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# NONPARAMETRIC ESTIMATION OF THE REGRESSION FUNCTION IN AN ERRORS-IN-VARIABLES MODEL

F. COMTE<sup>\*,1</sup> AND M.-L. TAUPIN<sup>2</sup>

**ABSTRACT.** We consider the regression model with errors-in-variables where we observe  $n$  i.i.d. copies of  $(Y, Z)$  satisfying  $Y = f(X) + \xi$ ,  $Z = X + \sigma\varepsilon$ , involving independent and unobserved random variables  $X, \xi, \varepsilon$ . The density  $g$  of  $X$  is unknown, whereas the density of  $\sigma\varepsilon$  is completely known. Using the observations  $(Y_i, Z_i)$ ,  $i = 1, \dots, n$ , we propose an estimator of the regression function  $f$ , built as the ratio of two penalized minimum contrast estimators of  $\ell = fg$  and  $g$ , without any prior knowledge on their smoothness. We prove that its  $\mathbb{L}_2$ -risk on a compact set is bounded by the sum of the two  $\mathbb{L}_2(\mathbb{R})$ -risks of the estimators of  $\ell$  and  $g$ , and give the rate of convergence of such estimators for various smoothness classes for  $\ell$  and  $g$ , when the errors  $\varepsilon$  are either ordinary smooth or super smooth. The resulting rate is optimal in a minimax sense in all cases where lower bounds are available.

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## 1. INTRODUCTION

We consider that we observe  $n$  independent and identically distributed (i.i.d.) copies of  $(Y, Z)$  satisfying the following errors-in-variables regression model

$$(1.1) \quad \begin{cases} Y = f(X) + \xi \\ Z = X + \sigma\varepsilon, \end{cases}$$

involving independent and unobserved, random variables  $X, \xi, \varepsilon$  and an unknown regression function  $f$ . The unobserved  $X_i$ 's, have common unknown density denoted by  $g$ . The errors  $\varepsilon_i$ 's have common known density  $f_\varepsilon$ , and  $\sigma$  is the known noise level. We assume moreover that all random variables have finite variance. Our aim is to estimate the regression function  $f$  on a compact set denoted by  $A$ , by using the observations  $(Y_i, Z_i)$  for  $i = 1, \dots, n$ , without any prior knowledge, neither on the smoothness of  $f$  nor on the smoothness of the density  $g$ .

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In nonparametric errors-in-variables regression models, two factors determine the estimation accuracy of the regression function: first, the smoothness of the function  $f$  to be estimated, and second the smoothness of the errors density  $f_\varepsilon$ . As in the deconvolution framework, the worst rates of convergence are obtained for the smoother errors density  $f_\varepsilon$ . In this context, two classes of errors are considered: first the so called ordinary smooth errors with polynomial decay of their Fourier transform and second, the super smooth errors with Fourier transform having an exponential decay.

Many papers deal with parametric or semi-parametric estimation in errors in variables models, but we only mention here previous known results in the general nonparametric case. In this context most of the proposed estimators are some Nadaraya-Watson kernel type estimators, constructed as the ratio of two deconvolution kernel type estimators, see e.g. Fan *et al.* (1991), Fan and Masry (1992), Fan and Truong (1993), Masry (1993), Truong (1991), Ioannides and Alevizos (1997). One assumption usually done in all those works, is that the regularity of the regression function  $f$  and the regularity of the density  $g$  of the design are equal. In particular, when the regression function  $f$  and the density  $g$  admit  $k$ th-order derivatives, Fan and Truong (1993) give upper and lower bounds of the minimax risk for quadratic pointwise risk and for  $\mathbb{L}_p$ -risk on compact sets for ordinary and super smooth errors  $\varepsilon$ .

In a slightly different way, Koo and Lee (1998) propose an estimation method based on  $B$ -spline, when the errors are ordinary smooth. This method also relates to estimation of the regression function as a ratio of two estimators.

To our knowledge, all previous papers consider that the regression function and the density  $g$  belong to the same smoothness class and that this common class is known.

We propose here an estimation procedure of  $f$ , that does not require any prior knowledge on the regularity of the unknown functions  $f$  and  $g$ . Our estimation procedure is based on the classical idea that the regression function  $f$  at point  $x$  can be written as the ratio

$$f(x) = \mathbb{E}(Y|X = x) = \frac{\int y f_{X,Y}(x, y) dy}{g(x)} = \frac{(fg)(x)}{g(x)},$$

with  $f_{X,Y}$  the joint density of  $(X, Y)$ . Hence  $f$  is estimated by a ratio of an adaptive estimator  $\tilde{\ell}$  of  $\ell = fg$  and of an adaptive estimator  $\tilde{g}$  of  $g$ , both of them being built by minimization of penalized contrast functions. The contrasts are determined by projection methods and the penalizations give an automatic choice of the relevant projection spaces.

We give upper bounds on the  $\mathbb{L}_2$ -risk on a compact set for the regression function  $f$  as well as for the  $\mathbb{L}_2(\mathbb{R})$ -risk of the density  $g$  when the errors are either ordinary or super smooth. We show in particular that the  $\mathbb{L}_2$ -risk on a compact set of our estimator  $\tilde{f}$  of  $f$  is bounded by the sum of the risks of  $\tilde{\ell}$  and  $\tilde{g}$ . The rate of convergence of  $\tilde{f}$  is thus given by the slower rate between the rate of the adaptive estimation of  $g$  and the rate of the adaptive estimation of  $\ell = fg$ . The resulting estimator automatically reaches the minimax rates in standard cases

where lower bounds are available. The other cases are intensively discussed. In other words, our procedure provides an adaptive estimator, in the sense that its construction does not require any prior knowledge on the smoothness of  $f$  nor  $g$ , which seems often optimal.

The paper is organized as follows. In Section 2, we describe the estimators. Section 3 is devoted to the presentation of the upper bounds for the resulting  $\mathbb{L}_2$ -risks with some discussions about the optimality in the minimax sense of the estimators. All proofs and technical lemmas are gathered in Section 4.

## 2. DESCRIPTION OF THE ESTIMATORS

For  $u$  and  $v$  in  $\mathbb{L}_2(\mathbb{R})$ ,  $u^*$  is the Fourier transform of  $u$  with  $u^*(x) = \int e^{itx}u(t)dt$ ,  $u * v$  is the convolution product,  $u * v(x) = \int u(y)v(x-y)dy$ , and  $\langle u, v \rangle = \int u(x)\bar{v}(x)dx$  with  $z\bar{z} = |z|^2$ . The quantities  $\|u\|_1$ ,  $\|u\|_2$ ,  $\|u\|_\infty$  and  $\|u\|_{\infty, K}$  denote  $\|u\|_1 = \int |u(x)|dx$ ,  $\|u\|_2^2 = \int |u(x)|^2dx$ ,  $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$ ,  $\|u\|_{\infty, K} = \sup_{x \in K} |u(x)|$ .

Subsequently we assume that  $f_\varepsilon \in \mathbb{L}_2(\mathbb{R})$ ,  $f_\varepsilon^* \in \mathbb{L}_2(\mathbb{R})$  with  $f_\varepsilon^*(x) \neq 0$  for all  $x \in \mathbb{R}$ .

**2.1. Projection spaces.** Consider  $\varphi(x) = \sin(\pi x)/(\pi x)$ , and  $\varphi_{m,j}(x) = \sqrt{D_m}\varphi(D_mx - j)$ . Here, we take  $D_m = m$  and  $m \in \mathcal{M}_n = \{1, \dots, m_n\}$ , but when  $D_m = 2^m$ , the basis  $\{\varphi_{m,j}\}_{j \in \mathbb{Z}}$  is known as the Shannon basis. It is well known (see for instance Meyer (1990), p.22), that  $\{\varphi_{m,j}\}_{j \in \mathbb{Z}}$  is an orthonormal basis of the space  $S_m$  of square integrable functions having a Fourier transform with compact support contained in  $[-\pi D_m, \pi D_m]$ , that is

$$S_m = \text{Vect}\{\varphi_{m,j}, j \in \mathbb{Z}\} = \{f \in \mathbb{L}_2(\mathbb{R}), \text{ with } \text{supp}(f^*) \text{ contained in } [-\pi D_m, \pi D_m]\}.$$

Since the orthogonal projection of  $g$  and  $\ell$  on  $S_m$ ,  $g_m$  and  $\ell_m$ ,  $g_m = \sum_{j \in \mathbb{Z}} a_{m,j}(g)\varphi_{m,j}$  and  $\ell_m = \sum_{j \in \mathbb{Z}} a_{m,j}(\ell)\varphi_{m,j}$  with  $a_{m,j}(g) = \langle \varphi_{m,j}, g \rangle$ , and  $a_{m,j}(\ell) = \langle \varphi_{m,j}, \ell \rangle$ , involve infinite sums, we consider in practice, the truncated spaces  $S_m^{(n)}$  defined as

$$S_m^{(n)} = \text{Vect} \{\varphi_{m,j}, |j| \leq k_n\}$$

where  $k_n$  is an integer to be chosen later. The family  $\{\varphi_{m,j}\}_{|j| \leq k_n}$  is an orthonormal basis of  $S_m^{(n)}$ , and the orthogonal projection of  $g$  and  $\ell$  on  $S_m^{(n)}$  denoted by  $g_m^{(n)}$  and  $\ell_m^{(n)}$ , are given by  $g_m^{(n)} = \sum_{|j| \leq k_n} a_{m,j}(g)\varphi_{m,j}$  and  $\ell_m^{(n)} = \sum_{|j| \leq k_n} a_{m,j}(\ell)\varphi_{m,j}$ .

**2.2. Construction of the minimum contrast estimators.** For  $r \in \mathbb{R}$  and  $d > 0$ , we denote by  $r^{(d)} = \text{sign}(r) \min(|r|, d)$ , and thus define the trimmed estimator of  $f$  by

$$(2.1) \quad \hat{f}_{\check{m}_\ell, \check{m}_g} = (\hat{\ell}_{\check{m}_\ell} / \hat{g}_{\check{m}_g})^{(a_n)},$$

with  $a_n$  being suitably chosen,  $\check{m}_\ell$  and  $\check{m}_g$  minimizing the  $\mathbb{L}_2(\mathbb{R})$  risks of  $\hat{\ell}_{\check{m}_\ell}$  the projection estimator on a space  $S_{\check{m}_\ell}^{(n)}$ , and of  $\hat{g}_{\check{m}_g}$  the projection estimator on a space  $S_{\check{m}_g}^{(n)}$ , defined as follows.

The estimator of  $\ell$ , is defined by

$$(2.2) \quad \hat{\ell}_m = \arg \min_{t \in S_m^{(n)}} \gamma_{n,\ell}(t),$$

with  $\gamma_{n,\ell}$  defined, for  $t \in S_m^{(n)}$ , by

$$(2.3) \quad \gamma_{n,\ell}(t) = \|t\|^2 - 2n^{-1} \sum_{i=1}^n (Y_i u_t^*(Z_i)) \text{ with } u_t(x) = (2\pi)^{-1} t^*(-x)/f_\varepsilon^*(-x),$$

that is  $\hat{\ell}_m = \sum_{|j| \leq k_n} \hat{a}_{m,j}(\ell) \varphi_{m,j}$  with  $\hat{a}_{m,j}(\ell) = n^{-1} \sum_{i=1}^n Y_i u_{\varphi_{m,j}}^*(Z_i)$ .

By using Parseval and inverse Fourier formulas, we get that

$$\mathbb{E}(Y_1 u_t^*(Z_1)) = \mathbb{E}(f(X_1) u_t^*(Z_1)) = \langle u_t^* * f_\varepsilon, fg \rangle = \frac{1}{2\pi} \langle f_\varepsilon^* t^* / f_\varepsilon^*, (fg)^* \rangle = \frac{1}{2\pi} \langle t^*, (fg)^* \rangle = \langle t, \ell \rangle.$$

Therefore, we find that  $\mathbb{E}(\gamma_{n,\ell}(t)) = \|t\|_2^2 - 2\langle \ell, t \rangle = \|t - \ell\|_2^2 - \|\ell\|_2^2$  which is minimal when  $t = \ell$ . This shows that  $\gamma_{n,\ell}(t)$  suits well for the estimation of  $\ell = fg$ .

By using the estimation procedure described in Comte *et al.* (2005a), the estimator of  $g$  on  $S_m^{(n)}$  is defined by  $\hat{g}_m = \sum_{|j| \leq k_n} \hat{a}_{m,j}(g) \varphi_{m,j}$  with  $\hat{a}_{m,j}(g) = n^{-1} \sum_{i=1}^n u_{\varphi_{m,j}}^*(Z_i)$ , that is

$$(2.4) \quad \hat{g}_m = \arg \min_{t \in S_m^{(n)}} \gamma_{n,g}(t)$$

with  $\gamma_{n,g}$  defined, for  $t \in S_m^{(n)}$  by  $\gamma_{n,g}(t) = \|t\|_2^2 - 2n^{-1} \sum_{i=1}^n u_t^*(Z_i)$ , with  $u_t$  defined in (2.3).

