New M-estimators in semiparametric regression with errors in variables
Cristina Butucea, Marie-Luce Taupin

To cite this version:
Cristina Butucea, Marie-Luce Taupin. New M-estimators in semiparametric regression with errors in variables. Annales de l’Institut Henri Poincaré (B) Probabilités et Statistiques, 2008, 44 (3), pp.393-421. hal-00013229v2

HAL Id: hal-00013229
https://hal.science/hal-00013229v2
Submitted on 4 Nov 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
NEW M-ESTIMATORS IN SEMIPARAMETRIC REGRESSION WITH ERRORS IN VARIABLES

CRISTINA BUTUCEA$^{1,2}$, MARIE-LUCE TAUPIN$^{3,4}$

Abstract. In the regression model with errors in variables, we observe $n$ i.i.d. copies of $(Y, Z)$ satisfying $Y = f_{\theta_0}(X) + \xi$ and $Z = X + \varepsilon$ involving independent and unobserved random variables $X, \xi, \varepsilon$ plus a regression function $f_{\theta_0}$, known up to some finite dimensional $\theta_0$. The common densities of the $X_i$’s and of the $\xi_i$’s are unknown whereas the distribution of $\varepsilon$ is completely known. We aim at estimating the parameter $\theta_0$ by using the observations $(Y_1, Z_1), \cdots, (Y_n, Z_n)$. We propose two estimation procedures based on the least square criterion $\tilde{S}_{\theta_0, g}(\theta) = E_{\theta_0, g}[(Y - f_{\theta}(X))^2w(X)]$ where $w$ is some weight function, to be chosen. In the first estimation procedure, $w$ does not depend on $\theta$ and the distribution of the $\xi$’s is unknown. The second estimation procedure is based on $S_{\theta_0, g}(\theta) = E_{\theta_0, g}[(Y - f_{\theta}(X))^2 - \sigma^2_\varepsilon]w_{\theta}(X)$ where $w_{\theta}$ is positive weight function, to be chosen, and requires the knowledge of $\sigma^2_\varepsilon = \text{Var}(\varepsilon)$. In both cases, we propose two estimators and derive upper bounds for the risk of those estimators, depending on the smoothness of the errors density $p_\varepsilon$ and on the smoothness properties of $w(x)f_\theta(x)$ or $w_{\theta}(x)f_\theta(x)$ with respect to $x$. Furthermore we give sufficient conditions that ensure that the parametric rate of convergence is achieved. We provide practical recipes for the choice of $w$ or $w_{\theta}$ in the case of nonlinear regression functions which are smooth on pieces allowing to gain in the order of the rate of convergence, up to the parametric rate in some cases.

1 Université Paris X, Modal’X, 200, ave. de la République, 92001 Nanterre Cedex, France
2 Université Paris VI, PMA, 175, rue de Chevaleret, 75013 Paris, France (butucea@ccr.jussieu.fr)
3 Université Paris Descartes, IUT de Paris, 143 ave. de Versailles, 75016 Paris, France
4 Université Paris-Sud, Bât. 425, Département de Mathématiques, 91405 Orsay Cedex, France (marie-luce.taupin@math.u-psud.fr)

November 4, 2005

Keywords: Asymptotic normality, consistency, deconvolution kernel estimator, errors-in-variables model, M-estimators, ordinary smooth and super-smooth functions, rates of convergence, semi-parametric nonlinear regression.

1. Introduction

We consider the regression model with errors in variables where one observe $n$ independent and identically distributed (i.i.d.) copies of $(Y, Z)$ satisfying

$$\begin{cases}
Y = f_{\theta_0}(X) + \xi \\
Z = X + \varepsilon,
\end{cases}$$

involving independent and unobserved random variables $X, \xi, \varepsilon$, plus a regression function $f_{\theta_0}$ known up to a finite dimensional parameter $\theta_0$, belonging to the interior of a compact set of $\Theta \subset \mathbb{R}^d$. The common densities of the $X_i$'s and of the $\xi_i$'s are unknown, with $\mathbb{E}(\xi) = 0$, whereas the density of the errors $\varepsilon_i$'s is completely known.

In this context we aim at estimating the finite dimensional parameter $\theta_0$ in presence of a functional nuisance parameter $g$, the density of the $X$.

Previous known results. This model has been widely studied with first results written in the 1950's (see for instance Reiersøl (1950) or Kiefer and Wolfowitz (1956)). Most of results deal with linear models where $\sqrt{n}$-consistency, asymptotic normality and efficiency have been studied. One can cite among the others Bickel and Ritov (1987), Bickel et al. (1993), Cheng and van Ness (1994), van der Vaart (1988), (1996), (2002), Murphy and van der Vaart (1996). The nonlinear models have been more recently considered starting with the case of repeated measurement data as in Fuller (1987), Wolter and Fuller (1982), under additional assumptions as in Gleser (1990), Hsiao (1989), Li (2002), Kukush and Schneeweiss (2005a), (2005b) or by simulation (see Carroll (1995), Hsiao et al. (2000), (1997), Li (2000)). To our knowledge the first consistent estimator in nonlinear regression models with errors in variables, under nonparametric assumptions on the design density $g$ has been proposed by Taupin (2001) when the errors $\varepsilon$ are Gaussian and by Comte, Taupin (2001) in the context of auto-regressive models with errors in variables for various types of errors density $p_\varepsilon$. In those papers, the estimation procedure is based on the estimation of the modified least square criterion $\mathbb{E} \left[ (Y - \mathbb{E}(f_{\theta}(X)|Z))^2 W(Z) \right]$ where the conditional expectation is estimated by using the observations $Z_1, \cdots, Z_n$ and where $W$ is some weight compactly supported function. It is also stated that the rate of convergence, which has not an explicit form, depends on the smoothness of the regression function as well as the smoothness of $p_\varepsilon$ through the increase of the ratio $(f_{\theta}(z-\cdot))^{*}(t)/p_\varepsilon^{*}(t)$ as $t$ goes to infinity. The parametric rate of convergence is achieved in some specific examples, such as polynomial or exponential regression functions. The main drawback of this estimator is that, besides its complexity, its rate of convergence has not an explicit form.

More recently, Hong and Tamer (2003) propose a consistent estimator in the specific case where $p_\varepsilon^{*}$ is of the form $p_\varepsilon^{*}(t) = 1/(1 + \sigma^2 t^2)$. Their estimation procedure strongly depends on this particular form of $p_\varepsilon^{*}$ through the ratio $1/p_\varepsilon^{*}$ always appearing in errors in variables techniques. The extension of the method for other errors density $p_\varepsilon$ seems thus not concluding.

Our results. We propose here new estimation procedures, more explicit, natural and tractable than the one proposed by Taupin (2001), that often provide better results, by using the weight function $w$ in order to smooth the regression function by a multiplication. Furthermore, this new estimation procedure allows to provide sufficient conditions to achieve the parametric rate.
Those estimation procedures are based on the least square criterion, both of them allowing to construct two estimators.

The first estimation procedure, is based on the estimation, using the observations \((Y_i, Z_i)\) for \(i = 1, \cdots, n\), of the least squares criterion \(\hat{S}_{\theta_0, g} := \mathbb{E}_{\theta_0, g}[(Y - f_\theta(X))^2 w(X)]\) where \(w\) is some positive weight function, to be chosen.

In this context, we define two estimators, and the first one is naturally defined by

\[
\hat{\theta}_1 = \arg \min_{\theta \in \Theta} \hat{S}_{n, 1}(\theta) \quad \text{with} \quad \hat{S}_{n, 1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \int [(Y_i - f_\theta(x))^2 w(x)] C_n K_n(C_n(x - Z_i))dx,
\]

where \(K_n\) is a deconvolution kernel such that its Fourier transform verifies \(K_n^*(t) = K^*(t)/p_\epsilon^*(tC_n)\), for \(K^*\) compactly supported, \(C_n\) a sequence \(C_n \to \infty\), and \(p^*\), defined by \(p^*(t) = \int e^{itx} f(x)dx\) for \(p\) an arbitrary square integrable function.

We show that under classical identifiability, moment and smoothness assumptions, this estimator \(\hat{\theta}_1\) is a consistent estimator of \(\theta^0\) with a rate of convergence depending on two factors, the smoothness of \((w f_\theta)\) and on the smoothness of the errors density \(p_\epsilon\). This partly comes from the fact that this estimation procedure is based on the estimation of the linear integral functional \(\mathbb{E}_{\theta_0, g}(Y w(X) f_\theta(X))\) and \(\mathbb{E}_{\theta_0, g}(w(X) f_\theta^2(X))\) using the observations \((Y_1, Z_1), \cdots, (Y_n, Z_n)\), that is by recovering some information on \((Y, X)\) using the observation \((Y, Z)\). More precisely, it depends on the smoothness of \(w(x)\partial f_\theta(x)/\partial \theta\) and \(w(x)\partial(f_\theta^2(x))/\partial \theta\) and the smoothness of the errors density \(p_\epsilon(x)\), as functions of \(x\) through the behavior of \((w\partial f_\theta/\partial \theta)^*(t)/p_\epsilon^*(t)\), \((w\partial(f_\theta^2)/\partial \theta)^*(t)/p_\epsilon^*(t)\) as function of \(t \to \infty\). From this construction we derive some sufficient conditions that ensure that \(\hat{\theta}_1\) achieves the parametric rate of convergence.

The second estimator, more simple, is based on conditions ensuring that \(\mathbb{E}_{\theta_0, g}(Y w(X) f_\theta(X))\) and \(\mathbb{E}_{\theta_0, g}(w(X) f_\theta^2(X))\) are estimated at the parametric rate \(\sqrt{n}\). In that case the parametric rate of convergence for the estimation of \(\theta^0\) can be achieved and the asymptotic normality of this estimator is stated.

The first estimator is more general and applicable in all setups, but for some specific regression functions, the use of deconvolution kernel is not required. In those simple cases, the second estimator is more simple. Nevertheless, the conditions ensuring that \(\mathbb{E}_{\theta_0, g}(Y w(X) f_\theta(X))\) and \(\mathbb{E}_{\theta_0, g}(w(X) f_\theta^2(X))\) are estimated at the parametric rate \(\sqrt{n}\) are not always fulfilled.

In the first estimation procedure, the weight function is used, first in order to make \((w f_\theta)\) integrable and second, in order to smooth \((w f_\theta)\) and to improve the rate of convergence of the estimators. For a large class of regression function, \(w\) not depending on \(\theta\) suits well, for instance for polynomial, exponential, cosines regression functions as well as for regression functions \(f_\theta\) of the form \(f_\theta(x) = \varphi(\theta)f(x)\). But sometimes, the smoothing properties of \(w\) will be improved by taking \(w\) depending on \(\theta\), e.g. in case where the regression function has to be smoothed at some point related to \(\theta\). That can be done, under an additional assumption, which is the knowledge of \(\sigma_2^2 = \text{Var}(\xi)\). In this context, the second estimation procedure, is based on the estimation, using the observations \((Y_1, Z_1), \cdots, (Y_n, Z_n)\), of the least squares criterion

\[
S_{\theta_0, g} := \mathbb{E}_{\theta_0, g}[(Y - f_\theta(X))^2 w_\theta(X)] - \sigma_2^2 \mathbb{E}_{\theta_0, g}[w_\theta(X)]
\]

where \(w_\theta\) is some positive weight function, depending on \(\theta\), to be chosen and where \(\sigma_2^2 = \text{Var}(\xi)\) is known. Using this criterion, we propose two other estimators. Analogously to the construction
of \( \hat{\theta}_1 \), we propose a first more general estimator \( \hat{\theta}_1 \) which is defined by

\[
\hat{\theta}_1 = \arg \min_{\theta \in \Theta} S_{n,1}(\theta) \text{ with } S_{n,1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \int [(Y_i - f_\theta(x))^2 - \sigma^2_{i,2}] w_\theta(x) C_n K_n(C_n(x - Z_i)) dx.
\]

We show that under classical identifiability, moment and smoothness assumptions, this estimator \( \hat{\theta}_1 \) is a consistent estimator of \( \theta^0 \) with a rate of convergence depending, as \( \hat{\theta}_1 \), on the smoothness of \( \partial(w_\theta(x)f_\theta(x))/\partial \theta \) and \( \partial(w_\theta(x)f_\theta^2(x))/\partial \theta \) and the smoothness of the errors density \( p_\varepsilon(x) \), as functions of \( x \), described by the behavior of \( (\partial w_\theta/\partial \theta)^*(t)/p_\varepsilon^*(t) \), \( (\partial(w_\theta f_\theta)/\partial \theta)^*(t)/p_\varepsilon^*(t) \), \( (\partial(f_\theta^2 w_\theta)/\partial \theta)^*(t)/p_\varepsilon^*(t) \) as \( t \to \infty \). This comes once again from the fact that this estimation procedure requires the estimation of the linear integral functional of the density \( g \), \( \mathbb{E}_{\theta_0,g}(w_\theta(X)f_\theta^2(X)) \) and \( \mathbb{E}_{\theta_0,g}(Yw_\theta(X)f_\theta(X)) \) as well as \( \mathbb{E}_{\theta_0,g}(Y^2w_\theta(X)) \), using the observations \( (Y_1, Z_1), \ldots, (Y_n, Z_n) \).

From this construction we derive some sufficient conditions that ensure that \( \hat{\theta}_1 \) achieves the parametric rate of convergence.

The second and more simple estimator, is based on conditions ensuring that \( \mathbb{E}_{\theta_0,g}(Y^2w_\theta(X)) \), \( \mathbb{E}_{\theta_0,g}(Yw_\theta(X)f_\theta(X)) \) and \( \mathbb{E}_{\theta_0,g}(w_\theta(X)f_\theta^2(X)) \) can be estimated at the parametric rate \( \sqrt{n} \). This induces that the parametric rate of convergence for the estimation of \( \theta^0 \) can be achieved. Once again, the conditions ensuring that \( \mathbb{E}_{\theta_0,g}(Yw_\theta(X)f_\theta(X)) \) and \( \mathbb{E}_{\theta_0,g}(w_\theta(X)f_\theta^2(X)) \) are estimated at the parametric rate \( \sqrt{n} \) are not always fulfilled.

The rates of convergence of the proposed estimators are thoroughly studied for various type of smoothness properties of \( w f_\theta \), \( w f_\theta^2 \) and their derivatives in \( \theta \) as functions of \( x \), respectively \( w_\theta \), \( w_\theta f_\theta \), \( w_\theta f_\theta^2 \) and their derivatives in \( \theta \) as functions of \( x \), and for various type of errors density \( p_\varepsilon \). It appears that as for the nonparametric estimation of the regression function in errors in variables models, the slower rates of convergence are obtained for smoother density \( p_\varepsilon \). In many practical cases we can improve this smoothness a lot by multiplication with a properly chosen weight function.

The conditions ensuring that the \( \sqrt{n} \)-consistency is achieved are also deeply studied. The main idea is that the actual shape of the regression function matters less than its smoothness (compared to the noise smoothness). Those conditions are illustrated through various examples of regression functions. The point is that these conditions allow us to include setups that were not known before.

Let us now compare our estimation procedure with the one proposed by Hong and Tamer (2003). It is noteworthy that in their special framework of noise densities satisfying \( p_\varepsilon^*(t) = c|t|^{-2}(1 + o(1)) \) as \( |t| \to \infty \) with regression functions \( f_\theta \) having derivatives in \( \theta \) up to order 3, twice continuously differentiable functions of \( x \), our estimation procedure also allows, to recover \( \sqrt{n} \)-consistency.

The drawback of smoothing by multiplication is that we can obtain infinitely differentiable functions but not analytic. In such a case and if the noise has an analytic density, the parametric rate of convergence can not be attained by our methods. Nevertheless the smoothing technique improves significantly the rate when using a clever choice of weight function \( w \).

The main question that remains open is : is it possible to construct a \( \sqrt{n} \)-consistent estimator of \( \theta^0 \) for all regression functions and independently of the smoothness of the density of the errors?

The paper is organized as follows. Sections \(^1\) and \(^2\) present the two estimation procedures and the associated estimators as well as their asymptotic properties. Those asymptotic properties
and practical recipes are illustrated through examples in Section 5. The proofs can be found in Section 6 with some technical lemmas presented in Appendix.

