\leq i\leq q_s}$ and heights $(h_i^{*s})_{1\leq i\leq q_s}$ are $\mathcal{W}_\infty^{s+1}$-measurable r.v.
Then there exists a sequence $(e_i^{*s'})_{0\leq i\leq q_s}$ of excursions of w^s before t, a constant such that supp$(e_1^{*s})\subset[0,t[$, verifying :

$$(3)\qquad \begin{cases} g_{q_s}^{*s'} < g_{q_s}^{*s} < g_{q_s-1}^{*s'} < ... < g_1^{*s} < g_0^{*s'} < g_t(w^s) \\ \\ h_{q_s}^{*s'} < h_{q_s-1}^{*s'} < ... < h_0^{*s'} < \varepsilon^* \end{cases}$$

for all constant $\varepsilon^* > 0$.

Proof of lemma 4.
We set : $g_{-1}^{*s'} := g_t(w^s)$, $h_{-1}^{*s'} := \varepsilon^*$, and $g_{q_s+1}^{*s} := 0$. We use lemma 3, so it suffices to choose, for $e_i^{*s'}, 0\leq i\leq q_s$, the higher (in absolute value) excursion of w^s among its excursions whose support is included in $[g_{i+1}^{*s}, g_i^{*s}]$ and height lower than $h_{i-1}^{*s'}$.
For these choices, the measurability required for the plug excursions, as well as (3), are obvious.

■

We introduce now the sequence $(e_i^{+s})_1\leq i\leq \nu_s$, obtained by the aggregation of :

- $(e_i^{*s})_{1\leq i\leq q_s}$
- $(e_i^{*s'})_{0\leq i\leq q_s}$
- $(e_i^{ys})_{1\leq i\leq p_s}$ the finite sequence of all the other excursions of w^s before g_t, whose height is greater than $h_{q_s}^{*s'}$, rearranged in the reverse chronological order.

(The reason for introducing the $(e_i^{ys})_{1\leq i\leq p_s}$ is the following :
when the plug excursions act their part of protecting the $(e_i^{*s})_{1\leq i\leq q_s}$, all the $(e_i^{ys})_{1\leq i\leq \nu_s}$ must be positive).
The following lemma is showing up the hereditary character of the construction of these excursions families :

Lemma 5.
Let be given two sequences $(e_i^{*s})_{1\leq i\leq q_s}$ *and* $(e_i^{+s})_{1\leq i\leq \nu_s}$ *verifying the hypothesis of the preceding lemma. Then, on the event :*
E_s *: "all the plug excursions and the excursions* $(e_i^{*s})_{1\leq i\leq \pi_s}$ *are* < 0*, all the other excursions of* $(e_i^{+s})_{1\leq i\leq \nu_s}$ *are* > 0 *" (where* π_s *is an integer valued constant lower than* q_s*, with the convention that, if* $\pi_s = 0$*, the condition on* (e_i^{*s}) *is empty; in the sequel,* $\pi_s = 0 or 1$*),*
then the excursions $(e_i^{*s})_{\pi_s+1\leq i\leq q_s}$ *are "preserved" at level* $s-1$*. This means that there exists a unique family of excursions of* w^{s-1}*, we note it* $(e_i^{*s-1})_{1\leq i\leq q_{s-1}}$*, with* $q_{s-1} := q_s - \pi_s$*, verifying :*

- $\mathrm{supp}\,(e_i^{*s-1}) \supset \mathrm{supp}\,(e_{\pi_s+i}^{*s}) \forall i \in [1, q_{s-1}]$
- *the supports of* e_i^{*s-1} *are disjointed two-by-two.*

Moreover, the beginnings of $(e_i^{*s-1})_{1\leq i\leq q_{s-1}}$ *are* $\mathcal{W}_\infty^s$*-measurable.*

Proof of lemma 5.
Following lemma 1, because on E_s each excursion of $(e_i^{*s})_{\pi_s+1\leq i\leq q_s}$ is positive and bordered by its two plug excursions, which are negative on E_s (for, the last minimum of w^s before the beginning of e_i^{*s} is attained on the support of its left plug, the following minimum being attained on the support of its right plug), $e_{i-\pi_s}^{*s-1}$ is the excursion of w^{s-1} which begins at the instant when for the last time,the left plug of e_i^{*s} reaches its minimum, and ends at the instant when, for the first time, the right plug excursion reaches the minimum of the left plug excursion.

■

In order to have the heredity to be effective, we choose the $(e_i^{*s-1})_{1\leq i\leq p_{s-1}}$ arbitrarily among the excursions of w^{s-1} on the complementary set of E_s, but in the respect of $\mathcal{W}_\infty^s$-measurability and disjointness of the support.

The following result establishes that the paths of different iterations of B, restrained to an interval where there is no intermediate zero, deduce from one another by isometry.

Lemma 6.
Let $(0, \overrightarrow{i}, \overrightarrow{j})$ *be an orthonormal basis of the plan in which we represent*

paths. Let τ_b^{a+} *be the vertical translation of vector* $(b-a)\overrightarrow{j}$ *and* τ_b^{a-} *the reflection along the horizontal axis of equation :* $y = \dfrac{a+b}{2}$.
Consider $(t,k,p) \in \mathbb{R}^+ \times \mathbb{N}^2$ *such that* $w_t^k = a$ *and* $w_t^{k+p} = b$ *and denote* γ_t *the first time posterior to* t *when at least one of the iterated Lévy transforms* w^s, $k \leq s \leq k+p-1$, *vanishes. Then we have :*

$$w^{k+p}_{|[t,\gamma_t]} = \begin{cases} \tau_b^{a+} \, o \, w^k_{|[t,\gamma_t]} & \text{if} \quad \prod_{i=k}^{k+p-1} w_t^i > 0 \\ \tau_b^{a-} \, o \, w^{-k}_{|[t,\gamma_t]} & \text{else} \end{cases}$$

We will denote $\tau^k_{k+p}(w)$ *the plan transformation, which transforms* $w^k_{|[t,\gamma_t]}$ *in* $w^{k+p}_{|[t,\gamma_t]}$.

Proof of lemma 6.
It is an immediate consequence of Tanaka's formula, when $p = 1$.
In general case, we break up the displacement τ which transforms $w^k_{/[t,\gamma_t]}$ in $w^{k+p}_{/[t,\gamma_t]}$ under the form $\tau = \tau_p \circ \tau_{p-1} \circ ... \circ \tau_1$ where τ_i transforms $w^{k+i-1}_{/[t,\gamma_t]}$ in $w^{k+i}_{/[t,\gamma_t]}$. From the preceding remark, each τ_i is a vertical translation or a reflection along an horizontal axis, according to the sign of w_t^{k+i-1}. Then we deduce the claim.

■

The notions to follow are just necessary for fine analysis of the Lévy raising. For our needs, we will call excursion each map $e : \mathbb{R}^+ \to \mathbb{R}$ whose support is a not empty segment and which doesn't vanish at any point of the interior of the support. In particular, for $w \in W$ and $t > 0$, we will call excursion straddling t, and denote it by : $e_t(w)$, the map so defined :

$$e_t(w) : \mathbb{R}^+ \to \mathbb{R},\ \forall u \in \mathbb{R}^+,\ e_t(w)(u) = \begin{cases} 0 & \text{if } u \in [0, g_t(w)] \cup [d_t(w), +\infty[\\ w_u & \text{else} \end{cases}$$

We will introduce the map too $de_t(w) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\forall u \in \mathbb{R}^+,\ de_t(w)(u) = \begin{cases} 0 & \text{if } u \in [0,t] \cup [R_t(w), +\infty[\\ & \text{where } R_t(w) = \sup\{u > t, \forall s \in]t,u], w_s \neq w_t\} \\ w_u - w_t & \text{else} \end{cases}$$

We denote them e_t and de_t when there is no ambiguity, and e_t^s and de_t^s in the case of excursions of w^s.

Lemma 7.
Let $w \in W$ and e a negative excursion of w, lower than all preceding it. Let γ be its beginning and δ its end. We set :

$$\gamma_1 = \arg\min_{[0,\gamma]} w^s,\ \gamma_2 = \inf\{t \in supp(e); e(t) = w^s_{\gamma_1}\},\ \gamma_3 = \arg\min_{[\gamma,\delta]} e$$

Then :

$$\begin{cases} de^s_{\gamma_1} \text{ coincides with an excursion of } |w^{s-1}|, \text{ and its support is } [\gamma_1, \gamma_2] \\ de^s_{\gamma_3} \text{ coincides with an excursion of } |w^{s-1}|, \text{ which begins at } \gamma_3 \text{ and whose} \\ \text{ support contains } [\gamma_3, \delta] \end{cases}$$

Furthermore, $\forall u \in [\gamma_2, \gamma_3]$, de^s_u coincides with an excursion of $|w^{s-1}|$, if, and only if :

$$\begin{cases} de^s_u \text{ is a positive excursion} \\ e(u) = \inf\{e(t),\ t \in [\gamma, u]\} \end{cases}$$

It is the case in particular when de^s_u is the first positive excursion of the form de^s_v, $v \in [\gamma_2, \gamma_3]$ to overflow a given value.

Proof of lemma 7.
From Tanaka's formula :

$$|w^{s-1}_t| = w^s_t + \sup\{-w^s_u, u \in [0,t]\}$$

Therefore,

$$|w^{s-1}_{\gamma_1}| = w^s_{\gamma_1} - w^s_{\gamma_1} = 0,$$

while, for all $t > \gamma_1$, sufficiently near :

$$w^s_t > w^s_{\gamma_1}.$$

So, $de^s_{\gamma_1}$ is a positive excursion of $|w^{s-1}_\cdot|$ which ends at γ_2.
In the same way, $w^s_t > w^s_{\gamma_3}$, for all $t \in [\gamma_3, \delta]$, therefore $de^s_{\gamma_3}$ is an excursion of $|w^{s-1}_\cdot|$ beginning at γ_3 whose support contains $[\gamma_3, \delta]$.
Let e' be an excursion of w^{s-1} with support included in $[\gamma_2, \gamma_3]$. Its beginning u, and its end v verify :

$$u = \arg\min_{[0,v[} w^s \qquad \text{and} \qquad v = \arg\min_{[0,v]} w^s$$

So we deduce : $de^s_u = |e'|$.
Reciprocally, let $u \in [\gamma_2, \gamma_3]$ such that de^s_u is a positive excursion and

$u = \underset{[0,u]}{\arg\min}\ w^s$.
Then, $u = \underset{[0,v[}{\arg\min}\ w^s$, where w is the end of de^s_u, because de^s_u is positive.
Thus, $w^{s-1}_u = w^{s-1}_v = 0$, and for all $t \in]u, v[$, $w^{s-1}_t \neq 0$.
Consequently, de^s_u is an excursion of $|w^{s-1}|$.
Let $h > 0$ be such that there exists $u \in [\gamma_2, \gamma_3]$ verifying de^s_u is the first positive excursion of the form de^s_v, $v \in [\gamma_2, \gamma_3]$, whose height overflows h. Then, for all $v < u$, the support of de^s_v can't contain this of de^s_u without denying the minimality of u.