**Remark 2.1.** The use of  $r^{(d)}$  avoids the problems that may occur when  $\hat{g}_{m_2}$  takes small values.

**2.3. Construction of the minimum penalized contrast estimators.** In order to construct the minimum penalized contrast estimators, and especially to define the penalty functions, we need to precise the behavior of  $f_\varepsilon^*$ , described as follows. We assume that, for all  $x$  in  $\mathbb{R}$ ,

$$(A_1) \quad \kappa_0(x^2 + 1)^{-\alpha/2} \exp\{-\beta|x|^\rho\} \leq |f_\varepsilon^*(x)| \leq \kappa'_0(x^2 + 1)^{-\alpha/2} \exp\{-\beta|x|^\rho\}.$$

Only the left-hand side of  $(A_1)$  is required to define the penalty function and for upper bounds. The right-hand side is needed when we consider lower bounds and the question of optimality in a minimax sense. When  $\rho = 0$ ,  $\alpha$  has to be such that  $\alpha > 1/2$ . When  $\rho = 0$  in  $(A_1)$ , the errors are usually called ‘‘ordinary smooth’’ errors, and ‘‘super smooth’’ errors when  $\rho > 0$ . The standard examples are the following: Gaussian or Cauchy distributions are super smooth of order  $(\alpha = 0, \rho = 2)$  and  $(\alpha = 0, \rho = 1)$  respectively, and the double exponential distribution is ordinary smooth  $(\rho = 0)$  of order  $\alpha = 2$ .

By convention, we set  $\beta = 0$  when  $\rho = 0$  and we assume that  $\beta > 0$  when  $\rho > 0$ . In the same way, if  $\sigma = 0$ , the  $X_i$ 's are directly observed without noise and we set  $\beta = \alpha = \rho = 0$ .

Under the assumption  $(A_1)$ , the regression function  $f$  is estimated by  $\tilde{f}$  defined as

$$(2.5) \quad \tilde{f} = (\tilde{\ell}/\tilde{g})^{(a_n)},$$

where  $\tilde{\ell}$  is the adaptive estimator defined by

$$(2.6) \quad \tilde{\ell} = \hat{\ell}_{\hat{m}_\ell} \text{ with } \hat{m}_\ell = \arg \min_{m \in \mathcal{M}_{n,\ell}} \left[ \gamma_{n,\ell}(\hat{\ell}_m) + \text{pen}_\ell(m) \right],$$

$\tilde{g}$  is the adaptive estimator defined as in Comte *et al.* (2005a), by

$$(2.7) \quad \tilde{g} = \hat{g}_{\hat{m}_g} \text{ with } \hat{m}_g = \arg \min_{m \in \mathcal{M}_{n,g}} \left[ \gamma_{n,g}(\hat{g}_m) + \text{pen}_g(m) \right],$$

where  $\mathcal{M}_{n,\ell}$  and  $\mathcal{M}_{n,g}$  are some restrictions of  $\mathcal{M}_n$  given below, and where  $\text{pen}_\ell$  and  $\text{pen}_g$  are data driven penalty functions given by

$$(2.8) \quad \text{pen}_\ell(m) = \kappa'(\lambda_1 + \mu_2)[1 + \hat{m}_2(Y)]\tilde{\Gamma}(m)/n, \quad \text{pen}_g(m) = \kappa(\lambda_1 + \mu_1)\tilde{\Gamma}(m)/n,$$

with

$$(2.9) \quad \hat{m}_2(Y) = \frac{1}{n} \sum_{i=1}^n Y_i^2, \quad \text{and} \quad \tilde{\Gamma}(m) = D_m^{2\alpha + \max(1-\rho, \min((1+\rho)/2, 1))} \exp\{2\beta\sigma^\rho(\pi D_m)^\rho\}.$$

The constants  $\lambda_1, \mu_1$  and  $\mu_2$  are some known constants, only depending on  $f_\varepsilon$  and  $\sigma$  (assumed to be known), to be defined later (see (3.4), (3.8) and (3.9)), and  $\kappa$  and  $\kappa'$  are some numerical constants.

**Remark 2.2.** First note that the penalty functions in (2.8) have the same form with different constants. More precisely, in both cases, the penalties are of order  $D_m^{2\alpha+1-\rho} \exp(2\beta\sigma^\rho(\pi D_m)^\rho)$  if  $0 \leq \rho \leq 1/3$ ,  $D_m^{2\alpha+(1+\rho)/2} \exp(2\beta\sigma^\rho(\pi D_m)^\rho)$  if  $1/3 \leq \rho \leq 1$  and of order  $D_m^{2\alpha+1} \exp(2\beta\sigma^\rho(\pi D_m)^\rho)$  if  $\rho \geq 1$ .

Second, the constants involve  $\kappa$  and  $\kappa'$ , universal numerical constants, as well as constants  $\lambda_1, \mu_1, \mu_2$  related to the known errors density  $f_\varepsilon$ . Any constant greater than any well chosen constant also suits for theoretical results. In practice, such constants are usually calibrated by some intensive simulation studies. We refer to Comte *et al.* (2005a, 2005b) for further details on penalty calibration as well as for details on the implementation of such estimators in density deconvolution problems.

### 3. RATES OF CONVERGENCE AND ADAPTIVITY

**3.1. Assumptions.** We consider Model (1.1) under  $(\mathbf{A}_1)$  and the following additional assumptions.

$$(\mathbf{A}_2) \quad \ell \in \mathbb{L}_2(\mathbb{R}) \text{ and } \ell \in \mathcal{L} = \left\{ \phi \text{ such that } \int x^2 \phi^2(x) dx \leq \kappa_{\mathcal{L}} < \infty \right\},$$

$$(\mathbf{A}_3) \quad f \in \mathcal{F}_G = \{ \phi \text{ such that } \sup_{x \in G} |\phi(x)| \leq \kappa_{\infty,G} < \infty \}, \text{ where } G \text{ is the support of } g.$$

$$(\mathbf{A}_4) \quad g \in \mathbb{L}_2(\mathbb{R}) \text{ and } g \in \mathcal{G} = \{ \phi, \text{ density, such that } \int x^2 \phi^2(x) dx < \kappa_{\mathcal{G}} < \infty \}.$$

$$(\mathbf{A}_5) \quad \text{There exist } g_0, g_1 \text{ positive constants such that for all } x \in A, g_0 \leq g(x) \leq g_1.$$

Note that we do not assume that  $g$  is compactly supported but only that  $f$  is bounded on the support of  $g$ . It follows that if  $g$  is compactly supported then  $f$  has to be bounded on a compact set. But if  $g$  has  $\mathbb{R}$  as support then the regression function has to be bounded on  $\mathbb{R}$ . We estimate  $f$  only on a compact set denoted by  $A$ . Hence, the assumption **(A<sub>5</sub>)** implies that  $A \subset G$  and therefore under **(A<sub>3</sub>)** and **(A<sub>5</sub>)**,  $f$  is bounded on  $A$ . The assumptions **(A<sub>3</sub>)** and **(A<sub>4</sub>)** imply that **(A<sub>2</sub>)** holds, with  $\kappa_{\mathcal{L}} = \kappa_{\infty, G}^2 \kappa_{\mathcal{G}}$ .

Classically, the slowest rate of convergence for estimating  $f$  and  $g$  are obtained for super smooth errors density. In particular, when  $f_{\varepsilon}$  is the Gaussian density the minimax rate of convergence obtained by Fan and Truong (1993) when  $f$  and  $g$  have the same Hölderian type regularity is of order a power of  $\ln(n)$ . Nevertheless, those rates can be improved by some additional regularity conditions on  $f$  and  $g$  described as follows.

$$\mathbf{(R_1)} \quad \mathcal{S}_{a,r,B}(C_1) = \{\psi \in \mathbb{L}_2(\mathbb{R}) : \text{such that } \int_{-\infty}^{+\infty} |\psi^*(x)|^2 (x^2 + 1)^a \exp\{2B|x|^r\} dx \leq C_1\},$$

for  $a, r, B, C_1$  some nonnegative real numbers. The smoothness class in **(R<sub>1</sub>)** is classically considered in nonparametric estimation, especially in deconvolution. When  $r = 0$ , this corresponds to Sobolev spaces of order  $a$ . The densities belonging to  $\mathcal{S}_{a,r,B}(C_1)$  with  $r > 0, B > 0$  are infinitely many times differentiable, admit analytic continuation on a finite width strip when  $r = 1$  and on the whole complex plane if  $r = 2$ .

**3.2. Risks bounds for the minimum contrast estimators.** We start by presenting some general bound for the risk.

**Proposition 3.1.** *Consider the estimators  $\hat{\ell}_{D_m} = \hat{\ell}_m$  and  $\hat{g}_{D_m} = \hat{g}_m$  of  $\ell$  and  $g$  defined by (2.2) and (2.4). Let  $\Delta(m) = D_m \pi^{-1} \int_0^{\pi D_m} |f_{\varepsilon}^*(D_m x \sigma)|^{-2} dx$ . Then, under **(A<sub>2</sub>)** and **(A<sub>4</sub>)**,*

$$(3.1) \quad \mathbb{E}(\|\ell - \hat{\ell}_m\|_2^2) \leq \|\ell - \ell_m\|_2^2 + 2\mathbb{E}(Y_1^2)\Delta(m)/n + (\kappa_{\mathcal{L}} + \|\ell\|_1)D_m^2/k_n$$

and

$$(3.2) \quad \mathbb{E}(\|g - \hat{g}_m\|_2^2) \leq \|g - g_m\|_2^2 + 2\Delta(m)/n + (\kappa_{\mathcal{G}} + 1)D_m^2/k_n.$$

As in deconvolution problems, the variance term  $\Delta(m)/n$  depends on the rate of decay of the Fourier transform  $f_{\varepsilon}^*$ , with larger variance for fast decreasing  $f_{\varepsilon}^*$ . Under **(A<sub>1</sub>)**, the variance term is bounded in the following way

$$(3.3) \quad \Delta(m) \leq \lambda_1 \Gamma(m) \quad \text{where} \quad \Gamma(m) = D_m^{2\alpha+1-\rho} \exp(2\beta\sigma^{\rho}(\pi D_m)^{\rho}),$$

with

$$(3.4) \quad \lambda_1 = (\sigma^2 \pi^2 + 1)^{\alpha} / (\pi^{\rho} \kappa_0^2 R(\beta, \sigma, \rho)) \quad \text{with} \quad R(\beta, \sigma, \rho) = \mathbf{1}_{\rho=0} + 2\beta\rho\sigma^{\rho} \mathbf{1}_{0<\rho\leq 1} + 2\beta\sigma^{\rho} \mathbf{1}_{\rho>1},$$

In order to ensure that  $\Gamma(m)/n$  is bounded, we only consider models such that  $\pi D_m = m \leq m_n$  in  $\mathcal{M}_n = \{1, \dots, m_n\}$  with

$$(3.5) \quad m_n \leq \begin{cases} \pi^{-1} n^{1/(2\alpha+1)} & \text{if } \rho = 0 \\ \pi^{-1} \left[ \frac{\ln(n)}{2\beta\sigma^\rho} + \frac{2\alpha+1-\rho}{2\rho\beta\sigma^\rho} \ln \left( \frac{\ln(n)}{2\beta\sigma^\rho} \right) \right]^{1/\rho} & \text{if } \rho > 0. \end{cases}$$

Lastly, the bias terms  $\|\ell - \ell_m\|_2^2$  and  $\|g - g_m\|_2^2$  depend, as usual, on the smoothness of the functions  $\ell$  and  $g$ . They have the expected order for classical smoothness classes since they relate to the distance between  $g$  and the classes of entire functions having Fourier transform compactly supported on  $[-\pi D_m, \pi D_m]$  (see Ibragimov and Hasminskii (1983)).

Since  $\ell_m$  and  $g_m$  are the orthogonal projections of  $\ell$  and  $g$  on  $S_m$ , when  $\ell$  belongs  $\mathcal{S}_{a_\ell, r_\ell, B_\ell}(\kappa_{a_\ell})$  and  $g$  belongs  $\mathcal{S}_{a_g, r_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$ , then

$$(3.6) \quad \|\ell - \ell_m\|_2^2 = (2\pi)^{-1} \int_{|x| \geq \pi D_m} |\ell^*(x)|^2 dx \leq [\kappa_{a_\ell}/(2\pi)] (D_m^2 \pi^2 + 1)^{-a_\ell} \exp\{-2B_\ell \pi^{r_\ell} D_m^{r_\ell}\},$$

and the same holds for  $\|g_m - g\|_2^2$  with  $(a_\ell, B_\ell, r_\ell)$  replaced by  $(a_g, B_g, r_g)$ .

**Corollary 3.1.** *Under  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  and  $(\mathbf{A}_4)$ , let  $\Gamma(m)$  and  $\lambda_1$  being defined in (3.3) and (3.4). Assume that  $k_n \geq n$ , that  $\ell$  belongs to  $\mathcal{S}_{a_\ell, r_\ell, B_\ell}(\kappa_{a_\ell})$  and that  $g$  belongs to  $\mathcal{S}_{a_g, r_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$ . Then*

$$\mathbb{E}(\|\ell - \hat{\ell}_m\|_2^2) \leq \frac{\kappa_{a_\ell}}{2\pi} (D_m^2 \pi^2 + 1)^{-a_\ell} e^{-2B_\ell \pi^{r_\ell} D_m^{r_\ell}} + 2\lambda_1 \mathbb{E}(Y_1^2) \Gamma(m)/n + D_m^2 (\kappa_{\mathcal{L}} + \|\ell\|_1)/n,$$

and

$$\mathbb{E}(\|g - \hat{g}_m\|_2^2) \leq \frac{\kappa_{a_g}}{2\pi} (D_m^2 \pi^2 + 1)^{-a_g} e^{-2B_g \pi^{r_g} D_m^{r_g}} + 2\lambda_1 \Gamma(m)/n + (\kappa_{\mathcal{G}} + 1) D_m^2/n.$$

**Remark 3.1.** We point out that the  $\{\varphi_{m,j}\}$  are  $\mathbb{R}$ -supported (and not compactly supported) and hence, we obtain estimations of  $\ell$  and  $g$  on the whole line and not only on a compact set as for usual projection estimators. This is a great advantage of this basis even if, due to the truncation  $|j| \leq k_n$ , it induces the residual terms  $D_m^2 (\kappa_{\mathcal{L}} + \|\ell\|_1)/k_n$  and  $D_m^2 (\kappa_{\mathcal{G}} + 1)/k_n$ , in the upper bounds of the risks. The most important thing is that the choice of  $k_n$  does not influence the other terms. Consequently, we can find a relevant choice of  $k_n$  ( $k_n \geq n$  under  $(\mathbf{A}_2)$  and  $(\mathbf{A}_4)$ ), that makes those additional terms unconditionally negligible with respect to the bias and variance terms. The condition  $k_n \geq n$  allows us to construct truncated spaces  $S_m^{(n)}$  using  $O(n)$  basis vectors and hence to use a tractable and fast algorithm. The choice of larger  $k_n$ , independent of  $\ell$  and  $g$ , does not change the efficiency of our estimator from a statistical point of view but will only change the speed of the algorithm from a practical point of view.