2. First estimation procedure

Notations
We denote by $x_-$ the negative part of $x$, $\| \varphi \|_2^2 = \int \varphi^2(x)dx$, $\| \varphi \|_\infty = \sup_{x \in \mathbb{R}} |\varphi(x)|$. In the same way $p \ast q(z) = \int p(z-x)q(x)dx$ denotes the convolution of two functions $p$ and $q$.

\[ \text{Var}(\xi) = E(\xi^2) \] is denoted by $\sigma_{\xi,2}^2$ and $E(\xi^4) = \sigma_{\xi,4}^4$. For some $\theta \in \mathbb{R}^d$, $\| \theta \|_2^2 = \sum_{k=1}^d \theta_k^2$ and $\theta^\top$ is the transpose matrix of $\theta$.

From now, $\mathbb{P}$, $E$ and $\text{Var}$ denote the probability $\mathbb{P}_{\theta^0, g}$, the expected value $E_{\theta^0, g}$, and respectively, the variance $\text{Var}_{\theta^0, g}$, when the underlying and unknown true parameters are $\theta^0$ and $g$.

The starting point of the first estimation procedure is to construct an estimator based on the observations $(Y_i, Z_i)$ for $i = 1, \ldots, n$, of the least square contrast

\begin{equation}
\tilde{S}_{\theta^0, g}(\theta) = E[(Y - f_\theta(X))^2 w(X)],
\end{equation}

where $w$ is some positive weight function to be suitably chosen.

This estimation procedure requires at least the following assumptions.

Smoothness assumption

(A) For any $\theta$ in $\Theta$, the function $\theta \mapsto f_\theta$ admits continuous derivatives with respect to $\theta$ up to the order 3.

Identifiability and moment assumptions

(I1) The quantity $\tilde{S}_{\theta^0, g}(\theta) = \sigma_{\xi,2}^2 E(w(X)) + E[(f_{\theta^0}(X) - f_\theta(X))^2 w(X)]$ admits one unique minimum at $\theta = \theta^0$.

(I2) For all $\theta \in \Theta$ the matrix $\tilde{S}^{(2)}_{\theta^0, g}(\theta) = \partial^2 S_{\theta^0, g}(\theta) / \partial \theta^2$ exists and

\[ \tilde{S}^{(2)}_{\theta^0, g}(\theta^0) = E \left[ w(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \bigg|_{\theta = \theta^0} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \bigg|_{\theta = \theta^0} \right)^\top \right] \]

is positive definite.

(I3) The quantities $E[w^2(X)(Y - f_\theta(X))^4]$ and their derivatives up to order 2 with respect to $\theta$ are finite.

We denote by $G$ the set of densities $g$ such that (I3) holds.

2.1. Construction and study of the first estimator $\tilde{\theta}_1$. We start by presenting an estimator, which is general, constructive, and which allows to give upper bounds in a general setting and to deduce some sufficient conditions to achieve the parametric rate of convergence.

2.1.1. Construction. Let us denote by $f_{Y,X}$ and by $f_{Y,Z}$ the joint densities of $(Y, X)$ and respectively of $(Y, Z)$ satisfying in this model,

\begin{equation}
f_{Y,X}(y, x) = g(x)p_\xi(y - f_{\theta^0}(x)) \quad \text{and} \quad f_{Y,Z}(y, z) = f_{Y,X}(y, \cdot) * p_\epsilon(z).
\end{equation}
Since \( \tilde{S}_{\theta_0,g}(\theta) \) is given by
\[
\tilde{S}_{\theta_0,g}(\theta) = \mathbb{E}[(Y - f_\theta(X))^2w(X)] = \int (y - f_\theta(x))^2w(x)f_{Y,X}(y,x)dy dx,
\]
according to (2.2), we naturally propose to estimate \( \tilde{S}_{\theta_0,g}(\theta) \) by
\[
(2.3) \quad \tilde{S}_{n,1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - f_\theta(x))^2w(x)K_{n,C_n}(x - Z_i)dx,
\]
where \( K_{n,C_n}(\cdot) = C_nK_n(C_n \cdot) \) is a deconvolution kernel defined via its Fourier transform, such that \( \int K_n(x)dx = 1 \), and
\[
(2.4) \quad K_{n,C_n}^*(t) = \frac{K_{n}^*(t/C_n)}{p^*_C(t)} = \frac{K^*(t/C_n)}{p^*_C(t)}.
\]
with \( K^* \) compactly supported satisfying \( |1 - K^*(t)| \leq 1_{|t| \geq 1} \).

Using this criterion we propose to estimate \( \theta_0^* \) by
\[
(2.5) \quad \hat{\theta}_1 = \arg \min_{\theta \in \Theta} \tilde{S}_{n,1}(\theta).
\]

2.1.2. Asymptotic properties of the first estimator \( \hat{\theta}_1 \).

(A2) \quad \sup_{g \in \mathcal{G}} \| f_{\theta_0}g \|_2^2 \leq C(f_{\theta_0}^2), \quad \sup_{g \in \mathcal{G}} \| f_{\theta_0}g \|_2^2 \leq C(f_{\theta_0}).

(A3) \quad \sup_{\theta \in \Theta} |w_{f_{\theta}}|, \ |w| \quad \text{and} \quad \sup_{\theta \in \Theta} |w_{f_{\theta}^2}| \quad \text{belong to} \ L_1(\mathbb{R})

As for density deconvolution, or for nonparametric regression function estimation in errors in variables models, the rate of convergence for estimating \( \theta_0^* \) is given by both the smoothness of the errors density \( p_\varepsilon \) and the smoothness of \( f_{\theta_0}w \), and more precisely by the smoothness of \( \partial (f_{\theta_0}w)/\partial \theta \) and \( \partial (f_{\theta_0}^2w)/\partial \theta \), as functions of \( x \).

Those smoothness properties are described in both cases, by the asymptotic behaviour of the Fourier transforms. We consider \( p_\varepsilon \) that satisfies the following assumption.

(N1) \quad The density \( p_\varepsilon \) belongs to \( L_2(\mathbb{R}) \) and for all \( x \in \mathbb{R}, \ p^*_C(x) \neq 0 \).

(N2) \quad There exist some positive constants \( C(p_\varepsilon), \overline{C}(p_\varepsilon), \beta, \rho, \alpha \) and \( u_0 \) such that
\[
\underline{C}(p_\varepsilon) \leq |p^*_C(u)| |u|^{\alpha} \exp (\beta |u|^{\rho}) \leq \overline{C}(p_\varepsilon) \text{ for all } |u| \geq u_0.
\]
The assumption \( (N3) \), quite usual in density deconvolution, ensures the existence of the deconvolution kernel in (2.4). If \( \rho > 0 \), then the noise is called exponential noise or super smooth noise. And if \( \rho = 0 \), then by convention \( \beta = 0, \alpha > 1 \) and the noise is called polynomial noise or ordinary smooth.

The smoothness properties of functions involving the regression function are also given by the asymptotic behaviour of the Fourier transforms described as follows.

(R1) \quad A function \( f \) satisfies \( (R1) \) if \( f \) belongs to \( L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \) and if there exist
\[
a, b, r, \quad \text{and} \quad u_0 \geq 0 \text{ such that } L(f) \leq |f^*(u)||u|^{\alpha} \exp (b|u|^r) \leq \overline{L}(f) < \infty \text{ for all } |u| \geq u_0.
\]
If \( r = 0 \), by convention \( b = 0 \), and the function \( f \) is called ordinary smooth. If \( r > 0 \), the function \( f \) is called super smooth.

**Theorem 2.1.** Let \( \hat{\theta}_1 = \tilde{\theta}_1(C_n) \) be defined by (2.3) under the assumptions (1.1), (1.3), (N2), (A1), (A2), and (A3). Assume moreover that for all \( \theta \in \Theta \), \( f_\theta w \) and \( f_\theta^2 w \) and their derivatives up to order 3, with respect to \( \theta \) satisfy (R1). Let \( C_n \) be a sequence such that

\[
C_n^{(2\alpha-2\rho+1)} \exp\{-2bC_n^{\rho} + 2\beta C_n^2\}/n = o(1) \quad \text{as} \quad n \to +\infty.
\]

1. Then, for all of the sequences satisfying (2.4), \( \mathbb{E}(\|\tilde{\theta}_1(C_n) - \theta^0\|_2^2) = o(1) \), as \( n \to \infty \) and \( \hat{\theta}_1(C_n) \) is a consistent estimator of \( \theta^0 \).
2. Moreover \( \mathbb{E}(\|\tilde{\theta}_1 - \theta^0\|_2^2) = O(\tilde{\varphi}_n^2) \) with \( \tilde{\varphi}_n = \|\tilde{\varphi}_{n,j}\|_{\ell_2} \) and \( \tilde{\varphi}_{n,j} = \tilde{B}_{n,j}^2(\theta^0) + \tilde{V}_{n,j}(\theta^0)/n, \]

\[
\tilde{B}_{n,j}^2(\theta) = \min \left\{ \left\| \left( \frac{\partial(wf_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - 1 \right\|_2^2 + \left\| \left( \frac{\partial(wf_\theta^2)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - 1 \right\|_1^2, \right. \\
\left. \left\| \left( \frac{\partial(wf_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - 1 \right\|_1^2 + \left\| \left( \frac{\partial(wf_\theta^2)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - 1 \right\|_1^2 \right\}, 
\]

and

\[
\tilde{V}_{n,j}(\theta) = \min \left\{ \left\| \left( \frac{\partial(wf_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - p_{\varepsilon}^* \right\|_2^2 + \left\| \left( \frac{\partial(wf_\theta^2)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - p_{\varepsilon}^* \right\|_2^2, \right. \\
\left. \left\| \left( \frac{\partial(wf_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - p_{\varepsilon}^* \right\|_1^2 + \left\| \left( \frac{\partial(wf_\theta^2)}{\partial \theta_j} \right)^* (K_{C_n}^*)^* - p_{\varepsilon}^* \right\|_1^2 \right\}. 
\]

**Comment** The terms \( \tilde{B}_{n,j} \) and \( \tilde{V}_{n,j}/n \) are respectively the bias and variance terms with, as in density deconvolution, bigger variance for smoother error density \( p_\varepsilon \) and smaller bias for smoother \( (wf_\theta) \). Contrary to density deconvolution, the smoothness of \( (wf_\theta) \) also appears in the variance term and not only in the bias. The resulting rate when \( (f_\theta w) \) and \( (f_\theta^2 w) \), as well as their derivatives with respect to \( \theta \) up to order 3 satisfy (R1), are given in the following corollary.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, assume that \( p_\varepsilon \) satisfies (N2) and that for all \( \theta \in \Theta \), \( (f_\theta w) \), \( (f_\theta^2 w) \) and their derivatives with respect to \( \theta_j, j = 1, \ldots, d \) up to order 3, satisfy (R1). Then \( \varphi_n^2 \) is given by Table 4, according to values of parameters \( a, b, r, \alpha, \beta \) and \( \rho \).

**Remark 2.1.** Let us comment those rates. First note that the slower rates are obtained for the smoother errors density \( p_\varepsilon \), for instance for Gaussian \( \varepsilon \)'s. Nevertheless, those rates are improved with the choice of \( w \) such that \( (wf_\theta), (wf_\theta^2) \) and their derivatives with respect to \( \theta \) are smooth enough.

Second, the parametric rate of convergence is achieved as soon as the derivatives with respect to \( \theta \), of \( (wf_\theta) \) and \( (wf_\theta^2) \) as functions of \( x \) are smoother than the errors density \( p_\varepsilon \).

**Remark 2.2.** It is noteworthy that the rate of convergence of the estimators could be improved by assuming some smoothness properties on the design density \( g \). Since \( g \) is unknown, we choose not to assume such smoothness conditions.
2.2. Consequence: a sufficient condition to obtain the parametric rate of convergence with $\tilde{\theta}_1$. We say that the conditions (C1)–(C3) hold if there exists some weight function $w$ such that for all $\theta \in \Theta$,

- **(C1)** the functions $(w f_{\theta})$ and $(w f_{\theta}^2)$ belong to $L_1(\mathbb{R})$, and the functions $\sup_{\theta} w^*/p_{\epsilon}^*$, $\sup_{\theta} (f_{\theta} w)/p_{\epsilon}^*$, $\sup_{\theta} (f_{\theta}^2 w)/p_{\epsilon}^*$ belong to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$;

- **(C2)** the functions $\sup_{\theta \in \Theta} \left( \frac{\partial (f_{\theta} w)}{\partial \theta} \right)^*/p_{\epsilon}^*$ and $\sup_{\theta \in \Theta} \left( \frac{\partial (f_{\theta}^2 w)}{\partial \theta} \right)^*/p_{\epsilon}^*$ belong to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$;

- **(C3)** the functions $\left( \frac{\partial^2 (f_{\theta} w)}{\partial \theta^2} \right)^*/p_{\epsilon}^*$ and $\left( \frac{\partial^2 (f_{\theta}^2 w)}{\partial \theta^2} \right)^*/p_{\epsilon}^*$ belong to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

<table>
<thead>
<tr>
<th>$w f_{\theta}$</th>
<th>$p_{\epsilon}$</th>
<th>$\rho$ = 0 in $(\mathbb{N}_2)$</th>
<th>$\rho$ &gt; 0 in $(\mathbb{N}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = r = 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sobolev</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a &lt; \alpha + 1/2$</td>
<td>$n^{-2a-1/2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a \geq \alpha + 1/2$</td>
<td>$n^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r &gt; 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C^\infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b &lt; \beta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = \beta$, $a &lt; \alpha + 1/2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = \beta$, $a \geq \alpha + 1/2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b &gt; \beta$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $A(a, r, \rho) = (-2a + 1 - r + (1 - r)_-)/\rho$.

**Table 1.** Rates of convergence $\varphi_{n}^p$ of $\tilde{\theta}_1$
Theorem 2.2. Consider Model (1.1) under the assumptions (A_i), (I_1)–(I_3) and the conditions (C_6)–(C_7). Assume that for all \( \theta \in \Theta, f\omega, f_\theta \omega \) and their derivatives up to order 3 satisfy (R_1). Then \( \hat{\theta}_1 \) defined by (2.3) is a \( \sqrt{n} \)-consistent estimator of \( \theta^0 \) which satisfies moreover that \( \sqrt{n}(\hat{\theta}_1 - \theta^0) \xrightarrow{\mathcal{L}} N(0, \Sigma_1) \), with \( \Sigma_1 \) that equals

\[
\left( \mathbb{E} \left[ w(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right] \biggr|_{\theta = \theta^0} \right)^{-1} \mathbb{E} \left[ w(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right] \biggr|_{\theta = \theta^0}^{-1}
\]

where \( \Sigma_{0,1} \) is given by

\[
\mathbb{E} \left\{ \left[ \int \left( \frac{\partial (f_\theta^2 w - 2Y f_\theta w)}{\partial \theta} \right)^* e^{-iuZ} (u) \frac{p_\theta^*(u)}{p_\theta^*(u)} du \right] \left[ \int \left( \frac{\partial (f_\theta^2 w - 2Y f_\theta w)}{\partial \theta} \right)^* e^{-iuZ} (u) \frac{p_\theta^*(u)}{p_\theta^*(u)} du \right]^\top \right\}.
\]

Comments Theorem 2.2 is an immediate consequence of Theorem 2.1 since conditions (C_6)–(C_7) ensure precisely that the variance term \( V_{n,j} = O(1) \) for \( j = 1, \ldots, d \).

2.3. Construction and study of the risk of the second estimator \( \hat{\theta}_2 \).

2.3.1. Construction. We now propose another estimator of \( \theta^0 \), based on some sufficient conditions allowing to construct directly a \( \sqrt{n} \)-consistent estimator of \( \hat{S}_{\rho_\omega, g} \) defined in (2.1), based on \( (Y_i, Z_i), i = 1, \ldots, n \).

