

Almost sure asymptotics for a diffusion process in a drifted Brownian potential

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Abstract

We study a one-dimensional diffusion process in a drifted Brownian potential. We characterize the upper functions of its hitting times in the sense of Paul Lévy, and determine the lower limits in terms of an iterated logarithm law.

KEY WORDS: *Random environment, diffusion in a random potential, Lévy class.*

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1 Introduction

We consider a diffusion process in random environment, defined as follows. For $\kappa \in \mathbb{R}$, we introduce the random potential

$$W_\kappa(x) := W(x) - \kappa x/2, \quad x \in \mathbb{R}, \quad (1)$$

where $(W(x), x \in \mathbb{R})$ is a standard two-sided Brownian motion. We define a diffusion process $(X(t), t \geq 0)$ in the random potential W_κ , as solution to the formal stochastic differential equation $dX(t) = d\beta(t) - \frac{1}{2}W'_\kappa(X(t))dt$, where $(\beta(t), t \geq 0)$ is a Brownian motion independent of W and $X(0) = 0$. More rigorously, X is a diffusion process such that $X(0) = 0$, and whose conditional generator given W_κ is $\frac{1}{2}e^{W_\kappa(x)} \frac{d}{dx} (e^{-W_\kappa(x)} \frac{d}{dx})$.

We denote by P_ω the law of X conditionally on the environment W_κ , and call it the quenched law. We also define $\mathbb{P}(\cdot) := \int P_\omega(\cdot) P(W_\kappa \in d\omega)$, and call it the annealed law.

The diffusion X , introduced by Schumacher (1985) and Brox (1986), is generally considered as the continuous time analogue of random walks in random environment (RWRE), which have many applications in physics and biology. For an account of general properties of RWRE, we refer to Zeitouni (2004).

In this paper, we are interested in the transient case, that is, we suppose $\kappa \neq 0$. We may assume without loss of generality that $\kappa > 0$. In this case, $X(t) \rightarrow_{t \rightarrow +\infty} +\infty$ \mathbb{P} -a.s.

Our goal is to study the almost sure asymptotics of X .

We denote by H the first hitting time of r by X , that is,

$$H(r) := \inf\{t \geq 0, \quad X(t) > r\}, \quad r \geq 0. \quad (2)$$

(See (27) for an analytic expression of $H(r)$). We recall that there are three different regimes for H :

Theorem A (*Kawazu and Tanaka (1997)*) *When r tends to infinity,*

$$\begin{aligned} H(r)/r^{1/\kappa} &\xrightarrow{\mathcal{L}} c_0 S_\kappa^{ca}, & 0 < \kappa < 1, \\ H(r)/(r \log r) &\xrightarrow{P} 4, & \kappa = 1, \\ H(r)/r &\xrightarrow{a.s.} 4/(\kappa - 1), & \kappa > 1, \end{aligned} \quad (3)$$

where $c_0 = c_0(\kappa) > 0$ is a finite constant, the symbols “ $\xrightarrow{\mathcal{L}}$ ”, “ \xrightarrow{P} ” and “ $\xrightarrow{a.s.}$ ” denote respectively convergence in law, in probability and almost sure convergence, with respect to the annealed probability \mathbb{P} . Moreover, S_κ^{ca} is a completely asymmetric stable variable of index κ , and is a positive variable for $0 < \kappa < 1$ (see (15) for its characteristic function).

In view of (3), we only need to study the case $\kappa \in (0, 1]$. We prove

Theorem 1.1 *Let $a(\cdot)$ be a positive nondecreasing function. If $0 < \kappa < 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \iff \limsup_{r \rightarrow \infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

If $\kappa = 1$, the statement holds under the additional assumption that $\limsup_{r \rightarrow +\infty} \frac{\log r}{a(r)} < \infty$.

Theorem 1.2 We have $(c_1(\kappa) \in (0, \infty)$ is given in equation (25)) :

$$\liminf_{r \rightarrow +\infty} \frac{H(r)}{r^{1/\kappa}/(\log \log r)^{(1/\kappa)-1}} = c_1(\kappa) \quad \mathbb{P}\text{-a.s.}, \quad 0 < \kappa < 1, \quad (4)$$

$$\liminf_{r \rightarrow +\infty} H(r)/(r \log r) = 4 \quad \mathbb{P}\text{-a.s.}, \quad \kappa = 1. \quad (5)$$

It was asked in Hu et al. (1999) whether the convergence in probability $H(r)/(r \log r) \rightarrow 4$ in Theorem A in the case $\kappa = 1$ can be strengthened into an almost sure convergence. Theorem 1.1 gives a negative answer.

We observe that in the case $0 < \kappa < 1$, the process $H(\cdot)$ has the same Lévy classes as κ -stable subordinators (see Bertoin (1996) p. 92).

Theorems 1.1 and 1.2 can be stated for the process X itself, by means of a standard argument.

These results are useful for the study of the maximum local time of X , see Devulder (2005).

The rest of the paper is organized as follows. We present in Section 2 some technical estimates which will be needed later on; the proof of one of the technical estimates (Lemma 2.3), is postponed until Section 4. In Section 3, we study the Lévy classes for the hitting times $H(r)$ and prove Theorems 1.1 and 1.2. Finally, we prove Lemma 2.3 in Section 4.

Throughout the paper, the letter c with a subscript, denotes unimportant constants that are finite and positive.

2 Technical estimates

We start by introducing $A(x) := \int_0^x e^{W_\kappa(y)} dy$ for $x \in \mathbb{R}$, which is a scaling function of X . We observe that, since $\kappa > 0$, $A(x) \xrightarrow{x \rightarrow +\infty} A_\infty < \infty$.

For technical reasons, we have to introduce the random function F as follows. Fix $r > 0$. Since the function $x \mapsto A_\infty - A(x)$ is almost surely continuous and (strictly) decreasing, there exists a unique $F(r) \in \mathbb{R}$, depending only on the process W_κ , such that

$$A_\infty - A(F(r)) = \exp(-\kappa r/2) =: \delta(r). \quad (6)$$

Our first technical estimate describes how close $F(r)$ is to r , for large r .

Lemma 2.1 Let $\kappa > 0$ and $0 < \delta_0 < 1/2$. Define for $r > 0$,

$$E_1(r) := \{(1 - 5r^{-\delta_0}/\kappa)r \leq F(r) \leq (1 + 5r^{-\delta_0}/\kappa)r\}. \quad (7)$$

Then for all large r , $\mathbb{P}(E_1(r)^c) \leq \exp(-r^{1-2\delta_0}/4)$. As a consequence, for any $\varepsilon > 0$, we have, almost surely, for all large r ,

$$(1 - \varepsilon)r \leq F(r) \leq (1 + \varepsilon)r. \quad (8)$$

Proof. Let $0 < \delta_0 < 1/2$, and fix $r > 0$. Define $\widetilde{W}_\kappa(u) := W(u+r) - W(r) - \kappa u/2$, and $\widetilde{A}_\infty := \int_0^\infty \exp(\widetilde{W}_\kappa(u)) du$. Hence, $\log[A_\infty - A(r)] = \log \widetilde{A}_\infty + W_\kappa(r)$. Let

$$E_2(r) := \{(-2r^{-\delta_0} - \kappa/2)r \leq \log[A_\infty - A(r)] \leq (2r^{-\delta_0} - \kappa/2)r\}.$$

Recall that $\tilde{A}_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ (see e.g. Dufresnes, (2000)), i.e., γ_κ has density $e^{-x}x^{\kappa-1}/\Gamma(\kappa)$ for positive x . Consequently,

$$\mathbb{P}(E_2(r)^c) \leq \mathbb{P}[\gamma_\kappa > 2e^{r^{1-\delta_0}}] + \mathbb{P}[\gamma_\kappa < 2e^{-r^{1-\delta_0}}] + \mathbb{P}[|W(r)| > r^{1-\delta_0}] \leq 3e^{-r^{1-2\delta_0}/2} \quad (9)$$

for r large enough. Recall that $A_\infty - A(F(r)) = \delta(r) = e^{-\kappa r/2}$. On $E_2[(1 + 5r^{-\delta_0}/\kappa)r]$,

$$\begin{aligned} \log\{A_\infty - A[(1 + 5r^{-\delta_0}/\kappa)r]\} &\leq (\kappa r/2)(-1 + 4r^{-\delta_0}/\kappa + o(r^{-\delta_0}))(1 + 5r^{-\delta_0}/\kappa) \\ &< \log[A_\infty - A(F(r))], \end{aligned}$$

where $f(r) = o(g(r))$ means $\lim_{r \rightarrow 0} f(r)/g(r) = 0$. This gives the second inequality in (7) by monotonicity of A . Similarly, the first inequality holds on $E_2[(1 - 5r^{-\delta_0}/\kappa)r]$. This yields $\mathbb{P}(E_1(r)^c) \leq \exp(-r^{1-2\delta_0}/4)$ in view of (9).

