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A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one

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Abstract. *In this paper, we will focus - in dimension one - on the SDEs of the type $dX_t = \sigma(X_t)dB_t^H + b(X_t)dt$ where B^H is a fractional Brownian motion. Our principal motivation is to describe one of the simplest theory - from our point of view - allowing to study this SDE, and this for any $H \in (0, 1)$. We will consider several definitions of solution and we will study, for each one of them, in which condition one has existence and uniqueness. Finally, we will examine the convergence or not of the canonical scheme associated to our SDE, when the integral with respect to fBm is defined using the Russo-Vallois symmetric integral.*

Key words: Stochastic differential equation, fractional Brownian motion, Russo-Vallois integrals, Newton-Côtes integral, approximation schemes, Doss-Sussmann transformation.

MSC 2000: 60G18, 60H05, 60H20.

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1 Introduction

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ of Hurst index $H \in (0, 1)$ is a centered Gaussian process verifying $B_0^H = 0$ a.s. and

$$E[(B_t^H - B_s^H)^2] = |t - s|^{2H} \quad (1.1)$$

for all $s, t \geq 0$. If $H = 1/2$, $B^{1/2}$ is nothing other than the standard Brownian motion. The equality (1.1) implies that the trajectories of B^H are $(H - \varepsilon)$ -Hölder continuous for any $\varepsilon > 0$ small enough. The fBm being selfsimilar (of index H) and having stationary increments, it is used as model in many fields (for example, in hydrology, economy, financial mathematics, etc.). In particular, the study of stochastic differential equations (SDEs) driven by a fBm is important in the sight of the applications. But, before putting the question of the existence and/or the uniqueness for this type of SDEs, the first difficulty is to give a sense to the integral with respect to a fBm. It is indeed well-known that B^H is not a semimartingale if $H \neq 1/2$. Thus the Itô or Stratonovich calculus does not apply in this case. There are several ways of building an integral with respect to the fBm and of obtaining a change of variables formula. Let us point out the contributions which appear to us most important:

1. *Regularization or discretization technics.* Since 1993, Russo and Vallois [29] have developed a regularization procedure, whose philosophy is similar to the discretization. They introduce forward (generalizing Itô), backward, symmetric (generalizing Stratonovich, see Definition 2.3 below) stochastic integrals and a generalized quadratic variation. The regularization, or discretization technique, for fBm and related processes have been performed by [14, 18, 30, 33], in the case of zero quadratic variation (corresponding to $H > 1/2$). Note also that Young [32] integral, which is often used in this case, coincides with the forward integral (but also with the backward or symmetric, since covariation between integrand and integrator is always zero). When the integrator has paths with finite p -variation for $p > 2$, there is no hope to make use of forward and backward integrals. In this case, we have to use some symmetric integrals introduced by Gradinaru, Nourdin, Russo and Vallois in [16] (see §2 below). We can also refer to [13, 26] for the specific case where $H > 1/3$.
2. *Rough paths.* An other approach was done by Lyons [21], who have considered, through an absolutely pathwise method based on Lévy stochastic area, integrators having p -variation for any $p > 1$, provided one could construct a canonical geometric rough paths associated with the process. We refer to the survey's article of Lejay [19] for more precise statements related to this theory. Let us note however that the case where the integrator is a fBm with index $H > 1/4$ has been studied by Coutin and Qian [9] (see also Feyel - de La Pradelle [15]).
3. *Malliavin calculus.* Since fBm is a Gaussian process, it is natural to use Skorohod approach. Integration with respect to fBm has been attacked by Decreusefond and Ustunel [10] for $H > 1/2$ and it has known then a very intensive study (see for instance [2, 3, 7]), even when the integrator is a more general Gaussian process. We refer to the survey's article of Nualart [24] for precise statements related to this theory.
4. *Wick products.* A new type of integral with zero mean defined using Wick products was introduced by Duncan, Hu and Pasik-Duncan in [12], assuming $H > 1/2$. This integral turns out to coincide with the divergence operator. In [4], Bender considers the case of arbitrary Hurst index $H \in (0, 1)$ and proves an Itô formula for generalized functionals of B^H .

In the sequel, we will focus, *only in dimension one*, on the SDEs of the type:

$$\begin{cases} dX_t = \sigma(X_t)dB_t^H + b(X_t)dt, & t \in [0, T] \\ X_0 = x_0 \in \mathbb{R} \end{cases} \quad (1.2)$$

where $\sigma, b : \mathbb{R} \rightarrow \mathbb{R}$ are two continuous functions, $x_0 \in \mathbb{R}$ and $H \in (0, 1)$. Our principal motivation is to describe one of the simplest theory - from our point of view - allowing to study SDE (1.2), and this for any $H \in (0, 1)$. Let us remark that it is more and more used and quoted in research articles (see for example [5, 22, 23, 24, 25]).

The paper is organized as follows. In the second part, we will consider several definitions of solution to (1.2) and we will study, for each one of them, in which condition one has existence and uniqueness. In the third part, we will examine the convergence or not of the canonical scheme associated to (1.2), when the integral with respect to fBm is defined using the Russo-Vallois symmetric integral.