We say that the conditions (C_6)–(C_7) hold if there exists some weight function \( w \) and there exist some functions \( \tilde{\Phi}_{\theta, \varepsilon,j}, j = 1, 2, 3 \) not depending on \( g \), such that for all \( \theta \in \Theta \) and for all \( g \)

\[
\int y^2 w(x) f_{Y,X}(y, x) dy dx = \int y^2 \tilde{\Phi}_{\theta, \varepsilon,3}(z) f_{Y,Z}(y, z) dy dz,
\]

\[
\int yf_\theta(x) w(x) f_{Y,X}(y, x) dy dx = \int y \tilde{\Phi}_{\theta, \varepsilon,2}(z) f_{Y,Z} dy dz
\]

and

\[
\int w(x) f_\theta^2(x) g(x) dx = \int \tilde{\Phi}_{\theta, \varepsilon,1}(z) h(z) dz;
\]

(C_5) For \( j = 1, 2, 3 \), \( \mathbb{E} \left[ \sup_{\theta \in \Theta} \left| \frac{\partial \tilde{\Phi}_{\theta, \varepsilon,j}(Z)}{\partial \theta} \right| \right] < \infty; \)

(C_6) For \( j = 1, 2, 3 \) and for all \( \theta \in \Theta \), \( \mathbb{E} \left[ \left| \frac{\partial \tilde{\Phi}_{\theta, \varepsilon,j}(Z)}{\partial \theta} \right|^2 \right] < \infty; \)

(C_7) For \( j = 1, 2, 3 \) and for all \( \theta \in \Theta \), \( \mathbb{E} \left[ \left| \frac{\partial^2 \tilde{\Phi}_{\theta, \varepsilon,j}(Z)}{\partial \theta^2} \right| \right] < \infty. \)

Note that \( \tilde{\Phi}_{\theta, \varepsilon,3} \) exists as soon as the chosen weight function \( w \) is smoother than \( p_\varepsilon \) in the way that \( w^*/p_\varepsilon^* \) belongs to \( L_1(\mathbb{R}) \). Furthermore, \( \tilde{\Phi}_{\theta, \varepsilon,3} \equiv \tilde{\Phi}_{\varepsilon,3} \) does not depend on \( \theta \). We refer to Section 4 for details on how to construct such functions \( \tilde{\Phi}_{\theta, \varepsilon,j} \).
Under (\(\mathbf{C}_1\))- (\(\mathbf{C}_7\)), we propose to estimate \(\tilde{S}_{\theta_0,g}(\theta)\) by
\[
\tilde{S}_{n,2}(\theta) = \frac{1}{n} \sum_{i=1}^{n} [(Y_i^2 \Phi_{\theta,\varepsilon,3}(Z_i) - 2Y_i \Phi_{\theta,\varepsilon,2}(Z_i) + \Phi_{\theta,\varepsilon,1}(Z_i)],
\]
and hence \(\theta^0\) is estimated by
\[
\tilde{\theta}_2 = \arg \min_{\theta \in \Theta} \tilde{S}_{n,2}(\theta).
\]

**Remark 2.3.** The main difficulty for finding such function \(\tilde{\Phi}_{\theta,\varepsilon,j}\) lies in the constraint that we expect that they do not depend on the unknown density \(g\). If we relax this constraint, obviously there are a lot of solutions.

2.3.2. Asymptotic properties of the second estimator \(\tilde{\theta}_2\).

**Theorem 2.3.** Consider Model (1.3) under the assumptions \(\{\mathbf{A}_1\}, \{\mathbf{B}_1\}-\{\mathbf{B}_3\}\), and the conditions \(\{\mathbf{C}_1\}-\{\mathbf{C}_7\}\). Then \(\tilde{\theta}_2\), defined by (2.8), is a \(\sqrt{n}\)-consistent estimator of \(\theta^0\) which satisfies moreover that \(\sqrt{n}(\tilde{\theta}_2 - \theta^0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma_2)\), with \(\Sigma_2\) that equals
\[
\left( \mathbb{E} \left[ w(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right] \right)_{\theta=\theta^0}^{-1} \sum_{0,2} \left( \mathbb{E} \left[ w(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right] \right)_{\theta=\theta^0}^{-1}
\]
where \(\sum_{0,2}\) equals
\[
\mathbb{E} \left[ \left( \frac{\partial \tilde{\Phi}_{\theta,\varepsilon,1}(Z)}{\partial \theta} - 2Y \tilde{\Phi}_{\theta,\varepsilon,2}(Z) \right) \left( \frac{\partial \tilde{\Phi}_{\theta,\varepsilon,1}(Z)}{\partial \theta} - 2Y \tilde{\Phi}_{\theta,\varepsilon,2}(Z) \right)^\top \right]_{\theta=\theta^0}.
\]

Note that, by denoting \(\tilde{\Phi}_{\theta,\varepsilon,1} = (w f_\theta)^*/p_\varepsilon, \tilde{\Phi}_{\theta,\varepsilon,2} = (w f_\theta)^*/p_\varepsilon\) and \(\tilde{\Phi}_{\theta,\varepsilon,3} = w^*/p_\varepsilon\), we get that \(\sum_{0,1} = \sum_{0,2}\) with \(\sum_{0,1}\) defined in Theorem 2.2 (See also Comment 1 in Section 4).

3. Second estimation procedure

In the first estimation procedure, the weight function is used in order to make \(w f_\theta, w f_\theta^2\) integrable and also in order to get smooth \(w f_\theta, w f_\theta^2\) and derivatives in \(\theta\). In a large class of regression functions, the weight function can smooth these functions without depending on \(\theta\). Sometimes, the smoothing properties and hence the rate of convergence are improved by making \(w\) to depend on \(\theta\). It appears in particular when the points where we need to smooth are related to \(\theta\).

The second estimation procedure uses an estimator based on the observations \((Y_i, Z_i), i = 1, \cdots, n\), of the least square contrast
\[
S_{\theta_0,g}(\theta) = \mathbb{E}[(Y - f_\theta(X))^2 - \sigma_{\xi,2}^2) w_\theta(X)],
\]
where \(w_\theta\) is some positive weight function to be suitably chosen. This criterion requires thus the knowledge of Var(\(\xi\)).

Subsequently we consider the following assumptions.
Identifiability and moment assumptions

(I21) The variance $\sigma^2_{\xi,2} = \text{Var}(\xi_i)$ is known.

(I22) The quantity $S_{\theta_0,g}(\theta) = \mathbb{E}[w_\theta(X)(Y - f_\theta(X))^2] - \sigma^2_{\xi,2}\mathbb{E}(w_\theta(X))$ admits one unique minimum at $\theta = \theta^0$.

(I23) For all $\theta \in \Theta$ the matrix $S_{\theta_0,g}^{(2)}(\theta) = \partial^2 S_{\theta_0,g}(\theta)/\partial \theta^2$ exists and

$$S_{\theta_0,g}^{(2)}(\theta^0) = \mathbb{E} \left[ w_\theta(X) \left( \frac{\partial (f_\theta(X))}{\partial \theta} \bigg|_{\theta = \theta^0} \right) \left( \frac{\partial (f_\theta(X))}{\partial \theta} \bigg|_{\theta = \theta^0} \right)^\top \right]$$

is positive definite.

(I24) The quantities $\mathbb{E}[w_\theta^2(X)(Y - f_\theta(X))^4]$ and their derivatives up to order 2 with respect to $\theta$ are finite.

Comments on identifiability assumptions: Easy calculations give that

$$\frac{\partial S_{\theta_0,g}(\theta)}{\partial \theta} = \mathbb{E}_{\theta_0,g} \left( f_{\theta_0}(X) - f_\theta(X) \right)^2 \left( \frac{\partial w_\theta(X)}{\partial \theta} \right) - 2\mathbb{E}_{\theta_0,g} \left[ w_\theta(X)(f_{\theta_0}(X) - f_\theta(X)) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \right],$$

$$S_{\theta_0,g}^{(2)}(\theta) = \mathbb{E}_{\theta_0,g} \left[ (f_{\theta_0}(X) - f_\theta(X))^2 \left( \frac{\partial^2 w_\theta(X)}{\partial \theta^2} \right) \right]$$

$$- 4\mathbb{E}_{\theta_0,g} \left[ (f_{\theta_0}(X) - f_\theta(X)) \left( \frac{\partial w_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right]$$

$$+ 2\mathbb{E}_{\theta_0,g} \left[ w_\theta(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right]$$

$$- 2\mathbb{E}_{\theta_0,g} \left[ w_\theta(X)(f_{\theta_0}(X) - f_\theta(X)) \left( \frac{\partial^2 f_\theta(X)}{\partial \theta^2} \right) \right].$$

Therefore $S_{\theta_0,g}(\theta)$ is minimum if and only if $\theta = \theta^0$ as soon as (I23) and (I24) hold.

3.1 Construction and study of the first estimator $\hat{\theta}_1$. As in the first estimation procedure, we start by presenting an estimator, which is general, constructive, and which allows to give upper bounds in a general setting and to deduce some sufficient conditions to achieve the parametric rate of convergence.

3.1.1 Construction. According to (2.2), $S_{\theta_0,g}(\theta)$ is naturally estimated by

$$S_{n,1}(\theta) = \frac{1}{n} \sum_{i=1}^n \int \left[ (Y_i - f_\theta(x))^2 - \sigma^2_{\xi,2} \right] w_\theta(x) K_{n,C_n}(C_n(x - Z_i)) dx$$

where $K_{n,C_n}(\cdot) = C_n K_n(C_n \cdot)$ is a deconvolution kernel satisfying (2.4). Using this criterion, under (N1), we propose to estimate $\theta^0$ by

$$\hat{\theta}_1 = \arg\min_{\theta \in \Theta} S_{n,1}(\theta).$$
3.1.2. Asymptotic properties of the first estimator \( \hat{\theta}_1 \).

**Theorem 3.1.** Let \( \hat{\theta}_1 = \hat{\theta}_1(C_n) \) be defined by (2.3) under the assumptions (2.1), (2.3), (2.4), (4.1)-(4.2), (A1), (A2) and (A3) with \( w \) replaced by \( w_0 \). Assume moreover that for all \( \theta \in \Theta \), \( f\omega \theta \) and \( f^2 \omega \) and their derivatives up to order 3 with respect to \( \theta \) satisfy (R1). Let \( C_n \) be a sequence such that (2.4) holds.

1) Then, for all of the sequences satisfying (2.4), \( \mathbb{E}(\| \hat{\theta}_1(C_n) - \theta \|_2^2) = o(1) \), as \( n \to \infty \) and \( \hat{\theta}_1(C_n) \) is a consistent estimator of \( \theta^0 \).

2) Moreover \( \mathbb{E}(\| \hat{\theta}_1 - \theta^0 \|_2^2) = O(\varphi_n^2) \) with \( \varphi_n \) given \( \varphi_n = \| (\varphi_{n,j}) \| \) with \( \varphi_{n,j} = B_{n,j}(\theta^0) + V_{n,j}(\theta^0)/n, j = 1, \ldots, d, \)

\[
B_{n,j}^2(\theta) = \min\left\{ \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_2^2 + \left\| \left( \frac{\partial (f \omega \theta)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_2^2 + \left\| \left( \frac{\partial (f^2 \omega)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_2^2, \right. \\
\left. \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_1^2 + \left\| \left( \frac{\partial (f \omega \theta)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_1^2 + \left\| \left( \frac{\partial (f^2 \omega)}{\partial \theta_j} \right)^* (K_{C_n}^* - 1) \right\|_1^2 \right\}
\]

and

\[
V_{n,j}(\theta) = \min\left\{ \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_2^2 + \left\| \left( \frac{\partial (f \omega \theta)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_2^2 + \left\| \left( \frac{\partial (f^2 \omega)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_2^2, \right. \\
\left. \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_1^2 + \left\| \left( \frac{\partial (f \omega \theta)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_1^2 + \left\| \left( \frac{\partial (f^2 \omega)}{\partial \theta_j} \right)^* \frac{K_{C_n}^*}{p^*_e} \right\|_1^2 \right\}
\]

**Comments:** The terms \( B_{n,j} \) and \( V_{n,j} \) are respectively the bias and variance terms with, as in density deconvolution, bigger variance for smoother error density \( p_e \) and smaller bias for smoother \( w_\theta f_\theta \) as function of \( x \). The resulting rate when \( f \omega \theta \) and \( f^2 \omega \), as well as their derivatives with respect to \( \theta \) up to order 3 satisfy (R1), are given in the following corollary.

**Corollary 3.1.** Under the assumptions of Theorem 3.1, assume that \( p_e \) satisfies (4.2) and that for all \( \theta \in \Theta \), \( f \omega \theta \) and \( f^2 \omega \) and their derivatives with respect to \( \theta \) up to order 3, satisfy (R1). Then \( \varphi_n^2 \) is given by the Corollary 2.1 with \( w_\theta f_\theta \) replaced by \( w_\theta f_\theta \).

**Remark 3.1.** The remarks 2.3 and 2.4 are still valid for the Corollary 3.1.

3.1.3. **Consequence:** a sufficient condition to obtain the parametric rate of convergence with \( \hat{\theta}_1 \). We say that the conditions (C8)-(C11) hold if there exists some weight function \( w_\theta \) such that
for all \( \theta \in \Theta \),

\[(C_8) \quad \text{the functions } (w_\theta f_\theta), (w_\theta) \text{ and } (w_\theta f_\theta^2) \text{ belong to } L_1(\mathbb{R}) \text{ and the functions } (w_\theta)/p_\varepsilon^*, \text{ sup}_{\theta}(f_\theta w_\theta)^*/p_\varepsilon^*, \text{ sup}_{\theta}(f_\theta^2 w_\theta)^*/p_\varepsilon^* \text{ belong to } L_1(\mathbb{R}) \cap L_2(\mathbb{R}); \]

\[(C_9) \quad \text{the functions sup}_{\theta \in \Theta} \left( \frac{\partial w_\theta}{\partial \theta} \right)^*/p_\varepsilon^*, \text{ sup}_{\theta \in \Theta} \left( \frac{\partial (f_\theta w_\theta)}{\partial \theta} \right)^*/p_\varepsilon^* \text{ and sup}_{\theta \in \Theta} \left( \frac{\partial (f_\theta^2 w_\theta)}{\partial \theta} \right)^*/p_\varepsilon^* \text{ belong to } L_1(\mathbb{R}) \cap L_2(\mathbb{R}); \]

\[(C_{10}) \quad \text{the functions } \left( \frac{\partial^2 w_\theta}{\partial \theta^2} \right)^*/p_\varepsilon^*, \left( \frac{\partial^2 (f_\theta w_\theta)}{\partial \theta^2} \right)^*/p_\varepsilon^* \text{ and } \left( \frac{\partial^2 (f_\theta^2 w_\theta)}{\partial \theta^2} \right)^*/p_\varepsilon^* \text{ belong to } L_1(\mathbb{R}) \cap L_2(\mathbb{R}). \]

**Theorem 3.2.** Consider Model (1.1) under the assumptions \((C_2), (C_3), (C_4), (C_5), (N_3), (A_1), (A_2), (A_3)\) for \( w \) replaced by \( w_\theta \), and under \((C_8)-(C_{10})\). Assume that for all \( \theta \in \Theta \), \((f_\theta w_\theta)\) and \((f_\theta^2 w_\theta)\) and their derivatives up to order 3 satisfy \((R_1)\). Then \( \hat{\theta}_1 \) defined by (3.3) is a \( \sqrt{n} \)-consistent estimator of \( \theta^0 \) which satisfies moreover that \( \sqrt{n}(\hat{\theta}_1 - \theta^0) \xrightarrow{L} N(0, \Sigma_1) \), with \( \Sigma_1 \) that equals

\[
\left( \mathbb{E} \left[ w_{\theta^0}(X) \left( \frac{\partial f_{\theta}(X)}{\partial \theta} \right) \left( \frac{\partial f_{\theta}(X)}{\partial \theta} \right)^\top \right] \right|_{\theta = \theta^0} \Sigma_{0,1}^{-1} \left( \mathbb{E} \left[ w_{\theta^0}(X) \left( \frac{\partial f_{\theta}(X)}{\partial \theta} \right) \left( \frac{\partial f_{\theta}(X)}{\partial \theta} \right)^\top \right] \right|_{\theta = \theta^0}^{-1}
\]

where

\[
\Sigma_{0,1} = \mathbb{E} \left\{ \int \left( \frac{\partial [f_{\theta}^2 w_{\theta} - 2Y f_{\theta} w_{\theta} + (Y^2 - \sigma_{\varepsilon,2}^2) w_{\theta}]}{\partial \theta} \right|_{\theta = \theta^0} e^{-iuZ} \frac{1}{p_\varepsilon^*(u)} du \right\} \times \int \left( \frac{\partial [f_{\theta}^2 w_{\theta} - 2Y f_{\theta} w_{\theta} + (Y^2 - \sigma_{\varepsilon,2}^2) w_{\theta}]}{\partial \theta} \right|_{\theta = \theta^0} e^{-iuZ} \frac{1}{p_\varepsilon^*(u)} du \right\}^{-1}.
\]

**Comments** The theorem 3.2 is an immediate consequence of Theorem 3.1 since conditions \((C_8)-(C_{10})\) ensure precisely that the variance term \( V_{n,j} = O(1) \) for \( j = 1, \ldots, d \).

### 3.2. Construction and study of the risk of the second estimator \( \hat{\theta}_2 \)

We now propose a second estimator whose construction is based on some sufficient conditions allowing to construct a direct and \( \sqrt{n} \)-consistent estimator of \( S_{\theta,g}(\theta) \) defined in (3.1), using the observations \( (Y_i, Z_i) \), \( i = 1, \ldots, n \).