Then, (8) follows from the Borel–Cantelli lemma and the monotonicity of $F(\cdot)$. \square

With an abuse of notation, for $r \geq 0$, we denote by $X \circ \Theta_{H(r)}$ the process $(X(H(r)+t) - r, t \geq 0)$, which, conditionally on W_κ , is a diffusion in the potential $(W_\kappa(x+r) - W_\kappa(r), x \in \mathbb{R})$, starting from 0. Define $H_{X \circ \Theta_{H(r)}}(s) = H(r+s) - H(r)$. Similarly, $F_{X \circ \Theta_{H(r)}}$, and $(H \circ F)_{X \circ \Theta_{H(r)}}$ denote respectively the processes F and $H \circ F$ for the diffusion $X \circ \Theta_{H(r)}$. The following lemma is a modification of the Borel–Cantelli lemma.

Lemma 2.2 *Let $\kappa > 0$. Let $(\Delta_n)_{n \geq 1}$ be a sequence of open sets in \mathbb{R} . Let $\alpha > 0$, $r_n := \exp(n^\alpha)$ and $R_n := \sum_{k=1}^n r_k$. If $\sum_{n \geq 1} \mathbb{P}\{(H \circ F)(r_{2n}) \in \Delta_n\} = +\infty$, then for any $\varepsilon > 0$, almost surely, there exist infinitely many n such that $H_{X \circ \Theta_{H(R_{2n-1})}}(t_n) \in \Delta_n$ for some $t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$.*

Proof. Let $n \geq 1$, $x_n := r_{2n-1}/2$, $\varepsilon_0 > 0$, $v_n := 2(\log n)/\kappa$ and

$$E_3(n) := \left\{ \inf_{t: H(R_{2n-1}) \leq t \leq H(R_{2n} + x_{n+1})} X(t) > R_{2n-2} + x_n \right\}.$$

First, notice that $P_\omega(E_3(n)^c) = (1 + [\int_{R_{2n-2}+x_n}^{R_{2n-1}} e^{W_\kappa(x)} dx] / [\int_{R_{2n-1}}^{R_{2n}+x_{n+1}} e^{W_\kappa(x)} dx])^{-1}$ a.s. Define

$$E_4 := \left\{ \sup_{0 \leq x \leq r_{2n-1} - x_n} |W_\kappa(x + R_{2n-2} + x_n) - W_\kappa(R_{2n-2} + x_n) + \kappa x/2| \leq \varepsilon_0(r_{2n-1} - x_n) \right\},$$

$$E_5 := \left\{ \sup_{x \geq 0} [W_\kappa(x + R_{2n-1}) - W_\kappa(R_{2n-1})] \leq v_n \right\}.$$

For large n , $\mathbb{P}(E_4^c) \leq 2 \exp[-\varepsilon_0^2(r_{2n-1} - x_n)/2]$ and $\mathbb{P}(E_5^c) = \exp(-\kappa v_n) = 1/n^2$ (see Borodin et al. (2002), formula 1.1.4 (1)). Moreover, we have for n large enough, on $E_4 \cap E_5$,

$$\begin{aligned} P_\omega(E_3(n)^c) &\leq \kappa \frac{(r_{2n} + x_{n+1}) \exp[v_n + W_\kappa(R_{2n-1})]}{\exp[W_\kappa(R_{2n-2} + x_n) - \varepsilon_0(r_{2n-1} - x_n)]} \\ &\leq \kappa(r_{2n} + x_{n+1}) \exp[v_n + (2\varepsilon_0 - \kappa/2)(r_{2n-1} - x_n)]. \end{aligned}$$

Integrating this over $E_4 \cap E_5$ yields $\sum_{n=1}^{+\infty} \mathbb{P}(E_3(n)^c) < \infty$ for $\varepsilon_0 < \kappa/4$.

To complete the proof of Lemma 2.2, we define

$$\begin{aligned} \mathcal{D}_n &:= \left\{ \exists t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}], H_{X \circ \Theta_{H(R_{2n-1})}}(t_n) \in \Delta_n \right\}, \\ \mathcal{E}_n &:= \left\{ (1 - 5r_{2n}^{-\delta_0}/\kappa)r_{2n} \leq F_{X \circ \Theta_{H(R_{2n-1})}}(r_{2n}) \leq (1 + 5r_{2n}^{-\delta_0}/\kappa)r_{2n} \right\}. \end{aligned}$$

Let $\tilde{t}_n := F_{X \circ \Theta_{H(R_{2n-1})}}(r_{2n})$. We have $\mathcal{D}_n \cap E_3(n) \supset \{H_{X \circ \Theta_{H(R_{2n-1})}}(\tilde{t}_n) \in \Delta_n\} \cap E_3(n) \cap \mathcal{E}_n$. By assumption, $\sum_n \mathbb{P}\{H_{X \circ \Theta_{H(R_{2n-1})}}(\tilde{t}_n) \in \Delta_n\} = \infty$. Moreover, $\mathbb{P}(\mathcal{E}_n) = \mathbb{P}(E_1(r_{2n}))$. Since $\sum_{n=1}^{+\infty} \mathbb{P}(E_3(n)^c) < \infty$, this and Lemma 2.1, yield $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{D}_n \cap E_3(n)) = +\infty$.

Since $\mathcal{D}_n \cap E_3(n)$, $n \geq 1$, are independent events, the Borel–Cantelli lemma yields Lemma 2.2. \square

In the rest of the paper, if $(\beta(s), s \geq 0)$ is a Brownian motion, we denote its local time by $(L_\beta(t, x) \ t \geq 0, x \in \mathbb{R})$, and define $\tau_\beta(x) := \inf\{t > 0, L_\beta(t, 0) = x\}$, $x \geq 0$. For $v > 0$, we define the Brownian motion $(\beta_v(s), s \geq 0)$ by $\beta_v(s) := (1/v)\beta(v^2s)$. We also introduce for $\delta_1 > 0$,

$$\lambda := 4(1 + \kappa), \quad c_2 := 2(\lambda/\kappa)^{\delta_1}, \quad \psi_\pm(r) := 1 \pm c_2/r^{\delta_1}, \quad t_\pm(r) := \kappa\psi_\pm(r)r/\lambda, \quad (10)$$

$$K_\beta(\kappa) := \int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) dx, \quad 0 < \kappa < 1, \quad (11)$$

$$C_\beta := \int_0^1 \frac{L_\beta(\tau_\beta(8), x) - 8}{x} dx + \int_1^{+\infty} \frac{L_\beta(\tau_\beta(8), x)}{x} dx. \quad (12)$$

We have the following approximation result.

Lemma 2.3 *Let $0 < \kappa \leq 1$ and $\varepsilon \in (0, 1)$. For $\delta_1 > 0$ small enough, there exist $c_3 > 0$ and a Brownian motion $(\beta(t), t \geq 0)$ such that for some $\alpha > 0$ and all large r , $\mathbb{P}\{E_6(r)\} \geq 1 - r^{-\alpha}$, where*

$$E_6(r) := \{(1 - \varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1 + \varepsilon)\widehat{I}_+(r)\}, \quad (13)$$

$$\widehat{I}_\pm(r) := \begin{cases} 4\kappa^{1/\kappa-2} t_\pm(r)^{1/\kappa} \{K_{\beta_{t_\pm(r)}}(\kappa) \pm c_3 t_\pm(r)^{1-1/\kappa}\}, & 0 < \kappa < 1, \\ 4t_\pm(r) \{C_{\beta_{t_\pm(r)}} + 8 \log t_\pm(r)\}, & \kappa = 1. \end{cases} \quad (14)$$

Proof. Postponed to Section 4. \square

3 Proof of Theorems 1.1 and 1.2

In this section, we assume $0 < \kappa \leq 1$, and prove Theorems 1.1 and 1.2.