■

2 One-time approximation

This paragraph and the following one are devoted to equip the reader with two settings, the first simpler and the second more intricate, of the method we are going to use in paragraph 4 where many difficulties are assembled. So, the technic used to prove proposition 2 isn't the most convenient, but the nearest to that used for proposition 4.
Our aim is, going from an iterated B.M. of level sufficiently deep, to construct a procedure, which we call *configuration process*, enabling the path raised at level 0 to approach, at precision ε , the value of an arbitrary fixed path φ at an arbitrary fixed time $t > 0$.
The configuration process elaborates a $B - m.r.(r) : \Gamma$, and this is done along two stages.
In the first stage, the paths of Γ are compelled to approach zero at time t, by working on the excursion of w^r straddling t. This necessitates a random number of raises. Before raising from level r, we choose excursions of w^r, with supports in $[0, t]$ and whose sum of heights approaches $|\varphi(t)|$. We call them the *building* excursions.
In the second stage, the purpose is to put in action the selected excursions to obtain the desired effect. But to do so, we must, from raise to raise, protect the selected excursions which haven't still been used. For this reason, we introduce the notion of *plug excursions* : namely two "small" excursions, which border the excursion to protect and separate it from the others, the height of the first in time being smaller than that of the second. We call the first one the *left* plug excursion, and the second one the *right* plug excursion. But in a sequence of excursions to protect, arranged in the chronological order, it suffices to equip each excursion excepted the last one, with a left plug excursion : the left plug excursion of the following excursion of the sequence

acting the part of a right plug excursion for the considered excursion. For the configuration process to do its work, we must straighten in the positive sense the excursions to protect, in the negative sense the plug excursions and, if we are in the building period, the first building excursion, and in the positive sense all the other excursions before t whose height is smaller than the first plug excursion. This necessitates only the choice of a finite number of signs at each raise. It is that we are doing in the following :

Proposition 2.
Let be given : $(\varphi, t, \varepsilon) \in W \times \mathbb{R}_+^* \times \mathbb{R}_+^*$. *Then there exists an integer* r *and a* B*-m.r.*$(r) : \Gamma$ *such that :*

$$P(|\Gamma(t) - \varphi(t)| > \varepsilon) < \varepsilon$$

Proof of Proposition 2.
We start by choosing an arbitrary integer r, large enough, that we will determine later. The proof consists essentially of four lemmas and an algorithm. In the first lemma, we accomplish the main task for level r.

Lemma 8.
Let ε' *and* β' *be constants* > 0 *to be specified later on.*

(1) The following r.v. are functionals of $|w^r|_{|[0,t]}$, $\mathcal{W}_t^{r+1}$*-measurable :*

- $t_0(w) := g_t(w^r)$, $m_0(w) := \sup\{|w_u^r|,\ u \in [t_0(w), t]\}$;
- $\forall k \in \mathbb{N}$, $m_{k+1}(w) := \sup\{m_k(w) - |w_u^r|,\ u \in [t_k(w), t]\}$;
- $t_{k+1}(w) := \arg\max_{[t_k(w),t]} (m_k(w) - |w_\cdot^r|)$, *the last time before* t *at which the previous sup is reached.*
  The sequence $(m_k(w))_{k\in\mathbb{N}}$ *is strictly decreasing of null limit while* $(t_k(w))_{k\in\mathbb{N}}$ *is strictly increasing of limit* t *a. s.*
  Let $K(w) := 1 + \inf\{k \in \mathbb{N} | 0 < m_k(w) < \varepsilon'\}$.
  Then K *is an a. s. finite* $\mathcal{W}_\infty^{r+1}$*-measurable r.v.*

(2) Let $Q_p(w)$ be the property : "there exists a sequence of excursions of w^r with supports in $[0, t_0(w)]$, $(e_i^r)_{1\leq i\leq p}$, whose beginnings $(G_i^r)_{1\leq i\leq p}$ is a strictly decreasing sequence with heights $(h_i^r)_{1\leq i\leq p}$ such that :

$$\sum_{i=1}^{p} h_i^r < |\varphi(t)| < \sum_{i=1}^{p} h_i^r + \beta' \quad " \tag{4}$$

Let $N(w)$ be the smallest integer p such that $Q_p(w)$ is true, else 0. Then N is an a. s. finite $\mathcal{W}_t^{r+1}$-measurable r.v. Moreover, for any finite $\mathcal{W}_\infty^{r+1}$-measurable integer valued r.v. N', on the set $[N' \geq N]$ there exists a sequence $(e_i^{r,N'})$ which verifies the requirements of $Q_{N'}(w)$ and whose beginnings $(G_i^{r,N'})_{1\leq i\leq N'}$ are a decreasing sequence of $\mathcal{W}_\infty^{r+1}$-measurable r.v.

Proof of lemma 8.

- By construction, the sequence $(m_k)_k$ and $(t_k)_k$ are respectively positive decreasing and increasing, bounded from above by t. So they converge. Set μ the limit of (m_k) and τ that of (t_k).
  Suppose $\mu > 0$. Then the oscillation of w^r is infinite at the neighbor of τ, which is in contradiction with the continuity of w^r. So $\mu = 0$.
  Suppose $\tau < t$. Then w^r is constant on $[\tau, t]$, which again is a. s. impossible. So $\tau = t$.

- Clearly, K is measurable and finite.

- Now we arrange the excursions of w^r before $t_0(w)$ by decreasing heights, obtaining thus the sequence $(\hat{e}_n)_{n\in\mathbb{N}}$ with heights $(\hat{h}_n)_{n\in\mathbb{N}}$.
  From lemma 3, we have :

$$\sum_{n\in\mathbb{N}} \hat{h}_n = \infty \ , \qquad \text{and } \lim_{n\to\infty} \hat{h}_n = 0 \ .$$

  Let $\mathcal{P}_f(\mathbb{N})$ be the set of finite parts of $\mathbb{N}$, and, for all $n \in \mathbb{N}$, $\mathcal{P}_n(\mathbb{N})$ the set of parts of $\mathbb{N}$ of cardinal n.

  For all $H \in \mathcal{P}_f(\mathbb{N})$, we set :

$$Z_H(w^r) := \left| w_t^r - w_{t_{K(w)}}^r \right| + \sum_{n\in H} \hat{h}_n$$

We introduce the event $C_H := [Z_H(w^r) < |\varphi(t)| < Z_H(w^r) + \beta']$.
Then we have, from lemma 3 : $C_H \in \mathcal{W}_\infty^{r+1}$ and $\bigcup_{H \in \mathcal{P}_f(\mathbb{N})} C_H = W$.

Let us set, for all $n \in \mathbb{N}$, $C_n := \bigcup_{H \in \mathcal{P}_n(\mathbb{N})} C_H$.
And $N(w) := \inf\{n \in \mathbb{N} / w^r \in C_n\}$.
Then, clearly, N is an a. s. finite ${\mathcal{W}^{\nabla+\infty}}_\infty$-measurable r.v.
Let us provide with lexicographic order the denumerable set $\mathcal{P}_n(\mathbb{N})$, after identifying its elements with the strictly increasing sequences of n integers.
Then, when $w^r \in C_n$, let H_n^0 be the smallest element H of $\mathcal{P}_n(\mathbb{N})$ such that $w^r \in C_H$.
We note $(e_i^{rn})_{1 \leq i \leq n}$ the sequence of the elements of $(\hat{e}_i)_{i \in H_n^0}$ arranged in the reverse chronological order.

- Consider at last N' an integral valued $\mathcal{W}_\infty^{r+1}$-measurable r.v.
  We set : $e_i^{rN'} := \sum_{n=1}^{\infty} 1_{[N'=n]} e_i^{rn}$, for all $i \in [0, N']$.
  We will omit, for sake of simplicity of notations, the superior index N' if there is no ambiguity. By construction, the sequence $(e_i^r)_{1 \leq i \leq N'}$ fulfill all the desired conditions of the statement.

■

Now, we are in position to take the definition of the families of r.v. $(Y_s)_{0 \leq s \leq r}$ and $(Y_{<s>})_{0 \leq s \leq r}$.
This definition is based on an algorithm which will conveniently call the preceding lemmas at appropriate times. To express this algorithm, we have chosen to use a programming language, near Pascal language. It is not necessary, in this paragraph, but doing so, we are preparing the ground for the more complex algorithms of the following paragraphs.
In the first phase, we accomplish the first K raises : these raises ought to plane the excursion straddling t. Then the building excursions which were protected during the preceding phase, are put to act one-by-one, for the N following raises : it is the second phase.
Then we start the third phase : the desired result being obtained, it remains to preserve it up to level 0. So, during the last raises, we are protecting the excursion straddling t.
Before beginning to write the "programm", let us introduce the following notation :

- t_j^{cN} designates the beginning of e_i^{rN}, for all i from 0 to N.

Variables
s, K, N, R of integers.
Y_s, $Y_{<s>}$ of finite sequences of signs.
$(e_j^{*s})_j$, $(e_j^{+s})_j$ of finite sequences of excursions
$M_\Box$ of elements of $\mathcal{W}_\infty$

Initialization
$s := r$
$(e_j^{*r})_j := (e_{t_i^{c_N}}(w^r))_i \vee (e_t(w^r))$
For all s, do :
$Y_s := (sgn(e_j^{+s})_j)$, $n_s(w) = \sharp(e_j^{+s})_j$.

End for.

For $i = 0$, to $K - 1$, do :
$(e_j^{+s})_j$ is the sequence associated to $(e_j^{*s})_j$ by lemma 4.
$Y_{<s>}$:= "the plug excursions and the excursion straddling t *(it begins at* $t_i(w^r)$*)* are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s \neq Y_{<s>}$ do triv (w, s)

End if.

let $(e_j^{*s-1})_j$ be associated to $(e_j^{+s})_j$ by lemma 5.
do :
$(e_j^{*s-1})_j := (e_j^{*s-1})_j \vee (e_t(w^{s-1}))$
$M_\Box := M_\Box \cap [Y_s = Y_{<s>}]$
$s := s - 1$, $i := i + 1$

End for.

(It is the end of the first phase : the excursion straddling t *is planed)*

For $i = 0$ to $N - 1$, do :
$(e_j^{+s})_j$ is associated to $(e{*}_j^s)_j$ by lemma 4.
$Y_{<s>}$:= "the plug excursions and the first building excursion are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s \neq Y_{<s>}$ do triv (w, s)

End if.

Let $(e_j^{*s-1})_j$ be the sequence associated to $(e_j^{+s})_j$ by lemma 5.
$M_\Box := M_\Box \cap [Y_s = Y_{<s>}]$
$s := s - 1$, $i := i + 1$

End for.

(It is the end of the second phase, the excursion straddling t has now the previewed value, just before the start of the configuration process : the building excursions have now do their work, i.e. the height of the excursion straddling t is at the present level close to $|\varphi(t)|$)

do $R(w) := s$ *($R(w)$ is keeping in memory the level from which the desired result is obtained)*
For $i := 0$ to $r - N - K - 1$, do :
$(e_j^{+s})_j$ is the sequence associated with $(e_j^{*s})_j$ by lemma 4.
(Notice that $(e_j^{*s})_j$ is the sequence reduced to the excursion straddling t)
$Y_{<s>} :=$ "$theplugexcusionsare < 0, alltheotherexcursionsof(e_j^{+s})_jare > 0$"
If $Y_s \neq Y_{<s>}$ do triv (w, s)

End if.

Let $(e_j^{*s-1})_j$ be associated to $(e_j^{+s})_j$ by lemma 5.
do $M_\square := M_\square \cap [Y_s = Y_{<s>}]$
$s := s - 1$, $i := i + 1$

End for.

(We are now at level 1, the third phase is nearly accomplished, it remains to give the good sign to the excursion straddling t)
do $(e_j^{+0})_j$:=$(e_t(w))$ *(This family is reduced to one element)*
$Y_{<0>} := (sgn(\varphi(t)))$, $M_\square := M_\square \cap [Y_0 = Y_{<0>}]$

End prog.

function triv(w, s)
for $i = s - 1$ to 0, do :
$(e_j^{+i}) := \emptyset$
$Y_{<i>} := \emptyset$

End function.

function arranged $(e_\alpha(\varphi))_{\alpha \in A}$
where $\varphi \in W$, and $(e_\alpha(\varphi))_{\alpha \in A}$ is a finite family of excursions of φ.
let g_α be the beginning of e_α
G the set of all the g_α of the finite family
γ the end of G, N its cardinality
$\Gamma := G$, $\gamma :=$ the end of Γ

while $\sharp(\Gamma) > 0$, do :
$i := N - \sharp(\Gamma) + 1$
e_i :="the excursion of the family which begins at γ"
$\Gamma := \Gamma \backslash \{\gamma\}$

End while.

do arranged $(e_\alpha(\varphi))_{\alpha \in A} := (e_1, e_2, ..., e_N)$

End function.

function $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B}$
do $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B}$:= arranged $(e_\gamma(\varphi))_{\gamma \in A \cup B}$

End function.

