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► **To cite this version:**

| Cédric Bonnafé. A note on the Grothendieck ring of the symmetric group. 2005. hal-00013014v1

HAL Id: hal-00013014

<https://hal.science/hal-00013014v1>

Preprint submitted on 2 Nov 2005 (v1), last revised 30 Nov 2005 (v2)

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A NOTE ON THE GROTHENDIECK RING OF THE SYMMETRIC GROUP

CÉDRIC BONNAFÉ

ABSTRACT. Let p be a prime number and let n be a non-zero natural number. We compute the descending Loewy series of the algebra $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{R}_n$, where \mathcal{R}_n denotes the ring of virtual ordinary characters of the symmetric group \mathfrak{S}_n .

Let p be a prime number and let n be a non-zero natural number. Let \mathfrak{S}_n be the symmetric group of degree n . Let \mathcal{R}_n denote the ring of virtual ordinary characters of the symmetric group \mathfrak{S}_n and let $\bar{\mathcal{R}}_n = \mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{R}_n$. The aim of this paper is to determine the descending Loewy series of $\bar{\mathcal{R}}_n$ (see Theorem A). In particular, we deduce that the Loewy length of $\bar{\mathcal{R}}_n$ is $[n/p] + 1$ (see Corollary B). Here, if x is a real number, $[x]$ denotes the unique $r \in \mathbb{Z}$ such that $r \leq x < r + 1$.

Let us introduce some notation. If $\varphi \in \mathcal{R}_n$, we denote by $\bar{\varphi}$ its image in $\bar{\mathcal{R}}_n$. The radical of $\bar{\mathcal{R}}_n$ is denoted by $\text{Rad } \bar{\mathcal{R}}_n$. If X and Y are two subspaces of $\bar{\mathcal{R}}_n$, we denote by XY the subspace of $\bar{\mathcal{R}}_n$ generated by the elements of the form xy , with $x \in X$ and $y \in Y$.

Compositions, partitions. A *composition* is a finite sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of non-zero natural numbers. We set $|\lambda| = \lambda_1 + \dots + \lambda_r$ and we say that λ is a *composition of $|\lambda|$* . The λ_i 's are called the *parts* of λ . If moreover $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, we say that λ is a *partition of $|\lambda|$* . The set of compositions (resp. partitions) of n is denoted by $\text{Comp}(n)$ (resp. $\text{Part}(n)$). We denote by $\hat{\lambda}$ the partition of n obtained from λ by reordering its parts. So $\text{Part}(n) \subset \text{Comp}(n)$ and $\text{Comp}(n) \rightarrow \text{Part}(n)$, $\lambda \mapsto \hat{\lambda}$ is surjective. If $1 \leq i \leq n$, we denote by $r_i(\lambda)$ the number of occurrences of i as a part of λ . We set

$$\pi_p(\lambda) = \sum_{i=1}^n \left[\frac{r_i(\lambda)}{p} \right].$$

Recall that λ is called *p -regular* (resp. *p -singular*) if and only if $\pi_p(\lambda) = 0$ (resp. $\pi_p(\lambda) \geq 1$). Note also that $\pi_p(\lambda) \in \{0, 1, 2, \dots, [n/p]\}$ and that $\pi_p(\hat{\lambda}) = \pi_p(\lambda)$. Finally, if $i \geq 0$, we set

$$\text{Part}_i^{(p)}(n) = \{\lambda \in \text{Part}(n) \mid \pi_p(\lambda) \geq i\}.$$

Young subgroups. For $1 \leq i \leq n - 1$, let $s_i = (i, i + 1) \in \mathfrak{S}_n$. Let $S_n = \{s_1, s_2, \dots, s_{n-1}\}$. Then (\mathfrak{S}_n, S_n) is a Coxeter group. We denote by $\ell : \mathfrak{S}_n \rightarrow \mathbb{N}$ the associated length function. If $\lambda = (\lambda_1, \dots, \lambda_r) \in \text{Comp}(n)$, we set

$$S_\lambda = \{s_i \mid \forall 1 \leq j \leq r, i \neq \lambda_1 + \dots + \lambda_j\}.$$

Let $\mathfrak{S}_\lambda = \langle S_\lambda \rangle$. Then $(\mathfrak{S}_\lambda, S_\lambda)$ is a Coxeter group: it is a standard parabolic subgroup of \mathfrak{S}_n which is canonically isomorphic to $\mathfrak{S}_{\lambda_1} \times \dots \times \mathfrak{S}_{\lambda_r}$. Note that

(1) \mathfrak{S}_λ and \mathfrak{S}_μ are conjugate in \mathfrak{S}_n if and only if $\hat{\lambda} = \hat{\mu}$.