We say that the conditions \((C_{11})-(C_{14})\) hold if there exists some weight function \( w_\theta \) and there exist some functions \( \Phi_{\theta,\varepsilon,j}, j = 1, \ldots, 3 \), not depending on \( g \) such that for all \( \theta \in \Theta \) and
for all $g$

\begin{align}
(C_{11}) & 
\int w_\theta(x)g(x)dx = \int \Phi_{\theta,\varepsilon,3}(z)h(z)dz \\
\int yf_\theta(x)w_\theta(x)fy(x,y)dx &= \int y\Phi_{\theta,\varepsilon,2}(z)f_YZ(y,z)dydz \\
\text{and } \int w_\theta(x)f_\theta^2(x)g(x)dx &= \int \Phi_{\theta,\varepsilon,1}(z)h(z)dz;
\end{align}

\begin{align}
(C_{12}) & 
\text{For } j = 1, 2, 3, \mathbb{E}\left[\sup_{\theta \in \Theta} \left| \frac{\partial \Phi_{\theta,\varepsilon,j}(Z)}{\partial \theta} \right| \right] < \infty; \\
(C_{13}) & 
\text{For } j = 1, 2, 3, \text{ and for all } \theta \in \Theta, \mathbb{E}\left[\left| \frac{\partial \Phi_{\theta,\varepsilon,j}(Z)}{\partial \theta} \right|^2 \right] < \infty; \\
(C_{14}) & 
\text{For } j = 1, 2, 3, \text{ and for all } \theta \in \Theta, \mathbb{E}\left[\left| \frac{\partial^2 \Phi_{\theta,\varepsilon,j}(Z)}{\partial \theta^2} \right| \right] < \infty.
\end{align}

Under $(C_{11})$–$(C_{14})$, we propose to estimate $S_{\theta_0,g}(\theta)$ by

\begin{equation}
S_{n,2}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left[ (Y_i^2 - \sigma_{x,2}^2)\Phi_{\theta,\varepsilon,3}(Z_i) - 2Y_i\Phi_{\theta,\varepsilon,2}(Z_i) + \Phi_{\theta,\varepsilon,1}(Z_i) \right].
\end{equation}

Using this criterion we propose to estimate $\theta^0$ by

\begin{equation}
\hat{\theta}_2 = \arg\min_{\theta \in \Theta} S_{n,2}(\theta).
\end{equation}

We refer to Section 4 for details on how to construct such functions $\Phi_{\theta,\varepsilon,j}$.

**Remark 3.2.** As in the first estimation procedure (see Remark 2.3)), the main difficulty for finding such function $\Phi_{\theta,\varepsilon,j}$ lies in the constraint that we expect that they do not depend on the unknown density $g$. If we relax this constraint, obviously there are a lot of solutions.

### 3.2.1. Asymptotic properties of the second estimator $\hat{\theta}_2$.

**Theorem 3.3.** Consider Model (1.1) under the assumptions $(\text{A}_1)$, $(\text{A}_2)$, $(\text{A}_3)$, $(\text{A}_4)$ and the conditions $(C_{11})$–$(C_{14})$. Then $\hat{\theta}_2$, defined by (3.3) is a $\sqrt{n}$-consistent estimator of $\theta^0$ which satisfies moreover that $\sqrt{n}(\hat{\theta}_2 - \theta^0) \xrightarrow{n \to \infty} N(0, \Sigma_2)$, with $\Sigma_2$ that equals

\begin{align}
\left( \mathbb{E}\left[ w_\theta(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right] \right)_{\theta = \theta^0}^{-1} \mathbb{E}\left[ w_\theta(X) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right) \left( \frac{\partial f_\theta(X)}{\partial \theta} \right)^\top \right]_{\theta = \theta^0}^{-1}
\end{align}
where $\Sigma_{0,2}$ equals
\[
\mathbb{E} \left[ \begin{pmatrix}
\partial (\Phi_{\theta,\varepsilon,1}(Z) - 2Y \Phi_{\theta,\varepsilon,2}(Z) + (Y^2 - \sigma_{\xi,2}^2) \Phi_{\theta^0,\varepsilon,3}(Z)) \\
\partial \theta
\end{pmatrix} \bigg|_{\theta = \theta^0} \right] + 
\mathbb{E} \left[ \begin{pmatrix}
\partial (\Phi_{\theta,\varepsilon,1}(Z) - 2Y \Phi_{\theta,\varepsilon,2}(Z) + (Y^2 - \sigma_{\xi,2}^2) \Phi_{\theta^0,\varepsilon,3}(Z)) \\
\partial \theta
\end{pmatrix} \bigg|_{\theta = \theta^0} \right]^{\top}.
\]

Also note that, by denoting $\Phi_{\theta,\varepsilon,1} = (w_{\theta f_0}^3)^*/p_{\varepsilon}^*$, $\Phi_{\theta,\varepsilon,2} = (w_{\theta f_0})^*/p_{\varepsilon}^*$ and $\Phi_{\theta,\varepsilon,3} = w_{\theta}/p_{\varepsilon}^*$, we get that $\Sigma_{0,1} = \Sigma_{0,2}$ with $\Sigma_{0,1}$ defined in Theorem 3.2 (See also Comment 2 in Section 4).

4. COMMENTS ON THE CONDITIONS ENSURING THE $\sqrt{n}$-CONSISTENCY

**Comment 1** Let us briefly compare the conditions (C1)-(C3) to the conditions (C4)-(C7). It is noteworthy that the conditions (C4)-(C7) are more general. For instance Condition (C1) implies (C4), with $\tilde{\Phi}_{\theta,\varepsilon}$ defined by $\tilde{\Phi}_{\theta,\varepsilon,1} = (w_{\theta f_0}^3)^*/p_{\varepsilon}^*$, $\tilde{\Phi}_{\theta,\varepsilon,2} = (w_{\theta f_0})^*/p_{\varepsilon}^*$, and $\tilde{\Phi}_{\theta,\varepsilon,3} = w_{\theta}/p_{\varepsilon}^*$. This comes from the following equalities $\mathbb{E}[\tilde{\Phi}_{\theta,\varepsilon,1}(Z)] = \mathbb{E}[(w_{\theta f_0}^3)(X)]$,
\[
\mathbb{E}[Y^2 \tilde{\Phi}_{\theta,\varepsilon,3}(Z)] = \mathbb{E}[f_{\theta_0}^3(X)\tilde{\Phi}_{\theta,\varepsilon,3}(Z)] + \sigma_{\xi,2}^2 \mathbb{E}[\tilde{\Phi}_{\theta,\varepsilon,3}(Z)]
\]
\[
= \left( f_{\theta_0}^2 g, \tilde{\Phi}_{\theta,\varepsilon,3} \right) + \sigma_{\xi,2}^2 \left( g, \tilde{\Phi}_{\theta,\varepsilon,3} \right)
\]
\[
= (2\pi)^{-1} \left( f_{\theta_0}^2 g, (\tilde{\Phi}_{\theta,\varepsilon,3})^* p_{\varepsilon}^* \right) + \sigma_{\xi,2}^2 (2\pi)^{-1} \left( g, (\tilde{\Phi}_{\theta,\varepsilon,3})^* p_{\varepsilon}^* \right)
\]
\[
= \left( f_{\theta_0}^2 g, w \right) + \sigma_{\xi,2}^2 \left( g, w \right) = \mathbb{E}[Y^2 w(X)],
\]
and
\[
\mathbb{E}[Y \tilde{\Phi}_{\theta,\varepsilon,2}(Z)] = \left( f_{\theta_0} g, \tilde{\Phi}_{\theta,\varepsilon,2} \right) = (2\pi)^{-1} \left( f_{\theta_0} g, (\tilde{\Phi}_{\theta,\varepsilon,2})^* p_{\varepsilon}^* \right)
\]
\[
= \left( f_{\theta_0} g, w \right) = \mathbb{E}[Y w_{\theta}(X)].
\]

But in the condition (C4), $f_{\theta} w, f_{\theta}^2 w$, and $w$ are not necessarily in $L_1(\mathbb{R})$. Nevertheless, the conditions (C1)-(C3) are more tractable and constructive conditions.

**Comment 2** The same comparison between (C11)-(C14) and (C8)-(C10) holds with $w$ replaced by $w_{\theta}$ and with the $\Phi_{\theta,\varepsilon,j}$ defined by $\Phi_{\theta,\varepsilon,1} = (w_{\theta f_0}^3)^*/p_{\varepsilon}^*$, $\Phi_{\theta,\varepsilon,2} = (w_{\theta f_0})^*/p_{\varepsilon}^*$, and $\Phi_{\theta,\varepsilon,3} = (f_{\theta} w_{\theta})^*/p_{\varepsilon}^*$. 

**Comment 3** The conditions (C4)-(C7) as well as (C11)-(C14) have to be related to the conditions given in Fazekas and Kukush (1998), Fazekas et al. (1999) and Baran (2000) in the context of the functional errors in variables model with $X_1, \ldots, X_n$ not random. In both mentioned papers, in order to construct $\sqrt{n}$-consistent estimator, they assume that there exist some functions $f_1$ and $f_2$ such that
\[
(4.1) \quad \mathbb{E}(\phi_1(x + \varepsilon, \theta)) = f_\theta(x), \quad \text{and} \quad \mathbb{E}(\phi_2(x + \varepsilon, \theta)) = f_\theta^2(x).
\]
Clearly our conditions (C4)-(C7) or (C11)-(C14) are less restrictive than the condition (4.1), and hence they allow to achieve the parametric rate of convergence for various type of regression functions, by using the possibility of the choice of the weight function $w_\theta$. For instance, if


\[ f_0(x) = \theta/(1 + x^2), \] and \( p_\varepsilon \) is the Gaussian density, then the conditions given in Fazekas and Kukush (1998) and Fazekas et al. (1999) are not fulfilled.

Whereas \( \mathbb{E}[f_0(X)w(X)] \) and \( \mathbb{E}[f_0^2(X)w(X)] \) can be estimated with the parametric rate of convergence, by taking \( w(x) = (1 + x^2)^4 \exp(-x^2/(4\beta)) \). It follows that the condition \((C_4)\) are fulfilled in this special example (see Section 3 for further details on this example). Nevertheless, such weight function are not always available and hence those conditions \((C_4)\) are not always fulfilled.

5. Examples and methodological advice

In this section, we propose a deep study of the asymptotic properties of the proposed estimators through various examples of regression functions. Moreover, we show that for many regression functions the practitioner may encounter there are a few simple smoothing weight functions to choose so that the rates improve significantly, up to parametric rate in many cases.

Example 1. Polynomial regression function Let \( f_0 \) be of the form \( f_0(x) = \sum_{k=1}^{p} \theta_k x^k \) and \( p_\varepsilon \) satisfying \((N_2)\) with \( \rho \leq 2 \). Assume that \( \mathbb{E}(Y^2) < \infty \) and that \( \mathbb{E}(Z^{2p}) < \infty \). Let \( K \) be such \( K^*(t) = \mathbb{I}_{|t| \leq 1} \) and let \( w_\varepsilon(x) = w(x) = \exp\{-x^2/(4\beta)\} \). Then conditions \((C_4)-(C_7)\) as well as conditions \((C_1)-(C_3)\) are satisfied. Consequently the estimators \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.

Remark 5.1. In this example, one can also choose \( w_\varepsilon \equiv 1 \), provided that the kernel \( K \) has finite absolute moments of order \( p \) and satisfies \( \int u^r K(u) du = 0 \), for \( r = 1, \ldots, p \). In this case \( \hat{\theta}_1 \) is again a \( \sqrt{n} \)-consistent and asymptotically Gaussian estimator of \( \theta^0 \).

The \( \sqrt{n} \)-consistency as well as the asymptotic normality was already achieved with different estimators, in the linear case (see e.g. Bickel (1987), Bickel et al. (1994), van der Vaart (1998) and (1988), Murphy and van der Vaart (1994)). In polynomial case, some \( \sqrt{n} \)-consistent estimator already exists, without proving the asymptotic normality (see e.g. Taupin (2001) and Comte and Taupin (2001)) or in the polynomial functional errors in variables model, with fixed and not random \( X_i \)'s (see e.g. Hausman et al. (1992), (1991) or Chan and Mak (1985)). It is noteworthy that these new estimation procedures, quite more simple and natural than the one proposed in Taupin (2001), also work in this simple case.

Example 2. Exponential regression function Let \( f_0 \) be of the form \( f_0(x) = \exp(\theta x) \) and \( p_\varepsilon \) satisfying \((N_2)\) with \( \rho \leq 2 \). Assume that \( \mathbb{E}(Y^2) < \infty \) and that \( \mathbb{E}(\exp(2\theta X)) < \infty \). Let \( K \) be such \( K^*(t) = \mathbb{I}_{|t| \leq 1} \) and let \( w_\varepsilon(x) = w(x) = \exp\{-x^2/(4\beta)\} \). Then the conditions \((C_4)-(C_7)\) as well as conditions \((C_1)-(C_3)\) are satisfied. Consequently the estimators \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.
Remark 5.2. In this example, one can also choose \( w \equiv 1 \) and use that
\[
E[\exp(\theta X)] = \frac{E[\exp(\theta Z)]}{E[\exp(\theta \varepsilon)]}.
\]
This implies that if we denote by
\[
\bar{\Phi}_{\theta,\varepsilon,1}(Z) = \frac{\exp(2\theta Z)}{E[\exp(2\theta \varepsilon)]} \quad \text{and} \quad \bar{\Phi}_{\theta,\varepsilon,2}(Z) = \frac{\exp(\theta Z)}{E[\exp(\theta \varepsilon)]}
\]
then
\[
E_h[\bar{\Phi}_{\theta,\varepsilon,1}(Z)] = E_y[f_\theta^2(X)] \quad \text{and} \quad E_{\theta^0, h}[Y \bar{\Phi}_{\theta,\varepsilon,2}(Z)] = E_{\theta^0}[Y f_\theta(X)].
\]
Consequently, \( \tilde{S}_{n,2} \) satisfies
\[
(5.1) \quad \tilde{S}_{n,2}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left[ Y_i - \frac{\exp(\theta Z_i)}{E[\exp(\theta \varepsilon)]]} \right]^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ Y_i^2 - 2Y_i \bar{\Phi}_{\theta,\varepsilon,2}(Z_i) + \bar{\Phi}_{\theta,\varepsilon,1}(Z_i) \right].
\]
In this case \( \tilde{\theta}_2 \) is also a \( \sqrt{n} \)-consistent and asymptotically Gaussian estimator of \( \theta^0 \).

Once again, these new estimation procedures allow to achieve the \( \sqrt{n} \)-consistency and the asymptotic normality in a simple example where a \( \sqrt{n} \)-consistent estimator was already known (see Taupin (2001)).

Example 3. Cosines regression function Let \( f_\theta \) be of the form \( f_\theta(x) = \sum_{j=1}^{d} \theta_j \cos(jx) \) and \( p_\varepsilon \) satisfying \((\mathbb{N}_3)\) with \( \rho \leq 2 \). Let \( K \) be such \( K^*(t) = I_{|t| \leq 1} \) and let \( w_\theta(x) = w(x) = \exp\{-x^2/(4\beta)\} \). Then the conditions \((\mathbb{C}_4)-(\mathbb{C}_7)\) as well as conditions \((\mathbb{C}_1)-(\mathbb{C}_3)\) are satisfied. Consequently the estimators \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.