		$f_\varepsilon$	
		$\rho = 0$ ordinary smooth	$\rho > 0$ super smooth
$g$	$r_\ell = 0$ Sobolev( $s$ )	$\pi D_{\check{m}_\ell} = O(n^{1/(2\alpha+2a_\ell+1)})$ rate = $O(n^{-2a_\ell/(2\alpha+2a_\ell+1)})$	$\pi D_{\check{m}_\ell} = [\ln(n)/(2\beta\sigma^\rho + 1)]^{1/\rho}$ rate = $O((\ln(n))^{-2a_\ell/\rho})$
	$r_\ell > 0$ $\mathcal{C}^\infty$	$\pi D_{\check{m}_\ell} = [\ln(n)/2B_\ell]^{1/r_\ell}$ rate = $O\left(\frac{\ln(n)^{(2\alpha+1)/r_\ell}}{n}\right)$	$\pi D_{\check{m}_\ell}$ implicit solution of $D_{\check{m}_\ell}^{2\alpha+2a_\ell+1-r_\ell} e^{2\beta\sigma^\rho(\pi D_{\check{m}_\ell})^\rho} + 2B(\pi D_{\check{m}_\ell})^{r_\ell}$ = $O(n)$

TABLE 1. Best choices of  $D_{\check{m}_\ell}$  minimizing  $\mathbb{E}(\|\ell - \hat{\ell}_m\|_2^2)$  and resulting rates for  $\hat{\ell}_{\check{m}_\ell}$ .

For the case  $r_\ell > 0$  and  $\rho > 0$ , the choice  $\pi D_{\check{m}_\ell} = [\ln(n)/(2\beta\sigma^\rho + 1)]^{1/\rho}$  leads to a rate which is faster than any power of  $\ln(n)$  and slower than any power of  $n$ . For instance if  $r_\ell = \rho$ , the rate is of order  $[\ln(n)]^b n^{-B_\ell/(B_\ell + \beta\sigma^\rho)}$  with  $b = [-2a_\ell\beta\sigma^\rho + (2\alpha - r_\ell + 1)B_\ell]/[r_\ell(\beta\sigma^\rho + B_\ell)]$ .

The same table holds for  $g$ , by replacing  $(a_\ell, B_\ell, r_\ell)$  by  $(a_g, B_g, r_g)$ . For  $D_{\check{m}_g}$  chosen in the same way as  $D_{\check{m}_\ell}$  in Table 1, the rate of convergence of  $\hat{g}_{\check{m}_g}$  is the minimax rate of convergence, as given in Fan (1991a) for  $r_g = 0$ , in Butucea (2004) for  $r_g > 0$  and  $\rho = 0$  and in Butucea and Tsybakov (2004) for  $0 < r_g < \rho$  and  $a_g = 0$ .

The rate of convergence of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  is given by the following proposition.

**Proposition 3.2.** *Under  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$ ,  $(\mathbf{A}_3)$ ,  $(\mathbf{A}_4)$ , and  $(\mathbf{A}_5)$ , assume that  $g$  belongs to some space  $\mathcal{S}_{a_g, r_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$  with  $a_g > 1/2$  if  $r_g = 0$ . Let  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  be defined by (2.1), with  $\check{m}_\ell$  and  $\check{m}_g$  such that  $D_{\check{m}_\ell}$  and  $D_{\check{m}_g}$  minimize the risks  $\mathbb{E}(\|\ell - \hat{\ell}_m\|_2^2)$  and  $\mathbb{E}(\|g - \hat{g}_m\|_2^2)$  respectively. If  $a_n = n^k$  for  $k > 0$ , and  $k_n \geq n^{3/2}$ , then, for  $n$  great enough and  $C_0 = Kg_0^{-2}(1 + g_1g_0^{-2}\kappa_{\infty, G})$ ,*

$$(3.7) \quad \mathbb{E}(\|\hat{f}_{\check{m}_\ell, \check{m}_g} - f\|_{\mathbf{A}}^2) \leq C_0[\mathbb{E}(\|\ell - \hat{\ell}_m\|_2^2) + \mathbb{E}(\|g - \hat{g}_m\|_2^2)] + o(n^{-1}).$$

If  $a_g \leq 1/2$  then we only have a result of type  $\|(f - \hat{f}_{\check{m}_\ell, \check{m}_g})\|_{\mathbf{A}}^2 = O_p(\|\ell - \hat{\ell}_m\|_2^2 + \|g - \hat{g}_m\|_2^2)$ . Also note that the result holds when the constant  $\kappa_{\infty, G}$  is replaced by  $\|f\|_{\infty, A}$  if  $f$  is bounded on the compact set  $A$ .

The performance of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  is given by the worst performance between the one of  $\hat{\ell}_{\check{m}_\ell}$  and the one of  $\hat{g}_{\check{m}_g}$ . Let us be more precise in some examples. Under the assumptions of Proposition 3.2:

- If the  $\varepsilon_i$ 's are ordinary smooth,

– If  $r_\ell = r_g = 0$  and  $\pi D_{\check{m}_\ell} = O(n^{1/(2a_\ell+2\alpha+1)})$  and  $\pi D_{\check{m}_g} = O(n^{1/(2a_g+2\alpha+1)})$ , then

$$\mathbb{E}(\|(f - \hat{f}_{\check{m}_\ell, \check{m}_g})\mathbf{I}_A\|_2^2) \leq O(n^{-2a^*/(2a^*+2\alpha+1)}) \quad \text{with } a^* = \inf(a_\ell, a_g).$$

– If  $r_\ell > 0$ ,  $r_g > 0$ ,  $\pi D_{\check{m}_\ell} = (\ln(n)/2B)^{1/r_\ell}$  and  $\pi D_{\check{m}_g} = (\ln(n)/2B)^{1/r_g}$ , then

$$\mathbb{E}(\|(f - \hat{f}_{\check{m}_\ell, \check{m}_g})\mathbf{I}_A\|_2^2) \leq O\left(\frac{(\ln(n))^{(2\alpha+1)/r^*}}{n}\right) \quad \text{with } r^* = \inf(r_\ell, r_g).$$

- If the  $\varepsilon_i$ 's are super smooth and  $r_\ell = r_g = 0$ ,  $\pi D_{\check{m}_\ell} = \pi D_{\check{m}_g} = [\ln(n)/(2\beta\sigma^\rho + 1)]^{1/\rho}$ , then

$$\mathbb{E}(\|(f - \hat{f}_{\check{m}_\ell, \check{m}_g})\mathbf{I}_A\|_2^2) \leq O([\ln(n)]^{-2a^*/\rho}) \quad \text{with } a^* = \inf(a_\ell, a_g).$$

Since  $\ell = fg$ , the smoothness properties of  $\ell$  are related to those of  $f$  and of  $g$ .

When  $\ell$  belongs to  $\mathcal{S}_{a_\ell, 0, B_\ell}(\kappa_{a_\ell})$  and  $g$  belongs to  $\mathcal{S}_{a_g, 0, B_g}(\kappa_{a_g})$  with  $a_\ell = a_g$ , then the resulting rate is the minimax rate given in Fan and Truong (1993) for Hölderian regression functions and densities with the same regularity. It follows that our estimator seems then optimal in that case. It is easy to see that the estimator is also optimal if  $a_g \geq a_\ell$ , that is when the density  $g$  is smoother than the regression function  $f$ . But the optimality of the rate of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  when  $a_\ell > a_g$ , that is when the regression function  $f$  is smoother than  $g$ , remains an open question. This is a known drawback of Nadaraya-Watson type estimators for regression functions, constructed as ratio of estimators. In “classical” regression models, when the  $X_i$ 's are observed, a lot of methods, like local polynomial estimators, mean square estimators..., avoid the need of regularity conditions on  $g$  for the estimation of  $f$ . The point is that standard methods solving the regression problem do not seem to work in the errors-in-variables model and it is an open problem to build an estimator of  $f$  that does not require the estimation of the density  $g$ .

From the above results we see that the choice of the dimensions  $D_{\check{m}_\ell}$  and  $D_{\check{m}_g}$  that realize the best trade-off between the squared bias and the variance terms depends on the unknown regularity coefficients of the functions  $\ell$  and  $g$ . In the next section we provide the upper bounds of the risks of the penalized estimators, constructed without such smoothness knowledge.

### 3.3. Risks bounds of the minimum penalized contrast estimators: adaptation.

**Theorem 3.1.** *Under the assumptions  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  and  $(\mathbf{A}_4)$ , let*

$$(3.8) \quad \mu_1 = \begin{cases} 0 & \text{if } \rho < 1/3 \\ \beta(\sigma\pi)^\rho \lambda_1^{1/2}(\alpha, \kappa_0, \beta, \sigma, \rho)(1 + \sigma^2\pi^2)^{\alpha/2} \kappa_0^{-1} (2\pi)^{-1/2} & \text{if } 1/3 \leq \rho \leq 1, \\ \beta(\sigma\pi)^\rho \lambda_1(\alpha, \kappa_0, \beta, \sigma, \rho) & \text{if } \rho > 1. \end{cases}$$

and

$$(3.9) \quad \mu_2 = \mu_1 \mathbf{I}_{\{0 \leq \rho < 1/3\} \cup \{\rho > 1\}} + \mu_1 \|f_\varepsilon\|_2 \mathbf{I}_{\{1/3 \leq \rho \leq 1\}}.$$

Let  $k_n \geq n$ ,  $\tilde{\ell} = \hat{\ell}_{\hat{m}_\ell}$  and  $\tilde{g} = \hat{g}_{\hat{m}_g}$  be defined by (2.6) and (2.7) and with  $\text{pen}_\ell$  and  $\text{pen}_g$  given by (2.8), for  $\kappa$  and  $\kappa'$  two universal numerical constants and  $1 \leq m \leq m_n$ ,  $m_n$  satisfying (3.5) and, if  $\rho > 0$ ,

$$(3.10) \quad m_n \leq \pi^{-1} \left[ \frac{\ln(n)}{2\beta\sigma^\rho} + \frac{2\alpha + \min[(1/2 + \rho/2), 1]}{2\rho\beta\sigma^\rho} \ln \left( \frac{\ln(n)}{2\beta\sigma^\rho} \right) \right]^{1/\rho}.$$

**1) Adaptive estimation of  $g$ .** (Comte et al. (2005a)).

Then  $\tilde{g}$  satisfies  $\mathbb{E}(\|g - \tilde{g}\|_2^2) \leq K \inf_{m \in \mathcal{M}_{n,g}} [\|g - g_m\|_2^2 + D_m^2(\kappa_G + 1)/n + \text{pen}_g(m)] + c/n$  where  $K$  is a constant and  $c$  is a constant depending on  $f_\varepsilon$  and  $A_g$ .

**2) Adaptive estimation of  $\ell$ .** Under the assumption  $(\mathbf{A}_3)$ , if  $\mathbb{E}|\xi_1|^8 < \infty$  then  $\tilde{\ell}$  satisfies

$$\mathbb{E}(\|\ell - \tilde{\ell}\|_2^2) \leq K' \inf_{m \in \mathcal{M}_{n,\ell}} [\|\ell - \ell_m\|_2^2 + D_m^2(\kappa_{\mathcal{L}} + \|\ell\|_1)/n + \mathbb{E}(\text{pen}_\ell(m))] + c'/n$$

where  $K'$  is a constant and  $c'$  is a constant depending on  $f_\varepsilon$ ,  $\kappa_{\mathcal{L}}$ , and  $\|\ell\|_1$ .

**Remark 3.2.** In Theorem 3.1, the penalty is random since it involves the term  $\hat{m}_2(Y)$ , instead of the unknown quantity  $\mathbb{E}(Y_1^2)$  which appears first. The only price to pay for this substitution is the moment condition  $\mathbb{E}|\xi_1|^8 < \infty$  instead of  $\mathbb{E}|\xi_1|^6 < \infty$  if  $\mathbb{E}(Y_1^2)$  was in the penalty. Moreover, the term  $\mathbb{E}(\text{pen}_\ell(m))$  in the bound is equal to  $\text{pen}_\ell(m)$  with  $\hat{m}_2(Y)$  replaced by  $\mathbb{E}(Y_1^2)$ .