Let S_κ^{ca} be a (positive) completely asymmetric stable variable of index κ , and C_8^{ca} a completely asymmetric Cauchy variable of parameter 8. Their characteristic functions are:

$$\mathbb{E} \exp(it S_\kappa^{ca}) = e^{-|t|^\kappa (1 - i \operatorname{sgn}(t) \tan(\frac{\pi\kappa}{2}))}, \quad \mathbb{E} \exp(it C_8^{ca}) = e^{-8(|t| + it \frac{2}{\pi} \log |t|)}. \quad (15)$$

Recall \widehat{I}_\pm from (14). By Biane and Yor (1987), for $\lambda > 0$,

$$\widehat{I}_\pm(r) \stackrel{\mathcal{L}}{=} t_\pm(r)^{1/\kappa} \{c_4 S_\kappa^{ca} \pm c_5 t_\pm(r)^{1-1/\kappa}\}, \quad 0 < \kappa < 1, \quad (16)$$

$$\widehat{I}_\pm(r) \stackrel{\mathcal{L}}{=} 4t_\pm(r) [8c_6 + (\pi/2)C_8^{ca} + 8 \log t_\pm(r)], \quad \kappa = 1, \quad (17)$$

$$\psi(\kappa) := \left(\frac{\pi\kappa}{4\Gamma^2(\kappa) \sin(\pi\kappa/2)} \right)^{1/\kappa}, \quad c_4 := 8\psi(\kappa)\lambda^{1/\kappa}\kappa^{-1/\kappa}, \quad (18)$$

and $c_5 > 0$ and $c_6 > 0$ are unimportant constants.

Proof of Theorem 1.1. Let $r_n := e^n$ and $R_n := \sum_{k=1}^n r_k$. Let $a(\cdot)$ be a positive nondecreasing function. Without loss of generality, we assume that $a(n) \rightarrow_{n \rightarrow +\infty} +\infty$.

We start with the case $0 < \kappa < 1$. By Samorodnitsky and Taqqu (1994) p. 16), $\mathbb{P}(S_\kappa^{ca} > x) \sim_{x \rightarrow +\infty} c_7 x^{-\kappa}$, where $f(x) \sim_{x \rightarrow +\infty} g(x)$ means $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$, and c_7 is a positive constant depending on κ .

Recall $t_\pm(\cdot)$ from (10). By Lemma 2.3 and (16), for large r , we have

$$\mathbb{P}[H(F(r)) > (a(e^{-2}r)t_+(r))^{1/\kappa}] \leq c_8/a(e^{-2}r) + (\log r)^{-2}. \quad (19)$$

Assume $\sum_{n \geq 1} \frac{1}{a(r_n)} < \infty$. By the Borel–Cantelli lemma, almost surely for n large enough, $H(F(r_n)) \leq [a(r_{n-2})t_+(r_n)]^{1/\kappa}$. On the other hand, by Lemma 2.1, almost surely for all large n , we have $r_{n+1} \leq F(r_{n+2})$, which implies that for $r \in [r_n, r_{n+1}]$,

$$H(r) \leq H[F(r_{n+2})] \leq [a(r_n)t_+(r_{n+2})]^{1/\kappa} \leq [\psi_+(r_{n+2})\kappa r_{n+2}a(r_n)/\lambda]^{1/\kappa} \leq c_9[ra(r)]^{1/\kappa}.$$

Therefore, $\limsup_{r \rightarrow +\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} \leq c_9$ \mathbb{P} -a.s., implying the “zero” part of Theorem 1.1, since we can replace $a(\cdot)$ by any constant multiple of $a(\cdot)$.

To prove the “infinity” part, we assume $\sum_{n \geq 1} \frac{1}{na(n)} = +\infty$, and observe that, by a similar argument leading to (19), we have, for r large enough,

$$\mathbb{P}[H(F(r)) > (a(e^2r)t_-(r))^{1/\kappa}] \geq c_{10}/a(e^2r) - (\log r)^{-2}. \quad (20)$$

It follows from Lemma 2.2 that $\sup_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} H_{X \circ \Theta_{H(R_{2n-1})}}(t) > [a(r_{2n+2})t_-(r_{2n})]^{1/\kappa}$ almost surely for infinitely many n , which implies, for these n ,

$$\sup_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} H(R_{2n-1} + t)/[a(R_{2n-1} + t)(R_{2n-1} + t)]^{1/\kappa} \geq c_{11}. \quad (21)$$

This gives $\limsup_{r \rightarrow +\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} \geq c_{11}$ \mathbb{P} -a.s., which proves Theorem 1.1 in the case $0 < \kappa < 1$.

It remains to treat the case $\kappa = 1$. We recall that there exists a constant $c_{12} > 0$ such that $\mathbb{P}(C_8^{ca} > x) \sim_{x \rightarrow +\infty} c_{12}/x$ (see e.g. Samorodnitsky et al. (1994) p. 16). Hence,

$$\mathbb{P}\{H(F(r)) > 4t_+(r)(1 + \varepsilon)[8c_6 + a(e^{-2}r) + 8 \log t_+(r)]\} \leq c_{12}\pi/a(e^{-2}r) + (\log r)^{-2} \quad (22)$$

by Lemma 2.3 and (17), for large r . Assume $\sum_{n \geq 1} \frac{1}{na(n)} < \infty$. Then by the Borel–Cantelli lemma, almost surely, for all large n , $H(F(r_n)) \leq 4t_+(r_n)(1 + \varepsilon)[8c_6 + a(r_{n-2}) + 8 \log(t_+(r_n))]$. Under the additional assumption $\limsup_{r \rightarrow +\infty} (\log r)/a(r) < \infty$, we have, almost surely, for all large n and $r \in [r_n, r_{n+1}]$ (thus $r \leq F(r_{n+2})$ by Lemma 2.1),

$$H(r) \leq H(F(r_{n+2})) \leq c_{13}r_{n+2}[a(r_n) + \log r_{n+2}] \leq c_{14}ra(r),$$

since $t_+(r) = \psi_+(r)\kappa r/\lambda$. This yields the “zero” part of Theorem 1.1 in the case $\kappa = 1$.

For the “infinity” part, we assume $\sum_{n \geq 1} \frac{1}{na(n)} = +\infty$. As in (22), we have, for large r ,

$$\mathbb{P}\{H(F(r)) > 4t_-(r)(1 - \varepsilon)a(e^2r)\} \geq \pi c_{12}/[4a(e^2r)] - (\log r)^{-2}.$$

As in the displays between (20) and (21), this yields the “infinity” part of Theorem 1.1 in the case $\kappa = 1$. \square

Proof of Theorem 1.2. We start with the case $0 < \kappa < 1$. Recall (see Bertoin (1996))

$$\log \mathbb{P}(S_\kappa^{ca} < x) \sim_{x \rightarrow 0, x > 0} -c_{15}x^{-\kappa/(1-\kappa)}, \quad (23)$$

where c_{15} is a constant depending only on κ . Consequently, for r large enough, by (16) and Lemma 2.3, for any (strictly) positive function f ,

$$\mathbb{P}[H(F(r)) < t_-(r)^{1/\kappa} f(r)] \leq \exp \left[-(c_{15} - \varepsilon) \left(\frac{(1-\varepsilon)c_4}{f(r) + c_{16}r^{1-1/\kappa}} \right)^{\kappa/(1-\kappa)} \right] + \log^{-2} r. \quad (24)$$

Let $s_n := \exp(n^{1-\varepsilon})$ and $f(r) := \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^{(1-\kappa)/\kappa} \frac{(1-\varepsilon)c_4(c_{15}-\varepsilon)^{(1-\kappa)/\kappa}}{(\log \log r)^{(1-\kappa)/\kappa}} - \frac{c_{16}}{r^{1/\kappa-1}}$. As a consequence, $\sum_n \mathbb{P}[H(F(s_n)) < t_-(s_n)^{1/\kappa} f(s_n)] < \infty$, which, by means of the Borel–Cantelli lemma, implies that, almost surely, for all large n , $H(F(s_n)) \geq t_-(s_n)^{1/\kappa} f(s_n)$.

Recall from Lemma 2.1 that, almost surely, for all large n , we have $F(s_n) \leq (1+\varepsilon)s_n$. Let r be large. There exists n (large) such that $(1+\varepsilon)s_n \leq r \leq (1+2\varepsilon)s_n$. Then

$$H(r) \geq H(F(s_n)) \geq t_-(s_n)^{1/\kappa} f(s_n) \geq t_-^{1/\kappa} [r/(1+2\varepsilon)] f[r/(1+2\varepsilon)].$$

Plugging the value of $t_-(\frac{r}{1+2\varepsilon})$ (defined in (10)), this yields inequality “ \geq ” of (4) with

$$c_1(\kappa) = 8\psi(\kappa)c_{15}^{(1/\kappa)-1} \quad (25)$$

where $c_{15} = c_{15}(\kappa)$ is defined in (23), and ψ in (18).