Remark 5.3. In this example, one can also choose \( w_\theta \equiv 1 \) and use that \( E[\exp(ijX)] = E[\exp(i\varepsilon Z)]/E[\exp(i\varepsilon \varepsilon)] \). This implies that if we denote by
\[
\bar{\Phi}_{\theta,\varepsilon,1}(Z) = \frac{1}{4} \left\{ 1 + \sum_{j=1}^{d} \theta_j^2 \left[ \frac{\exp(2ijZ)}{p_\varepsilon^2(2j)} + \frac{\exp(-2ijZ)}{p_\varepsilon^2(-2j)} \right] 
\right.
\]
\[
+ \sum_{j=1}^{d} \sum_{k \neq j} \theta_j \theta_k \left[ \frac{\exp(i(j + k)Z)}{p_\varepsilon^2(j + k)} + \frac{\exp(-i(j + k)Z)}{p_\varepsilon^2(-j - k)} + \frac{\exp(i(j - k)Z)}{p_\varepsilon^2(j - k)} + \frac{\exp(-i(j - k)Z)}{p_\varepsilon^2(-j + k)} \right] \right\}
\]
and
\[
\bar{\Phi}_{\theta,\varepsilon,2}(Z) = \frac{1}{2} \left[ \frac{\exp(ijZ)}{p_\varepsilon^2(j)} + \frac{\exp(-ijZ)}{p_\varepsilon^2(-j)} \right]
\]
then
\[
E[\bar{\Phi}_{\theta,\varepsilon,1}(Z)] = E_y[f_\theta^2(X)] \quad \text{and} \quad E[Y \bar{\Phi}_{\theta,\varepsilon,2}(Z)] = E[Y f_\theta(X)]
\]
Consequently, \( \tilde{S}_{n,2} \) satisfies
\[
(5.2) \quad \tilde{S}_{n,2}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left[ Y_i^2 - 2Y_i \bar{\Phi}_{\theta,\varepsilon,2}(Z_i) + \bar{\Phi}_{\theta,\varepsilon,1}(Z_i) \right].
\]
In this case \( \tilde{\theta}_2 \) with \( w \equiv 1 \) is again a \( \sqrt{n} \)-consistent and asymptotically Gaussian estimator of \( \theta^0 \). In the same way, \( \hat{\theta}_1 \) with \( w \equiv 1 \) is again a \( \sqrt{n} \)-consistent and asymptotically Gaussian estimator of \( \theta^0 \).

**Remark 5.4.** This example has already been considered in Taupin (2001) and Comte and Taupin (2001), but the \( \sqrt{n} \)-consistency as well as the asymptotic normality is new in this context. More precisely, the estimator constructed in Taupin (2001) has a rate of convergence of order \( \exp(\sqrt{\log n}/n \right) \) for Gaussian errors, and it was not clear that this rate can be improved by using the previous method.

**Example 4. Cauchy regression function 1** Consider Model (1.1) with \( f_\theta(x) = \theta/(1 + x^2) \) satisfying (R1) with \( a = 0 \), \( b = 1/2 \) and \( r = 1 \) and \( p_\varepsilon \) satisfying (N3) with \( \rho \leq 2 \). Let \( K \) be such \( K^*(t) = I_{|t| \leq 1} \) and let \( w_\theta(x) = w(x) = (1 + x^2)^4 \exp\{-x^2/(4\beta)\} \). With our choice of \( w \), the functions \( f_\theta w \), \( f_\theta^2 w \) and their derivatives in \( \theta \) up to order 3 satisfy (R1) with \( \rho < r = 2 \) or \( \rho = r = 2 \) and \( b > \beta \). Consequently, the conditions (C4)-(C7) as well as conditions (C1)-(C3) are satisfied and the estimators \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.

This simple example underlies the importance of the smoothing weight function \( w \) in the construction of \( \hat{\theta}_1 \) or \( \hat{\theta}_2 \), since without a smoothing function \( w \) in front of the regression function, \( f_\theta \), the estimator \( \hat{\theta}_1 \) or \( \hat{\theta}_2 \) would not provide the parametric rate of convergence since no weight function appears in front of the regression function, in order to smooth it. Moreover, it shows that in this case, the conditions (1.4) given in Fazekas and Kukush (1998) and Fazekas et al. (1999) are not satisfied when our conditions (C4)-(C7) or (C1)-(C3) hold and our estimators achieve the parametric rate.

**Example 5. Laplace regression function** Consider Model (1.1) with \( f_\theta(x) = \theta f(x) \) and \( f(x) = \exp\{-|x|/2\} \). The Fourier transform of \( f \) and hence of \( f_\theta \) is slowly decaying, like \( |u|^{-2} \) as \( |u| \to \infty \). The estimator \( \hat{\theta}_2 \) with \( w \equiv 1 \) would not provide a \( \sqrt{n} \)-consistent estimator for smoother noise densities (as soon as \( |p_\varepsilon^*(u)| \leq o(|u|^{-2}) \) with \( |u| \to \infty \)). A closer look tells us that \( f_\theta \) and its derivative in \( \theta \) is \( C^\infty \) except at one point \( x = 0 \). Therefore, a proper choice of \( w \) can smooth out at 0 and make \( w_\theta f \), \( w_\theta f^2 \) and their derivatives in \( \theta \) infinitely differentiable functions in \( x \).

With such a choice of the weight function \( w \), the estimator \( \hat{\theta}_1 \) attains the parametric rate of convergence for a large set of noise densities, such that \( p_\varepsilon \) satisfies (N2) for some \( 0 < \rho < 1 \). Even if the noise is smoother than that (say Gaussian), the rate of \( \hat{\theta}_1 \) is much faster when using our choice of \( w \) then it would be for \( w \equiv 1 \) or when using the estimator proposed in Taupin (2001).

Let us be more precise, and define

\[
\Psi_{a,b}(x) = \exp\left( -\frac{1}{(x-a)^R(b-x)^R} \right) I_{[a,b]}(x),
\]
Consequently the estimators \( \tilde{\theta} \) grow large and \( R/\epsilon \) and their derivatives up to order 3 satisfy (R_1) with the same \( r = R/(R+1) \) closer to 1 as \( R \) grows large and \( b > 0 \).

Another way to smooth without restraining to compact support is the following. Let \( w(x) = \exp(-1/|x|^{2R}) \) a weight function which smoothes at 0 as \( R > 0 \) is large.

We can vary the coefficient in the exponential in the expression of \( w \) and check that \( f_\theta w, f_\theta^2 w \) and their derivatives up to order 3 satisfy (R_1) with the same \( r = R/(R+1) \) closer to 1 as \( R \) grows large and \( b > 0 \).

If the noise satisfies (N_2) with \( 0 \leq \rho < 1 \), then we find \( R \) large enough such that \( r = R/(R+1) > \rho \) and thus conditions (C_4)-(C_7) as well as conditions (C_4)-(C_3) are satisfied. Consequently the estimators \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.

If \( \rho \geq 1 \), for \( w \) suitably chosen, then

\[
\mathbb{E} \| \tilde{\theta}_1 - \theta^0 \|^2_2 = O(1) (\log n)^{1-2 \alpha - \epsilon} \exp\left\{-2b(\log n/(2\beta))^{r/\rho}\right\}.
\]

Note that for the same regression functions with \( f(x) = \exp(|x|) \) we can multiply the previous weight function \( w \) by \( \exp(-4|x|) \) or by \( \exp(-x^2) \) in order to solve integrability problems without changing the previous conclusions.

**Example 6. Irregular regression function** Consider Model (1.1) with \( f_\theta(x) = \theta \mathbb{I}_{[-1,1]}(x) \) and \( p \) satisfying (N_2). Let \( K \) be such that \( K^*(t) = \mathbb{I}_{|t| \leq 1} \), and take \( w = \Psi_{-1,1} \) for some \( R > 0 \) defined by (5.3).

If \( \rho = 0 \) in (N_2), then the conditions (C_4)-(C_7) as well as conditions (C_4)-(C_3) are satisfied. Consequently the estimators \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \), with the same asymptotic variance.

If \( \rho > 0 \), then the best rate for estimating \( \theta^0 \) is obtained by choosing \( w = \Psi_{-1,1} \) with \( R > 0 \) sufficiently large such that \( w f_\theta \) and \( w f_\theta^2 \) satisfy (R_1) with \( 0 < r = R/(R+1) < 1 \) as close to 1 as needed.

It follows that if \( 0 < \rho < 1 \), then we can find \( w = \Psi_{-1,1} \) (with \( R \) large enough) satisfying (R_1) with \( r = R/(R+1) > \rho \) and hence the conditions (C_4)-(C_7) as well as conditions (C_4)-(C_3) are satisfied. Thus the estimators \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are \( \sqrt{n} \)-consistent and asymptotically Gaussian estimators of \( \theta^0 \). Whereas, if \( \rho \geq 1 \), for a suitably chosen \( w \), then

\[
\mathbb{E} \| \tilde{\theta}_1 - \theta^0 \|^2_2 = O(1) (\log n)^{1-2 \alpha - \epsilon} \exp\left\{-2b(\log n/(2\beta))^{r/\rho}\right\}.
\]

**Example 7. Polygonal regression function** Consider Model (1.1) with \( f_\theta(x) = \theta_0 + \theta_1 x + \theta_2(x-a)_+ + \theta_3|x-b|^3 \) and \( p \) satisfying (N_2). Let \( K \) be such \( K^*(t) = \mathbb{I}_{|t| \leq 1} \). This regression function is \( C^\infty \) except at points \( a \) and \( b \) where it is not differentiable. We suggest to use the smoothing weight function in (5.3) as follows. Let for some \( R > 0 \),

\[
w_\theta(x) = w(x) = \Psi_{a-100,a}(x) + \Psi_{a,b}(x) + \Psi_{b,b+100}(x)
\]
with $\Psi$ defined in (5.3). The idea is that we can truncate the regression function (say on the interval $[a - 100, b + 100]$, or larger) in order to smooth out at points $a$, $b$ and the end points of the support of $w$.

If the noise satisfies (N2) with $0 \leq \rho < 1$, then take $R$ large enough such that $r = R/(R+1) > \rho$ and thus conditions (C1)- (C7) as well as conditions (C9)-(C3) are satisfied. Consequently the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are $\sqrt{n}$-consistent and asymptotically Gaussian estimators of $\theta^0$, with the same asymptotic variance.

If $\rho \geq 1$ in (N2), according to Table II,

$$E \| \hat{\theta}_1 - \theta^0 \|^2 = O(1) \left( \log n \right) \frac{1-2a-x}{\rho} \exp\{-2b(\log n/(2\beta))^{r/\rho}\}.$$

Comments on the examples (8), (9) and (10)

In those three examples, $\hat{\theta}_1$ achieves the $\sqrt{n}$-rate of convergence provided that $p_x$ is ordinary smooth or super smooth with an exponent $\rho < 1$. But, $f_\theta w$ will satisfy (R1) with $r$ at most such that $r < 1$ and it seems therefore impossible to have $(w_0f_\theta)^r/p_x^r$ in $L_1(\mathbb{R})$ if the $\varepsilon_i$’s are Gaussian. For this regression function, if the $\varepsilon_i$’s are Gaussian, those least square criterion can not be estimated with the parametric rate of convergence and hence could probably, not provide a $\sqrt{n}$-consistent estimator of $\theta^0$ in this context. Nevertheless, even in cases where the parametric rate of convergence seems not achievable by such estimators, the resulting rate of the risk of $\hat{\theta}_1$ is clearly infinitely faster than the logarithmic rate we could have without such a choice of $w$ or by using the estimator proposed by Taupin (2001).

Example 8. Growth curves 1 Consider Model (1.1) with $f_\theta(x) = \theta_1/(1 + \theta_2 \exp(\theta_3 x))$ and $p_x$ satisfying (N2) with $\rho \leq 2$. Let $K^*(t) = I_{|t| \leq 1}$ and let $w_\theta(x) = (1 + \theta_2 \exp(\theta_3 x))^4 \exp\{-x^2/(4\beta)\}$. Then the conditions (C11)-(C14) as well as conditions (C8)-(C10) are satisfied. Consequently the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are $\sqrt{n}$-consistent and asymptotically Gaussian estimators of $\theta^0$, with the same asymptotic variance.

Example 9. Growth curve 2 Consider Model (1.1) with $f_\theta(x) = \theta_2 + (\theta_1 - \theta_2)/(1 + \exp(\theta_3 + \theta_4 x))$ and $p_x$ satisfying (N2) with $\rho \leq 2$. Let $K$ be such $K^*(t) = I_{|t| \leq 1}$ and let $w_\theta(x) = (1 + \exp(\theta_3 + \theta_4 x))^4 \exp\{-x^2/(4\beta)\}$. Then the conditions (C11)-(C14) as well as conditions (C8)-(C10) are satisfied. Consequently the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are $\sqrt{n}$-consistent and asymptotically Gaussian estimators of $\theta^0$, with the same asymptotic variance.

Example 10. Cauchy regression function 2 Consider Model (1.1) with $f_\theta(x) = 1/(1 + \theta x^2)$ and $p_x$ satisfying (N2) with $\rho \leq 2$. Let $K$ be such $K^*(t) = I_{|t| \leq 1}$ and let $w_\theta(x) = (1 + \theta x^2)^2 \exp\{-x^2/(4\beta)\}$. Then the conditions (C11)-(C14) as well as conditions (C8)-(C10) are satisfied. Consequently the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are $\sqrt{n}$-consistent and asymptotically Gaussian estimators of $\theta^0$, with the same asymptotic variance.

In these three last examples, we see the importance of the weight function $w_\theta$, with the improvement of the rate of convergence by taking $w_\theta$ depending on $\theta$. Clearly, for such regression function, the estimator in Taupin (2001) does not achieve the parametric rate of convergence whereas, $\hat{\theta}_1$ or $\hat{\theta}_2$ do.
6. Proofs of Theorems

We only detail the proof of Theorems 3.1 and 3.3 as well as Corollary 3.1 since Theorem 2.2 is an immediate consequence of 3.1 and also since the proofs of Theorems 2.1, 2.2 and 2.3 and Corollary 2.1 follow the same lines with $w$ replaced by $w_\theta$ and by putting $\sigma_\xi,2 = 0$ in the proofs. In the sequel, we denote by $C$ some absolute constant whose value may vary from one line to the other and we always mention the dependency of some constant $C$ with respect to some parameters. For instance $C(a,b)$ stands for a constant depending on $a$ and $b$.

6.1. Proof of 1) of Theorem 3.1. The main point of the proof consists in showing that for any $\theta \in \Theta$, $E[(S_{n,1}(\theta) - S_{\theta_0,g}(\theta))^2] = o(1)$, with $S_{\theta_0,g}(\theta)$ admitting a unique minimum in $\theta = \theta_0$.

The second part of the proof consists in studying $\omega_2(n, \rho)$ defined as

$$\omega_2(n, \rho) = \sup \{|S_{n,1}(\theta) - S_{n,1}(\theta')| : \|\theta - \theta'\|_2 \leq \rho\}.$$ 

By using the regularity assumptions on the regression function $f$, we state that there exist two sequences $\rho_k$ and $\epsilon_k$ tending to 0, such that for all $k \in \mathbb{N}$

(6.1) $$\lim_{n \to \infty} P[\omega_2(n, \rho_k) > \epsilon_k] = 0$$ and that $E[(\omega_2(n, \rho_k))^2] = O(\rho_k^2)$.

Let us start with the proof of

(6.2) $$E[(S_{n,1}(\theta) - S_{\theta_0,g}(\theta))^2] = o(1), \quad \text{for any } \theta \in \Theta \text{ and as } n \to \infty.$$ 

We have to check successively that for any $\theta \in \Theta$,

(6.3) $$E[S_{n,1}(\theta)] - S_{\theta_0,g}(\theta) = o(1) \quad \text{and} \quad \text{Var}(S_{n,1}(\theta)) = o(1), \quad \text{as } n \to \infty.$$ 

For both the bias and the variance, we give two upper bounds, based on the two following applications of the H"older’s inequality

(6.4) $$|<\varphi_1, \varphi_2> - |\leq \|\varphi_1\|_2\|\varphi_2\|_2 \quad \text{and} \quad |<\varphi_1, \varphi_2> - |\leq \|\varphi_1\|_\infty\|\varphi_2\|_1.$$ 

Proof of the first part of (6.3). According to Lemma 6.2, we have

$$E[S_{n,1}(\theta)] = E \left[(Y - f_\theta)^2 w_\theta - \sigma_\xi,2 w_\theta \right] \ast K_{n,C_n}(Z) = E \left[(Y^2 - \sigma_\xi,2)w_\theta \ast K_{C_n}(X) - 2E[Y(f_\theta w_\theta) \ast K_{C_n}(X)] + E[(f_\theta^2 w_\theta) \ast K_{C_n}(X)],\right.$$ 

and hence

$$E[S_{n,1}(\theta)] - S_{\theta_0,g}(\theta) = (2\pi)^{-1} \langle (f_\theta^2 g)^*, w_\theta (K_{C_n}^*(\cdot) - 1) \rangle - \pi^{-1} \langle (f_\theta g)^*, (f_\theta w_\theta)^*(K_{C_n}^* - 1) \rangle + (2\pi)^{-1} \langle g^*, (f_\theta^2 w_\theta)^*(K_{C_n}^* - 1) \rangle.$$ 

Consequently, under the assumption $\mathbb{A}_2$, $E[S_{n,1}(\theta)] - S_{\theta_0,g}(\theta)]^2 = o(1)$. Indeed, according to (6.4), a first upper bound of $|E[S_{n,1}(\theta)] - S_{\theta_0,g}(\theta)]$ is given by

$$(2\pi)^{-1} \| (f_\theta^2 g)^* \|_2 \| w_\theta^*(K_{C_n}^* - 1) \|_2 + \pi^{-1} \| (f_\theta g)^* \|_2 \| (f_\theta w_\theta)^*(K_{C_n}^* - 1) \|_2 + (2\pi)^{-1} \| g^* \|_2 \| (f_\theta^2 w_\theta)^*(K_{C_n}^* - 1) \|_2$$
Consequently, by combining the two bounds (6.5) and (6.6) we get that

\[ (6.6) \]

\[ C(f_{\theta^0}, \theta_0) \min \left\{ \|w_{\theta_0}(K_{C_n}^*) - 1\|_2^2 + \|w_{\theta_0}f_{\theta_0}^*(K_{C_n}^*) - 1\|_2^2 + \|w_{\theta_0}f_{\theta_0}^*(K_{C_n}^*) - 1\|_2^2, \right. \\
\left. \|w_{\theta_0}(K_{C_n}^*) - 1\|_1^2 + \|w_{\theta_0}f_{\theta_0}^*(K_{C_n}^*) - 1\|_1^2 + \|w_{\theta_0}f_{\theta_0}^*(K_{C_n}^*) - 1\|_1^2 \right\} \]

that is by applying Lemma 6.1,

\[ \|E[S_{n,1}(\theta)] - S_{\theta^0, \theta}(\theta)\| \leq C(f_{\theta^0}, \theta_0, b, r)C_n^{-2a+(1-r)+(1-r)-e^{-2bC_n}}, \]

and the first part of (6.3) follows.