**Remark 3.3.** According to Remark 2.2, the penalty functions are of order  $\Gamma(m)/n$  if  $0 \leq \rho \leq 1/3$ , of order  $D_m^{3\rho/2-1/2}\Gamma(m)/n$  if  $1/3 \leq \rho \leq 1$  and of order  $D_m^\rho\Gamma(m)/n$  if  $\rho \geq 1$ . When  $\rho > 1/3$ , the penalty functions  $\text{pen}_\ell(m)$  and  $\text{pen}_g(m)$  have not exactly the order of the variance  $\Gamma(m)/n$ , but a loss of order  $D_m^{\min[(3\rho/2-1/2), \rho]}$  occurs, that is of order  $D_m^{(3\rho-1)/2}$  if  $1/3 < \rho \leq 1$  and of order  $D_m^\rho$  if  $\rho > 1$ .

**Remark 3.4. Rates of convergence of  $\tilde{g}$ .** The rate of convergence of  $\tilde{g}$  is the rate of convergence of  $\hat{g}_{\hat{m}_g}$  when  $0 \leq \rho \leq 1/3$  or when  $\rho > 1/3$  and  $r_g = 0$  or  $r_g < \rho$ . And there is a logarithmic loss, as a price to pay for adaptation when  $r_g \geq \rho > 1/3$ . We refer to Comte et al. (2005a) for further comments on the optimality in a minimax sense of  $\tilde{g}$ .

**Remark 3.5. Rates of convergence of  $\tilde{\ell}$ .** The rates, similar to the rates of  $\tilde{g}$ , are easy to deduce from Theorem 3.1 as soon as  $\ell = fg$  belongs to some smoothness class, but the procedure can reach the rate of  $\hat{\ell}_{\hat{m}_\ell}$ , that uses the unknown smoothness parameter. If  $\text{pen}_\ell(m)$  has the same order as the variance order  $\Gamma(m)/n$ , then Theorem 3.1 guarantees an automatic trade-off between the squared bias term  $\|\ell - \ell_m\|_2^2$  and the variance term, up to some multiplicative constant. Else, there is some loss due to the adaptation. Let us be more precise.

If  $0 \leq \rho \leq 1/3$ , the errors  $\varepsilon_i$ 's are ordinary smooth or super smooth with  $\rho \leq 1/3$ . If  $\ell$  satisfies  $(\mathbf{R}_1)$ , the squared bias is bounded by applying (3.6) which combined with the value of  $\text{pen}_\ell(m)$ , of order  $\Gamma(m)/n$  (see (3.3)) gives that the estimator  $\tilde{g}$  automatically reaches the best rate achievable by the estimator  $\hat{\ell}_{\hat{m}_\ell}$ , as given in Table 1.

If  $\rho > 1/3$  the penalty function  $\text{pen}_\ell(m)$  is slightly bigger than the variance order  $\Gamma(m)/n$ . The rate of convergence remains the best rate if the bias  $\|\ell - \ell_m\|_2^2$  is the dominating term in the trade-off between  $\|\ell - \ell_m\|_2^2$  and  $\text{pen}_\ell(m)$ . When  $r_\ell = 0$  and  $\rho > 0$ , the rate of order  $(\ln(n))^{-2a_\ell/\rho}$  is given by the bias term, and the loss in the penalty function does not change the rate of the adaptive estimator  $\tilde{\ell}$ , which remains the best achievable rate  $\mathbb{E} \|\ell - \hat{\ell}_{\hat{m}_\ell}\|_2^2$ . In the same way, when  $0 < r_\ell < \rho$ , the rate is given by the bias term and thus this loss does not affect the rate of convergence of  $\tilde{\ell}$  either.

Let us now focus our discussion on the case where  $\text{pen}_\ell(m)$  can be the dominating term in the trade-off between  $\|\ell - \ell_m\|_2^2$  and  $\text{pen}_\ell(m)$ , that is when  $r_\ell \geq \rho > 1/3$ . In that case, there is a loss of order  $D_m^{\min[(3\rho/2-1/2), \rho]}$  in the penalty function, compared to the variance term. But this happens in cases where the order of the optimal  $D_m$  is less than  $(\ln n)^{1/\rho}$  and consequently the loss in the rate is at most of order  $\ln n$ , when the rate is faster than logarithmic: therefore the loss appears only in cases where it can be seen as negligible.

In particular, there is no price to pay for the adaptation if the  $\xi_i$ 's are Gaussian and the  $\varepsilon_i$ 's are ordinary smooth. Indeed, in that case, the rate of convergence of the penalized estimator  $\tilde{\ell}$ , without any knowledge on  $\ell$  or  $g$ , is the same as the rate given by the non penalized estimator  $\hat{\ell}_{\hat{m}_\ell}$ , requiring the knowledge of smoothness parameters. But, if both the  $\xi_i$ 's and the  $\varepsilon_i$ 's are Gaussian, then  $\rho = 2$  and a logarithmic negligible loss appears in the rate of  $\tilde{\ell}$  compared to the rate of  $\hat{\ell}_{\hat{m}_\ell}$ .

**Theorem 3.2. Adaptive estimation of  $f$ .** *Under the assumptions  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$ ,  $(\mathbf{A}_3)$ ,  $(\mathbf{A}_4)$  and  $(\mathbf{A}_5)$ , let  $\tilde{f}$  be defined by (2.5) with  $\tilde{g}$  and  $\tilde{\ell}$  be defined in (2.7) and (2.6) with  $\hat{m}_g \in \mathcal{M}_{n,g}$  satisfying (3.5) and (3.10),  $D_{m_{n,g}} \leq (n/\ln(n))^{1/(2\alpha+2)}$  and  $\hat{m}_\ell \in \mathcal{M}_{n,\ell}$  satisfying (3.5) and (3.10). Assume that  $g$  belongs to some space  $\mathcal{S}_{a_g, r_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$  with  $a_g > 1/2$  if  $r_g = 0$ , and that  $\mathbb{E}|\xi_1|^8 < \infty$ . If  $k_n \geq n^{3/2}$ ,  $a_n = n^k$  for  $k > 0$ , for  $n$  large enough,  $C_0 = 8Kg_0^{-2}$  and  $C_1 = 4K'g_0^{-2}(2g_1^2 + 1)\kappa_{\infty, G}^2$ , then*

$$(3.11) \quad \begin{aligned} \mathbb{E}(\|(f - \tilde{f})\mathbf{I}_A\|_2^2) &\leq C_0 \inf_{m \in \mathcal{M}_{n,\ell}} [\|\ell - \ell_m\|_2^2 + D_m^2(\kappa_{\mathcal{L}} + \|\ell\|_1)/n + \text{pen}_\ell(m)] \\ &+ C_1 \inf_{m \in \mathcal{M}_{n,g}} [\|g - g_m\|_2^2 + D_m^2(\kappa_{\mathcal{G}} + 1)/n + \text{pen}_g(m)] + c/n \end{aligned}$$

where  $K$  and  $K'$  are constants depending on  $f_\varepsilon$ , and  $c$  is a constant depending on  $f_\varepsilon$ ,  $f$  and  $g$ .

As in Theorem 2.1, if  $a_g \leq 1/2$  then it may happen that  $D_{\hat{m}_g} \geq n^{1/(2\alpha+2)}$ , and in this case we only have a result in probability:  $\|(f - \tilde{f})\mathbf{I}_A\|_2^2 = O_p(\|\ell - \tilde{\ell}\|_2^2 + \|g - \tilde{g}\|_2^2)$ . Moreover, the result holds when the constant  $\kappa_{\infty, G}$  is replaced by  $\|f\|_{\infty, A}$  if  $f$  is bounded on the compact set  $A$ . Also note that the remark 3.1 is still valid for all adaptive estimators.

**Comments about the resulting rates for estimating  $f$ .** First the rate of convergence of  $\tilde{f}$  is given by the worst rate of convergence between the rate of  $\tilde{\ell}$  and  $\tilde{g}$ . Obviously all the comments about  $\hat{f}_{\check{m}_\ell, \check{m}_g}$ , related to this fact keep holding here.

When  $0 \leq \rho < 1/3$  or when  $r_\ell \leq \rho$  and  $r_g \leq \rho$ , then  $\tilde{f}$  achieves the rate of convergence of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$ , given by the worst rate of convergence between  $\mathbb{E} \|\hat{\ell}_{\check{m}_\ell} - \ell\|_2^2$  and  $\mathbb{E} \|\hat{g}_{\check{m}_g} - g\|_2^2$ . And when  $r_g > \rho > 1/3$  or  $r_\ell > \rho > 1/3$ , there is a logarithmic loss in the rate of convergence of  $\tilde{f}$  compared to the rate of convergence of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$ .

Since the regularity of  $\ell$  is by definition the regularity of  $fg$ , the rate of convergence of  $\tilde{\ell}$  in fact depends on smoothness properties of  $f$  and  $g$ . As a consequence, if  $\ell$  and  $g$  belong respectively to  $\mathcal{S}_{a_\ell, r_\ell, B_f}(\kappa_{a_\ell})$  and  $\mathcal{S}_{a_g, r_g, B_g}(\kappa_{a_g})$ , then the rate of convergence of  $\tilde{f}$  is the rate of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  when  $0 \leq \rho \leq 1/3$ . According to Fan and Truong (1993), this rate seems the minimax rate when  $a_\ell \leq a_g$  and  $r_\ell = r_g = 0$ . In the other cases, the question of the optimality in a minimax sense remains open. Even if the regression function is smoother than  $g$  and  $0 \leq \rho \leq 1/3$ , the rate of convergence of  $\tilde{f}$  has the order of the rate of convergence of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$ , but we do not know if the rate of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$  is the minimax rate (see comments following Theorem 2.1). When  $\rho > 1/3$ , a loss appears between the rate of convergence of  $\tilde{f}$  and the rate of convergence of  $\hat{f}_{\check{m}_\ell, \check{m}_g}$ . This loss only appears, when  $r_\ell > \rho$  or  $r_g > \rho$  (see the comments after Theorem 3.1), in cases where it is negligible with respect to the rate.

**Remark 3.6.** Obviously, the resulting rates for all estimators depend on the noise level  $\sigma$ . The first point is to note that if  $\sigma = 0$ , then by convention  $B = 0 = \rho = 0$ ,  $\lambda = 1$ , and  $Z = X$  is observed. In that case,  $\Gamma(m)/n$  of order  $D_m/n$  has the expected order for the variance term in “usual regression”, when the explanatory variables are observed, and the same holds for the penalties  $\text{pen}_\ell$  and  $\text{pen}_g$ . This order  $D_m/n$  is the expected penalty order for density estimation and nonparametric regression estimation, when there is one model per dimension, as in our case.

The second point is to note that if  $\sigma$  is small, then the procedure automatically selects a dimension  $D_m$  closed to the dimension that would be selected in “usual” density estimation and nonparametric regression estimation.

### CONCLUDING REMARKS

Our estimation procedure provides an adaptive estimator in the sense that its construction does not require any prior knowledge on the smoothness parameters of the regression function  $f$  and of the density  $g$ . This estimation procedure allows to consider various smoothness classes for the regression function and for the density  $g$  when the errors are either ordinary smooth or super smooth, and to give upper bounds for the risk in all the cases.

The resulting rates of convergence for the estimation of  $f$  are given by the worst between the rate for the estimation of  $fg$  and the rate for the estimation of  $g$ . Nevertheless, they are

the minimax rates in cases where lower bounds are available. In the other cases, the resulting rates are in most cases the best rates achievable if the smoothness parameters were known. Some logarithmic loss, negligible compared to the order of the rate, appears, as a price to pay for the adaptation, when both the errors density  $f_\varepsilon$  and  $fg$  are super smooth with  $f_\varepsilon$  strictly smoother than  $fg$ . This logarithmic loss appears when the influence of the noise  $\sigma\varepsilon$  dominates the smoothness properties of  $f$  and  $g$ .