Now choose $\varepsilon' = \beta' := \frac{\varepsilon}{4}$.
To create this programm, we must suppose r finite. We are going to prove that it is possible.
Consider the r.v. R is $\mathbb{Z}$-valued (it suffices to work momentanously in the filtered probability space of $(B^n)_{n \in \mathbb{Z}}$).
So, because $r - R$ does not depend upon r, there exists a positive integer r such that :

$$P(A_1^\varepsilon) > 1 - \frac{\varepsilon}{2}$$

where

$$A_1^\varepsilon := [R > 0]$$

From now on, r is thus chosen (for instance, the smallest integer with this property).
Otherwise, the r.v. $n_s(w)$ being $\mathbb{N}$-valued, there exists a sequence of deterministic integers $(n_s^0)_{0 \leq s \leq r}$ verifying :

$$P(A_2^\varepsilon) > 1 - \varepsilon$$

where

$$A_2^\varepsilon := A_1^\varepsilon \cap \left(\cap_{s=0}^r [n_s \leq n_s^0] \right).$$

This enables us to "bring" the r.v. $Y_s, 0 \leq s \leq r$, at the cardinality $n_s^0 + 1$.
If necessary, we complete the $(e_j^{+s})_0 \leq j \leq n_s(w)$ to $(e_j^{+s})_0 \leq j \leq n_s$:

- By adding excursions of w^s with beginnings after time $1+t$ and chosen $\mathcal{W}_\infty^{s+1}$-measurably,

- Else by truncation, we preserve only the first $n_s^0 + 1$ excursions.

We do similarly with the $Y_{<}s>$ by adding as many " $+1$" as necessary, or by truncation.
Thus modified, the programm yields now sequences (Y_s) and $(Y_{<s>})$ which satisfy all the conditions of Proposition 2. In other words the construction of the $B - m.r.(r)$ is achieved. But, with arbitrary parameter ε^*, it does not fulfill the requirements we need.
So we define the parameter ε^* in the following way :

$$\varepsilon^* := \frac{\varepsilon'}{\sum_{j=0}^{r} j} (= \frac{\varepsilon'}{2r(r+1)})$$

Indeed the N building excursions are affected by as many plug excursions as their intervention order, and finally, during the last raises, the excursion straddling t is affected by one plug excursion at each raise.
We are now in a position to state the

Lemma 9.
For all $w \in A_2^{\varepsilon}$, we have with the preceding choice of ε^* :

$$|\Gamma(t) - \varphi(t)| < \varepsilon$$

Proof of lemma 9.
The choice of ε^* ensures that the intervention of the various plug excursions during the r raises doesn't affect the path from more than ε'. The difference from zero, when the first building excursion is put to act, is also lower than ε'. Finally the sum of heights of the building excursions approaches $|\varphi(t)|$ with precision ε'.

■

So the proof of proposition 1 is achieved.

3 d-times approximation

The raises we have realized on the Brownian path, in paragraph 1, can be called "elementary" raises : we prepare, at level r, the excursions which enable the path to approach $\varphi(t)$ at time t, then we put them in action successively while holding up the remaining excursions among the selected ones. Now we must proceed by induction. But a new difficulty appears : the necessity of protecting, which has been obtained at the first d-times. This time, we can't anymore be satisfied with elementary raises. First, only excursions with support in $[t_d, t_{d+1}]$ can be used, so it is necessary to rely on the density of zeroes [M]. Then, going from an iteration which vanishes somewhere between t_d and t_{d+1}, we achieve *"horizontal"* raises to correctly configure the path at time t_{d+1}, at the considered level, in the manner of lemmas 4 and 5 : during these raises, we maintain the main excursions in $[0, t_d]$ hold up to preserve the path up to time t_d, while we correctly configure the path at time t_{d+1}, obtaining thus a path of *"essential"* level, this of the beginning of the configuration process. Then we come back to usual raises which we call *"vertical"* raises. We give to the main excursions the sign they had before the horizontal raises (this compels us to replace the $B - m.r.$ by a disjoint sum of such processes to keep these signs in memory). But the estimated value of the path at time t_{d+1} is based upon the hypothesis that no later vertical raise will vanish on $[t_d, t_{d+1}]$. In the opposite case, the configuration process must return to its beginning. We are facing a complex algorithm which we treat in programming.
It's that we are doing in the following :

Proposition 3.
Let be given : $\varepsilon > 0, (t_1, ..., t_d) \in \mathbb{R}^d$ *such that* $0 < t_1 < ... < t_d$ *and* φ *a map from* $\mathbb{R}^+ \to \mathbb{R}$. *Then there exists an integer* r *and a disjoint sum* $(\Gamma^i)_{0 \leq i \leq r}$ *of* $B - m.r.(r)$ *such that*

$$P(||\Gamma^0(t_1, ..., t_d) - \varphi(t_1, ..., t_d)||_\infty > \varepsilon) < \varepsilon$$

(with the convention that $f(t_1, ..., t_d)$ *designates the vector of coordinates* $(f(t_1), ..., f(t_d))$, *for all map* f *from* $\mathbb{R} \to \mathbb{R}$.*)*

Proof of Proposition 3.
We proceed by induction on d.
Let $\mathcal{P}_d(\varepsilon)$ be the property stated in the Proposition.
At the first rank, the result yields from Proposition 2. In this case, the disjointed sum is nothing but a single $B - m.r.(r)$.

Suppose $\mathcal{P}_d(\varepsilon)$ true.
We will apply this hypothesis to the B.M. B^{s_0} for an integer s_0 to be determined later. We will denote $\widetilde{w}$ the generic path of B^{s_0} : $\widetilde{w} := w^{s_0}$.
Thus there exists a disjointed sum of $B^{s_0} - m.r.(r_d)$, $\widetilde{\Gamma}$, such that :

$$P(A_0^\varepsilon) > 1 - \frac{\varepsilon}{6},$$

where

$$A_0^\varepsilon \;:=\; [\|\widetilde{\Gamma}(t_1, ..., t_d) - \varphi(t_1, ..., t_d)\|_\infty < \varepsilon]$$

As usual, we will denote : $\forall i \in \mathbb{N}$, $\widetilde{\Gamma}^i = \widetilde{w}^i$.
By definition, $\forall i > r_d$, $\tilde{w}^i = w^{s_0+i}$.
Then we introduce the r.v. :

$$\varepsilon_2(w) := (\varepsilon - \|\widetilde{\Gamma}(t_1, ..., t_d) - \varphi(t_1, ..., t_d)\|_\infty).$$

This r.v. is strictly positive on A_0^ε.

Otherwise, from the theorem on density of zeroes, ([M]), there exists a.s. an integer ℓ such that $\widetilde{w}^\ell$ vanishes at least one time on $[t_d, t_{d+1}]$.
Let $L(w)$ be the smallest of these integers ℓ. L is a r.v. almost surely finite.
So there exists an integer ℓ_0 which we will choose $> r_d$ such that :

$$P(A_1^\varepsilon) > 1 - 2\frac{\varepsilon}{6},$$

where

$$A_1^\varepsilon \;:=\; A_0^\varepsilon \cap [L \leq \ell_0].$$

Our aim is to raise the path $\widetilde{w}^{\ell_0}$, which is nothing but $w^{s_0+\ell_0}$, from level $s_0+\ell_0$ to level 0, in such a way that the raised path approaches φ on $[t_d, t_{d+1}]$, while remaining at the proximity of φ on $[0, t_d]$ acquired at level s_0, with the usual measurability conditions.
For this to be realized without damage, we will introduce the family of protecting excursions of $\widetilde{w}^k$: $(\widetilde{ep}_j^{(k)})_{1\leq j\leq p_k}$. This family is constituted by the excursions of $\widetilde{w}^k$ with beginning before t_d and height greater than $\frac{\varepsilon_2(w)}{2\ell_0}$, and arranged in the reverse chronological order, for all k from 0 to ℓ_0.
The p_k, $0 \leq k \leq \ell_0$, are finite r.v. Let $(p_k^0)_{0\leq k\leq \ell_0}$ be a sequence of deterministic integers verifying :

$$P(A_2^\varepsilon) > 1 - 3\frac{\varepsilon}{6},$$

where

$$A_2^\varepsilon := A_1^\varepsilon \cap \left(\bigcap_{k=0}^{\ell_0} [p_k \leq p_k^0] \right).$$

Now, we can modify the protecting excursions : $(\widetilde{ep}_j^{(k)})_{0\leq j\leq p_k^0}$ is constituted of the excursion straddling t_d and of the p_k^0 highest excursions before t_d, arranged in the reverse chronological order.
We set :

$$\begin{aligned} \Lambda &:= \{(\lambda_k)_{0\leq k\leq \ell_0}\},\ \lambda_k \in \{-1,+1\}^{\{0,\ldots,p_k^0\}} = \prod_{k=0}^{\ell_0} \{-1,+1\}^{\{0,\ldots,p_k^0\}} \\ \Delta &:= \{[n\beta,\ (n+1)\beta[;\ n\in\mathbb{Z}\}^{\{0,\ldots,\ell_0\}} \end{aligned}$$

The partition (H_ν^{d+1}) from which we are going to construct Γ is so defined :
$N_{d+1} := \Lambda \times \Delta \times \mathbb{N}$ is denumerable ;
$\forall(\lambda,\ \delta,\ n) \in \Lambda\times\Delta\times\mathbb{N}^*$,

$$H_{\lambda,\,\delta,\,n}^{d+1} := \bigcap_{k=0}^{\ell_0} \Big([(sgn(\widetilde{ep}_j^k))_{0\leq j\leq p_k^0} = \lambda_k] \ \cap [\tilde{w}_{t_d}^k \in \delta(k)] \Big) \cap \ldots \cap \left[\tfrac{\varepsilon}{2n} < \varepsilon_2(w) \leq \tfrac{\varepsilon}{2^{n-1}}\right]$$

$$H_{\lambda,\,\delta,\,0}^{d+1} := \bigcap_{k=0}^{\ell_0} \Big([(sgn(\widetilde{ep}_j^k))_{0\leq j\leq p_k^0} = \lambda_k] \ \cap [\tilde{w}_{t_d}^k \in \delta(k)] \Big) \cap [\varepsilon_2(w) \leq 0]$$

$(H_{\lambda,\,\delta,\,0} \cap A_0^\varepsilon = \emptyset)$

To simplify the notations, we will omit the indices λ, δ, n.
Set : $b\delta(k)$, the beginning of the interval $\delta(k)$ and $\varepsilon(\lambda,k) := \prod_{i=0}^{k-1} \lambda_i(0)$.

Lemma 10.
On $H_{\lambda,\delta,n}^{d+1}$ the displacements $\tau_k^0(\widetilde{w})$ and $\tau_{b\delta(k)}^{b\delta(0)\ \varepsilon(\lambda,k)}$ differ from at most 2β.

Proof of lemma 10.
From lemma 6, these displacements are of the same nature , vertical translations or reflections along the horizontal axis.