2 Basic study of SDE (1.2)

In the sequel, we denote by B^H a fBm of Hurst parameter $H \in (0, 1)$.

Definition 2.1 *Let X, Y be two real continuous processes defined on $[0, T]$. The symmetric integral (in the sense of Russo-Vallois) is defined by*

$$\int_0^T Y_u d^\circ X_u = \lim_{\varepsilon \rightarrow 0} -\text{prob} \int_0^T \frac{Y_{u+\varepsilon} + Y_u}{2} \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du, \quad (2.3)$$

provided the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$.

Remark 2.2 If X, Y are two continuous semimartingales then $\int_0^T Y_u d^\circ X_u$ coincides with the standard Stratonovich integral (see [29]).

Let us recall an important result for our study:

Theorem 2.3 ([16], page 793). *The symmetric integral $\int_0^T f(B_u^H) d^\circ B_u^H$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^5 if and only if $H \in (1/6, 1)$. In this case, we have, for any primitive F of f :*

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^\circ B_u^H.$$

When $H \leq 1/6$, one can consider the so-called *m-order Newton-Côtes integral*:

Definition 2.4 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, X, Y be two continuous processes on $[0, T]$ and $m \geq 1$ be an integer. The m-order Newton-Côtes integral (in the sense of Russo-Vallois) of $f(Y)$ with respect to X is defined by*

$$\int_0^T f(Y_u) d^{\text{NC}, m} X_u = \lim_{\varepsilon \rightarrow 0} -\text{prob} \int_0^T \left(\int_0^1 f(Y_s + \beta(Y_{s+\varepsilon} - Y_s)) \nu_m(d\beta) \right) \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du,$$

provided the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$. Here, $\nu_1 = \frac{1}{2}(\delta_0 + \delta_1)$ and, for $m \geq 2$,

$$\nu_m = \sum_{j=0}^{2(m-1)} \left(\int_0^1 \prod_{k \neq j} \frac{2(m-1)u - k}{j - k} du \right) \delta_{j/(2m-2)}, \quad (2.4)$$

δ_a being the Dirac measure at point a .

Remark 2.5 • The 1-order Newton-Côtes integral is nothing other than the symmetric one, defined by (2.3).

- The terminology "Newton-Côtes integral" is due to the fact that the definition (2.4) of ν_m is related to the Newton-Côtes formula of numerical analysis.

The main interest of this concept is contained in the following theorem.

Theorem 2.6 ([16], page 793). *Let $m \geq 1$ be an integer. The m-order Newton-Côtes integral $\int_0^T f(B_u^H) d^{\text{NC}, m} B_u^H$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4m+1} if and only if $H \in (1/(4m+2), 1)$. In this case, we have, for any primitive F of f :*

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^{\text{NC}, m} B_u^H. \quad (2.5)$$

Remark 2.7 As immediate consequence of this result, we remark that $\int_0^T f(B_u^H) d^{\text{NC},m} B_u^H = \int_0^T f(B_u^H) d^{\text{NC},n} B_u^H = F(B_T^H) - F(0)$ as soon as $m > n$, f is C^{4m+1} and $H \in (1/(4n+2), 1)$. Then, for f regular enough, it is possible to define the so-called *Newton-Côtes integral* $\int_0^T f(B_u^H) d^{\text{NC}} B_u^H$ without ambiguity by:

$$\int_0^T f(B_u^H) d^{\text{NC}} B_u^H := \int_0^T f(B_u^H) d^{\text{NC},n} B_u^H \text{ if } H \in (1/(4n+2), 1). \quad (2.6)$$

In the sequel, we put $n_H = \inf\{n \geq 1 : H > 1/(4n+2)\}$. An immediate consequence of (2.5) and (2.6) is that, for any $H \in (0, 1)$ and any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4n_H+1} , we have:

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^{\text{NC}} B_u^H,$$

where F is a primitive of f .

In order to make a sense to $\int_0^t \sigma(X_s) d^{\text{NC}} B_s^H$ in (1.2), we can try to use the Newton-Côtes integral. But we are just able, for the moment, to consider integrands of the form $f(B^H)$ with f regular enough. That's why we first choose the following definition for solution to (1.2):

Definition 2.8 Assume that $\sigma \in C^{4n_H+1}$.

- i) Let \mathfrak{C}_1 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ verifying that there exist $f : \mathbb{R} \rightarrow \mathbb{R} \in C^{4n_H+1}$ such that, a.s., $\forall t \in [0, T]$, $X_t = f(B_t^H)$.
- ii) A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (1.2) if:

- $X \in \mathfrak{C}_1$,
- $\forall t \in [0, T]$, $X_t = x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s^H + \int_0^t b(X_s) ds$.

Remark 2.9 Let us note that the first point of the definition ii) is used to ensure that the integral $\int_0^t \sigma(X_s) d^{\text{NC}} B_s^H$ makes sense (a little like the adaptness condition in the Itô context).

We can now state the following result.