**Proof of the second part of (6.3).** Since the variables are i.i.d. random variables, the stochastic term on the left-hand side in (6.3) has variance

\[ \text{Var}[S_{n,1}(\theta)] \leq C/n \left\{ \left( \mathbb{E}[(Y - f_0)^2 \theta_0] + \sigma_{\xi,2}^2 \right) * K_{n,C_n}(Z) \right\}^2 \]

\[ \leq 3n^{-1} \left\{ \mathbb{E}[(Y^4 + \sigma_{\xi,2}^4) \theta_0 * K_{n,C_n}(Z)]^2 + 4\mathbb{E}[(f_0 \theta_0) * K_{n,C_n}(Z)]^2 \right\} + \mathbb{E}[(f_0^2 \theta_0) * K_{n,C_n}(Z)]^2 \]

\[ \leq C/n \left\{ \left( \mathbb{E}[f_0^2(X) \theta_0 * K_{n,C_n}(Z)]^2 \right) + \left( \mathbb{E}[\xi^4 + \sigma_{\xi,2}^4] \mathbb{E}[(\theta_0 * K_{n,C_n}(Z)]^2 \right) \right\} + \mathbb{E}[(f_0^2 \theta_0) * K_{n,C_n}(Z)]^2 \]

that is, according to Lemma 6.2,

\[ \text{Var}[S_{n,1}(\theta)] \leq C/n \left\{ \left( \mathbb{E}[(f_0^2 \theta_0) * p_\xi] + \left( \mathbb{E}[\xi^4] + \sigma_{\xi,2}^4 \right) \mathbb{E}[(g * p_\xi) \theta_0 * K_{n,C_n}(Z)]^2 \right) \right\} + \left( \mathbb{E}[(f_0 \theta_0) * p_\xi] + \left( \mathbb{E}[\xi^4] + \sigma_{\xi,2}^4 \right) \mathbb{E}[(g * p_\xi) * K_{n,C_n}(Z)]^2 \right) \]

\[ + \left( \mathbb{E}[(g * p_\xi) \theta_0 * K_{n,C_n}(Z)]^2 \right) \].
On one hand, according to (6.4), under the assumption $\mathbf{(A_2)}$,

\[
\text{Var}[S_{n,1}(\theta)] \leq C/n \left\{ \| f_{0\theta}^1 \| \times \| p_\varepsilon \|_\infty + \| \varepsilon \| \times \| g \times p_\varepsilon \|_\infty \right. + \| w_\theta \times K_{n,Cn} \|_2^2 \\
+ \| f_{0\theta}^2 \| \times \| p_\varepsilon \|_\infty + \| g \times p_\varepsilon \|_\infty \right\},
\]

and we get that

(6.7) \quad \text{Var}[S_{n,1}(\theta)] 
\leq C(\sigma_{\xi,2}^2, \sigma_{\xi,4}, f_{\theta}, w_\theta, p_\varepsilon)n^{-1} \left\{ \left\| (w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 + \left\| (f_{0\theta} w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 + \left\| (f_{0\theta}^2 w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 \right\}.

On the other hand, under the assumption $\mathbf{(A_2)}$,

\[
\text{Var}[S_{n,1}(\theta)] \leq C/n \left\{ \| f_{0\theta}^1 \| \times \| p_\varepsilon \|_1 + \| \varepsilon \| \times \| g \times p_\varepsilon \|_1 \right\},
\]

and hence we get that

(6.8) \quad \text{Var}[S_{n,1}(\theta)] 
\leq C(\sigma_{\xi,2}^2, \sigma_{\xi,4}, f_{\theta}, w_\theta, p_\varepsilon)n^{-1} \left\{ \left\| (w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_1^2 + \left\| (f_{0\theta} w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_1^2 + \left\| (f_{0\theta}^2 w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_1^2 \right\}.

By combining (6.7) and (6.8) and by applying Lemma 6.1 in Appendix, we get that, under assumption $\mathbf{(A_2)}$, $\text{Var}[S_{n,1}(\theta)]$ is bounded by

\[
n^{-1}C(\sigma_{\xi,2}^2, \sigma_{\xi,4}, f_{\theta}, w_\theta, p_\varepsilon) \times \min \left\{ \left\| (w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 + \left\| (f_{0\theta} w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 + \left\| (f_{0\theta}^2 w_\theta)^{i/2} \frac{K_{Cn}^*}{p_\varepsilon^*} \right\|_2^2 \right\},
\]

that is

\[
\text{Var}[S_{n,1}(\theta)] \leq C(\sigma_{\xi,2}^2, \sigma_{\xi,4}, f_{\theta}, w_\theta, p_\varepsilon) \max[1, C_n^{2n-2\alpha+(1-\rho)+(1-\rho)-2C_n^*}] \times \frac{1}{n}.
\]

and under (2.6), then (6.2) is proved.
It remains now to check that there exists two sequences \( \rho_k \) and \( \epsilon_k \) tending to 0, such that (6.1) holds. First write that

\[
|S_{n,1}(\theta) - S_{n,1}(\theta')| = \left| \frac{2}{n} \sum_{i=1}^{n} Y_i [(f_{\theta'} w_{\theta'} - f_{\theta} w_{\theta})] * K_{n,C_n}(Z_i) - \frac{1}{n} \sum_{i=1}^{n} [(f_{\theta'}^2 w_{\theta'} - f_{\theta}^2 w_{\theta})] * K_{n,C_n}(Z_i) \right| \\
= \left| \frac{2}{n} \sum_{i=1}^{n} (\theta' - \theta)^\top \left( \frac{\partial [Y_i f_{\theta} w_{\theta} + f_{\theta}^2 w_{\theta} - (Y_i^2 - \sigma_{\xi,2}^2) w_{\theta}]}{\partial \theta} \right|_{\theta=\hat{\theta}} \right| * K_{n,C_n}(Z_i) \right|
\]

where \( ||\theta||_{\ell^2} \leq ||\theta - \theta'||_{\ell^2} \), Consequently for \( ||\theta - \theta'||_{\ell^2} \leq \rho_k \),

\[
|S_{n,1}(\theta) - S_{n,1}(\theta')| \leq \frac{2\rho_k}{n} \left| \sum_{i=1}^{n} \left( \frac{\partial [Y_i f_{\theta} w_{\theta} + f_{\theta}^2 w_{\theta} + (Y_i^2 - \sigma_{\xi,2}^2) w_{\theta}]}{\partial \theta} \right|_{\theta=\hat{\theta}} \right| * K_{n,C_n}(Z_i) \left| \right|_{\ell^2}.
\]

It follows that (6.1) holds since for \( C_n \) satisfying (2.6), by using the same arguments as for the proof of (F.2), we have that for all \( \theta \in \Theta \)

\[
E[||n^{-1} \sum_{i=1}^{n} \left( \frac{\partial [Y_i f_{\theta} w_{\theta} + f_{\theta}^2 w_{\theta} + (Y_i^2 - \sigma_{\xi,2}^2) w_{\theta}]}{\partial \theta} \right) \right|_{\theta=\hat{\theta}} * K_{n,C_n}(Z_i) ||_{\ell^2}] = O(1),
\]

and hence for all \( k \in \mathbb{N} \),

\[
E[\sup_{||\theta' - \theta||_{\ell^2} \leq \rho_k} |S_{n,1}(\theta) - S_{n,1}(\theta')|] = O(\rho_k) \text{ as } n \to \infty. \quad \square
\]

6.2. Proofs of 2) of Theorem 3.1 and Corollary 3.1: If we denote by \( S_{n,1}^{(1)} \) and \( S_{n,1}^{(2)} \) the first and second derivatives of \( S_{n,1}(\theta) \) with respect to \( \theta \), by using classical Taylor expansion based on the smoothness properties of \( \theta \mapsto w_{\theta} f_{\theta} \) and the consistency of \( \hat{\theta}_1 \), we obtain that

\[
0 = S_{n,1}^{(1)}(\hat{\theta}_1) = S_{n,1}^{(1)}(\theta^0) + S_{n,1}^{(2)}(\theta^0)(\hat{\theta}_1 - \theta^0) + R_{n,1}(\hat{\theta}_1 - \theta^0),
\]

with \( R_{n,1} \) defined by

\[
R_{n,1} = \int_{0}^{1} [S_{n,1}^{(2)}(\theta^0) + s(\hat{\theta}_1 - \theta^0)] - S_{n,1}^{(2)}(\theta^0)] ds.
\]

This implies that

\[
\hat{\theta}_1 - \theta^0 = -[S_{n,1}^{(1)}(\theta^0) + R_{n,1}]^{-1}S_{n,1}^{(1)}(\theta^0),
\]

Consequently, we have to check the following four points.

i) \( E \left[ (S_{n,1}^{(1)}(\theta^0) - S_{\theta_{\rho,g}}^{(1)}(\theta^0)) (S_{n,1}^{(1)}(\theta^0) - S_{\theta_{\rho,g}}^{(1)}(\theta^0)) \right] = O(\varphi_n \varphi_n^\top) \)
ii) \( E \left[ ||S_{n,1}^{(2)}(\theta^0) - S_{\theta_{\rho,g}}^{(2)}(\theta^0)||_{\ell^2}^2 \right] = o(1) \)
iii) \( R_{n,1} \) defined in (6.3) satisfies \( E[||R_{n,1}||_{\ell^2}^2] = o(1) \) as \( n \to \infty \)
iv) \( E[||\hat{\theta}_1 - \theta^0||_{\ell^2}^2] \leq 4E \left[ (S_{n,1}^{(1)}(\theta^0)) \right] [(S_{\theta_{\rho,g}}^{(2)}(\theta^0))^{-1}]^\top (S_{\theta_{\rho,g}}^{(2)}(\theta^0))^{-1}S_{n,1}^{(1)}(\theta^0) + o(\varphi_n^2) \)

The rate of convergence of \( \hat{\theta}_1 \) is thus given by the order of \( S_{n,1}^{(1)}(\theta^0) - S_{\theta_{\rho,g}}^{(1)}(\theta^0) = S_{n,1}^{(1)}(\theta^0) \).
Proof of i)

Write that

\[
S_{n,1}^{(1)}(\theta) = \frac{\partial}{\partial \theta} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ ((Y_i - f_\theta)^2 - \sigma^2_{\xi,2})w_\theta \right] \ast K_{n,C_n}(Z_i) - \mathbb{E}[((Y - f_\theta(X))^2 - \sigma^2_{\xi,2})w_\theta(X)] \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\partial}{\partial \theta} \left[ ((Y_i - f_\theta)^2 - \sigma^2_{\xi,2})w_\theta \right] \right) \ast K_{n,C_n}(Z_i) - \mathbb{E} \left[ \frac{\partial}{\partial \theta} \left[ ((Y - f_\theta(X))^2 - \sigma^2_{\xi,2})w_\theta(X) \right] \right]
\]

Study of the bias

According to Lemma 6.2, \( \mathbb{E}[S_{n,1}^{(1)}(\theta^0)] \) is equal to

\[
-2\mathbb{E} \left[ f_{\theta^0}(X) \left( \frac{\partial(f_\theta w_\theta)}{\partial \theta} \ast K_{C_n}(X) - \frac{\partial(f_\theta w_\theta)}{\partial \theta}(X) \right) \right] + \mathbb{E} \left[ \frac{\partial}{\partial \theta} \left( f_{\theta^0}^2(X) \right) \ast K_{C_n}(X) - \frac{\partial(f_{\theta^0}^2 w_\theta)}{\partial \theta}(X) \right]
\]

that is

\[
\mathbb{E}[S_{n,1}^{(1)}(\theta^0)] = -2 \left\langle (f_{\theta^0}g)^*, \left( \frac{\partial(f_\theta w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\rangle
\]

+ \left\langle g^*, \left( \frac{\partial(f_\theta^2 w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\rangle + \left\langle (f_{\theta^0}^2 g)^*, \left( \frac{\partial w_\theta}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\rangle.
\]

On one hand, according to (6.4), under the assumption (A2), the bias is bounded in the following way

\[
\left| \mathbb{E} \left[ \frac{\partial S_{n,1}(\theta)}{\partial \theta} \right] \big|_{\theta=\theta^0} \right| \leq \pi^{-1} \left\| (f_{\theta^0}g)^* \right\|_2 \left\| \left( \frac{\partial(f_\theta w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\|_2
\]

+ \left(2\pi\right)^{-1} \left\| g^* \right\|_2 \left\| \left( \frac{\partial(f_\theta^2 w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\|_2
\]

and consequently,

\[
(6.11) \quad \left| \mathbb{E} \left[ \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right) \right] \right| \leq C(f_{\theta^0}, w_{\theta^0}, p_\varepsilon)
\]

\[
\times \left[ \left\| \left( \frac{\partial(w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\|_2 + \left\| \left( \frac{\partial(f_\theta f_\theta w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\|_2
\]+ \left\| \left( \frac{\partial(f_\theta^2 w_\theta)}{\partial \theta} \big|_{\theta=\theta^0} \right)^* (K_{C_n}^* - 1) \right\|_2 \right].
\]

And, on the other hand, the bias can also be bounded in the following way

\[
\left| \mathbb{E} \left[ \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \right] \right|_{\theta = \theta^0} \leq \pi^{-1} \| (f_{\theta^0} g)^* \|_\infty \left\| \left( \frac{\partial (f_{\theta^0} w_0)}{\partial \theta_j} \right) \bigg|_{\theta = \theta^0}^* (K_{C_n}^* - 1) \right\|_1 + (2\pi)^{-1} \| g^* \|_\infty \left\| \left( \frac{\partial (f_{\theta^0}^2 w_0)}{\partial \theta_j} \right) \bigg|_{\theta = \theta^0}^* (K_{C_n}^* - 1) \right\|_1 + (2\pi)^{-1} \| (f_{\theta^0}^2 g)^* \|_\infty \left\| \left( \frac{\partial w_0}{\partial \theta_j} \right) \bigg|_{\theta = \theta^0}^* (K_{C_n}^* - 1) \right\|_1
\]

and consequently,

\[
\left(6.12\right) \quad \left[ \mathbb{E} \left[ \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \right) \bigg|_{\theta = \theta^0}^* \right] \right] \leq C(f_{\theta^0}, w_{\theta^0}, p_\varepsilon) \times \left[ \left\| \left( \frac{\partial (w_0)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right)^* (K_{C_n}^* - 1) \right\|_1 + \left\| \left( \frac{\partial (w_0 f_0)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right)^* (K_{C_n}^* - 1) \right\|_1 \right.

\[
+ \left. \left\| \left( \frac{\partial (w_0 f_0^2)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right)^* (K_{C_n}^* - 1) \right\|_1 \right].
\]

By combining \(6.11\) and \(6.12\), we get that

\[
\mathbb{E}^2 \left[ \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right)^* \right] \leq C^{(1)}_{b_2}(f_{\theta^0}, w_{\theta^0}, p_\varepsilon, b, r)C_n^{-2a + (1-r) + (1-r) - e^{-2C_n^r}}.
\]