- In the case of a translation, $\varepsilon(\lambda,k) = +1$, we have :

$$\begin{aligned} \forall y\in\mathbb{R},\ |\tau_k^0(\tilde{w})(y) - \tau_{b\delta(k)}^{b\delta(0)\,\varepsilon(\lambda,k)}(y)| &= |\widetilde{w}_{t_d}^k - \widetilde{w}_{t_d}^0 - (b\delta(k) - b\delta(0))| \\ &= |\widetilde{w}_{t_d}^k - b\delta(k) - (\widetilde{w}_{t_d} - b\delta(0))| \\ &\leq \beta \end{aligned}$$

- In the case of a reflection, $\varepsilon(\lambda,k)=-1$, we have :

$$\begin{aligned}\forall y\in\mathbb{R}\ ,\ |\tau_k^0(\widetilde{w})(y)-\tau_{b\delta(k)}^{b\delta(0)-}(y)| &= |(\widetilde{w}_{t_d}^k+\widetilde{w}_{t_d}^0-y)-(b\delta(k)+b\delta(0)-y)|\\ &= |(\widetilde{w}_{t_d}^k-b\delta(k))+(\widetilde{w}_{t_d}-b\delta(0))|\\ &\leq 2\beta\end{aligned}$$

■

This lemma allows us to replace the value to be anticipated $\tau_k^0(\widetilde{w})(\varphi(t_{d+1}))$, which doesn't have the good measurability by $\tau_{b\delta(k)}^{b\delta(0)}(\varphi(t_{d+1}))$ on $H_{\lambda,\delta}$.
The purpose of the following lemma is to prepare, at level s, when the iterated Brownian motion vanishes on $]t_d,t_{d+1}[$, the excursions which will allow the correctly raised path to approach φ at level 0 on $]t_d,t_{d+1}[$. It is the analog of lemma 8.
We set : $Z_s:=\{w\in W/w^s vanishes at least one time on]t_d,t_{d+1}[\ \}$.

Lemma 11.
For all $w\in Z_s$, ε_s' being a $\mathcal{W}_\infty^{s+1}$-r.v. to be specified later, we have :
1) The following r.v. are functionals of $|w^s|_{|_{[t_d,t_{d+1}]}}$:

- $t_0^s(w):=g_{t_{d+1}}(w^s)$, $m_0^s(w):=\sup\{|w_u^s|,u\in[t_0^s(w),t_{d+1}]\}$
- $\forall n\in\mathbb{N}$, *while* $t_n^s<t_{d+1}$ *we set :*

$$\begin{cases} m_{n+1}^s(w) &:= \sup\{m_n^s-|w_u^s|,u\in[t_s^n,t_{d+1}]\}\\ t_{n+1}^s(w) &:= \underset{[t_s^n,t_{d+1}]}{\arg\max}(m_n^s-|w_\cdot^s|)\\ & \quad \textit{the last time before } t_{d+1} \textit{ at which the previous } \sup \textit{ is reached.}\end{cases}$$

The sequence $(m_k^s(w))_{k\in\mathbb{N}}$ is strictly decreasing of null limit, while $(t_k^s(w))_{k\in\mathbb{N}}$ is strictly increasing of limit t_{d+1} a.s.
Let $K_s(w):=1+\inf\{k\in\mathbb{N},0<m_k^s<\varepsilon_s'\}$.
Then K_s is an a.s. finite r.v. $\mathcal{W}_\infty^{s+1}$-measurable.

2) Let $Q_p(w)$ be the property : " there exists a sequence of excursions of w^s with supports in $[t_d,t_0^s]$, $(e_i^s)_{1\leq i\leq p}$, whose beginnings $(G_i^s)_i$ are a strictly decreasing sequence, with heights $(h_i^s)_i$ such that :

$$(5)\qquad \sum_{i=1}^p h_i^s \ <\ \left|\tau_{b\delta(esslevel(s))}^{b\delta(0)\,\varepsilon(\lambda,k)}(\varphi_{(t_{d+1})})\right|<\sum_{i=1}^p h_i^s+\beta' \qquad "$$

where the r.v. esslevel(s) will soon be defined, and $w\in H^{\lambda,\delta,n}$.
Let $N_s(w)$ be the smallest integer p such that $Q_p(w)$ is true when $w\in Z_s$,

else 0.
Then N_s *is an a.s. finite* $\mathcal{W}_\infty^{s+1}$*-measurable r.v.*

3) For any finite $\mathcal{W}_\infty^{s+1}$*-measurable integral valued r.v.* N'*, on the set* $[N' \geq N]$*, then there exists a sequence* $(e_i^{s,N'})_i$ *which verifies the requirements of* $Q_{N'}(w)$ *and whose beginnings* $(G_i^{s,N'})_{1\leq i\leq N'}$ *is a decreasing sequence of* $\mathcal{W}_\infty^{s+1}$*-measurable r.v.*

Proof of lemma 11.
Essentially, this lemma differs from lemma 8 only by notations. So the proof is exactly the same.

■

Now, with the help of lemmas 4 and 5, we are in a position to take on the definition of the families of r.v. $(Y^{d+1}_{\lambda,\delta,n,s})_{0\leq s\leq s_0+\ell_0}$ and $(Y^{d+1}_{\lambda,\delta,n,<s>})_{0\leq s\leq s_0+\ell_0}$. As in the preceding paragraph, this definition is based on the following algorithm, which will conveniently call the lemmas 4, 5, 11 at appropriate times. Before beginning to write the programm, let us introduce the notations :

- t_j^{p,s_0+k} designates the beginning of $\widetilde{ep}_j^{(k)}$, for all j from 0 to p_0^k.
- t_j^{c,N_s} designates the beginning of e_i^{s,N_s}, for all i from 0 to N_s.

The algorithm goes from level $r_{d+1} := s_0+\ell_0$. In the first stage, i.e. before the $B-m.r.(r_{d+1})$ vanishes on $]t_d, t_{d+1}[$, we just impose the signs of the protecting excursions $(\widetilde{ep}_j^s)$, namely the excursions of w^s of sufficient height, which are straddling times $t_j^{p,s}$. Thus we arrive at level s when the $B-m.r.(r_{d+1})$ vanishes : the second stage is beginning. For the first time, we put in action the configuration process. We start the K_s+N_s horizontal raises : the first K_s ones ought to plane the excursion straddling t_{d+1}; then the building excursions which were protected during the preceding phase are put to act, one-by-one, during the following N_s raises. Thus at level $s-K_s-N_s$, the path has nearly the same form on $[0,t_d]$ than at level s, and approaches the s-level desired value at time t_{d+1}. Therefore we qualify these K_s+N_s raises *horizontal* ones. And we say that the essential level corresponding to level $s-K_s-N_s$ is s. Now, it's the first stage. It is composed of vertical raises : the desired value at time t_{d+1} supposes that the path doesn't vanish on $[t_d, t_{d+1}]$, because of lemma 6, when it is the case, we raise the path. If it is not the case, we must stop, and begin again a new configuration process, and so on.

Variables
s, esslevel(s), $\sharp(s)$, n_s, K_s, N_s, R, level(M) (where $M \in \mathcal{W}_\infty$) of integers ($\sharp(s)$ represents the number of starts of the configuration process before level s).
$\square$, $\square_1$ of strings
Y_s^{d+1}, $Y_{<s>}^{d+1}$ of finite sequences of signs.
$(ep_j^s)_j$, $(ec_j^s)_j$, $(e_j^{*s})_j$, $(e_j^{+s})_j$ of finite sequences of excursions
M_Ψ, $M_\square$, $M_{\square_1}$ of elements of $\mathcal{W}_\infty$
$\mathcal{B}$, $\mathcal{A}$ of finite set of strings ($\mathcal{B}$ represents the set of levels where the starts took place, and $\mathcal{A}$ the genealogy of the path w).
Initialization
$s := s_0 + \ell_0$, esslevel $(s) := s_0 + \ell_0$
$M_\Psi := [B^{s_0+\ell_0}_{|_{[0,t_d]}} = \widetilde{\Gamma}^{\ell_0}_{|_{[0,t_d]}}]$

For all s, do :
$(e_j^{*s})_j := (ep_j^s)_j \vee (ec_j^s)_j$
(the first ones are the protecting excursions and the second the building excursions).
$Y_s^{d+1} := (sgn(e_j^{+s})_j)$, $n_s(w) = \sharp(e_j^{+s})_j$.

End for.

If $w \notin M_\Psi$, do triv $(w, s_0 + \ell_0)$

End if.

While no zero of w^s occurs on (t_d, t_{d+1}), do
$(e_j^{+s})_j := (e_{t_j^{ps}}(w^s))_j$
$Y_{<s>}^{d+1} := \lambda_s$
$s := s - 1$, esslevel $(s) := s - 1$

End while.

(It is the end of the first stage, we have encountered the first zero.)

do $\square := s - s_0$, $M_\square := M_\Psi \cap [\exists u \in (t_d, t_{d+1}), B_u^s = 0]$
level$(M_\square) := s$, $\mathcal{A} := \{s - s_0\}$, $\mathcal{B} := \{s\}$, $\sharp(s) := 1$
Label 1
do : $(ep_j^s)_j :=$ arranged $(e_{t_j^{p\,esslevel(s)}}(w^s))_j$, $\mathcal{B} := \mathcal{B} \cup \{\square\}$, $\sharp(s) := 1 + \sharp(s-1)$

$(ec_j^s)_j := (e_{t_i^{c_{N_s}}}(w^s))_i$
$\sigma := s$.
For $i = 0$, to $K_\sigma - 1$, do :
$(e_j^{+s})_j$ is the sequence associated to $(e_j^{*s})_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "the plug excursions and the excursion beginning at t_i^s are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

let $(e_j^{*s-1})_j$ be associated to $(e_j^{+s})_j$ by lemma 5
do :
$(e_j^{*s-1})_j := (e_j^{*s-1})_j \vee (e_{t_i^\sigma}(w^{s-1}))$
do $\square_1 := \square$, $\square := \square h$ (h like horizontal)
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$,
$s := s - 1$, esslevel$(s) := esslevel(\sigma)$, $i := i + 1$, $\sharp(s) := \sharp(\sigma)$

End for.

(It is the end of the first phase in the second stage : the excursion straddling t_{d+1} is planed.)

For $i = 0$ to $N_\sigma - 1$, do :
$(e_j^{+s})_j$ is associated with $(e*_j^s)_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "the plug excursions and the first building excursion are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

let $(e_j^{*s-1})_j$ be the sequence associated to $(e_j^{+s})_j$ by lemma 5
$\square_1 := \square$, $\square := \square h$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$
$s := s - 1$, esslevel$(s) := esslevel(\sigma)$, $i := i + 1$, $\sharp(s) := \sharp(\sigma)$

End for.

(It is the end of the second phase of the second stage : the excursion straddling t_{d+1} has now the desired value.)

do $(e_j^{+s})_j$ is the sequence associated with $(e_j^{*s})_j$ by lemma 4
$Y_{<s>}^{d+1} := \lambda_{esslevel(\sigma)-s_0}$
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$, (v like vertical)
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$, $\sharp(s) := \sharp(\sigma)$
while no zero of w^s occurs on (t_d, t_{d+1}) and esslevel $(s) > s_0$, do :
$s := s - 1$, esslevel(s) :=esslevel$(s+1) - 1$, $\sharp(s) := \sharp(\sigma)$
$(e_j^{+s})_j :=$ arranged $\left(e_{t_j^{p\ esslevel(s)(w^s)}}\right)_j$
$Y_{<s>}^{d+1} := \lambda_{esslevel(s)-s_0}$
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$

End while.

(At this stage, the configuration process must return to its debut.)

if esslevel$(s) \geq s_0$, go to **Label 1**

End if.