Theorem 2.10 Let $\sigma \in C^{4n_H+1}$ be a Lipschitz function, b be a continuous function and x_0 be a real. Then the equation (1.2) admits a solution X in the sense of Definition 2.8 if and only if b vanishes on $\mathfrak{S}(\mathbb{R})$, where \mathfrak{S} is the unique solution to $\mathfrak{S}' = \sigma \circ \mathfrak{S}$ with initial value $\mathfrak{S}(0) = x_0$. In this case, X is unique and is given by $X_t = \mathfrak{S}(B_t^H)$.

Remark 2.11 As a consequence of the mean value theorem, $\mathfrak{S}(\mathbb{R})$ is an interval. Moreover, it is easy to see that either \mathfrak{S} is constant or \mathfrak{S} is strictly monotonous, and that $\inf \mathfrak{S}(\mathbb{R})$ and $\sup \mathfrak{S}(\mathbb{R})$ are elements of $\{\sigma = 0\} \cup \{\pm\infty\}$. In particular, if σ does not vanish, then $\mathfrak{S}(\mathbb{R}) = \mathbb{R}$ and an immediate consequence of Theorem 2.10 is that (1.2) admits a solution in the sense of Definition 2.8 if and only if $b \equiv 0$.

Proof of Theorem 2.10. Assume that $X_t = f(B_t^H)$ is a solution to (1.2) in the sense of Definition 2.8. Then, we have

$$f(B_t^H) = x_0 + \int_0^t \sigma \circ f(B_s^H) d^{\text{NC}} B_s^H + \int_0^t b \circ f(B_s^H) ds = G(B_t^H) + \int_0^t b \circ f(B_s^H) ds, \quad (2.7)$$

where G is the primitive of $\sigma \circ f$ verifying $G(0) = x_0$. Put $h = f - G$ and denote by Ω^* the set of $\omega \in \Omega$ such that $t \mapsto B_t^H(\omega)$ is derivable in at least one point $t_0 \geq 0$ (it is well-known that $P(\Omega^*) = 0$). If $h'(B_{t_0}^H(\omega)) \neq 0$ for one $t_0 \in [0, T]$ and one $\omega \in \Omega$ then h is strictly monotonous in a neighborhood of $B_{t_0}^H(\omega)$ and, for $|t - t_0|$ sufficiently small, we have $B_t^H(\omega) = h^{-1}(\int_0^t b(X_s(\omega))ds)$ and, consequently, $\omega \in \Omega^*$. Then, a.s., $h'(B_t^H) = 0$ for all $t \in [0, T]$ and $h \equiv 0$. By uniqueness, we deduce $f = \mathfrak{S}$. Thus, if (1.2) admits a solution X in the sense of Definition 2.8, we have necessarily $X_t = \mathfrak{S}(B_t^H)$. Thanks to (2.7), we then have $b \circ \mathfrak{S}(B_t^H) = 0$ for all $t \in [0, T]$ a.s. and then b vanishes on $\mathfrak{S}(\mathbb{R})$. \square

Consequently, when SDE (1.2) has no drift b , one has a natural solution. But what can we do when $b \not\equiv 0$?

To answer this question, we begin by a lemma. Let us denote by \mathcal{A} the set of processes $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ having C^1 -trajectories and verifying that $E\left(e^{\lambda \int_0^T A_s^2 ds}\right) < \infty$ for some $\lambda > 1$.

Lemma 2.12 *Let $A \in \mathcal{A}$ and $m \in \mathbb{N}^*$. Then $\int_0^T f(B_u^H + A_u) d^{\text{NC}, m} B_u^H$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4n_H+1} if and only if $H > 1/(4m+2)$. In this case, we have, for any primitive F of f :*

$$F(B_T^H + A_T) = F(A_0) + \int_0^T f(B_u^H + A_u) d^{\text{NC}, m} B_u^H + \int_0^T f(B_u^H + A_u) A'_u du.$$

Proof. Set $\tilde{B}^H = B^H + A$. On the one hand, using Girsanov theorem in [27] and taking account of assumptions on A , we have that \tilde{B} is a fBm of index H under \mathbb{Q} , an equivalent probability to the initial probability \mathbb{P} . On the other hand, it is easy, while returning to Definition 2.4, to prove that $\int_0^T f(B_u^H + A_u) d^{\text{NC}, m} B_u^H$ exists if and only if $\int_0^T f(B_u^H + A_u) d^{\text{NC}, m} (\tilde{B}_u^H + A_u)$ exist, and in this case, we have

$$\int_0^T f(B_u^H + A_u) d^{\text{NC}, m} (\tilde{B}_u^H + A_u) = \int_0^T f(B_u^H + A_u) d^{\text{NC}, m} B_u^H + \int_0^T f(B_u^H + A_u) d^{\text{NC}, m} A'_u.$$

Then, since convergence under \mathbb{Q} or under \mathbb{P} is equivalent, the conclusion of Lemma is a direct consequence of Theorem 2.6. \square

Then, as previously, it is possible to define an integral (still called *Newton-Côtes integral*) verifying, for any $H \in (0, 1)$, for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4n_H+1} and any process $A \in \mathcal{A}$:

$$F(B_T^H + A_T) = F(A_0) + \int_0^T f(B_u^H + A_u) d^{\text{NC}} B_u^H + \int_0^T f(B_u^H + A_u) A'_u du,$$

where F is a primitive of f .