Date: November 2, 2005.

1991 Mathematics Subject Classification. Primary 20C30; Secondary 05E10.

We write $\lambda \subset \mu$ if $\mathfrak{S}_\lambda \subset \mathfrak{S}_\mu$ and we write $\lambda \leq \mu$ if \mathfrak{S}_λ is contained in a subgroup of \mathfrak{S}_n conjugate to \mathfrak{S}_μ . Then \subset is an order on $\text{Comp}(n)$ and \leq is a preorder on $\text{Comp}(n)$ which becomes an order when restricted to $\text{Part}(n)$.

Let $X_\lambda = \{w \in \mathfrak{S}_n \mid \forall x \in \mathfrak{S}_\lambda, \ell(wx) \geq \ell(w)\}$. Then X_λ is a cross-section of $\mathfrak{S}_n/\mathfrak{S}_\lambda$. Now, let $\mathcal{N}_\lambda = N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda)$ and $W(\lambda) = \mathcal{N}_\lambda \cap X_\lambda$. Then $W(\lambda)$ is a subgroup of \mathcal{N}_λ and $\mathcal{N}_\lambda = W(\lambda) \ltimes \mathfrak{S}_\lambda$. Note that

$$(2) \quad W(\lambda) \simeq \mathfrak{S}_{r_1(\lambda)} \times \cdots \times \mathfrak{S}_{r_n(\lambda)}.$$

Recall that, for a finite group G , the p -rank of G is the maximal rank of an elementary abelian p -subgroup of G . For instance, $[n/p]$ is the p -rank of \mathfrak{S}_n . So

$$(3) \quad \pi_p(\lambda) \text{ is the } p\text{-rank of } W(\lambda).$$

If $\lambda, \mu \in \text{Comp}(n)$, we set

$$X_{\lambda\mu} = (X_\lambda)^{-1} \cap X_\mu.$$

Then $X_{\lambda\mu}$ is a cross-section of $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu$. Moreover, if $d \in X_{\lambda\mu}$, there exists a unique composition ν of n such that $\mathfrak{S}_\lambda \cap {}^d\mathfrak{S}_\mu = \mathfrak{S}_\nu$. This composition will be denoted by $\lambda \cap {}^d\mu$ or by ${}^d\mu \cap \lambda$.

The ring \mathcal{R}_n . If $\lambda \in \text{Comp}(n)$, we denote by 1_λ the trivial character of \mathfrak{S}_λ and we set $\varphi_\lambda = \text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1_\lambda$. Then, by (1), we have $\varphi_\lambda = \varphi_{\hat{\lambda}}$. We recall the following well-known old result of Frobenius:

$$(4) \quad (\varphi_\lambda)_{\lambda \in \text{Part}(n)} \text{ is a } \mathbb{Z}\text{-basis of } \mathcal{R}_n.$$

Moreover, by the Mackey formula for tensor product of induced characters, we have

$$(5) \quad \varphi_\lambda \varphi_\mu = \sum_{d \in X_{\lambda\mu}} \varphi_{\lambda \cap {}^d\mu} = \sum_{d \in X_{\lambda\mu}} \widehat{\varphi_{\lambda \cap {}^d\mu}}.$$