(the horizontal raises have correctly configured the path if no following vertical raise vanishes on (t_d, t_{d+1}), else the procedure must return to the beginning).
do $R(w) := s$, $s := s - 1$, esslevel(s) :=essmevel$(s+1) - 1$
($R(w)$ represents the level from which the path takes a good form. The following raises have nothing to do but protecting this form.)
while $s > 0$, let :
$(e_j^{*s})_j$ be the sequence associated to $(e_j^{+(s+1)})_j$ by lemma 5
$(e_j^{+s})_j$ be the sequence associated to $(e_j^{*s})_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "all the plug excursions are < 0, the others > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$
$s := s - 1$, esslevel(s) :=esslevel$(s+1) - 1$, $\sharp(s) := \sharp(s+1)$

End while.

let $(e_j^{*0})_j$ be the sequence associated to $(e_j^{+1})_j$ by lemma 5
$(e_j^{+0})_j := (e_j^{*0})_j$
$Y_{<0>}^{d+1} := \lambda_0$
if $\varphi(t_d)\varphi(t_{d+1}) < 0$, do : $(e_j^{+0})_j := (e_j^{+0})_j \vee (e_{t_{d+1}(w^0)})$
$Y_{<0>}^{d+1} := Y_{<0>}^{d+1} \vee (sgn(\varphi(t_{d+1})))$
if $Y_0^{d+1} \neq Y_{<0>}^{d+1}$ do triv $(w, 0)$

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_0^{d+1} = Y_{<0>}^{d+1}]$, level$(M_\square) := 0$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$

End prog.

function triv(w, s)
for $i = s - 1$ to 0, do :
$(e_j^{+i}) := \emptyset$
$Y_{<i>}^{d+1} := \emptyset$

End function.

function arranged $(e_\alpha(\varphi))_{\alpha \in A}$
where $\varphi \in W$, and $(e_\alpha(\varphi))_{\alpha \in A}$ is a finite family of excursions of φ.
let g_α be the beginning of e_α
G the set of all the g_α of the finite family
γ the end of G, N its cardinality
$\Gamma := G$, $\gamma :=$ the end of Γ
while $\sharp(\Gamma) > 0$, do :
$i := N - \sharp(\Gamma) + 1$
$e_i :=$"the excursion of the family which begins at γ"
$\Gamma := \Gamma \backslash \{\gamma\}$

End while.

do arranged $(e_\alpha(\varphi))_{\alpha \in A} := (e_1, e_2, ..., e_N)$

End function.

function $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B}$
do $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B} :=$ arranged $(e_\gamma(\varphi))_{\gamma \in A \cup B}$

End function.

Now choose for all s, $\varepsilon'_s = 2\beta = \beta' := \frac{\varepsilon}{16} = \varepsilon'$.
Namely :

- ε' for the planning
- 2β for the estimation of the desired value at the level of the last start of the configuration process.
- β' for the margin of the building excursions
- and a last ε' for the total effect due to the plug excursions.

The parameters we have introduced during the whole construction are of two types : the preceding ones are acting only during the first ℓ_0 vertical raises, while the others concern only the plug excursions and are acting during all the raises. We don't choose the second ones before determining s_0.
(See $A_0^{varepsilon}$ we have a margin of $\varepsilon_2(w)$ for all the t_i, $1 \leq i \leq d$, and during the algorithm, four occasions of loosing precision because of the various starts of the configuration process).
To create this programm we must suppose s_0 finite. We are now going to prove that it is possible.
We begin by choosing a finite part Π of N_{d+1}, such that :

$$P(A_3^\varepsilon) > 1 - 4\frac{\varepsilon}{6}$$

where

$$A_3^\varepsilon := A_2^\varepsilon \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup} (H_{\lambda,\delta,n}) \right)$$

Then we consider that each r.v. $R^{\lambda,\delta,n}$ is $\mathbb{Z}$-valued. (It suffices momentarily to work in the filter probability space of $(B^n)_{n\in\mathbb{Z}}$).
So, because $s_0+\ell_0-R^{\lambda,\delta,n}$ doesn't depend on s_0, there exists a natural integer s_0 such that :

$$P(A_4^\varepsilon) > 1 - 5\frac{\varepsilon}{6}$$

where

$$A_4^\varepsilon := A_3^\varepsilon \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup} \left(H_{\lambda,\delta,n} \cap [R^{\lambda,\delta,n} \geq 0]\right) \right)$$

From now on, s_0 is thus determined.
Otherwise, the r.v. $n_s^{\lambda,\delta,n}(w)$ being $\mathbb{N}$-valued, there exists a sequence of deterministic integers $(n_s^0)_{0\leq s\leq r_{d+1}}$ verifying :

$$P(A_5^\varepsilon) \geq 1 - \varepsilon$$

where

$$A_5^\varepsilon := A_4^\varepsilon \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup} \left(H_{\lambda,\delta,n} \cap \cap_{s=0}^{s_0+\ell_0} (n_s^{\lambda,\delta,n} \leq n_s^{0,s_0}) \right) \right).$$

This enables us to bring the r.v. $Y^{d+1}_{\lambda,\delta,n,s}$ at the cardinality n_s^0+1 :

- If necessary, we complete the $(e_j^{+s})_{0\leq j\leq n_s^{\lambda,\delta,n}}$ to $(e_j^{+s})_{0\leq j\leq n_s^0}$ by adding excursions of w^s with beginnings after time t_{d+1} and chosen $\mathcal{W}_\infty^{s+1}$-measurably.
- Else, by truncation, we preserve only the first n_s^0+1 excursions.

We do similarly with the $Y^{d+1}_{\lambda,\delta,n,<s>}$ by adding as many $+1$ as necessary, or by truncation.
So modified, the programm yields now r.v. $(Y^{d+1}_{\lambda,\delta,n,s})$ and $(Y^{d+1}_{\lambda,\delta,n,<s>})$ which satisfy all the conditions of proposition 1. In other words, the construction of the disjointed sum of $B - m.r.(r_{d+1})$ is achieved. But, with arbitrary parameters of the second type, it doesn't fulfill the requirements we need. So we are going to define the ε_s^* in the following manner :
For all s from 0 to $s_0+\ell_0$,

$$\varepsilon_s^* := \frac{\varepsilon}{2.2^n \left(s_0 + \sum_{i=s_0+1}^{s_0+\ell_0} i \right)} \wedge \frac{\varepsilon'}{\sum_{j=1}^{r_{d+1}} j}$$

- The first term is intended to allow the path to remain close to φ at precision ε on $[0,t_d]$. For the path, from level $(s_0+\ell_0)$ is going to sustain ℓ_0 vertical raises during each of which it will receive a negative excursion, itself affected by as many plug excursions as horizontal raises soon accomplished. It is also going to sustain s_0 horizontal raises with each one plug excursion. The factor of 2.2^n at the denominator represents an upper bound for the total number of plug excursions which possibly affected the path from level $s_0+\ell_0$ to level 0. And, on $H_{\lambda,\delta,n}$, $\frac{\varepsilon}{2^{n+1}}$ is a number lower than $\varepsilon_2(w)$, the maximal total variation admissible for the path.

- The second term is intended to allow the path to approach at time t_{d+1} to $\varphi(t_{d+1})$ at precision $2\beta'$. The n_s^0 building excursions being affected by as many plug excursions as their intervention order, the following raises are horizontal and everyone brings one plug excursion more. So at most $(s_0+\ell_0-n_s^0)$ supplementary plug excursions.

We are now in a position to state the

Lemma 12.
With the preceding choice of ε_s^*, *we have, for all* s *from* 0 *to* $s_0+\ell_0$, $(\lambda,\delta,n)\in\Pi$ *and* $w\in A_5^\varepsilon\cap H_{(\lambda,\delta,n)}$ *:*

$$\|\Gamma^{\lambda,\delta,n}_{(t_1,...,t_{d+1})}(w)-\varphi(t_1,...,t_{d+1})\|_\infty<\varepsilon$$

Proof of lemma 12.
Let $\square\in B_{\lambda,\delta,n}$, $s:=\text{level}M_\square$, $w\in M_\square\cap\left(\cap_{i=0}^{K_s-1}[Y_{s-i}^{d+1}=Y_{<s-i>}^{d+1}]\right)$.
With the help of lemma 11, it is easily verified inductively that t_i^s is the beginning of the last excursion of w^{s-i}, before t_{d+1}, and the excursion of w^{s-K_s} straddling t_{d+1} is less high than ε'.
Then, the height of the excursion straddling t_{d+1} increases on the one hand, of $\sum_{i=1}^{N_s}h_i$ and on the other hand, of $\sum_{i=1}^{N_s}i=\frac{1}{2}N_s(N_s+1)$ plug excursions. Hence derives our choices of ε_s^* and ε' : so $\sum_{i=0}^{N_s}h_i\le|\Gamma(t_{d+1})|\le\sum_{i=0}^{N_s}h_i+2\varepsilon'+2\beta+2\beta'$, which entails :

$$|\varphi(t_{d+1})|-4\varepsilon'\le|\Gamma(t_{d+1})|\le|\varphi(t_{d+1})|+4\varepsilon'$$

which entails

$$|\Gamma(t_{d+1})-\varphi(t_{d+1})|\le4\epsilon'\le\frac{\varepsilon}{4}$$

For the values of Γ at times t_i, for all i from 1 to d, they differ from $\widetilde{\Gamma}(t_i)$ of at most : $\varepsilon_2(w)$ (see the first term in ε_s^*), so :

$$|\Gamma(t_i)-\varphi(t_i)|<\varepsilon$$

■

This achieves the proof of the proposition.

■

4 Density of orbits

In this paragraph, we want the raised path to approach the map φ uniformly on $[0,1]$. The additional difficulty is twofold : first, to protect what has already been obtained *on the whole* segment $[0,t_d]$; secondly, to correctly configure the path *on the whole* segment $[t_d,t_{d+1}]$. Only a more precise analysis of the Lévy Transform, obtained at lemma 7, enables a fruitful utilization of the methods settled in the preceding paragraphs. Through a best knowledge of raises, we can show at lemma 4 that the disjointed sum of $B-m.r.$ furnished by the algorithm come up to the requirements of the following :

Proposition 4.
Whatever ε strictly positive, and $\varphi \in W_{|_{[0,1]}}$, there exists a disjointed sum Γ of $B-m.r.$ such that :

$$P\left(\|\Gamma_{|_{[0,1]}} - \varphi_{|_{[0,1]}}\|_\infty < \varepsilon\right) > 1-2\varepsilon$$

Proof of Proposition 4.
We consider a modulus of uniform continuity α_0 associated to $\left(\frac{\varepsilon}{4},\varphi,[0,1]\right)$ and a real number α_1 such that $P(A_{0\,\varepsilon}) > 1-\frac{\varepsilon}{2}$ where

$$A_{0\,\varepsilon} = \left[\sup\{|B_t-B_u|,\ (t,u)\in[0,1]^2 \text{ and } |t-u|<\alpha_1\} < \frac{\varepsilon}{2}\right] ,$$

then we set $\alpha := \min(\alpha_0,\alpha_1)$, $d_0 := \left[\frac{1}{\alpha}\right]+1$, and for all $d\in\mathbb{N}$, $t_d = (d\alpha)\wedge 1$. We set again, for all integer $d\in[1,...,d_0]$,

$$A^d_\varepsilon := \left[\sup\{|B_t-B_u|,\ (t,u)\in[t_d,1]^2,\ |t-u|<\alpha\} < \frac{\varepsilon}{2}\right] .$$

Our aim is to show, by induction on d, the following property $\mathcal{P}_d(\varepsilon)$: "There exists an integer r_d and a disjointed sum Γ of $B-m.r.(r_d)$ such that :

$$\mathbb{P}\left(\left[\|\Gamma_{|_{[0,t_d]}} - \varphi_{|_{[0,t_d]}}\|_\infty < \varepsilon\right] \cap [|\Gamma_{t_d}-\varphi(t_d)|<\varepsilon_1] \cap A^d_\varepsilon\right) > 1-\varepsilon\left(1+\frac{d}{d_0}\right) \text{ "}$$

Notice that $\mathcal{P}_0(\varepsilon)$ immediately yields from the choice of α_1. We suppose now $\mathcal{P}_d(\varepsilon)$ true. We are going to apply this hypothesis to the Brownian motion B^{s_0}, for an integer s_0 which, as the real number ε_1, will be later specified.