Now, we can introduce a new definition for solution to (1.2):

Definition 2.13 *Assume that $\sigma \in C^{4n_H+1}$.*

- i) *Let \mathfrak{C}_2 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ verifying that there exist a function $f : \mathbb{R} \rightarrow \mathbb{R} \in C^{4n_H+1}$ and a process $A \in \mathcal{A}$ such that $A_0 = 0$ and, a.s., $\forall t \in [0, T]$, $X_t = f(B_t^H + A_t)$.*
- ii) *A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (1.2) if:*

- $X \in \mathfrak{C}_2$,
- $\forall t \in [0, T], X_t = x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s^H + \int_0^t b(X_s) ds$.

Theorem 2.14 *Let $\sigma \in C^{4n_H+1}$ be a Lipschitz function, b be a continuous function and x_0 be a real.*

- *If $\sigma(x_0) = 0$ then (1.2) admits a solution X in the sense of Definition 2.13 if and only if $b(x_0) = 0$. In this case, X is unique and is given by $X_t \equiv x_0$.*
- *If $\sigma(x_0) \neq 0$, then (1.2) admits a solution X . If moreover $\inf_{\mathbb{R}} |\sigma| > 0$ and $b \in \text{Lip}$ then X is unique.*

Proof. Assume that $X = f(B^H + A)$ is a solution to (1.2) in the sense of Definition 2.13. Then, we have

$$f(B_t^H + A_t) = G(B_t^H + A_t) - \int_0^t \sigma(X_s) A'_s ds + \int_0^t b(X_s) ds \quad (2.8)$$

where G is the primitive of $\sigma \circ f$ verifying $G(0) = x_0$. As in the proof of Theorem 2.10, we obtain that $f = \mathfrak{S}$ where \mathfrak{S} is defined by $\mathfrak{S}' = \sigma \circ \mathfrak{S}$ with initial value $\mathfrak{S}(0) = x_0$. Thanks to (2.8), we deduce that, a.s., $b \circ \mathfrak{S}(B_t^H + A_t) = A'_t \sigma \circ \mathfrak{S}(B_t^H + A_t)$ for all $t \in [0, T]$. Consequently:

- If $\sigma(x_0) = 0$ then $\mathfrak{S} \equiv x_0$ and $b(x_0) = 0$.
- If $\sigma(x_0) \neq 0$ then \mathfrak{S} is strictly monotonous and the ordinary integral equation $A_t = \int_0^t \frac{b \circ \mathfrak{S}}{\sigma \circ \mathfrak{S}}(B_s^H + A_s) ds$ admits a maximal (in fact, global since we know already that A is defined on $[0, T]$) solution thanks to Peano theorem. If moreover $\inf_{\mathbb{R}} |\sigma| > 0$ and $b \in \text{Lip}$ then $\frac{b \circ \mathfrak{S}}{\sigma \circ \mathfrak{S}} \in \text{Lip}$ and A is uniquely determined. \square

Since in the previous theorem, the discussion is with respect to x_0 , one could consider that it is not yet completely satisfactory. That's why we will finally introduce a last definition for solution to (1.2). Before that, we need to refine Definition 2.4.

Definition 2.15 *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function, A be a process on $[0, T]$ having C^1 -trajectories and $m \geq 1$ be an integer. The m -order Newton-Côtes integral (in the sense of Russo-Vallois) of $f(B^H, A)$ with respect to B^H is defined by*

$$\int_0^T f(B_u^H, A_u) d^{\text{NC}, m} B_u^H = \lim_{\varepsilon \rightarrow 0} -\text{prob} \int_0^T \left(\int_0^1 f(Z_s + \beta(Z_{s+\varepsilon} - Z_s)) \nu_m(d\beta) \right) \frac{B_{u+\varepsilon}^H - B_u^H}{\varepsilon} du,$$

provided the limit exists. Here, ν is as in Definition 2.4 and $Z = (B^H, A)$.

The interest of this concept is contained in the following theorem.

Theorem 2.16 *Let A be a process having C^1 -trajectories and $m \geq 1$ be an integer. If $H > 1/(2m+1)$ then the m -order Newton-Côtes integral $\int_0^T f(B_u^H, A_u) d^{\text{NC}, m} B_u^H$ exists for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m,1}$. In this case, we have, for any function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ verifying $F'_b = f$:*

$$F(B_T^H, A_T) = F(0, A_0) + \int_0^T f(B_u^H, A_u) d^{\text{NC}, m} B_u^H + \int_0^T F'_a(B_u^H, A_u) A'_u du.$$

Proof. The conclusion follows as a consequence of some Taylor expansions associated to F , related to the classical Newton-Côtes formula. Since only the result of Theorem 2.16 is important for the remainder of the paper, we do not make the proof with details but we return for instance to [23], Chapter 4. \square

Remark 2.17 • Remark that the condition is here $H > 1/(2m+1)$ and not $H > 1/(4m+2)$ as in Theorem 2.6 and Lemma 2.12. Thus, for instance, if $A \in \mathcal{A}$, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is C^5 and if $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $C^{5,1}$ then $\int_0^T g(B_s^H + A_s) d^\circ B_s^H$ exists if (and only if) $H > 1/6$ while $\int_0^T h(B_s^H, A_s) d^\circ B_s^H$ exists *a priori* only if $H > 1/3$.