Let us give another form of (5). If $d \in X_{\lambda\mu}$, we define $\Delta_d : \mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu \rightarrow \mathcal{N}_\lambda \times \mathcal{N}_\mu$, $w \mapsto (w, d^{-1}wd)$. Let $\bar{\Delta}_d : \mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu \rightarrow W(\lambda) \times W(\mu)$ be the composition of Δ_d with the canonical projection $\mathcal{N}_\lambda \times \mathcal{N}_\mu \rightarrow W(\lambda) \times W(\mu)$. Then the kernel of $\bar{\Delta}_d$ is $\mathfrak{S}_{\lambda \cap {}^d\mu}$, so $\bar{\Delta}_d$ induces an injective morphism $\tilde{\Delta}_d : W(\lambda, \mu, d) \hookrightarrow W(\lambda) \times W(\mu)$, where $W(\lambda, \mu, d) = (\mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu) / \mathfrak{S}_{\lambda \cap {}^d\mu}$. Now, $W(\lambda) \times W(\mu)$ acts on $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu$ and, if $d \in X_{\lambda\mu}$, then the stabilizer of $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$ in $W(\lambda) \times W(\mu)$ is $\tilde{\Delta}_d(W(\lambda, \mu, d))$. Moreover, if d and d' are two elements of $X_{\lambda\mu}$ such that $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$ and $\mathfrak{S}_\lambda d' \mathfrak{S}_\mu$ are in the same $(W(\lambda) \times W(\mu))$ -orbit, then $\mathfrak{S}_{\lambda \cap {}^d\mu}$ and $\mathfrak{S}_{\lambda \cap {}^{d'}\mu}$ are conjugate in \mathcal{N}_λ . Therefore,

$$(6) \quad \varphi_\lambda \varphi_\mu = \sum_{d \in X'_{\lambda\mu}} \frac{|W(\lambda)| \cdot |W(\mu)|}{|W(\lambda, \mu, d)|} \varphi_{\lambda \cap {}^d\mu},$$

where $X'_{\lambda\mu}$ denotes a cross-section of $\mathcal{N}_\lambda \backslash \mathfrak{S}_n / \mathcal{N}_\mu$ contained in $X_{\lambda\mu}$.

The Loewy series of $\bar{\mathcal{R}}_n$. We can now state the main results of this paper.

$$\text{THEOREM A. If } i \geq 0, \text{ we have } (\text{Rad } \bar{\mathcal{R}}_n)^i = \bigoplus_{\lambda \in \text{Part}_i^{(p)}(n)} \mathbb{F}_p \bar{\varphi}_\lambda.$$

$$\text{COROLLARY B. The Loewy length of } \bar{\mathcal{R}}_n \text{ is } [n/p] + 1.$$

Corollary B follows immediately from Theorem A. The end of this paper is devoted to the proof of Theorem A.

Proof of Theorem A. Let $\bar{\mathcal{R}}_n^{(i)} = \bigoplus_{\lambda \in \text{Part}_i^{(p)}(n)} \mathbb{F}_p \bar{\varphi}_\lambda$. Note that

$$0 = \bar{\mathcal{R}}_n^{([n/p]+1)} \subset \bar{\mathcal{R}}_n^{([n/p])} \subset \dots \subset \bar{\mathcal{R}}_n^{(1)} \subset \bar{\mathcal{R}}_n^{(0)} = \bar{\mathcal{R}}_n.$$

Let us first prove the following fact:

(♣) *If $i, j \geq 0$, then $\bar{\mathcal{R}}_n^{(i)} \bar{\mathcal{R}}_n^{(j)} \subset \bar{\mathcal{R}}_n^{(i+j)}$.*

Proof of (♣). Let λ and μ be two compositions of n such that $\pi_p(\lambda) \geq i$ and $\pi_p(\mu) \geq j$. Let $d \in X'_{\lambda\mu}$ be such that p does not divide $\frac{|W(\lambda)| \cdot |W(\mu)|}{|W(\lambda, \mu, d)|}$. By (6), we only need to prove that this implies that $\pi_p(\lambda \cap {}^d\mu) \geq i + j$. But our assumption on d means that $\tilde{\Delta}_d(W(\lambda, \mu, d))$ contains a Sylow p -subgroup of $W(\lambda) \times W(\mu)$. In particular, the p -rank of $W(\lambda, \mu, d)$ is greater than or equal to the p -rank of $W(\lambda) \times W(\mu)$. By (3), this means that the p -rank of $W(\lambda, \mu, d)$ is $\geq i + j$. Since $W(\lambda, \mu, d)$ is a subgroup of $W(\lambda \cap {}^d\mu)$, we get that the p -rank of $W(\lambda \cap {}^d\mu)$ is $\geq i + j$. In other words, again by (3), we have $\pi_p(\lambda \cap {}^d\mu) \geq i + j$, as desired. \square

By (♣), $\bar{\mathcal{R}}_n^{(i)}$ is an ideal of $\bar{\mathcal{R}}_n$ and, if $i \geq 1$, then $\bar{\mathcal{R}}_n^{(i)}$ is a nilpotent ideal of $\bar{\mathcal{R}}_n$. Therefore, $\bar{\mathcal{R}}_n^{(1)} \subset \text{Rad } \bar{\mathcal{R}}_n$. In fact:

(◇) $\text{Rad } \bar{\mathcal{R}}_n = \bar{\mathcal{R}}_n^{(1)}$.