As $A_\varepsilon^d \subset \left[\sup\{|B_t - B_u|,\, (t,u) \in [t_d, t_{d+1}]^2\} < \frac{\varepsilon}{2}\right] \cap A_\varepsilon^{d+1}$, and from the independence of the increments of Brownian motion, we can deduce the existence of a disjointed sum $\widetilde{\Gamma}$ of $\mathrm{B}^{s_0} - m.r.(r_d)$ such that :

$$P(A_0^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right)\ ,$$

where

$$\begin{aligned} A_0^\varepsilon \ :=\ & \left[\|\widetilde{\Gamma}_{|[0,t_d]} - \varphi_{|[0,t_d]}\|_\infty < \varepsilon\right] \cap \left[|\widetilde{\Gamma}_{t_d} - \varphi(t_d)| < \varepsilon_1\right] \quad ... \\ & ... \quad \cap \left[\sup\{|B_t^{s_0+r_d} - B_u^{s_0+r_d}|,\, (t,u) \in [t_d, t_{d+1}]^2\} < \frac{\varepsilon}{2}\right] \cap A_\varepsilon^{d+1} \end{aligned}$$

We will denote : $\forall i \in \mathbb{N}$, $\widetilde{\Gamma}^i = \widetilde{w}^i$. By definition, $\forall i > r_d$, $\tilde{w}^i = w^{s_0+i}$. Then :

$$\varepsilon_2(w) := \left(\varepsilon - \left\|\widetilde{\Gamma}_{|[0,t_d]} - \varphi_{|[0,t_d]}\right\|_\infty\right) \wedge \left(\varepsilon_1 - |\widetilde{\Gamma}_{t_d} - \varphi(t_d)|\right)$$

This r.v. is strictly positive.
Otherwise, from the theorem of density of zeroes ([M]), there exists a.s. an integer ℓ such that $\widetilde{w}^\ell$ vanishes at least one time on $[t_d, t_{d+1}]$.
Let $L(w)$ be the smallest of these integers ℓ. L is a r.v. almost surely finite. Then there exists an integer ℓ_0 which we will choose $> r_d$ such that :

$$P(A_1^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{\varepsilon}{5d_0} \quad \text{où} \quad A_1^\varepsilon := A_0^\varepsilon \cap [L \leq \ell_0]\,.$$

Our aim is to raise the path $\widetilde{w}^{\ell_0}$, which is nothing but $w^{s_0+\ell_0}$ from level $s_0+\ell_0$ to level 0, in such a way that the raised path approaches φ on $[t_d, t_{d+1}]$ while remaining at the proximity of φ on $[0, t_d]$ acquired at level s_0, with usual measurability condition. For this to be realized, without damage, we will introduce the family of protecting excursions of $\widetilde{w}^k$, $(\widetilde{ep}_j^{(k)})_{1 \leq j \leq p_k}$, constituted by the excursions of $\widetilde{w}^k$ with beginning before t_d and height greater than $\frac{\varepsilon_2(w)}{2\ell_0}$ and arranged in the reverse chronological order, for all k from 0 to ℓ_0.
The p_k, $0 \leq k \leq \ell_0$, are finite r.v. Let $(p_k^0)_{0 \leq k \leq \ell_0}$ be a sequence of deterministic integers verifying :

$$P(A_2^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{2\varepsilon}{5d_0}$$

where

$$A_2^\varepsilon := A_1^\varepsilon \cap \left(\bigcap_{k=0}^{\ell_0} [p_k \leq p_k^0]\right).$$

We set :

$$\Lambda := \{(\lambda_k)_{0\le k\le \ell_0}\}\,,\ \lambda_k \in \{-1,+1\}^{\{0,\dots,p_k^0\}} = \prod_{k=0}^{\ell_0}\{-1,+1\}^{\{0,\dots,p_k^0\}}$$
$$\Delta := \{[n\beta,\,(n+1)\beta[\,;\ n\in\mathbb{Z}\}^{\{0,\dots,\ell_0\}}\,.$$

The partition (H_ν^{d+1}) from which we are going to construct Γ is so defined :

$N_{d+1} := \Lambda\times\Delta\times\mathbb{N}$ is denumerable;
$\forall(\lambda,\,\delta,\,n)\in\Lambda\times\Delta\times\mathbb{N}^*$,

$$H^{d+1}_{\lambda,\,\delta,\,n} := \bigcap_{k=0}^{\ell_0}\Big([(sgn(\widetilde{ep}_j^k))_{0\le j\le p_k^0}=\lambda_k]\ \cap\ [\tilde{w}^k_{t_d}\in\delta(k)]\Big) \cap \dots \cap \left[\tfrac{\varepsilon}{2n} < \varepsilon_2(w) \le \tfrac{\varepsilon}{2^{n-1}}\right]$$

$$H^{d+1}_{\lambda,\,\delta,\,0} := \bigcap_{k=0}^{\ell_0}\Big([(sgn(\widetilde{ep}_j^k))_{0\le j\le p_k^0}=\lambda_k]\ \cap\ [\tilde{w}^k_{t_d}\in\delta(k)]\Big)\cap[\varepsilon_2(w)\le 0]$$

$(H_{\lambda,\,\delta,\,0}\cap A_0^\varepsilon=\emptyset)$

Set : $b\delta(k)$, the beginning of the interval $\delta(k)$ and $\varepsilon(\lambda,k) := \prod_{i=0}^{k-1}\lambda_i(0)$.

Lemma 13. *(analog of lemma 10)*
On $H^{d+1}_{\lambda,\delta,n}$ the displacements $\tau_k^0(\widetilde{w})$ and $\tau_{b\delta(k)}^{b\delta(0)\ \varepsilon(\lambda,k)}$ differ from at most 2β.

Proof of lemma 13.
See the proof of lemma 10.

■

This lemma allows us to replace the value to be anticipated $\tau_k^0(\widetilde{w})(\varphi(t_{d+1}))$, which doesn't have the good measurability by $\tau_{b\delta(k)}^{b\delta(0)}(\varphi(t_{d+1}))$ on $H_{\lambda,\delta}$.
The purpose of the following lemma is to prepare, at level s, when the iterated Brownian motion vanishes on $]t_d,t_{d+1}[$, the excursions which will allow the correctly raised path to approach φ at level 0 on $]t_d,t_{d+1}[$. It is the analogous of lemma ?.
We set : $Z_s := \{w\in W\,/\,w^s vanishes at least one time on]t_d,t_{d+1}[\,\}$.

Lemma 14.
For all $w\in Z_s$, ε'_s being a $\mathcal{W}_\infty^{s+1}$-r.v. to be specified later, we have :
1) The following r.v. are functionals of $|w^s|_{|_{[t_d,t_{d+1}]}}$:

- *$t_0^s(w) := g_{t_{d+1}}(w^s)$, $m_0^s(w) := \sup\{|w_u^s|, u\in[t_0^s(w),t_{d+1}]\}$ is the beginning of the first excursion of w^s with support in $[t_d,t_{d+1}]$, whose height is greater than ε'_s, or t_{d+1} if this set is empty.*

- $\forall n \in \mathbb{N}$, *while* $t^s_n < t_{d+1}$, *we set :*

$$t^s_{n+1} := \begin{cases} \inf\{u \in [t^s_n; \arg\max |de^s_{t^s_n}|[, h(de^s_u) > \varepsilon'_s \text{ and } sgn(de^s_u) = -sgn(de^s_{t^s_n})\} \\ \textit{if this set is not empty,} \\ \textit{else :} \\ \arg\max |de^s_{t^s_n}| \wedge t_{d+1} \end{cases}$$

The sequence $(t^s_n)_{n\in\ mathbbN}$ *is strictly increasing and finite. Let* $1 + K^{\lambda,\delta,n}_s$ *its cardinality.*

2) Let $Q^(_p w)$ *be the property :* "there exists a sequence of excursions of w^s with supports in $[t_d, t^s_0]$, $(e^s_i)_{1\le i\le p}$, whose beginnings $(G^s_i)_i$ is a strictly decreasing sequence, with heights $(h^s_i)_i$ such that :

$$(6) \qquad \sum_{i=1}^{p} h^s_i \;<\; \left|\tau^{b\delta(0)\,\varepsilon(\lambda,k)}_{b\delta(esslevel(s))}\,(\varphi_{(t_{d+1})})\right| < \sum_{i=1}^{p} h^s_i + \beta' \qquad "$$

where the r.v. esslevel(s) *will soon be defined, and* $w \in H^{\lambda,\delta,n}$.
Let $N_s(w)$ *be the smallest integer* p *such that* $Q_p(w)$ *is true when* $w \in Z_s$, *else* 0.
Then N_s *is an a.s. finite* $\mathcal{W}^{s+1}_\infty$*-measurable r.v.*

3) For any finite $\mathcal{W}^{s+1}_\infty$*-measurable integral valued r.v.* N', *on the set* $[N' \ge N]$, *then there exists a sequence* $(e^{s,N'}_i)_i$ *which verifies the requirements of* $Q_{N'}(w)$ *and whose beginnings* $(G^{s,N'}_i)_{1\le i\le N'}$ *is a decreasing sequence of* $\mathcal{W}^{s+1}_\infty$*-measurable r.v.*

Proof of lemma 14.
1) By construction, the sequence (t^s_n) is strictly increasing and lower than t_{d+1}. Suppose the number of its terms is infinite. In this case, it would admit a limit $t^s_* \le t_{d+1}$, and the oscillation of w^s at t^s_* would be infinite, so contradicting the continuity of w^s. Then $(t^s_n)_n$ is finite.
The measurability and the finiteness of K_s are immediate.
The remaining of the proof follows the same way as in the proof of lemma ?.

■

Now, with the help of lemmas 4 and 5, we are in a position to take on the definition of the families of r.v. $(Y^{d+1}_{\lambda,\delta,n,s})_{0\le s\le s_0+\ell_0}$ and $(Y^{d+1}_{\lambda,\delta,n,<s>})_{0\le s\le s_0+\ell_0}$. As in the preceding paragraph, this definition is based on the following algorithm, which will conveniently call the lemmas 4, 5, 14 at appropriate times. We

reproduce it, but it is exactly the same as this of the preceding paragraph. Before beginning to write the programm, let us introduce the notations :

- t_j^{p,s_0+k} designates the beginning of $\widetilde{ep}_j^{(k)}$, for all j from 0 to p_0^k.
- t_j^{c,N_s} designates the beginning of e_i^{s,N_s}, for all i from 0 to N_s.

The algorithm goes from level $r_{d+1} := s_0+\ell_0$. In the first stage, i.e. before the $B - m.r.(r_{d+1})$ vanishes on $]t_d, t_{d+1}[$, we just impose the signs of the protecting excursions $(\widetilde{ep}_j^s)$, namely the excursions of w^s of sufficient height, which are straddling times $t_j^{p,s}$. Thus we arrive at level s when the $B - m.r.(r_{d+1})$ vanishes : the second stage is beginning. For the first time, we put in action the configuration process. We start the $K_s + N_s$ horizontal raises : the first K_s ones ought to plane the excursions straddling t_{d+1}; then the building excursions which were protected during the preceding phase are put to act, one-by-one, during the following N_s raises. So at level $s - K_s - N_s$, the path has nearly the same form on $[0, t_d]$ than at level s, and approaches the s-level desired value at time t_{d+1}. Therefore we qualify these $K_s + N_s$ raises *horizontal* ones. And we say that the essential level corresponding to level $s - K_s - N_s$ is s. Now, it's the first stage. It is composed of vertical raises : the desired value at time t_{d+1} supposes that the path doesn't vanish on $[t_d, t_{d+1}]$, because of lemma 6, if it is the case, we raise the path. If it is not the case, we must stop, and begin again a new configuration process, and so on.
The program to follow is exactly the same as this of paragraph 3.