To prove the upper bound, let $g(r) := \varepsilon + \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^{(1-\kappa)/\kappa} \frac{(1+\varepsilon)c_4(c_{15}+\varepsilon)^{(1-\kappa)/\kappa}}{(\log \log r)^{(1-\kappa)/\kappa}} + \frac{c_{17}}{r^{1/\kappa-1}}$, $r_n := \exp(n^{1+\varepsilon})$ and $R_n := \sum_{k=1}^n r_k$. By means of an argument similar to the one leading to (24), and Lemma 2.2, there exist almost surely infinitely many n such that

$$\inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} H_{X \circ \Theta_{H(R_{2n-1})}}(u) < [t_+(r_{2n})]^{1/\kappa} g(r_{2n}).$$

In addition, by Theorem 1.1, $H(R_{2n-1}) < [R_{2n-1} \log^2 R_{2n-1}]^{1/\kappa} \leq \varepsilon [t_+(r_{2n})]^{1/\kappa} g(r_{2n})$, almost surely, for all large n , since $R_k \leq k \exp(-k^\varepsilon) r_{k+1}$ for all large k , which yields

$$\inf_{v \in [R_{2n-1} + (1-\varepsilon)r_{2n}, R_{2n-1} + (1+\varepsilon)r_{2n}]} H(v) < (1+\varepsilon) [t_+(r_{2n})]^{1/\kappa} g(r_{2n}).$$

This gives inequality “ \leq ” of (4), and thus yields Theorem 1.2 in the case $0 < \kappa < 1$.

We now assume $\kappa = 1$ (thus $\lambda = 8$). By Samorodnitsky and Taqqu (1994) Proposition 1.2.12), $\mathbb{E}[\exp(-C_8^{ca})] = 1$ (in the notation of Samorodnitsky and Taqqu (1994) $S_1(8, 1, 0)$). Hence,

$$\mathbb{P}[C_8^{ca} \leq -\varepsilon \log r] \leq r^{-\varepsilon} \mathbb{E}[\exp(-C_8^{ca})] = r^{-\varepsilon}. \quad (26)$$

By Lemma 2.3 and (17), we have, for all large r ,

$$\mathbb{P} \{ H(F(r)) \leq 32t_-(r)(1-2\varepsilon)[c_6 + \log t_-(r)] \} \leq \mathbb{P}(C_8^{ca} \leq -\frac{16\varepsilon \log t_-(r)}{\pi(1-\varepsilon)}) + \frac{1}{\log^2 r} \leq r^{-c_{18}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. Thus, by the Borel–Cantelli lemma, almost surely, for all large n , $H(F(s_n)) > 32t_-(s_n)(1-2\varepsilon)[c_6 + \log t_-(s_n)]$, which is greater than $4(1-3\varepsilon)s_n \log s_n$. In view of (the last part of) Lemma 2.1, this yields inequality “ \geq ” in (5). The inequality “ \leq ”, on the other hand, follows immediately from Theorem A (that $H(r)/(r \log r) \rightarrow 4$ in probability). Theorem 1.2 is proved. \square

4 Proof of Lemma 2.3

Let $\kappa > 0$. Recall $A(x) = \int_0^x e^{W_\kappa(u)} du$, and $A_\infty = \lim_{x \rightarrow +\infty} A(x) < \infty$, \mathbb{P} -a.s. For any Brownian motion β and any $r > 0$, we define $\sigma_\beta(r) := \inf\{t > 0, \beta(t) = r\}$.

Following Hu et al. (1999), there exists a Brownian motion B independent of W such that

$$H(r) = \int_{-\infty}^0 + \int_0^{A(r)} e^{-2W_\kappa[A^{-1}(x)]} L_B\{\sigma_B[A(r)], x\} dx := H_-(r) + H_+(r), \quad (27)$$

Recall F from (6) and notice that $F(r) > 0$ on $E_1(r)$. Let $\Delta(r) := \delta(r)^{-1}A(F(r))$. Following Hu et al. (1999) p.3930), there exists two Bessel processes R_2 and $R_{2+2\kappa}$, of dimensions 2 and $(2 + 2\kappa)$ respectively, starting from 0, such that

$$H_+(F(r)) = \int_0^{A(F(r))} \frac{16R_2^2(v)dv}{R_{2+2\kappa}^4(v + \delta(r))} = \int_0^{\Delta(r)} \frac{16R_2^2(\delta(r)v)dv}{R_{2+2\kappa}^4[\delta(r)(1+v)]} = \int_0^{\Delta(r)} \frac{16\tilde{R}_2^2(v)dv}{\tilde{R}_{2+2\kappa}^4(1+v)},$$

where $R_2 \stackrel{\mathcal{L}}{=} \tilde{R}_2$ and $R_{2+2\kappa} \stackrel{\mathcal{L}}{=} \tilde{R}_{2+2\kappa}$. As in Hu et al (1999), we define a Jacobi process of dimension (d_1, d_2) as the solution of

$$dY(t) = 2\sqrt{Y(t)(1-Y(t))}d\hat{\beta}(t) + [d_1 - (d_1 + d_2)Y(t)]dt, \quad (28)$$

where $\hat{\beta}$ is a standard Brownian motion. According to Warren and Yor (1997), there exists a Jacobi process $(Y(t), t \geq 0)$ of dimension $(2, 2 + 2\kappa)$, starting from 0, such that for any $u \geq 0$,

$$\frac{\tilde{R}_2^2(u)}{\tilde{R}_2^2(u) + \tilde{R}_{2+2\kappa}^2(u+1)} = Y \circ \Lambda_Y(u), \quad \Lambda_Y(u) := \int_0^u \frac{ds}{\tilde{R}_2^2(s) + \tilde{R}_{2+2\kappa}^2(s+1)}. \quad (29)$$

In particular, $(\Lambda_Y(t), t \geq 0)$ is independent of Y . As a consequence, for all $r \geq 0$,

$$\begin{aligned} H_+[F(r)] &= 16 \int_0^{\delta(r)^{-1}A(F(r))} \frac{[Y \circ \Lambda_Y(u)]\Lambda_Y'(u)du}{[1 - Y \circ \Lambda_Y(u)]^2} = 16 \int_0^{\gamma(r)} \frac{Y(u)}{(1-Y(u))^2} du \\ \gamma(r) &:= \Lambda_Y[\delta(r)^{-1}A(F(r))]. \end{aligned} \quad (30)$$

Let $\alpha_\kappa := 1/(4 + 2\kappa)$ and let $T_Y(\alpha_\kappa) := \inf\{t > 0, Y(t) = \alpha_\kappa\}$. Define

$$\bar{H}(r) := 16 \int_0^{T_Y(\alpha_\kappa)} \frac{Y(u)}{(1-Y(u))^2} du, \quad H_0(r) := 16 \int_{T_Y(\alpha_\kappa)}^{\gamma(r)} \frac{Y(u)}{(1-Y(u))^2} du, \quad (31)$$

and notice that

$$H_+(F(r)) = \bar{H}(r) + H_0(r), \quad \{T_Y(\alpha_\kappa) \leq 64 \log r\} \subset \{\bar{H}(r) \leq c_{19} \log r\}. \quad (32)$$

Observe that a scale function of Y is $S(y) := \int_{\alpha_\kappa}^y \frac{dx}{x(1-x)^{1+\kappa}}$. There exists a Brownian motion $(\beta(t), t \geq 0)$ such that for all $t \geq 0$,

$$Y[t + T_Y(\alpha_\kappa)] = S^{-1}\{\beta[U(t)]\}, \quad U(t) := 4 \int_0^t \frac{ds}{Y[s + T_Y(\alpha_\kappa)]\{1 - Y[s + T_Y(\alpha_\kappa)]\}^{1+2\kappa}}.$$

The rest of the proof of Lemma 2.3 requires some more estimates, stated as Lemmas 4.1–4.4 below. We admit these lemmas for the moment, and complete the proof of Lemma 2.3.