#### 4. PROOFS

**4.1. Proof of Proposition 3.1.** By applying Definition (2.2), for any  $m$  belonging to  $\mathcal{M}_n$ ,  $\hat{\ell}_m$  satisfies  $\gamma_{n,\ell}(\hat{\ell}_m) - \gamma_{n,\ell}(\ell_m^{(n)}) \leq 0$ . Denoting by  $\nu_n(t)$  the centered empirical process,

$$(4.1) \quad \nu_n(t) = \frac{1}{n} \sum_{i=1}^n (Y_i u_i^*(Z_i) - \langle t, \ell \rangle),$$

and by using that  $t \mapsto u_t^*$  is linear we get the following decomposition

$$(4.2) \quad \gamma_{n,\ell}(t) - \gamma_{n,\ell}(s) = \|t - \ell\|_2^2 - \|s - \ell\|_2^2 - 2\nu_n(t - s)$$

and therefore, since by Pythagoras Theorem,  $\|\ell - \ell_m^{(n)}\|_2^2 = \|\ell - \ell_m\|_2^2 + \|\ell - \ell_m^{(n)}\|_2^2$ , we infer that  $\|\ell - \hat{\ell}_m\|_2^2 \leq \|\ell - \ell_m\|_2^2 + \|\ell_m - \ell_m^{(n)}\|_2^2 + 2\nu_n(\hat{\ell}_m - \ell_m^{(n)})$ . Using that  $\hat{a}_{m,j}(\ell) - a_{m,j}(\ell) = \nu_n(\varphi_{m,j})$ , we get

$$(4.3) \quad \nu_n(\hat{\ell}_m - \ell_m^{(n)}) = \sum_{|j| \leq k_n} (\hat{a}_{m,j}(\ell) - a_{m,j}(\ell)) \nu_n(\varphi_{m,j}) = \sum_{|j| \leq k_n} [\nu_n(\varphi_{m,j})]^2,$$

and consequently

$$(4.4) \quad \mathbb{E}\|\ell - \hat{\ell}_m\|_2^2 \leq \|\ell - \ell_m\|_2^2 + \|\ell_m - \ell_m^{(n)}\|_2^2 + 2 \sum_{j \in \mathbb{Z}} \text{Var}[\nu_n(\varphi_{m,j})].$$

Now, since the  $(Y_i, Z_i)$ 's are independent,  $\text{Var}[\nu_n(\varphi_{m,j})] = n^{-1} \text{Var}[Y_1 u_{\varphi_{m,j}}^*(Z_1)]$ , and, arguing as in Comte *et al.* (2005a), by using Parseval's formula we get that

$$(4.5) \quad \sum_{j \in \mathbb{Z}} \text{Var}[\nu_n(\varphi_{m,j})] \leq n^{-1} \left\| \sum_{j \in \mathbb{Z}} |u_{\varphi_{m,j}}^*|^2 \right\|_\infty \mathbb{E}(Y_1^2) \leq \mathbb{E}(Y_1^2) \Delta(m)/n.$$

where  $\Delta$  is defined in Proposition (3.1). Let us study the residual term  $\|\ell_m - \ell_m^{(n)}\|_2^2$ , by simply writing that

$$\|\ell_m - \ell_m^{(n)}\|_2^2 = \sum_{|j| \geq k_n} a_{m,j}^2(\ell) \leq \left( \sup_j j a_{m,j}(\ell) \right)^2 \sum_{|j| \geq k_n} j^{-2}.$$

Now by definition

$$\begin{aligned}
ja_{m,j}(\ell) &= j\sqrt{D_m} \int \varphi(D_mx - j)\ell(x)dx \\
&\leq D_m^{3/2} \int |x|\varphi(D_mx - j)|\ell(x)|dx + \sqrt{D_m} \int |D_mx - j|\varphi(D_mx - j)|\ell(x)|dx \\
&\leq D_m^{3/2} \left( \int |\varphi(D_mx - j)|^2 dx \right)^{1/2} \kappa_{\mathcal{L}}^{1/2} + \sqrt{D_m} \sup_x |x\varphi(x)| \|\ell\|_1.
\end{aligned}$$

Consequently  $ja_{m,j} \leq D_m \|\varphi\|_2 \kappa_{\mathcal{L}}^{1/2} + \sqrt{D_m} \|\ell\|_1 / \pi$ , and  $\|\ell_m - \ell_m^{(n)}\|_2^2 \leq \kappa(\kappa_{\mathcal{L}} + \|\ell\|_1^2) D_m^2 / k_n$ .  $\square$

**4.2. Proof of Proposition 3.2.** The proof of Proposition 3.2 being rather similar to the proof of Theorem 3.2 is omitted. We refer to Comte and Taupin (2004) for further details.

**4.3. Proof of Theorem 3.1.** We only prove the result with  $\mathbb{E}(Y^2)$  in the penalty instead of  $\hat{m}_2(Y)$  and refer to Comte and Taupin (2004) for the complete proof with  $\hat{m}_2(Y)$ , as an application of Rosenthal's inequality (see Rosenthal (1970)).

For the study of  $\tilde{\ell}$ , the main difficulty compared to the study of  $\tilde{g}$  comes from the unbounded noise  $\xi_i$ . By definition,  $\tilde{\ell}$  satisfies that for all  $m \in \mathcal{M}_{n,\ell}$ ,  $\gamma_{n,\ell}(\tilde{\ell}) + \text{pen}_{\ell}(\hat{m}) \leq \gamma_{n,\ell}(\ell_m^{(n)}) + \text{pen}_{\ell}(m)$ . Therefore, by applying (4.2) we get that

$$(4.6) \quad \|\tilde{\ell} - \ell\|_2^2 \leq \|\ell - \ell_m^{(n)}\|_2^2 + 2\nu_n(\tilde{\ell} - \ell_m^{(n)}) + \text{pen}_{\ell}(m) - \text{pen}_{\ell}(\hat{m}).$$

Next, we use that if  $t = t_1 + t_2$  with  $t_1$  in  $S_m$  and  $t_2$  in  $S_{m'}$ , then  $t$  is such that  $t^*$  has its support in  $[-\pi D_{\max(m,m')}, \pi D_{\max(m,m')}]$  and therefore  $t$  belongs to  $S_{m^*}$  where  $m^* = \max(m, m')$ . Denote by  $B_{m,m'}(0, 1)$  the set

$$B_{m,m'}(0, 1) = \{t \in S_{\max(m,m')}^{(n)} / \|t\|_2 = 1\}.$$

It follows that

$$|\nu_n(\tilde{\ell} - \ell_m^{(n)})| \leq \|\tilde{\ell} - \ell_m^{(n)}\|_2 \sup_{t \in B_{m,\hat{m}}(0,1)} |\nu_n(t)|,$$

where  $\nu_n(t)$  is defined by (4.1). Consequently, by using that  $2ab \leq x^{-1}a^2 + xb^2$

$$\|\tilde{\ell} - \ell\|_2^2 \leq \|\ell_m^{(n)} - \ell\|_2^2 + \frac{1}{x} \|\tilde{\ell} - \ell_m^{(n)}\|_2^2 + x \sup_{t \in B_{m,\hat{m}}(0,1)} \nu_n^2(t) + \text{pen}_{\ell}(m) - \text{pen}_{\ell}(\hat{m})$$

and therefore, writing that  $\|\tilde{\ell} - \ell_m^{(n)}\|_2^2 \leq (1 + y^{-1})\|\tilde{\ell} - \ell\|_2^2 + (1 + y)\|\ell - \ell_m^{(n)}\|_2^2$ , with  $y = (x + 1)/(x - 1)$  for  $x > 1$ , we infer that

$$\|\tilde{\ell} - \ell\|_2^2 \leq \left( \frac{x + 1}{x - 1} \right)^2 \|\ell - \ell_m^{(n)}\|_2^2 + \frac{x(x + 1)}{x - 1} \sup_{t \in B_{m,\hat{m}}(0,1)} \nu_n^2(t) + \frac{x + 1}{x - 1} (\text{pen}_{\ell}(m) - \text{pen}_{\ell}(\hat{m})).$$

Choose some positive function  $p_\ell(m, m')$  such that  $xp_\ell(m, m') \leq \text{pen}_\ell(m) + \text{pen}_\ell(m')$ . Then, by denoting by  $\kappa_x = (x + 1)/(x - 1)$ ,

$$(4.7) \quad \begin{aligned} \|\tilde{\ell} - \ell\|_2^2 &\leq \kappa_x^2 \|\ell - \ell_m^{(n)}\|_2^2 + x\kappa_x \left[ \sup_{t \in B_{m, \hat{m}}(0,1)} |\nu_n|^2(t) - p(m, \hat{m}) \right]_+ \\ &\quad + \kappa_x (xp_\ell(m, \hat{m}) + \text{pen}_\ell(m) - \text{pen}_\ell(\hat{m})) \end{aligned}$$

that is

$$(4.8) \quad \|\tilde{\ell} - \ell\|_2^2 \leq \kappa_x^2 \|\ell - \ell_m^{(n)}\|_2^2 + 2\kappa_x \text{pen}_\ell(m) + x\kappa_x W_n(\hat{m}),$$

where

$$(4.9) \quad W_n(m') = \left[ \sup_{t \in B_{m, m'}(0,1)} |\nu_n(t)|^2 - p_\ell(m, m') \right]_+.$$

The main point of the proof lies in studying  $W_n(m')$ , more precisely in finding  $p_\ell(m, m')$  such that

$$(4.10) \quad \mathbb{E}(W_n(\hat{m})) \leq \sum_{m' \in \mathcal{M}_{n, \ell}} \mathbb{E}(W_n(m')) \leq C/n,$$

where  $C$  is a constant. In this case, combining (4.8) and (4.10) we infer that, for all  $m$  in  $\mathcal{M}_{n, \ell}$ ,

$$\mathbb{E}\|\ell - \tilde{\ell}\|_2^2 \leq \kappa_x^2 \|\ell - \ell_m^{(n)}\|_2^2 + 2\kappa_x \text{pen}_\ell(m) + x\kappa_x C/n,$$

which can also be written

$$(4.11) \quad \mathbb{E}\|\ell - \tilde{\ell}\|_2^2 \leq C_x \inf_{m \in \mathcal{M}_{n, \ell}} [\|\ell - \ell_m\|_2^2 + \text{pen}_\ell(m)] + C_x C'/n,$$

where  $C_x = \max(\kappa_x^2, 2\kappa_x)$  suits, when  $k_n \geq n$ , and (3.5) and (3.10) hold. It remains thus to find  $p_\ell(m, m')$  such that (4.10) holds.

The process  $W_n(m')$  is studied by using the decomposition of  $\nu_n(t) = \nu_{n,1}(t) + \nu_{n,2}(t)$  with

$$(4.12) \quad \nu_{n,1}(t) = \frac{1}{n} \sum_{i=1}^n (f(X_i)u_t^*(Z_i) - \langle t, \ell \rangle) \text{ and } \nu_{n,2}(t) = \frac{1}{n} \sum_{i=1}^n \xi_i u_t^*(Z_i).$$

It follows that  $W_n(m') \leq 2W_{n,1}(m') + 2W_{n,2}(m')$  where for  $i = 1, 2$ ,

$$(4.13) \quad W_{n,i}(m') = \left[ \sup_{t \in B_{m, m'}(0,1)} |\nu_{n,i}(t)|^2 - p_i(m, m') \right]_+, \text{ and } p_\ell(m, m') = 2p_1(m, m') + 2p_2(m, m').$$

- Study of  $W_{n,1}$ .

Since under **(A<sub>3</sub>)**,  $f$  is bounded on the support of  $g$ , we apply a standard Talagrand's (1996) inequality (see Lemma 4.1 below that can be a fortiori applied to identically distributed variables):



**Lemma 4.1.** *Let  $U_1, \dots, U_n$  be independent random variables and  $\nu_n(r) = (1/n) \sum_{i=1}^n [r(U_i) - \mathbb{E}(r(U_i))]$  for  $r$  belonging to a countable class  $\mathcal{R}$  of uniformly bounded measurable functions. Then for  $\epsilon > 0$*

$$(4.14) \quad \mathbb{E} \left[ \sup_{r \in \mathcal{R}} |\nu_n(r)|^2 - 2(1 + 2\epsilon)H^2 \right]_+ \leq \frac{6}{K_1} \left( \frac{v}{n} e^{-K_1 \epsilon \frac{nH^2}{v}} + \frac{8M_1^2}{K_1 n^2 C^2(\epsilon)} e^{-\frac{K_1 C(\epsilon) \sqrt{\epsilon} nH}{\sqrt{2} M_1}} \right),$$

with  $C(\epsilon) = \sqrt{1 + \epsilon} - 1$ ,  $K_1$  is a universal constant, and where

$$\sup_{r \in \mathcal{R}} \|r\|_\infty \leq M_1, \quad \mathbb{E} \left( \sup_{r \in \mathcal{R}} |\nu_n(r)| \right) \leq H, \quad \sup_{r \in \mathcal{R}} \frac{1}{n} \sum_{i=1}^n \text{Var}(r(U_i)) \leq v.$$

The inequality (4.14) is a straightforward consequence of Talagrand's (1996) inequality given in Ledoux (1996) (or Birgé and Massart (1998)). Therefore

$$(4.15) \quad \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}(t)|^2 - 2(1 + 2\epsilon_1)\mathbb{H}_1^2 \right]_+ \leq \kappa_1 \left( \frac{v_1}{n} e^{-K_1 \epsilon_1 \frac{n\mathbb{H}_1^2}{v_1}} + \frac{M_1^2}{n^2} e^{-K_2 \sqrt{\epsilon_1} C(\epsilon_1) \frac{n\mathbb{H}_1}{M_1}} \right),$$

where  $K_2 = K_1/\sqrt{2}$  and  $\mathbb{H}_1$ ,  $v_1$  and  $M_1$  are defined by  $\mathbb{E}(\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}(t)|^2) \leq \mathbb{H}_1^2$ ,

$$\sup_{t \in B_{m,m'}(0,1)} \text{Var}(f(X_1)u_t^*(Z_1)) \leq v_1, \quad \text{and} \quad \sup_{t \in B_{m,m'}(0,1)} \|f(X_1)u_t^*(Z_1)\|_\infty \leq M_1.$$

According to (3.3) and (4.5), we propose to take

$$(4.16) \quad M_1 = M_1(m, m') = \kappa_{\infty, G} \sqrt{\lambda_1 \Gamma(m^*)}.$$

For  $v_1$ , denoting by  $P_{j,k}$ , the quantity  $P_{j,k}(m) = \mathbb{E} \left[ f^2(X_1)u_{\varphi_{m,j}}^*(Z_1)u_{\varphi_{m,k}}^*(-Z_1) \right]$ , write

$$\sup_{t \in B_{m,m'}(0,1)} \text{Var}(f(X_1)u_t^*(Z_1)) \leq \left( \sum_{j,k \in \mathbb{Z}} |P_{j,k}(m^*)|^2 \right)^{1/2}.$$