Proof of (◇). First, note that $\text{Rad } \bar{\mathcal{R}}_n$ consists of the nilpotent elements of $\bar{\mathcal{R}}_n$ because $\bar{\mathcal{R}}_n$ is commutative. Now, let φ be a nilpotent element of $\bar{\mathcal{R}}_n$. Write $\varphi = \sum_{\lambda \in \text{Part}(n)} a_\lambda \bar{\varphi}_\lambda$ and let $\lambda_0 \in \text{Part}(n)$ be maximal (for the order \leq on $\text{Part}(n)$) such that $a_{\lambda_0} \neq 0$. Then, by (6), the coefficient of φ_{λ_0} in φ^r is equal to $a_{\lambda_0}^r |W(\lambda_0)|^{r-1}$. Therefore, since φ is nilpotent and $a_{\lambda_0} \neq 0$, we get that p divides $|W(\lambda_0)|$, so that $\lambda_0 \in \text{Part}_1^{(p)}(n)$ (by (3)). Consequently, $\varphi - a_{\lambda_0} \bar{\varphi}_{\lambda_0}$ is nilpotent and we can repeat the argument to find finally that $\varphi \in \bar{\mathcal{R}}_n^{(1)}$. \square

We shall now establish a special case of (5) (or (6)). We need some notation. If $\alpha = (\alpha_1, \dots, \alpha_r)$ is a composition of n' and $\beta = (\beta_1, \dots, \beta_s)$ is a composition of n'' , let $\alpha \sqcup \beta$ denote the composition of $n' + n''$ equal to $(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s)$. If $1 \leq j \leq n$ and if $0 \leq k \leq [n/j]$, we denote by $\nu(n, j, k)$ the composition $(n - jk, j, j, \dots, j)$ of n , where j is repeated k times (if $n = jk$, then the part $n - jk$ is omitted). If $\lambda \in \text{Comp}(n)$, we set

$$M(\lambda) = \{0\} \cup \{1 \leq j \leq n \mid p \text{ does not divide } r_j(\lambda)\},$$

$$\mathbf{m}(\lambda) = \max M(\lambda),$$

$$J(\lambda) = \{0\} \cup \{1 \leq j \leq n \mid r_j(\lambda) \geq p\},$$

$$\mathbf{j}(\lambda) = \min J(\lambda)$$

and

$$\mathbf{jm}(\lambda) = (\mathbf{j}(\lambda), \mathbf{m}(\lambda)).$$

Let $I = \{0, 1, \dots, [n/p]\}$. Then $\mathbf{jm}(\lambda) \in I \times I$. Let us now introduce an order \preceq on $I \times I$. If $(j, m), (j', m')$ are two elements of $I \times I$, we write $(j, m) \preceq (j', m')$ if one of the following two conditions is satisfied:

- (a) $j < j'$.
- (b) $j = j'$ and $m \geq m'$.

Now, let $i \geq 1$ and let $\lambda \in \text{Part}_{i+1}^{(p)}(n)$. Let $(j, m) = \mathbf{jm}(\lambda)$. Then $\lambda = \widehat{\alpha \sqcup \nu_0}$, where α is a partition of $n - m - jp$ and $\nu_0 = \nu(m + jp, j, p)$. Let $\tilde{\lambda} = \alpha \sqcup (m + jp)$. Then $\pi_p(\tilde{\lambda}) = i$ (indeed, $r_{m+jp}(\tilde{\lambda}) = 1 + r_{m+jp}(\alpha) = 1 + r_{m+jp}(\lambda)$ and, by the maximality of m , we have that p divides $r_{m+jp}(\lambda)$) and

$$(\heartsuit) \quad \bar{\varphi}_{\nu(n,j,p)} \bar{\varphi}_{\tilde{\lambda}} \in \bar{\varphi}_{\lambda} + \left(\bigoplus_{\substack{\mu \in \text{Part}_{i+1}^{(p)}(n) \\ \mathbf{jm}(\mu) \prec (j,m)}} \mathbb{F}_p \bar{\varphi}_{\mu} \right).$$