**Study of the variance**

For the variance term, it is easy to see that

\[
\text{Var} \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right) \leq \frac{C + o(1)}{n} \mathbb{E} \left[ \left( \frac{\partial [-2Y_i f_{\theta^0} w_0 + f_{\theta^0}^2 w_0 + (Y_i^2 - \sigma_{\xi,2}^2) w_\xi]}{\partial \theta_j} \bigg|_{\theta = \theta^0}^* \right) * K_{n,C_n}(Z_i) \right]^2
\]

that is, according to Lemma 6.2, \( \text{Var} (\partial S_{n,1}(\theta)/\partial \theta_j |_{\theta = \theta^0}) \) equals

\[
\frac{C + o(1)}{n} \left\{ \left( (f_{\theta^0}^4 + \sigma_{\xi,2}^2) g + (Q_{\xi,4} + 4f_{\theta^0}^2 \sigma_{\xi,2} - \sigma_{\xi,2}^4 + 4f_{\theta^0} \sigma_{\xi,3}) g) \right) * p_\varepsilon \right\} \left\{ \left( \frac{\partial (f_{\theta^0} w_0)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right) * K_{n,C_n} \right\}^2
\]

\[
+ \left\{ g * p_\varepsilon \right\} \left\{ \left( \frac{\partial (f_{\theta^0}^2 w_0)}{\partial \theta_j} \bigg|_{\theta = \theta^0} \right) * K_{n,C_n} \right\}^2 \}.
\]
It follows that, according to \((6.4)\), \(\text{Var}\left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} |_{\theta=\theta^0} \right)\) is less than

\[
\frac{C}{n} \left\{ \sigma_{\xi,2}^2 \left\| g \ast p_\varepsilon \right\|_{\infty} + \left\| (f_{\theta}^2 \varepsilon) \ast p_\varepsilon \right\|_{\infty} \right\} \left\| \left( \frac{\partial (f_{\theta} w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|_2 \times K_{n,C_n} \right\}^2
\]

\[
+ \left\| (f_{\theta}^1 + \sigma_{\xi,4} + 4f_{\theta}^2 \sigma_{\xi,2}^2 - \sigma_{\xi,2}^4 + 4f_{\theta} \sigma_{\xi,3}) g \ast p_\varepsilon \right\|_{\infty} \left\| \left( \frac{\partial w_\theta}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|_2 \times K_{n,C_n} \right\}^2
\]

\[
+ \left\| g \ast p_\varepsilon \right\|_{\infty} \left\| \left( \frac{\partial (f_{\theta}^2 \varepsilon) w_\theta}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|_2 \times K_{n,C_n} \right\}^2,
\]

that is

\[
\text{Var}\left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} |_{\theta=\theta^0} \right) \leq \frac{C(\sigma_{\xi,2}^2, f_{\theta}^1, f_{\theta}^2, w_\theta, p_\varepsilon)}{n} \times \left[ \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*} \right] + \left\| \left( \frac{\partial (f_{\theta} w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*} + \left\| \left( \frac{\partial (f_{\theta}^2 \varepsilon) w_\theta}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*}.
\]

And once again, according to \((6.4)\), another bound the variance term can be obtained to get that \(\text{Var}\left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} |_{\theta=\theta^0} \right)\) is bounded by

\[
\frac{C}{n} \left\{ \sigma_{\xi,2}^2 \left\| g \ast p_\varepsilon \right\|_{\infty} + \left\| (f_{\theta}^2 \varepsilon) g \ast p_\varepsilon \right\|_{\infty} \right\} \left\| \left( \frac{\partial (f_{\theta} w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|_2 \times K_{n,C_n} \right\}^2
\]

\[
+ \left\| g \ast p_\varepsilon \right\|_{\infty} \left\| \left( \frac{\partial (f_{\theta}^2 \varepsilon) w_\theta}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|_2 \times K_{n,C_n} \right\}^2,
\]

that is

\[
\text{Var}\left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} |_{\theta=\theta^0} \right) \leq \frac{C(\sigma_{\xi,2}^2, f_{\theta}^1, f_{\theta}^2, w_\theta, p_\varepsilon)}{n} \times \left[ \left\| \left( \frac{\partial (w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*} \right] + \left\| \left( \frac{\partial (f_{\theta} w_\theta)}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*} + \left\| \left( \frac{\partial (f_{\theta}^2 \varepsilon) w_\theta}{\partial \theta_j} \right) |_{\theta=\theta^0} \right\|^2_{p_\varepsilon^*}.
\]
By combining (6.13) and (6.14), we get that
\[
\text{Var} \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right) \leq \frac{C(\sigma_{\xi,2}^2, f_{\theta_0}, f_{\theta_0}^{(1)}, w_{\theta_0}, p_\varepsilon)}{n} \min \left\{ \left\| \frac{\partial (w_{\theta_0})}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right\|_2^2 + \left\| \frac{\partial (f_{\theta_0} w_{\theta_0})}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right\|_2^2, \right. \\
\left. \left\| \frac{\partial (w_{\theta_0})}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right\|_1^2 + \left\| \frac{\partial (f_{\theta_0} w_{\theta_0})}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right\|_1^2 \right\},
\]
that is, by applying Lemma 3.1
\[
\text{Var} \left( \frac{\partial S_{n,1}(\theta)}{\partial \theta_j} \bigg|_{\theta = \theta_0} \right) \leq \frac{C v_2(\sigma_{\xi,2}^2, f_{\theta_0}, f_{\theta_0}^{(1)}, w_{\theta_0}, p_\varepsilon)}{n} \max \{1, C_n^{2a-2a+(1-\rho)+(1-\rho)-\exp{-2bC_n^r+2\beta C_n^\rho}} \}.
\]

The rate of convergence of \( \hat{\theta}_1 \) denoting \( \varphi_n \varphi_n^\top \) corresponds to the best choice for the sequence \( C_n^* \), minimizing the sum of the variance \( \text{Var}[S_{n,1}^{(1)}(\theta)] \) and the square of the bias \( \langle (\mathbb{E}[S_{n,1}^{(1)}(\theta)] - S^{(1)}(\theta))^2 \rangle = (\mathbb{E}[S_{n,1}^{(1)}(\theta)])(\mathbb{E}[S_{n,1}^{(1)}(\theta)]^\top) \).

Polynomial noise (see (N2) with \( \beta = \rho = 0 \))

\( \bullet \) If for \( j = 1, \cdots, m \), \( \partial (f_\theta w_\theta)/\partial \theta_j \bigg|_{\theta = \theta_0} \), \( w_\theta \bigg|_{\theta = \theta_0} \) and \( \partial (f_\theta w_\theta)/\partial \theta_j \bigg|_{\theta = \theta_0} \) satisfy (R1) with \( r = 0 \), then
\[
|\mathbb{E} [\partial S_{n,1}(\theta)]/\partial \theta_j \bigg|_{\theta = \theta_0}\|^2 \leq C b_2(f_{\theta_0}, w_{\theta_0}, b, \sigma_{\xi,2}^2) C_n^{-2a+1},
\]
and
\[
\text{Var}(\partial S_{n,1}(\theta)/\partial \theta_j \bigg|_{\theta = \theta_0}) \leq C v_2(\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, f_{\theta_0}^{(1)}, f_{\theta_0}, w_{\theta_0}, p_\varepsilon) \max \{1, C_n^{2a-2a+1}/n \}
\]
It follows that if \( r = 0 \) and \( a < \alpha + 1/2 \) then which implies that
\[
C_n^* = n^{1/(2\alpha)} \quad \text{and} \quad \varphi_n^2 = O(n^{(1-2a)/(2\alpha)}).
\]
If \( r = 0 \), and \( a \geq \alpha + 1/2 \), then
\[
C_n^* = n^{1/(2a-1)} \quad \text{and} \quad \varphi_n = O(n^{-1})
\]

\( \bullet \) If for \( j = 1, \cdots, m \), \( \partial (f_\theta w_\theta)/\partial \theta_j \bigg|_{\theta = \theta_0} \), \( w_\theta \bigg|_{\theta = \theta_0} \) and \( \partial (f_\theta w_\theta)/\partial \theta_j \bigg|_{\theta = \theta_0} \) satisfy (R1) with \( r > 0 \), then
\[
|\mathbb{E} [\partial S_{n,1}(\theta)]/\partial \theta_j \bigg|_{\theta = \theta_0}\|^2 \leq C b_2(f_{\theta_0}, \sigma_{\xi,2}^2, f_{\theta_0}^{(1)}, w_{\theta_0}) C_n^{-2a+(1-r)+(1-\rho)-\exp{-2bC_n^r}},
\]
and
\[
\text{Var}(\partial S_{n,1}(\theta)/\partial \theta_j \bigg|_{\theta = \theta_0}) \leq C v_2(f_{\theta_0}, \sigma_{\xi,2}^2, f_{\theta_0}^{(1)}, w_{\theta_0}, p_\varepsilon)/n
\]
which implies that
\[
C_n^* = \begin{cases} 
\log n/2b + \frac{-2a + (1-r) + (1-r)}{2br} \log \left( \log n/2b \right) \right)^{1/r} & \text{and} \quad \varphi_n = O(n^{-1}).
\end{cases}
\]
Exponential noise (see (N2) with $\rho > 0$)

- If for $j = 1, \ldots, m$, $\partial (f_{\theta} w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$, $\partial w_{\theta} / \partial \theta_j |_{\theta = \theta_0}$ and $\partial (f_{\theta}^2 w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$ satisfy (R1) with $r = 0$, then

$$\left| \mathbb{E}[\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}] \right|^2 \leq C_{b_2} (f_{\theta_0}, \sigma_{\xi,2}', f_{\theta_0}^{(1)} w_{\theta_0}) C_n^{-2a+1},$$

and

$$\text{Var}(\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}) \leq C_{v_2} (f_{\theta_0}, \sigma_{\xi,2}', f_{\theta_0}^{(1)} w_{\theta_0}, p_{\varepsilon}) C_n^{-2a+(1-r)+(1-r)} \exp\{2\beta C_n^\rho\} / n$$

which implies that

$$C_n^* = \left[\frac{\log n}{2\beta} - \frac{2a + (1 - r) - (1 - r)}{2\rho} \log \left(\frac{\log n}{2b}\right)\right]^{1/r}$$

and $\varphi_n^2 = O((\log n / (2\beta))^{(1-2a)/\rho})$.

- If for $j = 1, \ldots, m$, $\partial (f_{\theta} w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$, $\partial w_{\theta} / \partial \theta_j |_{\theta = \theta_0}$ and $\partial (f_{\theta}^2 w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$ satisfy (R1) with $r > 0$ and $\{r > \rho\}$ or $\{r = \rho\}$ or $b = \beta$ and $a \geq \alpha + 1/2$ then

$$\left| \mathbb{E}[\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}] \right|^2 \leq C_{b_2} (\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, L(f_{\theta_0}^{(1)} w_{\theta_0}), L(f_{\theta_0} f_{\theta_0}^{(1)} w_{\theta_0}))$$

$$\times C_n^{-2a+(1-r)+(1-r)} \exp\{-2bC_n^\rho\},$$

and

$$\text{Var}(\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}) \leq C_{v_2} (\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, L(f_{\theta_0}^{(1)} w_{\theta_0}), L(f_{\theta_0} f_{\theta_0}^{(1)} w_{\theta_0})) / n$$

which implies that

$$C_n^* = \left[\frac{\log n}{2\beta} + \frac{2a + (1 - r) - (1 - r)}{2br} \log \left(\frac{\log n}{2b}\right)\right]^{1/r}$$

and $\varphi_n^2 = O((\log n / (2\beta))^{1/r})$.

- If for $j = 1, \ldots, m$, $\partial (f_{\theta} w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$, $\partial w_{\theta} / \partial \theta_j |_{\theta = \theta_0}$ and $\partial (f_{\theta}^2 w_{\theta}) / \partial \theta_j |_{\theta = \theta_0}$ satisfy (R1) with $r > 0$, and $\{r \leq \rho\}$ or $\{r = \rho\}$ or $b < \beta$ then

$$\left| \mathbb{E}[\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}] \right|^2 \leq C_b (\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, L(f_{\theta_0}^{(1)} w_{\theta_0}), L(f_{\theta_0} f_{\theta_0}^{(1)} w_{\theta_0}))$$

$$\times C_n^{-2a+(1-r)+(1-r)} \exp\{-2bC_n^\rho\},$$

and

$$\text{Var}(\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}) \leq C_{v_2} (\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, L(f_{\theta_0}^{(1)} w_{\theta_0}), L(f_{\theta_0} f_{\theta_0}^{(1)} w_{\theta_0}))$$

$$\times C_n^{2\alpha-2a+(1-r)+(1-r)} \exp\{-2bC_n^\rho + 2\beta C_n^\rho\} / n$$

which implies that the bias is by a logarithmic factor larger than the variance

$$C_n^* = \left[\frac{\log n}{2\beta} - \frac{2a + (1 - r) - (1 - r)}{2\rho} \log \left(\frac{\log n}{2\beta}\right)\right]^{1/r}$$

and $\varphi_n^2 = O(\log n / (2\beta))^{r/\rho} \exp\{-2b \left(\frac{\log n}{2\beta}\right)^{r/\rho}\}$.

If $r = \rho$, $b = \beta$ and $a < \alpha + 1/2$ then

$$\text{Var}(\partial S_n,1(\theta) / \partial \theta_j |_{\theta = \theta_0}) \leq C_{v_2} (\mathbb{E}(f_{\theta_0}^2(X)), \sigma_{\xi,2}^2, L(f_{\theta_0}^{(1)} w_{\theta_0}), L(f_{\theta_0} f_{\theta_0}^{(1)} w_{\theta_0})) C_n^{2\alpha-2a+1} / n.$$
giving the rate $\varphi_n^2 = O((\log n)^{2\alpha-2\alpha+1}/n)$.

**Proof of ii)**

By using that

\[
(S_{n,1}^{(2)}(\theta))_{j,k} = -\frac{2}{n} \sum_{i=1}^{n} \left( -2Y_i \frac{\partial^2 (f_\theta w \theta)}{\partial \theta_j \partial \theta_k} + \frac{\partial^2 (f_\theta^2 w \theta)}{\partial \theta_j \partial \theta_k} + (Y_i^2 - \sigma_{\xi,2}^2) \frac{\partial^2 w \theta}{\partial \theta_j \partial \theta_k} \right) |_{\theta=\theta^0} * K_{n,C_n}(Z_i),
\]

we write $S_{n,1}^{(2)}(\theta^0) - S_{\theta^0,g}(\theta^0) = A_1 + A_2 + A_3$ with

\[
A_1 = -\frac{2}{n} \sum_{i=1}^{n} Y_i \left( \frac{\partial^2 (f_\theta w \theta)}{\partial \theta_j \partial \theta_k} |_{\theta=\theta^0} \right) * K_{n,C_n}(Z_i) - \mathbb{E} \left[ Y_i \left( \frac{\partial^2 (f_\theta w \theta)}{\partial \theta_j \partial \theta_k} (X) |_{\theta=\theta^0} \right) \right]
\]

\[
A_2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\partial^2 (f_\theta w \theta)}{\partial \theta_j \partial \theta_k} |_{\theta=\theta^0} \right) * K_{n,C_n}(Z_i) - \mathbb{E} \left[ \left( \frac{\partial^2 (f_\theta^2 w \theta)}{\partial \theta_j \partial \theta_k} (X) |_{\theta=\theta^0} \right) \right]
\]

\[
A_3 = \frac{1}{n} \sum_{i=1}^{n} (Y_i^2 - \sigma_{\xi,2}^2) \left( \frac{\partial^2 w \theta}{\partial \theta_j \partial \theta_k} |_{\theta=\theta^0} \right) * K_{n,C_n}(Z_i) - \mathbb{E} \left[ (Y_i^2 - \sigma_{\xi,2}^2) \left( \frac{\partial^2 w \theta}{\partial \theta_j \partial \theta_k} (X) |_{\theta=\theta^0} \right) \right].
\]

As soon as $w_\theta f_\theta$ and $w_\theta f_\theta^2$ and their derivatives up to order 2, satisfy \(\mathbb{R}_1\), then for $C_n$ satisfying that \(\mathbb{F}_0\), we get that $A_j = o_{p}(1)$ as $n \to \infty$ for $j = 1, \ldots, 3$, and ii) is proved.

**Proof of iii)**

Again using \(6.19\), the smoothness properties of the derivatives of $w_\theta f_\theta$ and $w_\theta f_\theta^2$ up to order 3 and the consistency of $\hat{\theta}_1$, we get that $\|R_{n,1}\|_{\ell_2} = o_{p}(1)$ as $n \to \infty$.