Arguing as in Comte *et al.* (2005a), let us define  $\Delta_2(m, \Psi)$  by

$$\Delta_2(m, \Psi) = D_m^2 \int \int \left| \frac{\varphi^*(x)\varphi^*(y)}{f_\epsilon^*(D_m x)f_\epsilon^*(D_m y)} \Psi^*(D_m(x-y)) \right|^2 dx dy \leq \lambda_2^2(\|\Psi\|_2) \Gamma_2^2(m^*),$$

with

$$(4.17) \quad \Gamma_2(m^*) = D_m^{2\alpha + \min[(1/2 - \rho/2), (1 - \rho)]} \exp\{2\beta\sigma^\rho(\pi D_m^*)^\rho\}$$

and  $\lambda_2(\|\Psi\|_2) = \lambda_2(\alpha, \kappa_0, \beta, \sigma, \rho, \|\Psi\|_2)$  given by

$$(4.18) \quad \lambda_2(\|\Psi\|_2) = \begin{cases} \lambda_1(\alpha, \kappa_0, \beta, \sigma, \rho) & \text{if } \rho > 1, \\ \kappa_0^{-1}(2\pi)^{-1/2} \lambda_1^{1/2}(\alpha, \kappa_0, \beta, \sigma, \rho) (1 + \sigma^2 \pi^2)^{\alpha/2} \|\Psi\|_2 & \text{if } \rho \leq 1. \end{cases}$$

Now, write  $P_{j,k}$  as

$$P_{j,k}(m) = \iint f^2(x)u_{\varphi_{m,j}}^*(x+y)u_{\varphi_{m,k}}^*(-(x+y))g(x)f_\epsilon(y)dx dy$$

that is

$$\begin{aligned}
P_{j,k}(m) &= D_m \iint f^2(x) \iint e^{-i(x+y)uD_m} \frac{\varphi^*(u)e^{iju}}{f_\varepsilon^*(D_mu)} e^{i(x+y)vD_m} \frac{\varphi^*(v)e^{ikv}}{f_\varepsilon^*(D_mv)} dudvg(x)f_\varepsilon(y)dx dy \\
&= D_m \iint \frac{e^{iju+ikv} \varphi^*(u)\varphi^*(v)}{f_\varepsilon^*(D_mu)f_\varepsilon^*(D_mv)} \left( \iint e^{-i(x+y)(u-v)D_m} f^2(x)g(x)f_\varepsilon(y)dx dy \right) dudv \\
&= D_m \iint \frac{e^{iju+ikv} \varphi^*(u)\varphi^*(v)}{f_\varepsilon^*(D_mu)f_\varepsilon^*(D_mv)} [(f^2g) * f_\varepsilon]^*((u-v)D_m) dudv.
\end{aligned}$$

By applying Parseval's formula we get that  $\sum_{j,k} |P_{j,k}(m)|^2$  equals

$$D_m^2 \int \int \left| \frac{\varphi^*(u)\varphi^*(v)}{f_\varepsilon^*(D_mu)f_\varepsilon^*(D_mv)} [(f^2g) * f_\varepsilon]^*((u-v)D_m) \right|^2 dudv = \Delta_2(m, (f^2g) * f_\varepsilon).$$

Since  $\|(f^2g) * f_\varepsilon\|_2 \leq \|f^2g\|_2 \|f_\varepsilon\|_2 = \mathbb{E}^{1/2}(f^2(X_1)) \|f_\varepsilon\|_2$ , and  $\lambda_2(\|f^2g\|_2 \|f_\varepsilon\|_2) \leq \mu_2$ , by using the definition of  $\mu_2$  given in (3.8), we propose to take

$$(4.19) \quad v_1 = v_1(m, m') = \mu_2 \Gamma_2(m^*).$$

Lastly, we have  $\mathbb{E}[\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}(t)|^2] \leq \mathbb{E}(f^2(X_1)) \lambda_1 \Gamma(m^*)/n$  and thus we propose to take

$$(4.20) \quad \mathbb{H}_1^2 = \mathbb{H}_1^2(m, m') = \mathbb{E}(f^2(X_1)) \lambda_1 \Gamma(m^*)/n.$$

It follows from (4.15), (4.16), (4.19) and (4.20) that if

$$p_1(m, m') = 2(1 + 2\epsilon_1) \mathbb{H}_1^2 = 2(1 + 2\epsilon_1) \mathbb{E}(f^2(X_1)) \lambda_1 \Gamma(m^*)/n$$

then

$$(4.21) \quad \mathbb{E}(W_{n,1}(m')) \leq \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} |\nu_{n,1}(t)|^2 - 2(1 + 2\epsilon_1) \mathbb{H}_1^2 \right]_+ \leq A_1(m^*) + B_1(m^*)$$

with

$$(4.22) \quad A_1(m) = K_3 \frac{\mu_2 \Gamma_2(m)}{n} \exp \left( -K_1 \epsilon_1 \mathbb{E}(f^2(X_1)) \frac{\lambda_1 \Gamma(m)}{\mu_2 \Gamma_2(m)} \right)$$

$$(4.23) \quad \text{and } B_1(m) = K_3 \frac{\kappa_{\infty,G}^2 \lambda_1 \Gamma(m)}{n^2} \exp \left\{ -K_2 \sqrt{\epsilon_1} C(\epsilon_1) \frac{\sqrt{\mathbb{E}(f^2(X_1))}}{\kappa_{\infty,G}} \sqrt{n} \right\}.$$

Since  $\forall m \in \mathcal{M}_{n,\ell}$ ,  $\Gamma(m) \leq n$  and  $|\mathcal{M}_{n,\ell}| \leq n$ , there exist some constants  $K_4$  and  $c$  such that

$$\sum_{m \in \mathcal{M}_{n,\ell}} B_1(m^*) \leq K_3 \|f\|_{\infty,G}^2 \lambda_1 \exp[-K_4 \sqrt{\mathbb{E}(f^2(X_1))} \sqrt{n}/\kappa_{\infty,G}] \leq c/n.$$

Let us now come to the study of  $A_1(m^*)$ .

1) **Case**  $0 \leq \rho < 1/3$ . In that case,  $\rho \leq (1/2 - \rho/2)_+$  and the choice  $\epsilon_1 = 1/2$  ensures the convergence of  $\sum_{m' \in \mathcal{M}_{n,\ell}} A_1(m^*)$ . Indeed, if we denote by  $\psi = 2\alpha + \min[(1/2 - \rho/2), (1 - \rho)]$ ,  $\omega = (1/2 - \rho/2)_+$ ,  $K' = \kappa_2 \lambda_1 / \mu_2$ , then for  $a, b \geq 1$ , we infer that

$$\max(a, b)^\psi e^{2\beta\sigma^\rho \pi^\rho \max(a,b)^\rho} e^{-K'\xi^2 \max(a,b)^\omega} \leq (a^\psi e^{2\beta\sigma^\rho \pi^\rho a^\rho} + b^\psi e^{2\beta\sigma^\rho \pi^\rho b^\rho}) e^{-(K'\xi^2/2)(a^\omega + b^\omega)}$$

is bounded by

$$(4.24) \quad a^\psi e^{2\beta\sigma^\rho \pi^\rho a^\rho} e^{-(K'\xi^2/2)a^\omega} e^{-(K'\xi^2/2)b^\omega} + b^\psi e^{2\beta\sigma^\rho \pi^\rho b^\rho} e^{-(K'\xi^2/2)b^\omega}.$$

Since the function  $a \mapsto a^\psi e^{2\beta\sigma^\rho \pi^\rho a^\rho} e^{-(K'\xi^2/2)a^\omega}$  is bounded on  $\mathbb{R}^+$  by a constant, depending on  $\alpha$ ,  $\rho$  and  $K'$  only, and since  $Ak^\rho - \beta k^\omega \leq -(\beta/2)k^\omega$  for any  $k \geq 1$ , it follows that  $\sum_{m' \in \mathcal{M}_{n,\ell}} A_1(m^*) \leq C/n$ .

2) **Case**  $\rho = 1/3$ . In that case,  $\rho = (1/2 - \rho/2)_+$ , and  $\omega = \rho$ . We choose  $\epsilon_1 = \epsilon_1(m, m')$  such that  $2\beta\sigma^\rho \pi^\rho D_{m^*}^\rho - K'\mathbb{E}(f^2(X_1))\epsilon_1 D_{m^*}^\rho = -2\beta\sigma^\rho \pi^\rho D_{m^*}^\rho$  that is, since  $K' = K_1 \lambda_1 / \mu_2$ ,  $\epsilon_1 = \epsilon_1(m, m') = (4\beta\sigma^\rho \pi^\rho \mu_2) / (K_1 \lambda_1 \mathbb{E}(f^2(X_1)))$ .

3) **Case**  $\rho > 1/3$ . In that case,  $\rho > (1/2 - \rho/2)_+$ . Bearing in mind the inequality (4.24) we choose  $\epsilon_1 = \epsilon_1(m, m')$  such that  $2\beta\sigma^\rho \pi^\rho D_{m^*}^\rho - K'\mathbb{E}(f^2(X_1))\epsilon_1 D_{m^*}^\omega = -2\beta\sigma^\rho \pi^\rho D_{m^*}^\rho$  that is, since  $K' = K_1 \lambda_1 / \mu_2$ ,  $\epsilon_1 = \epsilon_1(m, m') = (4\beta\sigma^\rho \pi^\rho \mu_2) / (K_1 \lambda_1 \mathbb{E}(f^2(X_1))) D_{m^*}^{\rho-\omega}$ .

These choices ensure that  $\sum_{m' \in \mathcal{M}_{n,\ell}} A_1(m^*)$  is less than  $C/n$ .

- Study of  $W_{n,2}$ .

Denote by

$$(4.25) \quad \mathbb{H}_\xi^2(m, m') = (n^{-1} \sum_{i=1}^n \xi_i^2) \lambda_1 \Gamma(m^*) / n,$$

with  $(n^{-1} \sum_{i=1}^n \xi_i^2) \lambda_1 \Gamma(m) / n = (n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2) \lambda_1 \Gamma(m) / n + \sigma_\xi^2 \lambda_1 \Gamma(m) / n$  bounded by

$$(n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2) \mathbf{1}_{\{n^{-1} \sum_{i=1}^n (\xi_i^2 - \sigma_\xi^2) > \sigma_\xi^2/2\}} \lambda_1 \Gamma(m) / n + 3\sigma_\xi^2 \lambda_1 \Gamma(m) / (2n).$$

Consequently  $\mathbb{H}_\xi^2(m, m') \leq \mathbb{H}_{\xi,1}(m, m') + \mathbb{H}_{\xi,2}(m, m')$  where

$$\mathbb{H}_{\xi,1}(m, m') = (n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2) \mathbf{1}_{\{n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2 > \sigma_\xi^2/2\}} \lambda_1 \Gamma(m^*) / n \text{ and } \mathbb{H}_{\xi,2}(m, m') = 3\sigma_\xi^2 \lambda_1 \Gamma(m^*) / (2n).$$

By applying (4.12) we infer that  $\mathbb{E}[\sup_{t \in B_{m,m'}(0,1)} |\nu_{n,2}(t)|^2 - p_2(m, m')]_+$  is bounded by

$$\begin{aligned} \mathbb{E}[2 \sup_{t \in B_{m,m'}(0,1)} (n^{-1} \sum_{i=1}^n \xi_i (u_t^*(Z_i) - \langle t, g \rangle))^2 - 4(1 + 2\epsilon_2) \mathbb{H}_\xi^2(m, m')]_+ + 2\|g\|_2^2 \mathbb{E}[(n^{-1} \sum_{i=1}^n \xi_i^2)^2] \\ + \mathbb{E}[4(1 + 2\epsilon_2) \mathbb{H}_\xi^2(m, m') - p_2(m, m')]_+, \end{aligned}$$

that is

$$\begin{aligned}
(4.26) \quad & \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} |\nu_{n,2}(t)|^2 - p_2(m, m') \right]_+ \\
& \leq 2 \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} \left( n^{-1} \sum_{i=1}^n \xi_i (u_t^*(Z_i) - \langle t, g \rangle) \right)^2 - 2(1 + 2\epsilon_2) \mathbb{H}_\xi^2(m, m') \right]_+ + 2 \|g\|_2^2 \sigma_\xi^2 / n \\
& \quad + 4(1 + 2\epsilon_2) \mathbb{E} |\mathbb{H}_{\xi,1}(m, m')| + \mathbb{E} [4(1 + 2\epsilon_2) \mathbb{H}_{\xi,2}(m, m') - p_2(m, m')]_+.
\end{aligned}$$

Since we only consider dimensions  $D_m$  such that  $\Gamma(m)/n$  is bounded by some constant  $\kappa$ , we get that for some  $p \geq 2$ ,  $\mathbb{E} |\mathbb{H}_{\xi,1}(m, m')|$  is bounded by

$$\kappa \lambda_1 \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2 \mathbf{1}_{\{n^{-1} |\sum_{i=1}^n (\xi_i^2 - \sigma_\xi^2)| > \sigma_\xi^2 / 2\}} \right] \leq \kappa \lambda_1 2^{p-1} \mathbb{E} \left[ |n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2|^p \right] / \sigma_\xi^{2(p-1)}$$

According to Rosenthal's inequality (see Rosenthal (1970)), we find that, for  $\sigma_{\xi,p}^p := \mathbb{E} (|\xi|^p)$ ,  $\sigma_{\xi,2}^2 = \sigma_\xi^2$ ,

$$\mathbb{E} \left| n^{-1} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2 \right|^p \leq C'(p) \left( \sigma_{\xi,2p}^{2p} n^{1-p} + \sigma_{\xi,4}^{2p} n^{-p/2} \right).$$

Now, the assumption  $(\mathbf{A}_1)$  implies that  $\alpha > 1/2$ , therefore  $|\mathcal{M}_n| \leq \sqrt{n}$  and consequently, by choosing  $p = 3$  this leads to  $\sum_{m' \in \mathcal{M}_n} \mathbb{E} |\mathbb{H}_{\xi,1}(m, m')| \leq C(\sigma_{\xi,6}, \sigma_\xi) / n$ . The last term of the inequality (4.26) vanishes as soon as

$$(4.27) \quad p_2(m, m') = 4(1 + 2\epsilon_2) \mathbb{H}_{\xi,2}(m, m') = 6(1 + 2\epsilon_2) \lambda_1 \sigma_\xi^2 \Gamma(m^*) / n.$$

For this choice of  $p_2(m, m')$ , the inequality (4.26) becomes  $\mathbb{E} [\sup_{t \in B_{m,\hat{m}}(0,1)} |\nu_{n,2}(t)|^2 - p_2(m, \hat{m})]_+$  is less than

$$\begin{aligned}
2 \sum_{m' \in \mathcal{M}_{n,\ell}} \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} \left( n^{-1} \sum_{i=1}^n \xi_i (u_t^*(Z_i) - \langle t, g \rangle) \right)^2 - 2(1 + 2\epsilon_2) \mathbb{H}_\xi^2(m, m') \right]_+ \\
+ 2 \|g\|_2^2 \sigma_\xi^2 / n + 4C(1 + 2\epsilon_2) / n.
\end{aligned}$$

Then we apply the following Lemma to reach the same kind of result as (4.15) for  $W_{n,1}$ .