- We define $m_H = \inf\{m \geq 1 : H > 1/(2m+1)\}$. As in the Remark 2.7, it is possible to consider, for any $H \in (0, 1)$ and without ambiguity, an integral (still called *Newton-Côtes integral*) which verifies, for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m_H,1}$ and any process A having C^1 -trajectories:

$$F(B_T^H, A_T) = F(0, A_0) + \int_0^T f(B_u^H, A_u) d^{\text{NC}} B_u^H + \int_0^T F'_a(B_u^H, A_u) A'_u du,$$

where F is such that $F'_b = f$.

Finally, we can introduce a last definition for solution to (1.2):

Definition 2.18 Assume that $\sigma \in C^{2m_H}$.

- Let \mathfrak{C}_3 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ verifying that there exist a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m_H,1}$ and a process $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ having C^1 -trajectories such that $A_0 = 0$ and, a.s., $\forall t \in [0, T]$, $X_t = f(B_t^H, A_t)$.
- A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (1.2) if:

- $X \in \mathfrak{C}_3$,
- $\forall t \in [0, T]$, $X_t = x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s^H + \int_0^t b(X_s) ds$.

Theorem 2.19 Let $\sigma \in C_b^2$, b be a Lipschitz function and x_0 be a real. Then the equation (1.2) admits a solution X in the sense of Definition 2.18. Moreover, if σ is analytic, then X is the unique solution of the form $f(B^H, A)$ with f analytic (resp. of class C^1) in the first (resp. second) variable and A a process having C^1 -trajectories and verifying $A_0 = 0$.

Remark 2.20 • If $H > 1/3$, we can improve Theorem 2.19. Indeed, in [25], it is shown that we have uniqueness without supplementary condition on σ . Moreover, they give an other meaning to (1.2) than Definition 2.18 using the concept of Lévy area.

- In [25], one studies the problem of the absolute continuity in equation (1.2), where the solution is in the sense of Definition 2.18, and one proves that, if $\sigma(x_0) \neq 0$, then $\mathcal{L}(X_t)$ is absolutely continuous with respect to the Lebesgue measure for all $t \in]0, T]$. More precisely, one shows that the Bouleau-Hirsch criterion holds: if $x_t = x_0 + \int_0^t b(x_s) ds$ and $t_x = \sup\{t \in [0, T] : x_t \notin \text{Int}J\}$ where $J = \sigma^{-1}(\{0\})$ then $\mathcal{L}(X_t)$ is absolutely continuous if and only if $t > t_x$.
- Among the m -order Newton-Côtes integrals, only the first (that is the symmetric integral, defined by (2.3)) is a "true" integral (indeed, for $m \geq 2$, one should rather write

$I_m(f, Y, X)$ instead of $\int_0^T f(Y) d^{\text{NC}, m} X$ - see Definition 2.4). For those which just prefer use symmetric integral to give a sense to (1.2), the main results contained in this paper are summarized in the following table (where f denotes a regular enough function and A a process having C^1 -trajectories):

If we use Definition	we have to choose $H \in$	X is then of the form	and we have existence if	and uniqueness if moreover	See Theorem
2.8	$(1/6, 1)$	$f(B^H)$	$\sigma \in C^5 \cap \text{Lip}$, $b \in C^0$ and $b _{\mathfrak{S}(\mathbb{R})} \equiv 0$	-	2.10
2.13	$(1/6, 1)$	$f(B^H + A)$	$\sigma \in C^5 \cap \text{Lip}$, $b \in C^0 +$ i) $\sigma(x_0) = 0$ $b(x_0) = 0$ or ii) $\sigma(x_0) \neq 0$	i) - ii) $\inf_{\mathbb{R}} \sigma > 0$ and $b \in \text{Lip}$	2.14
2.18	$(1/3, 1)$	$f(B^H, A)$	$\sigma \in C_b^2$ and $b \in \text{Lip}$	-	2.19 and [25]

Table 1. *Existence and uniqueness in SDE $X_t = x_0 + \int_0^t \sigma(X_s) d^\circ B_s^H + \int_0^t b(X_s) ds$*

Proof of Theorem 2.19. Let us remark that the classical Doss[11]-Sussman[31] method gives a natural solution X of the form $f(B^H, A)$. Then, in the remainder of the proof, we will concentrate on the uniqueness. Assume that $X = f(B^H, A)$ is a solution to (1.2) in the sense of Definition 2.18. On the one hand, we have

$$\begin{aligned} X_t &= x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s^H + \int_0^t b(X_s) ds \\ &= x_0 + \int_0^t \sigma \circ f(B_s^H, A_s) d^{\text{NC}} B_s^H + \int_0^t b \circ f(B_s^H, A_s) ds. \end{aligned} \quad (2.9)$$