Lemma 4.1 Let $(R(t), t \geq 0)$ be a Bessel process of dimension $d > 4$, starting from $R_0 \stackrel{\mathcal{L}}{=} \tilde{R}_{d-2}(1)$, where $(\tilde{R}_{d-2}(t), t \in [0, 1])$ is a $(d-2)$ -dimensional Bessel process. For any $\delta_2 \in (0, \frac{1}{2})$ and all large t ,

$$\mathbb{P} \left\{ \left| \frac{1}{\log t} \int_0^t \frac{ds}{R^2(s)} - \frac{1}{d-2} \right| > \frac{1}{(\log t)^{(1/2)-\delta_2}} \right\} \leq \exp \left(-c_{20} (\log t)^{2\delta_2} \right).$$

Lemma 4.2 If $\delta_1 > 0$ is small enough, then for all large v , $\mathbb{P}(E_7^c) \leq v^{-1/4+5\delta_1}$, where

$$E_7 := \{ \tau_\beta[(1 - v^{-\delta_1})\lambda v] \leq U(v) \leq \tau_\beta[(1 + v^{-\delta_1})\lambda v] \}. \quad (33)$$

Lemma 4.3 Let $\kappa > 0$ and define $H_-(+\infty) := \lim_{r \rightarrow +\infty} H_-(r)$. There exists $c_{21} > 0$ such that for all large z ,

$$\mathbb{P}(H_-(+\infty) > z) \leq c_{21} [(\log z)/z]^{\kappa/(\kappa+2)}. \quad (34)$$

Lemma 4.4 Let $(\beta(t), t \geq 0)$ be a Brownian motion, and let $\lambda = 4(1 + \kappa)$. We define

$$J_\beta(\kappa, t, \lambda) := \int_0^1 y(1-y)^{\kappa-2} L_\beta(\tau_\beta(\lambda), \frac{S(y)}{t}) dy, \quad 0 < \kappa \leq 1, t \geq 0. \quad (35)$$

Let $0 < d < 1$ and let $0 < \varepsilon < 1$. There exists $c_{22} > 0$ such that for t large enough, on an event E_8 of probability greater than $1 - c_{22}/t^d$,

(i) Case $0 < \kappa < 1$: (recall $K_\beta(\kappa)$ from (11))

$$(1 - \varepsilon)K_\beta(\kappa) - c_{23}t^{1-1/\kappa} \leq \kappa^{2-1/\kappa}t^{1-1/\kappa}J_\beta(\kappa, t, \lambda) \leq (1 + \varepsilon)K_\beta(\kappa) + c_{23}t^{1-1/\kappa}. \quad (36)$$

(ii) Case $\kappa = 1$: (recall C_β from (12))

$$(1 - \varepsilon)[C_\beta + 8 \log t] \leq J_\beta(1, t, 8) \leq (1 + \varepsilon)[C_\beta + 8 \log t]. \quad (37)$$

By admitting Lemmas 4.1–4.4, we can now complete the proof of Lemma 2.3.

Proof of Lemma 2.3. Notice that

$$S(y) \sim_{y \rightarrow 1} (1-y)^{-\kappa}/\kappa, \quad y/(1-y) \sim_{y \rightarrow 1} [\kappa S(y)]^{1/\kappa}. \quad (38)$$

We first look for an estimate of $U[\gamma(r) - T_Y(\alpha_\kappa)]$. Since $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ , and $A(F(r)) \leq A_\infty$, we have $\mathbb{P}[A(F(r)) > r^{2/\kappa}] \leq \frac{2^\kappa r^{-2}}{\kappa \Gamma(\kappa)}$. On the other hand, by definition, $A(F(r)) = A_\infty - \delta(r) = A_\infty - e^{-\kappa r/2}$ (see (6)). Hence,

$$\mathbb{P}[A(F(r)) < 1/(2 \log r)] \leq \mathbb{P}[2/\gamma_\kappa < 1/(2 \log r) + \delta(r)] \leq r^{-2}/\Gamma(\kappa).$$

Recall that $\gamma(r) = \Lambda_Y[\delta(r)^{-1}A(F(r))]$, see (30). Thus, for large r ,

$$\mathbb{P} \{ \Lambda_Y[\exp(\kappa r/2 - 2 \log \log r)] \leq \gamma(r) \leq \Lambda_Y[\exp(\kappa r/2 + (2/\kappa) \log r)] \} \geq 1 - c_{24}r^{-2}.$$

By definition, $\Lambda_Y(u) = \int_0^u \frac{ds}{\tilde{R}_2^2(s) + \widehat{R}_{2+2\kappa}^2(s+1)}$. Since $(\tilde{R}_2^2(t) + \widehat{R}_{2+2\kappa}^2(t+1), t \geq 0)$ is a $(4 + 2\kappa)$ -dimensional squared Bessel process starting from $\tilde{R}_{2+2\kappa}^2(1)$, it follows from Lemma 4.1 that there exist constants $\delta_3 \in (0, \frac{1}{2})$, $c_{25} > 0$ and $c_{26} > 0$, such that for large r , with $\lambda = 4(1 + \kappa)$ as before,

$$\mathbb{P} \left\{ \kappa r / \lambda - c_{25} r^{1/2+\delta_2} \leq \gamma(r) \leq \kappa r / \lambda + c_{25} r^{1/2+\delta_2} \right\} \geq 1 - c_{26} r^{-2}. \quad (39)$$

To study the behaviour of $T_Y(\alpha_\kappa)$, we notice that Y satisfies (28) with $d_1 = 2$ and $d_2 = 2 + 2\kappa$. By the Dubins–Schwarz theorem, there exists a Brownian motion $(\widehat{B}(t), t \geq 0)$ such that $Y(t) = \widehat{B} \left(4 \int_0^t Y(s)(1 - Y(s)) ds \right) + \int_0^t [2 - (4 + 2\kappa)Y(s)] ds$ for $t \geq 0$. Recall that $\alpha_\kappa = 1/(4 + 2\kappa)$. Let $t \geq 2\alpha_\kappa$. We have, on the event $\{T_Y(\alpha_\kappa) \geq t\}$,

$$\inf_{0 \leq s \leq 4t} \widehat{B}(s) \leq \widehat{B} \left(4 \int_0^t Y(s)(1 - Y(s)) ds \right) \leq \alpha_\kappa - t \leq -\frac{t}{2},$$

since $Y(s) \in (0, 1)$ for any $s \geq 0$. As a consequence, for $t \geq 2\alpha_\kappa$,

$$\mathbb{P}(T_Y(\alpha_\kappa) > t) \leq \mathbb{P}[\inf_{0 \leq s \leq 4t} \widehat{B}(s) \leq -t/2] \leq 2 \exp[-t/32]. \quad (40)$$

In particular, $\mathbb{P}[T_Y(\alpha_\kappa) > 64 \log r] \leq \frac{2}{r^2}$ for large r . Plugging this into (39), and introducing $\underline{\gamma} = \underline{\gamma}(r) := \frac{\kappa}{\lambda} r - 2c_{25} r^{1/2+\delta_2}$ and $\overline{\gamma} = \overline{\gamma}(r) := \frac{\kappa}{\lambda} r + c_{25} r^{1/2+\delta_2}$ yields that for large r ,

$$\mathbb{P} \left\{ U(\underline{\gamma}) \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq U(\overline{\gamma}) \right\} \geq 1 - c_{27} r^{-2}.$$

By Lemma 4.2, for small $\delta_1 > 0$ and all large r ,

$$\mathbb{P} \left\{ \tau_\beta \left[(1 - \underline{\gamma}^{-\delta_1}) \lambda \underline{\gamma} \right] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta \left[(1 + \overline{\gamma}^{-\delta_1}) \lambda \overline{\gamma} \right] \right\} \geq 1 - r^{-c_{28}}.$$

We choose δ_1 so small that $\delta_1 < 1/2 - \delta_2$. Then for large r , we have $(1 - \frac{1}{\underline{\gamma}^{\delta_1}}) \lambda \underline{\gamma} \geq [1 - 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \kappa r = \lambda t_-(r)$, and $(1 + \frac{1}{\overline{\gamma}^{\delta_1}}) \lambda \overline{\gamma} \leq [1 + 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \kappa r = \lambda t_+(r)$ (see (10)). Hence,

$$\mathbb{P} \left\{ \tau_\beta [\lambda t_-(r)] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta [\lambda t_+(r)] \right\} \geq 1 - r^{-c_{28}}. \quad (41)$$

As in Hu et al. (1999), p. 3923), (31) leads to $H_0(r) = 4 \int_0^1 x(1-x)^{\kappa-2} L_\beta[U(\gamma(r) - T_Y(\alpha_\kappa)), S(x)] dx$. By (41), $\mathbb{P} [I'_-(r) \leq H_0(r) \leq I'_+(r)] \geq 1 - r^{-c_{28}}$, for large r , where

$$I'_\pm(r) := 4 \int_0^1 x(1-x)^{\kappa-2} L_\beta \{ \tau_\beta [\lambda t_\pm(r)], S(x) \} dx = 4 t_\pm(r) J_{\beta_{t_\pm(r)}} [\kappa, t_\pm(r), \lambda],$$

and, as before, $t_\pm(r) = [1 \pm 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \frac{\kappa}{\lambda} r$, $\beta_v(s) = \beta(v^2 s)/v$ and J_β is defined in (35).