Variables
s, esslevel(s), $\sharp(s)$, n_s, K_s, N_s, R, level(M) (where $M \in \mathcal{W}_\infty$) of integers ($\sharp(s)$ represents the number of starts of the configuration process before level s).
$\square$, $\square_1$ of strings
Y_s^{d+1}, $Y_{<s>}^{d+1}$ of finite sequences of signs.
$(ep_j^s)_j$, $(ec_j^s)_j$, $(e_j^{*s})_j$, $(e_j^{+s})_j$ of finite sequences of excursions
M_Ψ, $M_\square$, $M_{\square_1}$ of elements of $\mathcal{W}_\infty$
$\mathcal{B}$, $\mathcal{A}$ of finite set of strings ($\mathcal{B}$ represents the set of levels where the starts took place, and $\mathcal{A}$ the genealogy of the path w).
Initialization
$s := s_0 + \ell_0$, esslevel $(s) := s_0 + \ell_0$
$M_\Psi := [B^{s_0+\ell_0}_{|_{[0,t_d]}} = \widetilde{\Gamma}^{\ell_0}_{|_{[0,t_d]}}]$

For all s, do :
$(e_j^{*s})_j := (ep_j^s)_j \vee (ec_j^s)_j$
(the first ones are the protecting excursions and the second the building excursions).
$Y_s^{d+1} := (sgn(e_j^{+s})_j)$, $n_s(w) = \sharp(e_j^{+s})_j$.

End for.

If $w \notin M_\Psi$, do triv $(w, s_0 + \ell_0)$

End if.

While no zero of w^s occurs on (t_d, t_{d+1}), do
$(e_j^{+s})_j := (e_{t_j^{ps}}(w^s))_j$
$Y_{<s>}^{d+1} := \lambda_s$
$s := s - 1$, esslevel $(s) := s - 1$

End while.

(It is the end of the first stage, we have encountered the first zero.)

do $\square := s - s_0$, $M_\square := M_\Psi \cap [\exists u \in (t_d, t_{d+1}), B_u^s = 0]$
level$(M_\square) := s$, $\mathcal{A} := \{s - s_0\}$, $\mathcal{B} := \{s\}$, $\sharp(s) := 1$
Label 1
do : $(ep_j^s)_j :=$ arranged $(e_{t_j^{p\,esslevel(s)}}(w^s))_j$, $\mathcal{B} := \mathcal{B} \cup \{\square\}$, $\sharp(s) := 1 + \sharp(s-1)$

$(ec_j^s)_j := (e_{t_i^{c_{N_s}}}(w^s))_i$
$\sigma := s$.
For $i = 0$, to $K_\sigma - 1$, do :
$(e_j^{+s})_j$ is the sequence associated to $(e_j^{*s})_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "the plug excursions and the excursion beginning at t_i^s are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

let $(e_j^{*s-1})_j$ be associated to $(e_j^{+s})_j$ by lemma 5
do :
$(e_j^{*s-1})_j := (e_j^{*s-1})_j \vee (e_{t_i^\sigma}(w^{s-1}))$
do $\square_1 := \square$, $\square := \square h$ (h like horizontal)
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$,
$s := s - 1$, esslevel$(s) := esslevel(\sigma)$, $i := i + 1$, $\sharp(s) := \sharp(\sigma)$

End for.

(It is the end of the first phase in the second stage : the excursion straddling t_{d+1} is planed.)

For $i = 0$ to $N_\sigma - 1$, do :
$(e_j^{+s})_j$ is associated with $(e*_j^s)_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "the plug excursions and the first building excursion are < 0, all the other excursions of $(e_j^{+s})_j$ are > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

let $(e_j^{*s-1})_j$ be the sequence associated to $(e_j^{+s})_j$ by lemma 5
$\square_1 := \square$, $\square := \square h$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$
$s := s - 1$, esslevel$(s) := esslevel(\sigma)$, $i := i + 1$, $\sharp(s) := \sharp(\sigma)$

End for.

(It is the end of the second phase of the second stage : the excursion straddling t_{d+1} has now the desired value.)

do $(e_j^{+s})_j$ is the sequence associated with $(e_j^{*s})_j$ by lemma 16
$Y_{<s>}^{d+1} := \lambda_{esslevel(\sigma)-s_0}$
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$, (v like vertical)
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$, $\sharp(s) := \sharp(\sigma)$
while no zero of w^s occurs on (t_d, t_{d+1}) and esslevel $(s) > s_0$, do :
$s := s - 1$, esslevel(s) :=esslevel$(s+1) - 1$, $\sharp(s) := \sharp(\sigma)$
$(e_j^{+s})_j :=$ arranged $\left(e_{t_j^{p\ esslevel(s)(w^s)}}\right)_j$
$Y_{<s>}^{d+1} := \lambda_{esslevel(s)-s_0}$
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$

End while.

(At this stage, the configuration process must return to its debut.)

if esslevel$(s) \geq s_0$, go to **Label 1**

End if.

(the horizontal raises have correctly configured the path if no following vertical raise vanishes on (t_d, t_{d+1}), else the procedure must return to the beginning).
do $R(w) := s$, $s := s - 1$, esslevel(s) :=essmevel$(s+1) - 1$
($R(w)$ represents the level from which the path takes a good form. The following raises have nothing to do but protecting this form.)
while $s > 0$, let :
$(e_j^{*s})_j$ be the sequence associated to $(e_j^{+(s+1)})_j$ by lemma 5
$(e_j^{+s})_j$ be the sequence associated to $(e_j^{*s})_j$ by lemma 4
$Y_{<s>}^{d+1}$:= "all the plug excursions are < 0, the others > 0"
If $Y_s^{d+1} \neq Y_{<s>}^{d+1}$ do triv (w, s)

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_s^{d+1} = Y_{<s>}^{d+1}]$, level$(M_\square) := s$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$
$s := s - 1$, esslevel(s) :=esslevel$(s+1) - 1$, $\sharp(s) := \sharp(s+1)$

End while.

let $(e_j^{*0})_j$ be the sequence associated to $(e_j^{+1})_j$ by lemma 5
$(e_j^{+0})_j := (e_j^{*0})_j$
$Y_{<0>}^{d+1} := \lambda_0$
if $\varphi(t_d)\varphi(t_{d+1}) < 0$, do : $(e_j^{+0})_j := (e_j^{+0})_j \vee (e_{t_{d+1}(w^0)})$
$Y_{<0>}^{d+1} := Y_{<0>}^{d+1} \vee (sgn(\varphi(t_{d+1})))$
if $Y_0^{d+1} \neq Y_{<0>}^{d+1}$ do triv $(w, 0)$

End if.

do $\square_1 := \square$, $\square := \square v$
$M_\square := M_{\square_1} \cap [Y_0^{d+1} = Y_{<0>}^{d+1}]$, level$(M_\square) := 0$, $\mathcal{A} := \mathcal{A} \cup \{\square\}$

End prog.

function triv(w, s)
for $i = s - 1$ to 0, do :
$(e_j^{+i}) := \emptyset$
$Y_{<i>}^{d+1} := \emptyset$

End function.

function arranged $(e_\alpha(\varphi))_{\alpha \in A}$
where $\varphi \in W$, and $(e_\alpha(\varphi))_{\alpha \in A}$ is a finite family of excursions of φ.
let g_α be the beginning of e_α
G the set of all the g_α of the finite family
γ the end of G, N its cardinality
$\Gamma := G$, $\gamma :=$ the end of Γ
while $\sharp(\Gamma) > 0$, do :
$i := N - \sharp(\Gamma) + 1$
$e_i :=$"the excursion of the family which begins at γ"
$\Gamma := \Gamma \backslash \{\gamma\}$

End while.

do arranged $(e_\alpha(\varphi))_{\alpha \in A} := (e_1, e_2, ..., e_N)$

End function.

function $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B}$
do $(e_\alpha(\varphi))_{\alpha \in A} \vee (e_\beta(\varphi))_{\beta \in B} :=$ arranged $(e_\gamma(\varphi))_{\gamma \in A \cup B}$

End function.

Now choose for all s, $\varepsilon'_s = 2\beta = \beta' := \frac{\varepsilon}{16} = \varepsilon'$.
Namely :

- ε' for the planing
- 2β for the estimation of the desired value at the level of the last start of the configuration process.
- β' for the margin of the building excursions
- and a last ε' for the total effect due to the plug excursions.

The parameters we have introduced during the whole construction are of two types : the preceding ones are acting only during the first ℓ_0 vertical raises, while the others concern only the plug excursions and are acting during all the raises. We don't choose the second ones before determining s_0.
(See $A_0^{varepsilon}$ we have a margin of $\varepsilon_2(w)$ for all the t_i, $1 \leq i \leq d$, and during the algorithm, four occasions of loosing precision because of the various starts of the configuration process).
To create this programm we must suppose s_0 finite. We are now going to prove that it is possible.
We begin by choosing a finite part Π of N_{d+1}, such that :

$$P(A_3^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{3\varepsilon}{5d_0}$$

where

$$A_3^\varepsilon := A_2^\varepsilon \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup}(H_{\lambda,\delta,n})\right)$$

Then we consider that each r.v. $R^{\lambda,\delta,n}$ is $\mathbb{Z}$-valued. (It suffices momentarily to work in the filter probability space of $(B^n)_{n\in\mathbb{Z}}$).
So, because $s_0+\ell_0-R^{\lambda,\delta,n}$ doesn't depend on s_0, there exists a natural integer s_0 such that :

$$P(A_4^\varepsilon) > 1 - \varepsilon\left(1 + \frac{d}{d_0}\right) - \frac{4\varepsilon}{5d_0}$$

where

$$A_4^\varepsilon := A_3^\varepsilon \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup}\left(H_{\lambda,\delta,n} \cap [R^{\lambda,\delta,n} \geq 0]\right)\right)$$

From now on, s_0 is thus determined.
Otherwise, the r.v. $n_s^{\lambda,\delta,n}(w)$ being $\mathbb{N}$-valued, there exists a sequence of deterministic integers $(n_s^0)_{0\leq s\leq r_{d+1}}$ verifying :

$$P(A_5^\varepsilon) \geq 1 - \varepsilon\left(1 + \frac{d+1}{d_0}\right)$$

where

$$A_5^{\varepsilon} := A_4^{\varepsilon} \cap \left(\underset{(\lambda,\delta,n)\in\Pi}{\cup} \left(H_{\lambda,\delta,n} \cap \cap_{s=0}^{s_0+\ell_0} (n_s^{\lambda,\delta,n} \leq n_s^{0,s_0}) \right) \right).$$

This enables us to bring the r.v. $Y_{\lambda,\delta,n,s}^{d+1}$ at the cardinality $n_s^0 + 1$:

- If necessary, we complete the $(e_j^{+s})_{0\leq j\leq n_s^{\lambda,\delta,n}}$ to $(e_j^{+s})_{0\leq j\leq n_s^0}$ by adding excursions of w^s with beginnings after time t_{d+1} and chosen $\mathcal{W}_\infty^{s+1}$-measurably.
- Else, by truncation, we preserve only the first $n_s^0 + 1$ excursions.

We do similarly with the $Y_{\lambda,\delta,n,<s>}^{d+1}$ by adding as many $+1$ as necessary, or by truncation.
Thus modified, the programm yields now r.v. $(Y_{\lambda,\delta,n,s}^{d+1})$ and $(Y_{\lambda,\delta,n,<s>}^{d+1})$ which satisfy all the conditions of proposition 1. In other words, the construction of the disjointed sum of $B - m.r.(r_{d+1})$ is achieved. But, with arbitrary parameters of the second type, it doesn't fulfill the requirements we need. So we are going to define the ε_s^* in the following manner :
For all s from 0 to $s_0 + \ell_0$,

$$\varepsilon_s^* := \frac{\varepsilon}{2.2^n \left(s_0 + \sum_{i=s_0+1}^{s_0+\ell_0} i \right)} \wedge \frac{\varepsilon'}{\sum_{j=1}^{r_{d+1}} j}$$

- The first term is intended to allow the path to remain close to φ at precision ε on $[0, t_d]$. For the path, since level $(s_0 + \ell_0)$ is going to sustain ℓ_0 vertical raises during each of which it will receive a negative excursion, itself affected by as many plug excursions as horizontal raises soon accomplished. It is going to sustain also s_0 horizontal raises with each one plug excursion. The factor of 2.2^n at the denominator represents an upper bound for the total number of plug excursions which possibly affected the path from level $s_0 + \ell_0$ to level 0. And, on $H_{\lambda,\delta,n}$, $\frac{\varepsilon}{2^{n+1}}$ is a number lower than $\varepsilon_2(w)$, the maximal total variation admissible for the path.