On the other hand, using the change of variables formula, we can write

$$X_t = x_0 + \int_0^t f'_b(B_s^H, A_s) d^{\text{NC}} B_s^H + \int_0^t f'_a(B_s^H, A_s) A'_s ds. \quad (2.10)$$

Using (2.9) and (2.10), we deduce that $t \mapsto \int_0^t \varphi(B_s^H, A_s) d^{\text{NC}} B_s^H$ has C^1 -trajectories where $\varphi := f'_b - \sigma \circ f$. As in the proof of Theorem 2.14, we show that, a.s.,

$$\forall t \in]0, T[, \varphi(B_t^H, A_t) = 0. \quad (2.11)$$

Similarly, we can obtain that, a.s.,

$$\forall k \in \mathbb{N}, \forall t \in]0, T[, \frac{\partial^k \varphi}{\partial b^k}(B_t^H, A_t) = 0.$$

If σ and $f(\cdot, y)$ are analytic, then $\varphi(\cdot, y)$ is analytic and

$$\forall t \in]0, T[, \forall x \in \mathbb{R}, \varphi(x, A_t) = f'_b(x, A_t) - \sigma \circ f(x, A_t) = 0. \quad (2.12)$$

By uniqueness, we deduce

$$\forall t \in [0, T], \forall x \in \mathbb{R}, f(x, A_t) = u(x, A_t),$$

where u is the unique solution to $u'_b = \sigma(u)$ with initial value $u(0, y) = y$ for any $y \in \mathbb{R}$. In particular, we obtain a.s.

$$\forall t \in [0, T], X_t = f(B_t^H, A_t) = u(B_t^H, A_t). \quad (2.13)$$

Identity (2.9) can then be rewritten as:

$$X_t = x_0 + \int_0^t \sigma \circ u(B_s^H, A_s) d^{\text{NC}} B_s^H + \int_0^t b \circ u(B_s^H, A_s) ds,$$

while change of variables formula yields:

$$X_t = x_0 + \int_0^t u'_b(B_s^H, A_s) d^{\text{NC}} B_s^H + \int_0^t u'_a(B_s^H, A_s) A'_s ds.$$

Since $u'_b = \sigma \circ u$, we obtain a.s.:

$$\forall t \in [0, T], b \circ u(B_t^H, A_t) = u'_a(B_t^H, A_t) A'_t. \quad (2.14)$$

But we have existence and uniqueness in (2.14). Then the proof of Theorem is done. \square

3 Convergence of some approximation schemes associated to SDE (1.2)

Approximation of stochastic differential equations driven by a fBm is studied only in few articles. In [20], Lin shows that the Euler approximation of (1.2) - in the case where $\sigma(X_t)$ is replaced by $\sigma(t)$ - converges uniformly in probability. In [6, 24], some Wong-Zakai-type approximations for (1.2) are given. In [24], upper bounds for approximation schemes related to (1.2) are given for any $H \in (0, 1)$, when the integral with respect to fBm is defined using the Newton-Côtes integral. In [22], Neuenkirch derives, by means of the Malliavin calculus, the *exact* rate of convergence of the Euler scheme associated to (1.2) in the case where $\sigma(X_t)$ is replaced by $\sigma(t)$ and for $H > 1/2$.

Here, we are only interested in the convergence or not (according to the value of H) of the canonical implicit scheme associated to (1.2) with step $1/n$ - and classically called *Crank-Nicholson* scheme in the literature, when the integral with respect to fBm is in the sense of symmetric integral (2.3). More precisely, we consider

$$\begin{cases} \hat{X}_0^{(n)} = x_0 \\ \hat{X}_{(k+1)/n}^{(n)} = \hat{X}_{k/n}^{(n)} + \frac{1}{2} \left(\sigma(\hat{X}_{k/n}^{(n)}) + \sigma(\hat{X}_{(k+1)/n}^{(n)}) \right) (B_{(k+1)/n}^H - B_{k/n}^H) + \frac{1}{n} b(\hat{X}_{k/n}^{(n)}), \end{cases}$$

$k = 0, 1, \dots, n-1$ and the goal of this section is to try to answer the following question:

$$\begin{aligned} & \text{"One has the convergence of the sequence } \{\hat{X}_1^{(n)}\} \text{ for which values of } H \in (0, 1)? \\ & \text{In this case, which is the limit?"} \end{aligned} \quad (3.15)$$

Remark 3.1 • First, let us mention that, if $H > 1/3$, if $\sigma \in C_b^2$ and if $b \in \text{Lip}$, then it is possible, without too many difficulties, to prove that $\widehat{X}_1^{(n)} \rightarrow X$ a.s., where X is the solution given by Theorem 2.19 (it suffices, for instance, to use the same technics as in [24]). Thus, we can remark that we obtain in this case X for limit, although here we do not need to precise a particular form for $\widehat{X}^{(n)}$ as for X in Definition 2.18. It suggests then that the class \mathfrak{C}_3 is not completely arbitrary!