Applying Lemma 4.4 to $d = 1/2$ yields that, for large r , recalling $\widehat{I}_\pm(r)$ from (14),

$$\mathbb{P} \{ (1 - \varepsilon) \widehat{I}_-(r) \leq H_0(r) \leq (1 + \varepsilon) \widehat{I}_+(r) \} \geq 1 - r^{-c_{29}}. \quad (42)$$

In the case $0 < \kappa < 1$, (40) and (32) give $\mathbb{P}[\overline{H}(r) \leq c_{30} \log r] \geq 1 - 2r^{-2}$ for some c_{30} and all large r . On the other hand, by Lemma 4.3, $\mathbb{P}[H_-(F(r)) \leq \varepsilon r] \geq \mathbb{P}[H_-(+\infty) \leq \varepsilon r] \geq 1 - \frac{c_{31}}{r^{(1-\delta_1)\kappa/(\kappa+2)}}$, for all large r . Consequently, by (42) and (32), for large r ,

$$\mathbb{P}[(1 - \varepsilon) \widehat{I}_-(r) \leq H(F(r)) \leq (1 + \varepsilon) \widehat{I}_+(r) + (4\varepsilon \lambda / \kappa) t_+(r)] \geq 1 - r^{-c_{32}}.$$

This proves Lemma 2.3 in the case $0 < \kappa < 1$.

Now we turn to the case $\kappa = 1$. We again have $\mathbb{P}[H_-(F(r)) + \overline{H}(r) \leq 2\varepsilon r] \geq 1 - r^{-c_{33}}$ (for large r). By Biane and Yor (1987) $C_{\beta_{t_{\pm}(r)}} \stackrel{\mathcal{L}}{=} \pi C_8^{ca}/2 + c_{34}$, where $c_{34} > 0$. Hence, by (26), $\mathbb{P}[C_{\beta_{t_{\pm}(r)}} > -\pi \log r] \geq 1 - r^{-2}$. Thus (14) gives $\mathbb{P}\{\widehat{I}_+(r) \geq 16t_+(r) \log r\} \geq 1 - r^{-2}$. This, together with (42), yields that, for large r ,

$$\mathbb{P}[(1 - \varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1 + 2\varepsilon)\widehat{I}_+(r)] \geq 1 - r^{-c_{35}}.$$

This proves Lemma 2.3 in the case $\kappa = 1$. □

The rest of the section is devoted to the proof of Lemmas 4.1–4.4.

Proof of Lemma 4.1. Let $d > 4$ and $R_0 \stackrel{\mathcal{L}}{=} \widetilde{R}_{d-2}(1)$, where \widetilde{R} is a $(d - 2)$ -dimensional Bessel process. We consider a d -dimensional Bessel process R , starting from R_0 . Let $\theta(t) := \int_0^t R^{-2}(s) ds$. Itô's formula gives $\log R(t) = \log R_0 + M(t) + \frac{d-2}{2}\theta(t)$, where $M(t) := \int_0^t R(s)^{-1} d\widehat{\beta}(s)$ and $(\widehat{\beta}(t), t \geq 0)$ is a Brownian motion. By the Dubins–Schwarz theorem, there exists a Brownian motion $(\beta(t), t \geq 0)$ such that

$$(d - 2)\theta(t)/2 = \log R(t) - \log R_0 - \widetilde{\beta}(\theta(t)), \quad t \geq 0. \quad (43)$$

Let $\delta_3 \in (0, \frac{1}{2})$, and let $x = x(t) := \frac{d-2}{6} \frac{1}{(\log t)^{(1/2)-\delta_3}}$. For large t , we have

$$\mathbb{P}\left(\left|\frac{\log R_0}{\log t}\right| > x\right) \leq \mathbb{P}\left(\frac{\log R_0}{\log t} > x\right) + \mathbb{P}\left(\frac{\log R_0}{\log t} < -x\right) \leq \exp\left(- (1 - \varepsilon) \frac{t^{2x}}{2}\right) + c_{36} t^{-\frac{x}{d-2}}. \quad (44)$$

Let $n := \lceil d \rceil$ be the smallest integer such that $n \geq d$. Since an n -dimensional Bessel process can be realized as the Euclidean modulus of an \mathbb{R}^n -valued Brownian motion, it follows from the triangular inequality that $R(t) \leq_{\mathcal{L}} R_0 + \widehat{R}_n(t)$, where $(\widehat{R}_n(t), t \geq 0)$ is an n -dimensional Bessel process starting from 0. Consequently, $\mathbb{P}(R(t) > t^{(1/2)+x}) \leq \exp(-(1 - \varepsilon)t^{2x}/4)$ for large t , and $\mathbb{P}(R(t) < t^{(1/2)-x}) \leq c_{36} t^{-x/d}$. Therefore, for large t ,

$$\mathbb{P}(|(\log R(t))/\log t - 1/2| > x) \leq \exp(-(1 - \varepsilon)t^{2x}/4) + c_{36} t^{-x/d}. \quad (45)$$

Define $E_9 := \{\sup_{0 \leq s \leq 2(\log t)/(d-2)} |\widetilde{\beta}(s)| \leq x \log t\}$ and

$$E_{10} := \left\{ \left| \frac{\log R(t)}{\log t} - \frac{1}{2} \right| \leq x \right\} \cap \left\{ \left| \frac{\log R_0}{\log t} \right| \leq x \right\}, \quad E_{11} := \left\{ \frac{d-2}{2} \theta(t) < \log t \right\}.$$

By (44) and (45), we have, for large t ,

$$\mathbb{P}(E_{10}^c) \leq 2 \exp(-(1 - \varepsilon)t^{2x}/4) + c_{37} t^{-x/d}. \quad (46)$$

We now estimate $\mathbb{P}(E_{10} \cap E_{11}^c)$. We first observe that $\left| \widetilde{\beta}(\theta(t)) + (d - 2)\theta(t)/2 - (\log t)/2 \right| \leq 2x \log t$ on E_{10} , by (43). We claim that $E_{10} \cap E_{11}^c \subset \{|\widetilde{\beta}(\theta(t))| > \frac{d-2}{6}\theta(t)\}$ for large t . Indeed, on the event $E_{10} \cap E_{11}^c \cap \{|\widetilde{\beta}(\theta(t))| \leq \frac{d-2}{6}\theta(t)\}$,

$$(d - 2)\theta(t)/2 \leq (2x + 1/2) \log t - \widetilde{\beta}(\theta(t)) \leq (2x + 1/2) \log t + (d - 2)\theta(t)/6,$$

which implies $\frac{d-2}{2}\theta(t) \leq (\frac{3}{4} + 3x) \log t$. This, for large t , contradicts $\frac{d-2}{2}\theta(t) > \log t$ on E_{11}^c . Therefore, $E_{10} \cap E_{11}^c \subset \{|\tilde{\beta}(\theta(t))| > \frac{d-2}{6}\theta(t)\}$ holds for all large t , from which it follows that

$$\mathbb{P}(E_{10} \cap E_{11}^c) \leq \mathbb{P}\left(\sup_{s \geq 2(\log t)/(d-2)} |\tilde{\beta}(s)|/s > (d-2)/6\right) \leq \exp[-(1-\varepsilon)(d-2)(\log t)/36].$$

Since $\mathbb{P}(E_9^c) \leq \exp[-(1-\varepsilon)\frac{d-2}{4}x^2 \log t]$ (for large t), this and (46) give for large t ,

$$\mathbb{P}(E_{10}^c \cup E_{11}^c \cup E_9^c) \leq \mathbb{P}(E_{10}^c) + \mathbb{P}(E_{10} \cap E_{11}^c) + \mathbb{P}(E_{10} \cap E_{11} \cap E_9^c) \leq \exp(-c_{38} x^2 \log t).$$

Since $E_{10} \cap E_{11} \cap E_9 \subset \{|\frac{\theta(t)}{\log t} - \frac{1}{d-2}| \leq \frac{6x}{d-2}\}$, this completes the proof of Lemma 4.1. \square

Proof of Lemma 4.2. Let $v > 0$. Recall that for $x > 0$, $\beta_v(x) = (1/v)\beta(v^2x)$, and $v^2\tau_{\beta_v}(x) = \tau_{\beta}(xv)$ a.s. Then,

$$E_7 = \left\{ \tau_{\beta_v}[(1-v^{-\delta_1})\lambda] \leq U(v)/v^2 \leq \tau_{\beta_v}[(1+v^{-\delta_1})\lambda] \right\}. \quad (47)$$