**Proof of iv)**

Let us introduce the random event $E_n = \cap_{j,k} E_{n,j,k}$, where

\[
E_{n,j,k} = \left\{ \omega \text{ such that } \left| \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} \right|_{\theta=\theta^0} - \frac{\partial^2 S_{n,1}(\theta, \omega)}{\partial \theta_j \partial \theta_k} \right|_{\theta=\theta^0} + (R_{n,1})_{j,k}(\omega) \right| \leq \frac{1}{2} \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} \right|_{\theta=\theta^0} \right\}.
\]

According to \(6.10\) and \(6.9\) we have

\[
\mathbb{E}\|\hat{\theta}_1 - \theta^0\|_{\ell_2} \leq \mathbb{E}\|\hat{\theta}_1 - \theta^0\|_{\ell_2}^2 \mathbb{E}_{E_n} + \mathbb{E}\|\hat{\theta}_1 - \theta^0\|_{\ell_2}^2 \mathbb{E}_{E_n} + 2 \sup_{\theta \in \Theta} \|\theta\|_{\ell_2}^2 \mathbb{P}_{\theta^0,g}(E_n^c)
\]

\[
\leq \mathbb{E}\left[ (S_{n,1}^{(1)}(\theta^0))^\top \left( (S_{n,1}^{(2)}(\theta^0) + R_{n,1})^{-1} (S_{n,1}^{(2)}(\theta^0) + R_{n,1})^{-1} S_{n,1}^{(1)}(\theta^0) \right) \right]_{\ell_2} E_{n}
\]

\[
+ 2 \sup_{\theta \in \Theta} \|\theta\|_{\ell_2}^2 \mathbb{P}_{\theta^0,g}(E_n^c)
\]

\[
\leq C_4^2 \sup_{j,k} \left| \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} \right|_{\theta=\theta^0}^{-2} \mathbb{E}\left( (S_{n,1}^{(1)}(\theta^0))^\top S_{n,1}^{(1)}(\theta^0) \right) + 2 \sup_{\theta \in \Theta} \|\theta\|_{\ell_2}^2 \mathbb{P}_{\theta^0,g}(E_n^c)
\]

\[
 \leq C_4^2 \sup_{j,k} \left| \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} \right|_{\theta=\theta^0}^{-2} \varphi_n^2 + 2 \sup_{\theta \in \Theta} \|\theta\|_{\ell_2}^2 \mathbb{P}_{\theta^0,g}(E_n^c).
\]
It remains thus to show that \( \mathbb{P}_{\theta^0,g}(E_n^C) = o\left(\varphi_n^2\right) \) with
\[
\sup_{j,k} \mathbb{E} \left[ \left( \frac{\partial^2 (S_{\theta_1}(\theta) - S_{\theta^0}(\theta))}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right)^2 \right] \leq \varphi_n^2.
\]
By Markov’s inequality, for some \( p > 2 \),
\[
\mathbb{P}_{\theta^0,g}(E_n^C) \leq \sum_{j=1}^{m} \sum_{k=1}^{m} \mathbb{P}_{\theta^0,g}(E_{n,j,k}^C),
\]
with
\[
\mathbb{P}_{\theta^0,g}(E_{n,j,k}^C) \leq \mathbb{E} \left[ \left( \frac{\partial^2 (S_{\theta^0,g}(\theta) - S_{\theta_1}(\theta))}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) + (R_{n,j,k}) \right]^{p} \leq 2^{p-1} \left( \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) - \mathbb{E} \left[ \left( \frac{\partial^2 S_{\theta_1}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) \right]^{p} + 2^{p-1} \mathbb{E} \left[ \left( \frac{\partial^2 S_{\theta^0,g}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) - \mathbb{E} \left[ \left( \frac{\partial^2 S_{\theta_1}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) \right] \right]^{p} \leq 2^{p-2} \sum_{i=1}^{n} W_{n,i,j,k}\]
Now we apply the Rosenthal’s inequality to the sum of centered variables
\[
\left( \frac{\partial^2 S_{\theta_1}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) - \mathbb{E} \left[ \left( \frac{\partial^2 S_{\theta_1}(\theta)}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) \right] = n^{-1} \sum_{i=1}^{n} W_{n,i,j,k}
\]
where \( W_{n,i,j,k} \) equals
\[
\left( \frac{\partial^2 [-2Y_i f_\theta w_\theta + f_\theta^2 w_\theta + (Y_i^2 - \sigma^2_\theta) w_\theta]}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) * K_{n,C_n}(Z_i)
\]
\[
-2 \mathbb{E} \left[ \left( \frac{\partial^2 [-2Y_i f_\theta w_\theta + f_\theta^2 w_\theta + (Y_i^2 - \sigma^2_\theta) w_\theta]}{\partial \theta_j \partial \theta_k} |_{\theta = \theta^0} \right) * K_{n,C_n}(Z_i) \right].
\]

**Lemma 6.1.** *Rosenthal’s inequality* (Rosenthal (1970), Petrov (1995)). For \( U_1, \ldots, U_n \), be \( n \) independent centered random variables, there exists a constant \( C(p) \) such that for \( p \geq 1 \),
\[
(6.20) \quad \mathbb{E}[\sum_{i=1}^{n} U_i^p] \leq C(p) \left[ \sum_{i=1}^{n} \mathbb{E}[|U_i|^p] + (\sum_{i=1}^{n} \mathbb{E}[U_i^2])^{p/2} \right].
\]
Consequently
\[
\mathbb{E}\left[\left(\frac{\partial^2 S_{n,1}(\theta)}{\partial \theta_j \partial \theta_k} \bigg|_{\theta = \theta^0}\right) - \mathbb{E}\left(\left(\frac{\partial^2 S_{n,1}(\theta)}{\partial \theta_j \partial \theta_k} \bigg|_{\theta = \theta^0}\right)\right)^p\right] 
\leq C(p) \left[ n^{1-p}\mathbb{E}|W_{n,1,j,k}|^p + n^{-p/2}\mathbb{E}^{p/2}|W_{n,1,j,k}|^2 \right].
\]

Take \( p = 4 \) to get that
\[
\mathbb{E}\left[\left(\frac{\partial^2 S_{n,1}(\theta)}{\partial \theta_j \partial \theta_k} \bigg|_{\theta = \theta^0}\right) - \mathbb{E}\left(\left(\frac{\partial^2 S_{n,1}(\theta)}{\partial \theta_j \partial \theta_k} \bigg|_{\theta = \theta^0}\right)\right)^4\right] 
\leq C(4) \left[ n^{-3}\mathbb{E}|W_{n,1}|^4 + n^{-2}\mathbb{E}|W_{n,1}|^2 \right].
\]

Therefore under the conditions ensuring that \( (\mathbb{E}[S_{\theta^0,g}^{(2)}(\theta^0) - S_{n,1}^{(2)}(\theta^0)]_{j,k} = o(1) \) we have
\[
(\mathbb{E}[(S_{n,1}^{(2)}(\theta^0) - S_{\theta^0,g}^{(2)}(\theta^0)])^4]_{j,k} = O(\nu_n^4) = o(\nu_n^4).
\]

Now, by using the definition of \( R_{n,1} \) combined with (6.19), and the smoothness properties of the derivatives of \( (w_{\theta^0} f_0) \) and \( (w_{\theta^0} f_2^2) \) up to order 3, we get that \( \mathbb{E}((R_{n,1})^4_{j,k}) = o(||\hat{\theta}_1 - \theta^0||^4_{L^2}) \), and we conclude that
\[
\mathbb{E}||\hat{\theta}_1 - \theta^0||^2_{L^2} \leq 4\mathbb{E}\left[(S_{n,1}^{(1)}(\theta^0))^\top[(S_{\theta^0,g}^{(2)}(\theta^0))^{-1}]^\top(S_{\theta^0,g}^{(2)}(\theta^0))^{-1}S_{n,1}^{(1)}(\theta^0)\right] + o(\nu_n^2) + o(\mathbb{E}[||\hat{\theta}_1 - \theta^0||^4_{L^2}]).
\]

6.3. **Proof of Theorem 3.3**: As for the proof of Theorem 3.1, the main point of the proof consists in showing that for any \( \theta \in \Theta \), \( \mathbb{E}[(S_{n,2}(\theta) - S_{\theta^0,g}(\theta))^2] = o(1) \), with \( S_{\theta^0,g}(\theta) \) admitting a unique minimum in \( \theta = \theta^0 \). The second part of the proof consists in studying \( \omega_2(n, \rho) = \sup \{|S_{n,2}(\theta) - S_{\theta^0,g}(\theta)| : ||\theta - \theta^0||_2 \leq \rho\} \). By using the regularity assumptions on the regression function \( f \), we state that there exists two sequences \( \rho_k \) and \( \epsilon_k \) tending to 0, such that for all \( k \in \mathbb{N} \),
\[
(6.21) \quad \lim_{n \to \infty} \mathbb{P}[\omega_2(n, \rho_k) > \epsilon_k] = 0 \quad \text{and} \quad \mathbb{E}[(\omega_2(n, \rho_k))^2] = O(\rho_k^2).
\]

Under the conditions (C11)-(C14) and by applying the Law of Large numbers, we get that for any \( \theta \in \Theta \), \( \mathbb{E}[(S_{n,2}(\theta) - S_{\theta^0,g}(\theta))^2] = o(1) \) as \( n \to \infty \) and \( \mathbb{E}[(S_{n,2}(\theta) - S_{n,2}(\theta'))^2] = o(1) \) and consequently \( \hat{\theta}_2 \) is a consistent estimator of \( \theta^0 \). By using classical Taylor expansion based on the smoothness properties of the regression function, with respect to \( \theta \) and the consistency of \( \hat{\theta}_2 \), we get that
\[
0 = S_{n,2}^{(1)}(\hat{\theta}_1) = S_{n,2}^{(1)}(\theta^0) + S_{n,2}^{(2)}(\theta^0)\hat{\theta}_2 - \theta^0 + R_{n,2}(\hat{\theta}_2 - \theta^0),
\]

with \( R_{n,2} \) defined by
\[
R_{n,2} = \int_0^1 [S_{n,2}^{(2)}(\theta^0) + s(\hat{\theta}_2 - \theta^0))\hat{\theta} = S_{n,2}^{(2)}(\theta^0)]ds.
\]

This implies that
\[
(6.22) \quad \hat{\theta}_2 - \theta^0 = -[S_{n,2}^{(2)}(\theta^0) + R_{n,2}]^{-1}S_{n,2}^{(1)}(\theta^0),
\]

where \( S_{n,2}^{(1)}(\theta) \) and \( S_{n,2}^{(2)}(\theta) \) denote the first and second derivatives of \( S_{n,2}(\theta) \) with respect to \( \theta \). The \( \sqrt{n} \)-consistency follows by applying the Central limit theorem to \( S_{n,2}^{(1)}(\theta^0) \) to get that
\[
(6.24) \quad \sqrt{n}S_{n,2}^{(1)}(\theta^0) \xrightarrow{\mathcal{L}} N(0, \Sigma_0,2)
\]
with
\[
\Sigma_{0,2} = \mathbb{E} \left[ \left( \Phi_{\theta^0, \epsilon, 1}^{(1)}(Z) - 2Y \Phi_{\theta^0, \epsilon, 2}^{(1)}(Z) + (Y^2 - \sigma^2_{\epsilon, 2}) \Phi_{\theta^0, \epsilon, 3}^{(1)}(Z) \right) \times \left( \Phi_{\theta^0, \epsilon, 1}^{(1)}(Z) - 2Y \Phi_{\theta^0, \epsilon, 2}^{(1)}(Z) + (Y^2 - \sigma^2_{\epsilon, 2}) \Phi_{\theta^0, \epsilon, 3}^{(1)}(Z) \right)^\top \right].
\]

**APPENDIX : TECHNICAL LEMMAS**

**Lemma 6.1.** Let \( \varphi \) a function such that \( \varphi \) belongs to \( L_2(\mathbb{R}) \) satisfying \( (K) \). Then
\[
\int_{|u| \geq C_n} |\varphi^*(u)| du \leq \frac{L(\varphi)}{R(a, b, r)} C_n^{-a+1-r} \exp\{ -bC_n^p \}.
\]
Furthermore, if \( p_\varepsilon \) satisfies \( (N_2) \), then
\[
\int_{|u| \leq C_n} \frac{|\varphi^*(u)|}{p_\varepsilon(u)} du \leq \frac{L(\varphi)}{R(\alpha, \beta, \rho, a, b, r)C(p_\varepsilon)} \max\{ 1, C_n^{a-1-\rho} \exp\{ -bC_n^p + \beta C_n^p \} \}.
\]

**Lemma 6.2.** Let \( \varphi \) such that \( \mathbb{E}(|\varphi(Y, X)|) \) is finite and let \( \Phi \) such that \( \mathbb{E}(|\Phi(U)|) \) is finite. Then
\[
\mathbb{E}[\varphi(Y, X) \Phi * K_{n,C_n}(U)] = \mathbb{E}[\varphi(Y, X) \Phi * K_{C_n}(X)] = \langle \varphi(\cdot, \cdot) f_{Y, X}(y, \cdot), \Phi * K_{C_n} \rangle
\]
and
\[
\mathbb{E} [\varphi(Y, X) \Phi * K_{n,C_n}(U)]^2 = \int \langle \varphi(x, \cdot) f_{Y, X}(x, \cdot) * f_\varepsilon, (\Phi * K_{n,C_n})^2 \rangle \, dx.
\]

**Proof of Lemma 6.2:** If we denote by \( f_{Y,Z,X} \) the joint distribution of \( (Y, Z, X) \), then
\[
f_{Y,Z,X}(y, z, x) = f_{Y,X}(y, x) f_\varepsilon(z - x) \quad \text{and} \quad f_{Y,Z}(y, z) = f_{Y,X}(y, \cdot) * f_\varepsilon(z).
\]
Since \( K_{n,C_n} * p_\varepsilon = K_{C_n} \) by construction, we get by the Parseval’s formula that
\[
\mathbb{E} [\varphi(Y, X) \Phi * K_{n,C_n}(Z)] = \iint \int \varphi(y, x) \Phi * K_{n,C_n}(z) f_{Y,Z,X}(y, z, x) dy \, dz \, dx
\]
\[
= \iiint \varphi(y, x) \Phi * K_{n,C_n}(z) f_{Y,X}(y, x) f_\varepsilon(z - x) dy \, dz \, dx
\]
\[
= \iint \varphi(y, x) f_{Y,X}(y, x) \int \Phi * K_{n,C_n}(z) f_\varepsilon(z - x) dz \, dx
\]
\[
= (2\pi)^{-1} \iiint \varphi(y, x) f_{Y,X}(y, x) \int \Phi^*(u) K_{n,C_n}^*(u) f_\varepsilon^*(u) e^{-iux} du \, dy \, dx
\]
\[
= (2\pi)^{-1} \iiint \varphi(y, x) f_{Y,X}(y, x) \int \Phi^*(u) K_{n,C_n}^*(u) f_\varepsilon^*(u) e^{-iux} du \, dy \, dx
\]
\[
= (2\pi)^{-1} \iiint \varphi(y, x) f_{Y,X}(y, x) \int \Phi^*(u) K_{n,C_n}^*(u) e^{-iux} du \, dy \, dx
\]
\[
= \iint \varphi(y, x) f_{Y,X}(y, x) \int \Phi(z) K_{C_n}(x - z) dz \, dy \, dx
\]
\[
= \iint \varphi(y, x) f_{Y,X}(y, x) \Phi * K_{C_n}(x) \, dy \, dx.
\]
The second equality follows by writing that

$$
\mathbb{E} [\varphi(Y, X) \Phi \star K_{n,C_n}(Z)]^2 = \iiint \varphi(y, x)(\Phi \star K_{n,C_n}(z))^2 f_{Y,Z,X}(y, z, x) dy \, dz \, dx \\
= \iiint \varphi(y, x)(\Phi \star K_{n,C_n}(z))^2 f_{Y,X}(y, x) f_\varepsilon(z - x) dy \, dz \, dx \\
= \iiint \varphi(y, x)(\Phi \star K_{n,C_n}(z))^2 f_{Y,X}(y, x) f_\varepsilon(z - x) dy \, dz \, dx \\
= \int \langle (\varphi(y, \cdot) f_{Y,X}(y, \cdot)) \star f_\varepsilon, (\Phi \star K_{n,C_n})^2 \rangle dy.
$$

\[\square\]

References


References