- When $H \leq 1/3$, to answer this question seems to be much more difficult. Indeed, it is closely related to the convergence or not of quantities of the type $u_n(f) := \sum_{k=0}^{n-1} f(\widehat{X}_{k/n}^{(n)}) (B_{(k+1)/n}^H - B_{k/n}^H)^3$ and $v_n(g) := \sum_{k=0}^{n-1} g(\widehat{X}_{k/n}^{(n)}) (B_{(k+1)/n}^H - B_{k/n}^H)^5$ (see the proof of Theorem 3.2 below) whereas, at the present time, we only know (see Lemma 3.4 below) that

$$u_n(1) \text{ (resp. } v_n(1)) \text{ converge if and only if } H > 1/6 \text{ (resp. } H > 1/10) \quad (3.16)$$

and, in this case, the limit is zero.

(I think that, in general, one has again $u_n(f) \rightarrow 0$ when $H > 1/6$ and $v_n(g) \rightarrow 0$ when $H > 1/10$ but it remains an open question). However, thanks to (3.16), one will be able to answer the question (3.15) for particular σ and when $b \equiv 0$:

Theorem 3.2 *Let a, b, c be three reals satisfying $a > 0$ and $b^2 - 4ac < 0$ and let $\varepsilon \in \{\pm 1\}$. Define $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ by $\sigma(x) = \varepsilon \sqrt{ax^2 + bx + c}$ and consider*

$$\begin{cases} \widehat{X}_0^{(n)} = x_0 \\ \widehat{X}_{(k+1)/n}^{(n)} = \widehat{X}_{k/n}^{(n)} + \frac{1}{2} \left(\sigma(\widehat{X}_{k/n}^{(n)}) + \sigma(\widehat{X}_{(k+1)/n}^{(n)}) \right) (B_{(k+1)/n}^H - B_{k/n}^H), \quad k = 0, 1, \dots, n-1. \end{cases} \quad (3.17)$$

Then the sequence $\{\widehat{X}_1^{(n)}\}$ converges in L^2 if and only if $H > 1/6$. In this case, the limit is the unique solution at time 1 of the SDE $X_t = x_0 + \int_0^t \sigma(X_s) d^\circ B_s^H$, in the sense of Definition 2.8 and given by Theorem 2.10.

Remark 3.3 When $\sigma(x) = x$ (which is roughly speaking the case where $(a, b, c) = (1, 0, 0)$ in the previous Theorem), we can easily show that, once again, $\{\widehat{X}_1^{(n)}\}$ converges in L^2 if and only if $H > 1/6$. Indeed, setting $\Delta_k^n = B_{(k+1)/n}^H - B_{k/n}^H$, we have

$$\widehat{X}_1^{(n)} = x_0 \prod_{k=0}^{n-1} \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n} = x_0 \exp \left\{ \sum_{k=0}^{n-1} \ln \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n} \right\};$$

but

$$\ln \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n} = \Delta_k^n + \frac{1}{12}(\Delta_k^n)^3 + \frac{1}{80}(\Delta_k^n)^5 + O((\Delta_k^n)^6),$$

and, since $\sum_{k=0}^{n-1} \Delta_k^n = B_1^H$ and using Lemma 3.4, one has that $\{\widehat{X}_1^{(n)}\}$ converges if and only if $H > 1/6$ and that, in this case, the limit is $x_0 \exp(B_1^H)$.

As a preliminary of the proof of Theorem 3.2, we need:

Lemma 3.4 *Let $m \geq 1$ be an integer.*

- We have

$$\sum_{k=0}^{n-1} (B_{(k+1)/n}^H - B_{k/n}^H)^{2m} \text{ converges in } L^2 \text{ as } n \rightarrow \infty \text{ if and only if } H \geq \frac{1}{2m}.$$

In this case, the limit is zero if $H > 1/2m$ and equals $(2m)!/(2^m m!)$ if $H = 1/2m$.

- We have

$$\sum_{k=0}^{n-1} (B_{(k+1)/n}^H - B_{k/n}^H)^{2m+1} \text{ converges in } L^2 \text{ as } n \rightarrow \infty \text{ if and only if } H > \frac{1}{4m+2}.$$

In this case, the limit is zero.

Idea of the proof of Lemma 3.4. It suffices to adapt, to this discret setting, the proof of [16], Theorem 4.1, point 3), page 793. \square

Proof of Theorem 3.2. Consider $\phi(x, t)$ the flow associated to σ , that is $\phi(x, \cdot)$ is the unique solution to $y' = \sigma(y)$ with initial value $y(0) = x$. Recall the classical flow property verified by ϕ :

$$\forall x, s, t \in \mathbb{R} : \phi(\phi(x, t), s) = \phi(x, t + s). \quad (3.18)$$

Since σ does not vanish, $\phi(x, \cdot)$ is a bijection from \mathbb{R} to himself for any x and we can consider $\varphi(x, \cdot)$ such that

$$\forall x, t \in \mathbb{R} : \varphi(x, \phi(x, t)) = t \text{ and } \phi(x, \varphi(x, t)) = t. \quad (3.19)$$