- The second term is intended to allow the path to approach at time t_{d+1} to $\varphi(t_{d+1})$ at precision $2\beta'$. The n_s^0 building excursions being affected by as many plug excursions as their intervention order, the following raises are horizontal and every one brings one plug excursion more. Thus a maximum of $(s_0 + \ell_0 - n_s^0)$ supplementary plug excursions.

We are now in a position to state the

Lemma 15.
With the choice of $\varepsilon_1 := \frac{\varepsilon}{4}$ *and the preceding choice of* ε_s^**, we have, for all* s *from* 0 *to* $s_0 + \ell_0$*,* $(\lambda, \delta, n) \in \Pi$ *and* $w \in A_5^\varepsilon \cap H^{(\lambda,\delta,n)}$ *:*

$$\left\| \Gamma^{\lambda,\delta,n}|_{[0,t_{d+1}]} - \varphi|_{[0,t_{d+1}]} \right\|_\infty \leq \varepsilon$$

and

$$\left| \Gamma^{\lambda,\delta,n}_{t_{d+1}}(w) - \varphi(t_{d+1}) \right| \leq \varepsilon_1$$

Proof of lemma 15.
Being given the choice of the ε_σ^*) and the help of lemma 14, it is easily shown by induction, that t_i^s is the beginning of the first excursion of w^{s-i} before t_{d+1}, with height greater than ε', $0 \leq i \leq K_s - 1$. Indeed, the intermediate excursions (those whose support lies in $[t_d, t_i^s]$ and height is greater than the height of the smallest plug excursion), during the K_s raises, being all positive, receive at each raise a plug excursion whose height has been estimated such that the height of the compound excursion never goes beyond ε'.
Thus, at the issue of the K_s first raises, the path between t_0^s and t_{d+1} has been planned, its height never overflowing ε'.
In the same manner, during the N_s following raises, for $w \in M_\square \cap (\cap_{i=0}^{K_s+N_s-1}[Y_{s-i}^{d+1} = Y_{<s-i>}^{d+1}])$, the excursion straddling t_{d+1} receives one after the other the building excursions, and so, succeeds in being close at time t_{d+1} to $\tau_{s-i}^0(\varphi(t_{d+1}))$, at the precision ε_1.
Then, the height of the excursion straddling t_{d+1} increases on the one hand, of $\sum_{i=1}^{N_s} h_i$ and on the other hand, of $\sum_{i=1}^{N_s} i = \frac{1}{2} N_s(N_s+1)$ plug excursions. Hence derive our choices of ε_s^* and ε' : so $\sum_{i=0}^{N_s} h_i \leq |\Gamma(t_{d+1})| \leq \sum_{i=0}^{N_s} h_i + 2\varepsilon' + 2\beta + 2\beta'$, which entails :

$$|\varphi(t_{d+1})| - 4\varepsilon' \leq |\Gamma(t_{d+1})| \leq |\varphi(t_{d+1})| + 4\varepsilon'$$

which entails

$$|\Gamma(t_{d+1}) - \varphi(t_{d+1})| \leq 4\epsilon' \leq \frac{\varepsilon}{4}$$

For the protecting excursions $(ep_j^{s-i})_{0 \leq j \leq p_{s-i}^0}$ their variation is controlled during the $K_s + N_s$ raises.

Otherwise, as the excursions of w^s they don't include are so low, that their intervention will not allow the path to move from φ for more than ε.
So now, we can claim that $\Gamma^{\lambda,\delta,n}$, for all $(\lambda,\delta,n) \in N_{d+1}$, is correctly defined as a $B-m.r.(r_{d+1})$, and that, for all $(\lambda,\delta,n) \in \Pi$, $w \in A_5^\varepsilon \bigcap H_{(\lambda,\delta,n)}$, $\Gamma^{\lambda,\delta,n}(w)$ approaches φ at precision ε on $[0,t_d]$, and at precision ε_1 at time t_{d+1}.
We are going now to analyze more in details its behavior on $[t_d,t_{d+1}]$. Let us denote γ_{t_d} the first time after t_d at which one of the w^σ, $0 \leq \sigma \leq s_0+\ell_0$, vanishes on $[t_d,t_{d+1}]$, and σ_0 the corresponding level.
Set $\sigma_1 := \sigma_0 - K_{\sigma_0} - N_{\sigma_0}$ the rectangle $Rect_{\sigma_1}$ defined by the four straight lines with equations :
$x = t_d$, $x = t_{d+1}$, $y = \inf \tau^{b\delta(0)\,\varepsilon(\lambda,esslevel(\sigma_0))}_{b\delta(esslevel(\sigma_0))}\ (\varphi)|_{[t_d,t_{d+1}]}$ *and*
$y = \sup \tau^{b\delta(0)\,\varepsilon(\lambda,esslevel(\sigma_0))}_{b\delta(esslevel(\sigma_0))}\ (\varphi)|_{[t_d,t_{d+1}]}$
$Rect_{\sigma_1}$ contains by definition the path of $\tau^{b\delta(0)\,\varepsilon(\lambda,esslevel(\sigma_0))}_{b\delta(esslevel(\sigma_0))}\ (\varphi)|_{[t_d,t_{d+1}]}$ and, from the choice of α_0, its height is lower than $\dfrac{\varepsilon}{4}$.
Now consider the path of $w^{\sigma_1}|_{[t_d,t_{d+1}]}$, it takes one of the two forms :
(diagram)

In the two cases by hypothesis, the total variation of $w^{s_0+r_d}$ is on $[t_d,t_{d+1}]$ lower than $\dfrac{\varepsilon}{2}$, and by lemma 6, and the definition of γ_{t_d}, it is equal to that of w^{σ_0}, on $[t_d,\gamma_{t_d}]$, the path w^{σ_1} can move again from $Rect_{\sigma_1}$ but at most from ε'. And rapidly, it is bound to join φ in $Rect_{\sigma_1}$ by the building excursions, the flat part remaining flat.
Therefore, the rectangle RR_{σ_1} with the same center and vertical straight lines bordering it, and height that of $Rect_{\sigma_1} + \varepsilon$, contains the path $w^{\sigma_1}|_{[t_d,t_{d+1}]}$.
During the following raises, the rectangle $Rect_{\sigma_1}$, according to lemma 6, moves by isometry. We call R_σ its new positions, and likewise RR_σ that of RR_{σ_1}.
We can easily check that, for all $\sigma > \sigma_1$ corresponding to a vertical raise, the path $w^\sigma|_{[t_d,t_{d+1}]}$ is contained in RR_σ.
Finally, for $w \in A_5^\varepsilon \cap H^{(\lambda,\delta,n)}$, at level 0 we have the desired property :

$$\left\|\Gamma^{\lambda,\delta,n}|_{[t_d,t_{d+1}]}(w) - \varphi|_{[t_d,t_{d+1}]}\right\|_\infty \leq\ \varepsilon$$

■

We immediately deduce that $\mathcal{P}_{d+1}(\varepsilon)$ is true. Hence the property $\mathcal{P}_d(\varepsilon)$ is established by induction, in particular for $d = d_0$, which achieves the proof of Proposition 5.

We denote the :

Theorem 1.
For almost every $w \in W$, the orbit of w

$$orb(w) = \{T^n w;\ n \in \mathbb{N}\}$$

is dense in W, equipped with the topology of uniform convergence on compact sets.

Proof of theorem 1.
We use the :

Lemma 16.
Let Γ be a disjointed sum of $B.m - r(r)$.
Then, $\forall F \in \mathcal{W}_\infty$, $P(F) > 0 \Rightarrow P(\Gamma(F)) > 0$.

Proof of lemma 16.
Clearly, we can suppose that Γ is a $B.m - r(r)$. Then we show, with the help of the definition, and by induction, that for all s from r to 0, $\Gamma^s(F)$ is not negligible.

■

Let us now reason by *reductio ad absurdum* :
Let $\varphi \in W|_{[0,1]}$ and $\beta > 0$.
Suppose $P(E) = \alpha > 0$ where $E = \left\{w \in W / o(w)|_{[0,1]} \cap B(\varphi, \beta) = \emptyset\right\}$.
Let $\varepsilon := \min\left(\frac{\alpha}{2}, \beta\right)$.
Then, with the notations of Proposition 4, we have :

$$P(F) > 0\ , \quad F := E \cap \left[||\ \Gamma|_{[0,1]} - \varphi|_{[0,1]}||_\infty < \varepsilon\right]$$

From lemma 16, $P(\Gamma(F)) > 0$.
But, E is invariant under T (immediate verification), and $\Gamma(F) \subset \Gamma(E) \subset T^{-r}T^r(E) = E[\text{mod}0]$.
Thus, $P(\Gamma(F) \cap E) > 0$, which is contradictory, because in $\Gamma(F)$, all the paths belong to $B(\varphi, \beta)$, while in E, the orbits of paths don't encounter $B(\varphi, \beta)$.

■

Let us notice that if, in place of restrain ourselves with the open sets B, we have shown :
$\forall B$ closed set in W,

$$P(B) > 0 \Rightarrow P(orb(w) \cap B \neq \emptyset) = 1$$

Then every set A T-invariant, measurable and not negligible, would contain the event $[orb(w) \cap B \neq \emptyset]$ and so, would be almost sure. Therefore, T would be ergodic.
To end, we are going to claim in an equivalent way, following thus an interesting suggestion of J.P Thouvenot :

$$\forall (\varphi, \varepsilon) \in W|_{[0,1]} \times \mathbb{R}_*^+ ,$$

the reverse martingale $P(w \in B(\varphi, \varepsilon)|\mathcal{W}_\infty^n)$ admits a regular conditional version $P(w \in B(\varphi, \varepsilon)|w^n)$, and we have :

Theorem 2.

$$P \text{ a.s.}, \lim_{n\to\infty} P(w \in B(\varphi, \varepsilon)|w^n) > 0$$

Proof of theorem 2.
Suppose the contrary, and let :

$$A := [w \in W, \lim_{n\to\infty} P(w \in B(\varphi, \varepsilon)|w^n = 0]$$

As $P(w \in B(\varphi, \varepsilon)|w^1 = P(w \in B(\varphi, \varepsilon)|w^{n+1}$, because T is measure-preserving.
So we have :

$$w \in A \Leftrightarrow Tw \in A$$

So A is T-invariant. Consequently :

$$E\left(1_A P(w \in B(\varphi, \varepsilon)|w^n)\right) = P\left(A \cap [w \in B(\varphi, \varepsilon)]\right) = P\left(A \cap [w^n \in B(\varphi, \varepsilon)]\right)$$

But by hypothesis :

$$\lim_{n\to\infty} E\left(1_A P(w \in B(\varphi, \varepsilon)|w^n)\right) = 0$$

Therefore,

$$P\left(A \cap [orb(w) \cap B(\varphi, \varepsilon) \neq \emptyset]\right) = 0$$

which, from theorem 1, entails that $P(A) = 0$

■

Finally, let us remark that, if we could show :

$$\lim_{n\to\infty} P\left(\left[w \in B(\varphi,\varepsilon)\right]|w^n\right) = P\left(\left[w \in B(\varphi,\varepsilon)\right]\right) \ ,$$

Than, not only T would be ergodic but exact which means :

$$\mathcal{W}_\infty^\infty := \underset{n\in\mathbb{N}}{\cap} \mathcal{W}_\infty^n \text{ would be trivial.}$$

■

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