Proof of (\heartsuit) . Let $\nu = \nu(n, j, p)$. Since $\pi_p(\lambda) = i + 1 \geq 2$, there exists $j' \in \{1, 2, \dots, [n/p]\}$ such that $r_{j'}(\lambda) \geq p$. Then $j' \geq j$ (by definition of j), so $n \geq 2pj$. In particular, $n - pj > j$, so $W(\nu) \simeq \mathfrak{S}_p$. Now, if $m' > m$, then $r_{m'}(\alpha) = r_{m'}(\lambda)$, so

$$(\heartsuit') \quad \forall m' > m, r_{m'}(\alpha) \equiv 0 \pmod{p}.$$

Also

$$(\heartsuit'') \quad \forall l \neq m + jp, r_l(\tilde{\lambda}) = r_l(\alpha)$$

and

$$(\heartsuit''') \quad r_{m+jp}(\tilde{\lambda}) = r_{m+jp}(\alpha) + 1.$$

Now, keep the notation of (6). We may, and we will, assume that $1 \in X'_{\nu \tilde{\lambda}}$. First, note that $\nu \cap \tilde{\lambda} = \alpha \sqcup \nu_0$ and that the image of $\tilde{\Delta}_1$ in $W(\nu) \times W(\tilde{\lambda})$ is equal to $W(\nu) \times W(\alpha)$. But, by (\heartsuit') , (\heartsuit'') and (\heartsuit''') , the index of $W(\alpha)$ in $W(\tilde{\lambda})$ is $\equiv 1 \pmod{p}$. Thus, by (6), we have

$$\bar{\varphi}_{\nu} \bar{\varphi}_{\tilde{\lambda}} = \bar{\varphi}_{\lambda} + \sum_{d \in X'_{\nu \tilde{\lambda}} - \{1\}} \frac{|W(\nu)| \cdot |W(\tilde{\lambda})|}{|W(\nu, \tilde{\lambda}, d)|} \bar{\varphi}_{\nu \cap d \tilde{\lambda}}.$$

Now, let d be an element of $X_{\nu \tilde{\lambda}}$ such that p does not divide $\frac{|W(\nu)| \cdot |W(\tilde{\lambda})|}{|W(\nu, \tilde{\lambda}, d)|} = x_d$ and such that $\mathbf{jm}(\nu \cap d \tilde{\lambda}) \succ \mathbf{jm}(\lambda)$. It is sufficient to show that $d \in \mathcal{N}_{\nu} \mathcal{N}_{\tilde{\lambda}}$. Write $\alpha = (\alpha_1, \dots, \alpha_r)$. Then

$$\nu \cap d \tilde{\lambda} = (n_1, \dots, n_r, n_0) \sqcup j^{(1)} \sqcup \dots \sqcup j^{(p)},$$

where $n_k \geq 0$ and $j^{(l)}$ is a composition of j with at most $r + 1$ parts. Since p does not divide x_d , the image of $\mathcal{N}_{\nu} \cap d \mathcal{N}_{\tilde{\lambda}}$ in $W(\nu)$ contains a Sylow p -subgroup of $W(\nu) \simeq \mathfrak{S}_p$. Let $w \in \mathcal{N}_{\nu} \cap d \mathcal{N}_{\tilde{\lambda}}$ be such that its image in $W(\nu)$ is an element of order p . Then there exists $\sigma \in \mathcal{N}_{\nu}$ such that $w\sigma$ normalizes $\mathfrak{S}_{\nu \cap d \tilde{\lambda}}$. In particular, $\widehat{j^{(1)}} = \dots = \widehat{j^{(p)}}$. So, if $j^{(1)} \neq (j)$, then $\mathbf{j}(\nu \cap d \tilde{\lambda}) < \mathbf{j}(\lambda)$, which contradicts our hypothesis. So $j^{(1)} = \dots = j^{(p)} = (j)$. Therefore,