On the one hand, thanks to (3.19), it is a little long but easy to compute (personally, I used a Maple procedure to make all the computations contained in this proof)

$$\begin{aligned} \varphi(x, x) &= 0, \\ \varphi'_t(x, x) &= 1/\sigma(x), \\ \varphi''_{tt}(x, x) &= [-\sigma'/\sigma^2](x), \\ \varphi^{(3)}_{ttt}(x, x) &= [(2\sigma'^2 - \sigma\sigma'')/\sigma^3](x), \\ \varphi^{(4)}_{tttt}(x, x) &= [(-6\sigma'^3 + 6\sigma\sigma'\sigma'' - \sigma^2\sigma''')/\sigma^4](x) \\ \varphi^{(5)}_{ttttt}(x, x) &= [(24\sigma'^4 - 36\sigma\sigma'^2\sigma'' + 8\sigma^2\sigma'\sigma''' + 6\sigma^2\sigma''^2 - \sigma^3\sigma^{(4)})/\sigma^5](x). \end{aligned}$$

Then, for u sufficiently small, we have

$$\begin{aligned} \varphi(x, x + u) &= \frac{1}{\sigma}(x)u - \frac{\sigma'}{2\sigma^2}(x)u^2 + \frac{2\sigma'^2 - \sigma\sigma''}{6\sigma^3}(x)u^3 + \frac{-6\sigma'^3 + 6\sigma\sigma'\sigma'' - \sigma^2\sigma'''}{24\sigma^4}(x)u^4 \\ &\quad + \frac{24\sigma'^4 - 36\sigma\sigma'^2\sigma'' + 8\sigma^2\sigma'\sigma''' + 6\sigma^2\sigma''^2 - \sigma^3\sigma^{(4)}}{\sigma^5}(x)u^5 + O(u^6). \end{aligned}$$

We put $\Delta_k^n = B_{(k+1)/n}^H - B_{k/n}^H$. On the other hand, using (3.17) and some basic Taylor expansions, we can write, for $k \in \{0, 1, \dots, n-1\}$:

$$\begin{aligned} \widehat{X}_{(k+1)/n}^{(n)} &= \widehat{X}_{k/n}^{(n)} + \sigma(\widehat{X}_{k/n}^{(n)})\Delta_k^n + \frac{\sigma\sigma'}{2}(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^2 + \frac{\sigma\sigma'^2 + \sigma^2\sigma''}{4}(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^3 \\ &\quad + \left(\frac{\sigma\sigma'^3}{8} + \frac{3\sigma^2\sigma'\sigma''}{8} + \frac{\sigma^3\sigma'''}{12} \right) (\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^4 \\ &\quad + \left(\frac{\sigma\sigma'^4}{16} + \frac{3\sigma^2\sigma'^2\sigma''}{8} + \frac{\sigma^3\sigma'\sigma'''}{6} + \frac{\sigma^3\sigma''^2}{8} + \frac{\sigma^4\sigma^{(4)}}{48} \right) (\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^5 \\ &\quad + O((\Delta_k^n)^6). \end{aligned}$$

Then, we have

$$\begin{aligned}
\varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{(k+1)/n}^{(n)}) &= \varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{k/n}^{(n)} + [\widehat{X}_{(k+1)/n}^{(n)} - \widehat{X}_{k/n}^{(n)}]) \\
&= \Delta_k^n + \frac{\sigma'^2 + \sigma\sigma'}{12} (\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^3 + \left(\frac{\sigma\sigma'\sigma''}{8} + \frac{\sigma^2\sigma'''}{24} \right) (\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^4 \\
&\quad + \left(\frac{\sigma'^4}{80} + \frac{\sigma^2\sigma'\sigma'''}{15} + \frac{3\sigma\sigma'^2\sigma''}{40} + \frac{\sigma^2\sigma''^2}{20} + \frac{\sigma^3\sigma^{(4)}}{80} \right) (\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^5 + O((\Delta_k^n)^6).
\end{aligned} \tag{3.20}$$

But $\sigma(x) = \varepsilon\sqrt{ax^2 + bx + c}$ and we can simplify in (3.20) to obtain

$$\varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{(k+1)/n}^{(n)}) = \Delta_k^n + \frac{a}{12} (\Delta_k^n)^3 + \frac{a^2}{80} (\Delta_k^n)^5 + O((\Delta_k^n)^6).$$

Since $\widehat{X}_{(k+1)/n}^{(n)} = \phi(\widehat{X}_{k/n}^{(n)}, \varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{(k+1)/n}^{(n)}))$, we deduce from the flow property (3.18) that

$$\widehat{X}_1^{(n)} = \phi(x_0, B_1^H + \frac{a}{12} \sum_{k=0}^{n-1} (\Delta_k^n)^3 + \frac{a^2}{80} \sum_{k=0}^{n-1} (\Delta_k^n)^5 + O(\sum_{k=0}^{n-1} (\Delta_k^n)^6)).$$

Thus, the conclusion of Theorem 3.2 follows now easily as a conclusion of Lemma 3.4. \square

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