**Lemma 4.2.** *Under the assumptions of Theorem 3.1, if  $\mathbb{E} |\xi_1|^6 < \infty$ , then for some given  $\epsilon_2 > 0$ :*

$$\begin{aligned}
(4.28) \quad & \sum_{m' \in \mathcal{M}_{n,\ell}} \mathbb{E} \left[ \sup_{t \in B_{m,m'}(0,1)} \left( \frac{1}{n} \sum_{i=1}^n \xi_i (u_t^*(Z_i) - \langle t, g \rangle) \right)^2 - 2(1 + 2\epsilon_2) \mathbb{H}_\xi^2(m, m') \right]_+ \\
& \leq K_1 \left\{ \sum_{m' \in \mathcal{M}_{n,\ell}} \left[ \frac{\sigma_\xi^2 \mu_2 \Gamma_2(m^*)}{n} \exp \left( -K_1 \epsilon_2 \frac{\lambda_1 \Gamma(m^*)}{\mu_2 \Gamma_2(m^*)} \right) \right] + \left( 1 + \frac{\ln^4(n)}{\sqrt{n}} \right) \frac{1}{n} \right\},
\end{aligned}$$

where  $\mu_2$  and  $\Gamma_2(m)$  are defined by (3.8) and (4.17) and  $K_1$  is a constant depending on the moments of  $\xi$ . The constant  $\mu_2$  can be replaced by  $\lambda_2(\|h\|_2)$  where  $\lambda_2$  is defined by (4.18).

By analogy with (4.22) we denote by

$$(4.29) \quad A_2(m^*) = \frac{K_1 \sigma_\xi^2}{n} \mu_2 \Gamma_2(m^*) \exp\left(-K_1 \epsilon_2 \frac{\lambda_1 \Gamma(m^*)}{\mu_2 \Gamma_2(m^*)}\right) = \frac{K_1 \sigma_\xi^2 \mu_2 \Gamma_2(m^*)}{n} \exp\left(-\kappa_2 \epsilon_2 \frac{\lambda_1}{\mu_2} D_{m^*}^{(1/2-\rho/2)_+}\right).$$

With  $p_2(m, m')$  given by (4.27), by gathering (4.15) and (4.28), we find, for  $W_{n,2}$  defined by (4.13),

$$\mathbb{E}(W_{n,2}(\hat{m})) \leq K \sum_{m' \in \mathcal{M}_n} A_2(m^*) + C(1 + \ln(n)^6/n)/n + K'/n.$$

The sum  $\sum_{m' \in \mathcal{M}_n} A_2(m^*)$  is bounded in the same way as the sum  $\sum_{m' \in \mathcal{M}_n} A_1(m^*)$  with  $\epsilon_2 = \epsilon_1 = 1/2$  if  $0 \leq \rho < 1/3$  and  $\epsilon_1(m, m')$  replaced by  $\epsilon_2 = \epsilon_2(m, m') = \mathbb{E}(f^2(X_1))\epsilon_1(m, m')$ , when  $\rho \geq 1/3$  that is  $\epsilon_2(m, m') = (4\beta\sigma^\rho\pi^\rho\mu_2)/(K_1\lambda_1)D_{m^*}^{\rho-\omega}$ . These choices ensure that  $\sum_{m' \in \mathcal{M}_{n,\ell}} A_2(m^*)$  is less than  $C/n$ . The result follows by taking as announced in (4.13),  $p_\ell(m, m') = 2p_1(m, m') + 2p_2(m, m')$ , that is  $p_\ell(m, m') = 4[(1 + 2\epsilon_1(m, m'))\mathbb{E}(f^2(X_1)) + 3(1 + 2\epsilon_2(m, m'))\sigma_\xi^2]\lambda_1\Gamma(m^*)/n$ , and more precisely if  $0 \leq \rho < 1/3$ ,

$$(4.30) \quad p_\ell(m, m') = 24\mathbb{E}(Y_1^2)\lambda_1\Gamma(m^*)/n,$$

and if  $\rho \geq 1/3$ ,

$$(4.31) \quad p_\ell(m, m') = 4[3\mathbb{E}(Y_1^2) + 32\beta\sigma^\rho\pi^\rho\mu_2 D_{m^*}^{\rho-\omega}/k_1\lambda_1]\lambda_1\Gamma(m^*)/n.$$

Consequently if  $0 \leq \rho < 1/3$ , we take  $\text{pen}_\ell(m) = \kappa\mathbb{E}(Y_1^2)\lambda_1\Gamma(m)/n$ , and if  $\rho \geq 1/3$  we take  $\text{pen}_\ell(m) = \kappa[\mathbb{E}(Y_1^2) + \beta\sigma^\rho\pi^\rho\mu_2 D_m^{\rho-\omega}/k_1\lambda_1]\lambda_1\Gamma(m)/n$ , for some numerical constants  $\kappa$ . Note that for  $\rho = 1/3$ ,  $\rho - \omega = 0$  and the second penalty has the same order as the first one with a different multiplicative constant.  $\square$

**4.4. Proof of Lemma 4.2, by using a conditioning argument.** We work conditionally to the  $\xi_i$ 's and  $\mathbb{E}_\xi$  and  $\mathbb{P}_\xi$  denote the conditional expectations and probability for fixed  $\xi_1, \dots, \xi_n$ .

We apply Lemma 4.1 with  $f_t(\xi_i, Z_i) = \xi_i u_t^*(Z_i)$ , conditionally to the  $\xi_i$ 's to the random variables  $(\xi_1, Z_1), \dots, (\xi_n, Z_n)$  which are independent but non identically distributed since the  $\xi_i$ 's are fixed constants. Let  $Q_{j,k} = \mathbb{E}[u_{\varphi_{m,j}}^*(Z_1)u_{\varphi_{m,k}}^*(-Z_1)]$ . Straightforward calculations give that for  $\mathbb{H}_\xi(m, m')$  defined in (4.25) we have

$$\mathbb{E}_\xi^2\left[\sup_{t \in B_{m,m'}(0,1)} n^{-1} \sum_{i=1}^n \xi_i (u_t^*(Z_i) - \langle t, g \rangle)\right] \leq \mathbb{H}_\xi^2(m, m').$$

Again, arguing as in Comte *et al.* (2005a),  $\sum_{j,k} |Q_{j,k}|^2 \leq \Delta_2(m, h) \leq \lambda_2(\|h\|_2)\Gamma_2(m, \|f_\varepsilon\|_2)$  with  $\|h\|_2 \leq \|f_\varepsilon\|_2$ , where  $\Delta_2(m, h)$  is defined by (4.17),  $\lambda_2$  by (4.18),  $\Gamma_2(m)$  by (4.17),  $\mu_2$  by

(3.8). We now write that

$$\sup_{t \in B_{m,m'}(0,1)} \left( n^{-1} \sum_{i=1}^n \text{Var}_{\xi}(\xi_i u_t^*(Z_i)) \right) \leq \left( n^{-1} \sum_{i=1}^n \xi_i^2 \right) \mu_2 \Gamma_2(m^*, \|f_{\varepsilon}\|_2)$$

and thus we take

$$v_{\xi}(m, m') = \left( n^{-1} \sum_{i=1}^n \xi_i^2 \right) \mu_2 \Gamma_2(m^*, \|f_{\varepsilon}\|_2).$$

Lastly, since

$$\sup_{t \in B_{m,m'}(0,1)} \|f_t\|_{\infty} \leq 2 \max_{1 \leq i \leq n} |\xi_i| \sqrt{\Delta(m^*)} \leq 2 \max_{1 \leq i \leq n} |\xi_i| \sqrt{\lambda_1 \Gamma(m^*)}$$

we take  $M_{1,\xi}(m, m') = 2 \max_{1 \leq i \leq n} |\xi_i| \sqrt{\lambda_1 \Gamma(m^*)}$ . By applying Lemma 4.1, we get for some constants  $\kappa_1, \kappa_2, \kappa_3$

$$\begin{aligned} \mathbb{E}_{\xi} \left[ \sup_{t \in B_{m,m'}(0,1)} \nu_{n,1}^2(t) - 2(1 + 2\epsilon) \mathbb{H}_{\xi}^2 \right]_{+} &\leq \kappa_1 \left[ \frac{\mu_2 \Gamma_2(m^*)}{n^2} \left( \sum_{i=1}^n \xi_i^2 \right) \exp \left\{ -\kappa_2 \epsilon \frac{\lambda_1 \Gamma(m^*)}{\mu_2 \Gamma_2(m^*)} \right\} \right. \\ &\quad \left. + \frac{\lambda_1 \Gamma(m^*)}{n^2} \left( \max_{1 \leq i \leq n} \xi_i^2 \right) \exp \left\{ -\kappa_3 \sqrt{\epsilon} C(\epsilon) \frac{\sqrt{\sum_{i=1}^n \xi_i^2}}{\max_i |\xi_i|} \right\} \right] \end{aligned}$$

To relax the conditioning, it suffices to integrate with respect to the law of the  $\xi_i$ 's the above expression. The first term in the bound simply becomes:

$$\sigma_{\xi}^2 \mu_2 \Gamma_2(m^*) \exp[-\kappa_2 \epsilon \lambda_1 \Gamma(m^*) / (\mu_2 \Gamma_2(m^*))] / n$$

and has the same order as in the case of bounded variables. The second term is bounded by

$$(4.32) \quad \frac{\lambda_1 \Gamma(m^*)}{n^2} \mathbb{E} \left[ \left( \max |\xi_i|^2 \right) \exp \left( -\kappa_3 \sqrt{\epsilon} C(\epsilon) \frac{\sqrt{\sum_{i=1}^n \xi_i^2}}{\max_{1 \leq i \leq n} |\xi_i|} \right) \right].$$

Since we only consider dimensions  $D_m$  such that the penalty term is bounded, we have  $\Gamma(m)/n \leq K$  and the sum of the above terms for  $m \in \mathcal{M}_{n,\ell}$  and  $|\mathcal{M}_{n,\ell}| \leq n$  is less than

$$\lambda_1 \mathbb{E} \left[ \left( \max_{1 \leq i \leq n} \xi_i^2 \right) \exp \left( -\kappa_3 \sqrt{\epsilon} C(\epsilon) \frac{\sqrt{\sum_{i=1}^n \xi_i^2}}{\max_{1 \leq i \leq n} |\xi_i|} \right) \right].$$

We need to study when such a term is less than  $c/n$  for some constant  $c$ . We bound  $\max_i |\xi_i|$  by  $b$  on the set  $\{\max_i |\xi_i| \leq b\}$  and the exponential by 1 on the set  $\{\max_i |\xi_i| \geq b\}$  and by

denoting  $\mu_\epsilon = \kappa_3 \sqrt{\epsilon} C(\epsilon)$ , this yields

$$\begin{aligned}
& \mathbb{E} \left[ \max_{1 \leq i \leq n} \xi_i^2 \exp \left( -\mu_\epsilon \sqrt{\frac{\sum_{i=1}^n \xi_i^2}{\max_{1 \leq i \leq n} \xi_i^2}} \right) \right] \\
& \leq b^2 \mathbb{E} \left( \exp(-\mu_\epsilon \frac{\sqrt{\sum_{i=1}^n \xi_i^2}}{b}) \right) + \mathbb{E} \left( \max_{1 \leq i \leq n} \xi_i^2 \mathbf{1}_{\{\max_{1 \leq i \leq n} |\xi_i| \geq b\}} \right) \\
& \leq b^2 \left[ \mathbb{E} \left( \exp(-\mu_\epsilon \sqrt{n\sigma_\xi^2/(2b^2)}) \right) + \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2 \right| \geq \sigma_\xi^2/2 \right) \right] + b^{-r} \mathbb{E}(\max_{1 \leq i \leq n} |\xi_i|^{r+2}) \\
& \leq b^2 e^{-\mu_\epsilon \sqrt{n}\sigma_\xi/(\sqrt{2}b)} + b^2 2^p \sigma_\xi^{-2p} \mathbb{E} \left( \left| \frac{1}{n} \sum_{i=1}^n \xi_i^2 - \sigma_\xi^2 \right|^p \right) + b^{-r} \mathbb{E}(\max_{1 \leq i \leq n} |\xi_i|^{r+2}).
\end{aligned}$$