$$\tilde{\lambda} \cap d^{-1} \nu = \nu(\alpha_1, j, k_1) \sqcup \dots \sqcup \nu(\alpha_r, j, k_r) \sqcup \nu(m + jp, j, k_0),$$

where $0 \leq k_i \leq p$ and $\sum_{i=0}^r k_i = p$. Note that $(n_1, \dots, n_r, n_0) = (\alpha_1 - k_1 j, \dots, \alpha_r - k_r j, \alpha_0 - k_0 j)$ where, for simplification, we denote $\alpha_0 = m + jp$. Also, $\mathbf{j}(\tilde{\lambda} \cap d^{-1} \nu) \leq j$ and, since $\mathbf{jm}(\tilde{\lambda} \cap d^{-1} \nu) \succ (j, m)$, we have that $\mathbf{m}(\tilde{\lambda} \cap d^{-1} \nu) \leq m$. Recall that $d^{-1} w d \in \mathcal{N}_{\tilde{\lambda}}$. So two cases may occur:

- If the image of $d^{-1} w d$ in $W(\tilde{\lambda})$ has order p , this means that there exists a sequence $0 \leq i_1 < \dots < i_p \leq r$ such that $0 \neq k_{i_1} = \dots = k_{i_p} (= 1)$ and such that $\alpha_{i_1} = \dots = \alpha_{i_p}$. So $r_l(\tilde{\lambda} \cap d^{-1} \nu) \equiv r_l(\tilde{\lambda}) \pmod{p}$ for every $l \geq 1$. In particular, $r_{m+jp}(\tilde{\lambda} \cap d^{-1} \nu) \equiv 1 + r_{m+jp}(\alpha) \equiv 1 \pmod{p}$ by (\heartsuit') and (\heartsuit''') . Thus, $\mathbf{m}(\tilde{\lambda} \cap d^{-1} \nu) \geq m + jp > m$, which contradicts our hypothesis.

- Therefore, the image of $d^{-1} w d$ in $W(\tilde{\lambda})$ has order 1. So there exists a unique $i \in \{0, 1, \dots, r\}$ such that $k_i = p$. Consequently, $k_{i'} = 0$ if $i' \neq i$. If $\alpha_i > m + jp$, then $r_{\alpha_i}(\tilde{\lambda} \cap d^{-1} \nu) =$

$r_{\alpha_i}(\tilde{\lambda}) - 1 = r_{\alpha_i}(\alpha) - 1$ (by (\heartsuit'')), so p does not divide $r_{\alpha_i}(\tilde{\lambda} \cap d^{-1}\nu)$ (by (\heartsuit')), which implies that $\mathbf{m}(\tilde{\lambda} \cap d^{-1}\nu) \geq \alpha_i > m$, contrarily to our hypothesis. If $\alpha_i < m + jp$, then $r_{m+jp}(\tilde{\lambda} \cap d^{-1}\nu) = r_{m+jp}(\tilde{\lambda}) = r_{m+jp}(\alpha) + 1$ (by (\heartsuit''')), so p does not divide $r_{m+jp}(\tilde{\lambda} \cap d^{-1}\nu)$ (by (\heartsuit')), contrarily to our hypothesis. This shows that $\alpha_i = m + jp$. In other words, $d \in \mathcal{N}_{\tilde{\lambda}}\mathcal{N}_{\nu}$, as desired. \square

By (\diamond) , Theorem A follows immediately from the next result: if $i \geq 0$, then

$$(\spadesuit) \quad \bar{\mathcal{R}}_n^{(1)} \bar{\mathcal{R}}_n^{(i)} = \bar{\mathcal{R}}_n^{(i+1)}.$$

Proof of (\spadesuit) . We may assume that $i \geq 1$. By (\clubsuit) , we have $\bar{\mathcal{R}}_n^{(1)} \bar{\mathcal{R}}_n^{(i)} \subset \bar{\mathcal{R}}_n^{(i+1)}$. So we only need to prove that, if $\lambda \in \text{Part}_{i+1}^{(p)}(n)$ then $\bar{\varphi}_\lambda \in \bar{\mathcal{R}}_n^{(1)} \bar{\mathcal{R}}_n^{(i)}$. But this follows from (\heartsuit) and an easy induction on $\mathbf{jm}(\lambda) \in I \times I$ (for the order \preceq). \square

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