For $\delta_1 > 0$, define $E_{12} := \{\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} |\varepsilon_1(v, s)| < v^{-\delta_1}\}$, where $\varepsilon_1 = \varepsilon_1(v, s) := \frac{1}{4} \int_0^1 (1-x)^\kappa [L_{\beta_v}(s, S(x)/v) - L_{\beta_v}(s, 0)] dx$ for $s \geq 0$. By Hu et al. (1999) p. 3924, $E_{12} \subset E_7$. Thus it remains to prove that for δ_1 small enough, $\mathbb{P}(E_{12}^c) \leq 1/v^{1/4-5\delta_1}$ for large v . Notice that for $s \geq 0$,

$$\begin{aligned} |\varepsilon_1| &\leq \left(\int_{\{S(x) > \sqrt{v}\}} + \int_{\{S(x) < -\sqrt{v}\}} + \int_{\{|S(x)| \leq \sqrt{v}\}} \right) \frac{(1-x)^\kappa}{4} |L_{\beta_v}(s, S(x)/v) - L_{\beta_v}(s, 0)| dx \\ &=: \varepsilon_2(v, s) + \varepsilon_3(v, s) + \varepsilon_4(v, s). \end{aligned} \quad (48)$$

By (38), we have, for all large v (and all $s \geq 0$)

$$\begin{aligned} \sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} \varepsilon_2(v, s) &\leq \frac{1}{4} \int_{1-(\frac{2}{\kappa\sqrt{v}})}^1 (1-x)^\kappa \sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} \sup_{u \geq 0} [L_{\beta_v}(s, u) + L_{\beta_v}(s, 0)] dx \\ \sup_{u \geq 0} [L_{\beta_v}(\tau_{\beta_v}(2\lambda), u) + 2\lambda]. \end{aligned}$$

By the second Ray–Knight theorem, $Z := (L_{\beta_v}(\tau_{\beta_v}(2\lambda), u), u \geq 0)$ is a 0-dimensional squared Bessel process starting from 2λ . Hence, for large v ,

$$\mathbb{P}[\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} \varepsilon_2(v, s) \geq [2/(\kappa\sqrt{v})]^{\frac{1}{\kappa}+1} (\sqrt{v} + 2\lambda)] \leq \mathbb{P}(\sup_{u \geq 0} Z(u) \geq \sqrt{v}) = 2\lambda/\sqrt{v}. \quad (49)$$

Similarly (this time, using $S(x) \sim \log x$, $x \rightarrow 0$), we have, for large v ,

$$\mathbb{P}[\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} \varepsilon_3(v, s) \geq \exp(-\sqrt{v}/2) (\sqrt{v} + 2\lambda)] \leq 2\lambda/\sqrt{v}. \quad (50)$$

To estimate $\varepsilon_4(v, s)$, we note that $\varepsilon_4(v, s) \leq \sup_{|u| \leq 1/\sqrt{v}} |L_{\beta_v}(s, u) - L_{\beta_v}(s, 0)|$. Let $\varepsilon \in (0, 1/2)$, $t_v > 0$, $\gamma \geq 1$ and define $(\beta_v)_{t_v}^* := \sup_{0 \leq s \leq t_v} |\beta_v(s)|$. Applying Barlow and Yor (1982) (ii) to the continuous martingale $\beta_v(\cdot \wedge t_v)$, we see that for some constant $C_{\gamma, \varepsilon} > 0$,

$$\|\sup_{0 \leq s \leq t_v, a \neq b} |L_{\beta_v}(s, b) - L_{\beta_v}(s, a)| / |b - a|^{1/2-\varepsilon}\|_\gamma \leq C_{\gamma, \varepsilon} \|[(\beta_v)_{t_v}^*]^{1/2+\varepsilon}\|_\gamma.$$

Then, by Chebyshev's inequality, for $\alpha > 0$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t_v, a \neq b} \frac{|L_{\beta_v}(s, b) - L_{\beta_v}(s, a)|}{|b - a|^{1/2 - \varepsilon}} \geq \alpha\right) \leq \frac{(\sqrt{t_v})^{(1/2 + \varepsilon)\gamma}}{\alpha^\gamma} \left[C_{\gamma, \varepsilon} \|[(\beta_v)_1^*]^{1/2 + \varepsilon}\|_\gamma \right]^\gamma. \quad (51)$$

On $E_{13} := \{\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda), a \neq b} |L_{\beta_v}(s, b) - L_{\beta_v}(s, a)|/|b - a|^{1/2 - \varepsilon} \leq v^{\frac{1}{2}(\frac{1}{2} - 2\varepsilon)}\}$, we have

$$\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} \varepsilon_4(v, s) \leq v^{\frac{1}{2}(-\frac{1}{2} + \varepsilon)} v^{\frac{1}{2}(\frac{1}{2} - 2\varepsilon)} = v^{-\varepsilon/2}. \quad (52)$$

We choose $\gamma := 2$ and $t_v := v^{\frac{1/4 - \varepsilon}{1/2 + \varepsilon}}$ to see that for all large v (if ε is small enough),

$$\mathbb{P}(E_{13}(v)^c) \leq \mathbb{P}(\tau_{\beta_v}(2\lambda) > t_v) + (\sqrt{t_v})^{(1/2 + \varepsilon)\gamma} \left[C_{\gamma, \varepsilon} \|[(\beta_v)_1^*]^{1/2 + \varepsilon}\|_\gamma \right]^\gamma (v^{1/4 - \varepsilon})^{-\gamma} \leq v^{-1/4 + 2\varepsilon}/2.$$

Combining this with (48), (49), (50) and (52), we obtain that, for $\varepsilon > 0$ small enough,

$$\mathbb{P}(\sup_{0 \leq s \leq \tau_{\beta_v}(2\lambda)} |\varepsilon_1(v, s)| \geq 2v^{-\varepsilon/2}) \leq v^{-1/4 + 2\varepsilon}.$$

This gives, with the choice of $\delta_1 := 2\varepsilon/5$, $\mathbb{P}(E_{12}^c) \leq v^{-1/4 + 5\delta_1}$ (for large v). \square

Proof of Lemma 4.3. For $a > 0$, $\alpha > 0$ and $b > 0$, let

$$E_{14} := \{\sup_{x < 0} e^{-W_\kappa(x)} \leq a\}, \quad E_{15} := \{A_\infty \leq \alpha\}, \quad E_{16} := \{\sup_{y < 0} L_B[\sigma_B(\alpha), y] \leq b\},$$

$$L_X^*(-\infty) := \sup_{r \geq 0} \sup_{x < 0} \left\{ e^{-W_\kappa(x)} L_B[\sigma_B(A(r)), A(x)] \right\} \leq \left(\sup_{x < 0} e^{-W_\kappa(x)} \right) \sup_{y < 0} L_B[\sigma_B(A_\infty), y].$$

It follows from the second Ray–Knight theorem that $\mathbb{P}(E_{16}^c) \leq c_{39}\alpha/b$. Now, let $a := z^{\frac{1}{\kappa+2}}$, $\alpha := z^{\frac{1}{\kappa+2}}$ and $b := z^{\frac{\kappa+1}{\kappa+2}}$. Notice that $L_X^*(-\infty) \leq z$ on $E_{14} \cap E_{15} \cap E_{16}$, and recall $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ . We have for z large enough,

$$\mathbb{P}(L_X^*(-\infty) > z) \leq \mathbb{P}(E_{14}^c) + \mathbb{P}(E_{15}^c) + \mathbb{P}(E_{16}^c) \leq a^{-\kappa} + c_{39}(2/\alpha)^\kappa + c_{39}\alpha/b \leq c_{40}z^{-\frac{\kappa}{\kappa+2}}. \quad (53)$$

Define for $c > 0$, $E_{17} := \{\min_{0 \leq s \leq \sigma_B(A_\infty)} B(s) > -A_\infty z^{\frac{\kappa+1}{\kappa+2}}\}$, $E_{18} := \{|A^{-1}(-z)| \leq c \log z\}$. On $E_{14} \cap \dots \cap E_{18}$, $H_-(+\infty) \leq \lim_{r \rightarrow +\infty} \int_{A^{-1}(\min_{0 \leq s \leq \sigma_B(A(r)}) B(s))}^0 L_X^*(-\infty) dx$ for $r \geq 0$. Hence,

$$H_-(+\infty) \leq |A^{-1}(\min_{0 \leq s \leq \sigma_B(A_\infty)} B(s))| L_X^*(-\infty) \leq |A^{-1}(-z)| L_X^*(-\infty) \leq cz \log z. \quad (54)$$