Again by applying Rosenthal's inequality (see Rosenthal (1970)), we get that

$$\begin{aligned}
& \mathbb{E} \left[ \max_{1 \leq i \leq n} \xi_i^2 \exp \left( -\mu_\epsilon \sqrt{\frac{\sum_{i=1}^n \xi_i^2}{\max_{1 \leq i \leq n} \xi_i^2}} \right) \right] \\
& \leq b^2 e^{-\mu_\epsilon \sqrt{n}\sigma_\xi/(\sqrt{2}b)} + b^2 \frac{2^p}{\sigma_\xi^{2p}} \frac{C(p)}{n^p} [n \mathbb{E}(|\xi_1^2 - \sigma_\xi^2|^p) + (n \mathbb{E}(\xi_1^4))^{p/2}] + n \mathbb{E}(|\xi_1|^{r+2}) b^{-r}
\end{aligned}$$

also bounded by

$$b^2 e^{-\mu_\epsilon \sqrt{n}\sigma_\xi/(\sqrt{2}b)} + C'(p) b^2 \sigma_{\xi,2p}^{2p} 2^p \sigma_\xi^{-2p} [n^{1-p} + n^{-p/2}] + n \sigma_{\xi,r+2}^{r+2} b^{-r}.$$

Since  $\mathbb{E}|\xi_1|^6 < \infty$ , we take  $p = 3$ ,  $r = 4$ ,  $b = \sigma_\xi \sqrt{\epsilon} C(\epsilon) \kappa_3 \sqrt{n} / [2\sqrt{2}(\ln(n) - \ln \ln n)]$  and for any  $n \geq 3$ , and for  $C_1$  and  $C_2$  some constants depending on the moments of  $\xi$ , we find that

$$\mathbb{E} \left\{ \left( \max_{1 \leq i \leq n} \xi_i^2 \right) \exp \left( -\kappa_3 \sqrt{\epsilon} C(\epsilon) \sqrt{\frac{\sum_{i=1}^n \xi_i^2}{\max_{1 \leq i \leq n} \xi_i^2}} \right) \right\} \leq \frac{C_1}{\sqrt{n}} + C_2 \left( \frac{\ln^4(n)}{\sqrt{n}} \right) \frac{1}{\sqrt{n}}.$$

Then the sum over  $\mathcal{M}_{n,\ell}$  with cardinality less than  $\sqrt{n}$  of the terms in (4.32) is bounded by  $C(1 + \ln(n)^4/\sqrt{n})/n$  for some constant  $C$ , by using again that  $\Gamma(m^*)/n$  is bounded.

**4.5. Proof of Theorem 3.2.** Let  $\tilde{E}_n$  be the event  $\tilde{E}_n = \{\|g - \tilde{g}\|_{\infty, A} \leq g_0/2\}$ . Since  $g(x) \geq g_0$  for any  $x$  in  $A$ , then, on  $\tilde{E}_n$ ,  $\tilde{g}(x) \geq g_0/2$  also for any  $x$  in  $A$ . It follows that

$$(4.33) \quad \mathbb{E} \|(f - \tilde{f}) \mathbf{1}_A \mathbf{1}_{\tilde{E}_n}\|_2^2 \leq 8g_0^{-2} \mathbb{E} \|\tilde{\ell} - \ell\|_2^2 + 8\|\ell\|_{\infty, A}^2 g_0^{-4} \mathbb{E} \|\tilde{g} - g\|_2^2,$$

where  $\|\ell\|_{\infty, A} \leq g_1 \kappa_{\infty, G}$ . Using that  $\|\tilde{f}\|_{\infty, A} \leq a_n$ , we obtain

$$(4.34) \quad \mathbb{E} \|(f - \tilde{f}) \mathbf{1}_A \mathbf{1}_{\tilde{E}_n^c}\|_2^2 \leq 2(a_n^2 + \|f\|_{\infty, A}^2) \lambda(A) \mathbb{P}(\tilde{E}_n^c),$$

where  $\lambda(A) = \int_A dx$ . It follows that for  $\hat{m}_\ell = \hat{m}_\ell(n)$ ,  $\hat{m}_g = \hat{m}_g(n)$ , if  $a_n \mathbb{P}(\tilde{E}_n^c) = o(n^{-1})$ , then (3.11) is proved by applying Theorem 3.1. We now come to the study of  $\mathbb{P}(\tilde{E}_n^c)$  by writing that  $\mathbb{P}(\tilde{E}_n^c) = \mathbb{P}(\|g - \tilde{g}\|_{\infty} > g_0/2) = \mathbb{P}(\|g - g_{\hat{m}_g}^{(n)} + g_{\hat{m}_g}^{(n)} - \tilde{g}\|_{\infty} > g_0/2)$ . By applying Lemma 4.3:

**Lemma 4.3.** *Let  $g$  belongs to  $\mathcal{S}_{a_g, \nu_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$  with  $a_g > 1/2$ . Then for  $t \in S_m$ ,  $\|t\|_\infty \leq \sqrt{D_m} \|t\|_2$  and  $\|g - g_m\|_\infty \leq (2\pi)^{-1} \sqrt{\pi D_m} ((\pi D_m)^2 + 1)^{-a_g/2} \exp(-B_g |\pi D_m|^{r_g}) A_g^{1/2}$ .*

and by arguing as for  $\|\ell_m - \ell_m^{(n)}\|_2^2$ , we get that  $\|g - g_{\hat{m}_g}^{(n)}\|_\infty \leq \|g - g_{\hat{m}_g}\|_\infty + \|g_{\hat{m}_g} - g_{\hat{m}_g}^{(n)}\|_\infty$  also bounded by

$$\sqrt{\kappa(\kappa_{\mathcal{G}} + 1)} D_{\hat{m}_g}^{3/2} / \sqrt{k_n} + (2\pi)^{-1} \sqrt{\pi D_{\hat{m}_g}} ((\pi D_{\hat{m}_g})^2 + 1)^{-a_g/2} \exp(-B_g |\pi D_{\hat{m}_g}|^{\nu_g}) A_g^{1/2}.$$

Consequently,  $\|g - g_{\hat{m}_g}^{(n)}\|_\infty$  tends to zero as soon as  $g$  belongs to some space  $\mathcal{S}_{a_g, \nu_g, B_g}(\kappa_{a_g})$  defined by  $(\mathbf{R}_1)$  with  $a_g > 1/2$  if  $r_g = 0$  and since  $k_n \geq n^{3/2}$  and  $D_{\hat{m}_g} = o(\sqrt{n})$  for  $\alpha > 1/2$ . It follows that for  $n$  large enough,  $\|g - g_{\hat{m}_g}^{(n)}\|_\infty \leq g_0/4$  and consequently  $\mathbb{P}(\tilde{E}_n^c) \leq \mathbb{P}[\|g_{\hat{m}_g}^{(n)} - \tilde{g}\|_\infty > g_0/4]$ . By applying again Lemma 4.3, since  $g_{\hat{m}_g}^{(n)} - \tilde{g}$  belongs to  $S_{\hat{m}_g}$ , we get that

$$(4.35) \quad \mathbb{P}(\tilde{E}_n^c) \leq \mathbb{P}[\|g_{\hat{m}_g}^{(n)} - \tilde{g}\|_2 > g_0 / (4\sqrt{D_{\hat{m}_g}})].$$

In this context, we have

$$(4.36) \quad \|g_{\hat{m}_g}^{(n)} - \tilde{g}_{\hat{m}_g}\|_2^2 = \sum_{|j| \leq k_n} (\hat{a}_{\hat{m}_g, j} - a_{\hat{m}_g, j})^2 = \sum_{|j| \leq k_n} \nu_{n, g}^2(\varphi_{\hat{m}_g, j}) = \sup_{t \in B_{\hat{m}_g}(0, 1)} \nu_{n, g}^2(t).$$

Consequently,

$$\begin{aligned} \mathbb{P}(\tilde{E}_n^c) &\leq \mathbb{P}\left[\sup_{t \in B_{\hat{m}_g}(0, 1)} |\nu_{n, g}(t)| \geq g_0 / (4\sqrt{D_{\hat{m}_g}})\right] \leq \sup_{m \in \mathcal{M}_n} \mathbb{P}\left[\sup_{t \in B_{\hat{m}_g}(0, 1)} |\nu_{n, g}(t)| \geq g_0 / (4\sqrt{D_m})\right] \\ &\leq \sum_{m \in \mathcal{M}_n} \mathbb{P}\left[\sup_{t \in B_{\hat{m}_g}(0, 1)} |\nu_{n, g}(t)| \geq g_0 / (4\sqrt{D_m})\right]. \end{aligned}$$

We apply Talagrand's (1996) inequality as given in Birgé and Massart (1998), to get that if we take  $\lambda = g_0 / (8\sqrt{D_m})$  and if we ensure  $2H < g_0 / (8\sqrt{D_m})$ , then  $\mathbb{P}[\sup_{t \in B_m(0, 1)} |\nu_{n, g}(t)| \geq g_0 / (4\sqrt{D_m})] \leq 3 \exp[-K'_1 n (\min[(D_m v)^{-1}, (M_1 \sqrt{D_m})^{-1})]]$ . This yields

$$(4.37) \quad \mathbb{P}(\tilde{E}_n^c) \leq K \sum_{m \in \mathcal{M}_n} \{\exp[-K'_1 n / (M_1 \sqrt{D_m})] + \exp[-K'_1 n / (D_m v)]\}.$$

Since we only consider  $D_m$  such that  $D_m \leq \sqrt{n}$ ,

$$a_n |\mathcal{M}_n| \exp[-K'_1 n / (M_1 \sqrt{D_m})] \leq a_n |\mathcal{M}_n| \exp(-K'' n^{1/4}) = o(n^{-1}).$$

We only consider  $D_m$  such that  $\Gamma(m)/n$  tends to zero. Consequently, when  $\rho > 0$  then  $D_m \leq (\ln n / (2\beta\sigma^\rho + 1))^{1/\rho}$  which combined with the fact that  $v \leq D_m^{2\alpha+1-\rho} \exp(2\beta\sigma^\rho \pi^\rho D_m^\rho)$  gives that  $a_n |\mathcal{M}_n| \exp(-K'_1 n / (D_m v)) = o(1/n)$ .

When  $\rho = 0$ , then  $v = \mu_1 D_m^{2\alpha+1/2}$  and consequently, as  $D_m \leq (n / \ln(n))^{1/(2\alpha+1)} \leq n^{1/(2\alpha+1)}$ ,

$$\exp(-K'_1 n / (D_m v)) \leq \exp(-K'' n / (D_m^{2\alpha+3/2})) \leq \exp(-K'' n^{1/(4(\alpha+1))}).$$

Analogously,  $\sqrt{D_m} H \leq 1 / \sqrt{\ln(n)}$  in the worst case corresponding to  $\rho = 0$ , for  $D_m \leq (n / \ln(n))^{1/(2\alpha+2)}$ , tends to zero and therefore is bounded by  $g_0/8$  for  $n$  great enough. We



conclude that if we only consider  $D_m$  such that  $D_m \leq n^{1/(2\alpha+2)}$  then  $a_n \mathbb{P}(\tilde{E}_n^c) = o(1/n)$ , and the result follows by applying the inequalities (4.33) and (4.34).  $\square$

**Proof of Lemma 4.3.** For  $t \in S_m$ , written as  $t(x) = \sum_{j \in \mathbb{Z}} \langle t, \varphi_{m,j} \rangle \varphi_{m,j}(x)$  and  $|t(x)|^2 \leq \sum_{j \in \mathbb{Z}} |\langle t, \varphi_{m,j} \rangle|^2 \sum_{j \in \mathbb{Z}} |(\varphi_{m,j}^*)^*(-x)|^2 / (2\pi)^2$  with by applying Parseval's Formula

$$\sum_{j \in \mathbb{Z}} |\langle t, \varphi_{m,j} \rangle|^2 \sum_{j \in \mathbb{Z}} |(\varphi_{m,j}^*)^*(-x)|^2 / (2\pi)^2 = \|t\|_2^2 D_m \int \varphi^*(u)^2 du / (2\pi) = D_m \|t\|_2^2.$$

Let  $b$  such that  $1/2 < b < a_g$ . Since  $\|g - g_m\|_\infty \leq (2\pi)^{-1} \int_{|x| \geq \pi D_m} |g^*(x)| dx$  we get that

$$\|g - g_m\|_\infty \leq (2\pi)^{-1} ((\pi D_m)^2 + 1)^{-(a_g-b)/2} e^{-B_g |\pi D_m|^{r_g}} \int_{|x| \geq \pi D_m} |g^*(x)| (x^2 + 1)^{(a_g-b)/2} e^{B_g |x|^{r_g}} dx$$

also bounded by

$$\begin{aligned} & \frac{1}{2\pi} ((\pi D_m)^2 + 1)^{-(a_g-b)/2} \exp(-B_g |\pi D_m|^{r_g}) \kappa_{a_g}^{1/2} \sqrt{\int_{|x| \geq \pi D_m} (x^2 + 1)^{-b} dx} \\ & \leq (2\pi)^{-1} ((\pi D_m)^2 + 1)^{-(a_g-b)/2} \exp(-B_g |\pi D_m|^{r_g}) \kappa_{a_g}^{1/2} (\pi D_m)^{1/2-b} \\ & \leq (2\pi)^{-1} \sqrt{\pi D_m} ((\pi D_m)^2 + 1)^{-a_g/2} \exp(-B_g |\pi D_m|^{r_g}) \kappa_{a_g}^{1/2}. \end{aligned}$$

$\square$

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