Moreover, for $c > 2/\kappa$, $\varepsilon > 0$, and z large enough,

$$\begin{aligned} \mathbb{P}(E_{18}^c) &= \mathbb{P}\left(z > \int_0^{c \log z} e^{W(u) + \kappa u/2} du\right) \leq \mathbb{P}\left[z > \exp\left(\inf_{0 \leq u \leq c \log z} W(u)\right) \frac{2}{\kappa}(z^{\kappa c/2} - 1)\right] \\ &\leq 2z^{-\frac{1}{2c}(\frac{\kappa c}{2} - 1 - \varepsilon)^2}. \end{aligned} \quad (55)$$

Since B is independent of A_∞ , we have $\mathbb{P}(E_{17}^c | A_\infty) = A_\infty / [A_\infty + A_\infty z^{\frac{\kappa+1}{\kappa+2}}] \leq z^{-\frac{\kappa+1}{\kappa+2}}$. Choosing c large enough, this, together with (53), (54) and (55), gives (34). \square

Proof of Lemma 4.4. Assume $0 < \kappa \leq 1$. Consider a Brownian motion β , a small constant $\varepsilon > 0$, and $0 < d < 1$. Recall $S(y) = \int_{\alpha_\kappa}^y \frac{dx}{x(1-x)^{1+\kappa}}$ and notice that $1 - S^{-1}(u) \sim_{u \rightarrow +\infty} (\kappa u)^{-1/\kappa}$. Therefore, there exists $x_\varepsilon > 0$ such that for all $u \geq x_\varepsilon$, $[1 - S^{-1}(u)]^{2\kappa-1}/(\kappa u)^{1/\kappa-2} \in (1 - \varepsilon, 1 + \varepsilon)$ and $S^{-1}(u) \geq (1 - \varepsilon)$.

Let $g(t) := t^{\varepsilon-1}$, and write

$$\begin{aligned} & J_\beta(\kappa, t, \lambda) \\ = & \left(\int_{\{S(y) \leq -tg(t)\}} + \int_{\{-tg(t) < S(y) \leq 0\}} + \int_{\{0 < S(y) \leq x_\varepsilon\}} + \int_{\{x_\varepsilon < S(y)\}} \right) \frac{y}{(1-y)^{2-\kappa}} L_\beta(\tau_\beta(\lambda), \frac{S(y)}{t}) dy \\ := & J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Since $S(x) \sim \log x$, $x \rightarrow 0$, we have for large t , $J_1 \leq \exp\left(-\frac{tg(t)}{2}\right) (\sup_{s \geq 0} Z(s))$, where Z is a 0-dimensional squared Bessel process starting from λ (by the second Ray–Knight theorem). Hence, we get $\mathbb{P}[J_1 \geq e^{-t^\varepsilon/2} t^d] \leq \lambda/t^d$.

Fix a large constant $\gamma > 0$, and define

$$E_{19} := \{\tau_\beta(\lambda) \leq t^{2d}\}, \quad E_{20} := \left\{ \sup_{0 \leq s \leq t^{2d}, a \neq b} |L_\beta(s, b) - L_\beta(s, a)| / |b - a|^{1/2-\varepsilon} \leq t^{d(1/2+\varepsilon+1/\gamma)} \right\}.$$

Recall that $S(\alpha_\kappa) = 0$. On the event $E_{19} \cap E_{20}$ and for all large t ,

$$\begin{aligned} \kappa^{2-1/\kappa} t^{1-1/\kappa} J_3 & \leq \kappa^{2-1/\kappa} t^{1-1/\kappa} \sup_{0 \leq x \leq x_\varepsilon/t} L_\beta(\tau_\beta(\lambda), x) \int_{\alpha_\kappa}^{S^{-1}(x_\varepsilon)} y(1-y)^{\kappa-2} dy \\ & \leq c_{41} t^{1-1/\kappa} [\lambda + t^{d(1/2+\varepsilon+1/\gamma)} (x_\varepsilon/t)^{\frac{1}{2}-\varepsilon}] \leq 2\lambda c_{41} t^{1-1/\kappa}, \end{aligned}$$

$$J_2 \leq \sup_{-g(t) \leq s \leq 0} L_\beta(\tau_\beta(\lambda), s) \int_0^{\alpha_\kappa} y(1-y)^{\kappa-2} dy \leq c_{42} \left[\lambda + t^{d(1/2+\varepsilon+1/\gamma)} (t^{\varepsilon-1})^{\frac{1}{2}-\varepsilon} \right] \leq 2c_{42}.$$

Since $\mathbb{P}(E_{19}^c) \leq c_{43}/t^d$ and $\mathbb{P}(E_{20}^c) \leq c_{44}/t^d$ (see (51)), we obtain, for large t , $\mathbb{P}(J_3 \leq c_{45}) \geq 1 - c_{46}/t^d$ and $\mathbb{P}(J_2 \leq 2c_{42}) \geq 1 - c_{47}/t^d$.

Now, we write $J_4 = \kappa^{1/\kappa-2} t^{1/\kappa-1} \int_{x_\varepsilon/t}^{+\infty} (S^{-1}(tx))^2 \frac{(1-S^{-1}(tx))^{2\kappa-1}}{(\kappa t)^{1/\kappa-2}} L_\beta(\tau_\beta(\lambda), x) dx$. Therefore

$$(1 - \varepsilon)^3 \int_{x_\varepsilon/t}^{+\infty} x^{\frac{1}{\kappa}-2} L_\beta(\tau_\beta(\lambda), x) dx \leq \kappa^{2-\frac{1}{\kappa}} t^{1-\frac{1}{\kappa}} J_4 \leq (1 + \varepsilon) \int_{x_\varepsilon/t}^{+\infty} x^{\frac{1}{\kappa}-2} L_\beta(\tau_\beta(\lambda), x) dx. \quad (56)$$

We first assume $0 < \kappa < 1$. On $E_{19} \cap E_{20}$, for large t , we have $\int_0^{x_\varepsilon/t} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) dx \leq c_{48} t^{1-1/\kappa}$. Recall $K_\beta(\kappa)$ from (11). By (56), for large t ,

$$\mathbb{P} \left[(1 - \varepsilon)^3 K_\beta(\kappa) - (1 - \varepsilon)^3 c_{48} t^{1-1/\kappa} \leq \kappa^{2-1/\kappa} t^{1-1/\kappa} J_4 \leq (1 + \varepsilon) K_\beta(\kappa) \right] \geq 1 - c_{49}/t^d,$$

Since $J_\beta(\kappa, t, \lambda) = J_1 + J_2 + J_3 + J_4$, we get

$$\mathbb{P} \left\{ (1 - \varepsilon)^3 K_\beta(\kappa) - c_{23} t^{1-1/\kappa} \leq \kappa^{2-1/\kappa} t^{1-1/\kappa} J_\beta(\kappa, t, \lambda) \leq (1 + \varepsilon) K_\beta(\kappa) + c_{23} t^{1-1/\kappa} \right\} \geq 1 - \frac{c_{50}}{t^d},$$

proving the lemma in the case $0 < \kappa < 1$.

We now assume $\kappa = 1$. By the definition of C_β (see (12)),

$$\int_{x_\varepsilon/t}^{\infty} \frac{L_\beta(\tau_\beta(8), x)}{x} dx = C_\beta - \int_0^{x_\varepsilon/t} \frac{L_\beta(\tau_\beta(8), x) - 8}{x} dx + 8 \log t - 8 \log x_\varepsilon.$$

On $E_{19} \cap E_{20}$, for large t , $\int_0^{x_\varepsilon/t} \frac{|L_\beta(\tau_\beta(8), x) - 8|}{x} dx \leq \int_0^{x_\varepsilon/t} \frac{t^{d(1/2+\varepsilon+1/\gamma)} x^{1/2-\varepsilon}}{x} dx \leq \varepsilon$. As in (26), $\mathbb{P}(C_\beta + 8 \log t < \log t) \leq r^{-7}$. Therefore, by (56), we have, for large t ,

$$\mathbb{P}\{(1-\varepsilon)^4[C_\beta + 8 \log t] \leq J_4 \leq (1+\varepsilon)^2[C_\beta + 8 \log t]\} \geq 1 - c_{51}/t^d.$$

Since $J_\beta(1, t, 8) = J_1 + J_2 + J_3 + J_4$, this yields that for large t ,

$$\mathbb{P}\{(1-\varepsilon)^4[C_\beta + 8 \log t] \leq J_\beta(1, t, 8) \leq (1+\varepsilon)^3[C_\beta + 8 \log t]\} \geq 1 - c_{52}/t^d. \quad \square$$

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