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► **To cite this version:**

| Djalil Chafai. Binomial-Poisson entropic inequalities and the $M/M/\infty$ queue. 2005. hal-00012429v1

HAL Id: hal-00012429

<https://hal.science/hal-00012429v1>

Preprint submitted on 23 Oct 2005 (v1), last revised 26 Mar 2006 (v2)

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Binomial-Poisson entropic inequalities and the $M/M/\infty$ queue

Djalil CHAFAÏ

October 2005

Abstract

This article provides entropic inequalities for binomial-Poisson distributions, derived from the two points space. They describe in particular the exponential dissipation of Φ -entropies along the $M/M/\infty$ queue. This simple queueing process appears as a model of “constant curvature”, and plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for Brownian Motion. These inequalities are exactly the local inequalities of the $M/M/\infty$ process. Some of them are recovered by semigroup interpolation. Additionally, we explore the behaviour of these entropic inequalities under a particular scaling, which sees the Ornstein-Uhlenbeck process as a fluid limit of $M/M/\infty$ queues. Proofs are elementary and rely essentially on the development of a “ Φ -calculus”.

1 Introduction

In this article, we consider the $M/M/\infty$ queueing process, which is actually a very special case of the model studied in [CP04]. We show that this elementary continuous time Markov process on \mathbb{N} plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for Brownian motion. In particular, its law at time t is explicitly given by a binomial-Poisson Mehler like formula, and the associated semigroup commutes with the discrete gradient operator, up to a time decreasing exponential factor. We derive general entropic inequalities for binomial-Poisson measures from the two points space, essentially by convexity. They hold in particular for the law at fixed time of the process, as for Ornstein-Uhlenbeck. In particular, these entropic inequalities contain as special cases Poincaré inequalities and various modified logarithmic Sobolev inequalities, which appear for instance in [BL98], [AL00], and [CP04].

It is known that the lack of a chain rule and of a good notion of curvature in discrete space settings make difficult the derivation of entropic inequalities for discrete space Markov processes. Poincaré inequalities are exceptional, due to their Hilbertian nature. Their derivation does not need the diffusion property. Lévy processes and Poisson space are also exceptional, since their i.i.d. underlying structure makes them “flat” in a way. This nature is translated on the infinitesimal Markov generator as a commutation with translations. The M/M/∞ queue has non-homogeneous independent increments, and is thus beyond this framework. The reader may find various entropic inequalities for finite space Markov processes in [DSC96, SC97] and [Ané01], and for infinite countable space Markov processes in [Mic99], [AL00], [Wu00], [Mao02], [Che03], [CP04], [Cha04], and [BCDPP05].

Birth and death processes are the discrete space analogue of diffusion processes. However, they are not diffusions, and specific diffusion tools like Bakry-Émery \mathbf{I}_2 calculus are of difficult usage for such processes. It follows from our study that the M/M/∞ queueing process can serve as a model of “constant curvature” on \mathbb{N} . It is known that convexity may serve as an alternative to the diffusion property, as presented for instance in [Wu00] and [Cha04]. In this article, we circumvent the lack of chain rule by elementary convexity bounds for germs of discrete Dirichlet forms. This work can be seen as a continuation of [Cha04], and was initially motivated by the time inhomogeneous M/M/∞ queue which appears in the biological problem studied in [CC04]. The notion of queueing process is widely used in applied probability. The reader may find a modern introduction to queueing processes in the book [Rob03] by Robert, in connection with random networks, general Markov processes, martingales, and point processes. This large family of Markov processes contains, as particular cases, the simple Poisson process, the continuous time simple random walk on \mathbb{N} , and more generally all continuous time birth and death processes on \mathbb{N} .

The approach and results of this article may be extended by various ways. The first step is to consider birth and death processes on \mathbb{N} or \mathbb{Z} , and then on \mathbb{N}^d or \mathbb{Z}^d with interactions. Some versions of such models were already considered in the statistical mechanics literature, at least for Poincaré and modified logarithmic Sobolev inequalities, see for instance [CP04, BCDPP05] and references therein. These extensions concern continuous time processes on a discrete space E with generator of the form

$$\mathbf{L}(f)(x) = \int_E (f(y) - f(x)) d\gamma_x(y),$$

where γ_x is the “jump measure” at point x , which is a finite Borel measure on E . Another possibility is to consider Volterra processes driven by a simple Poisson process, possibly together with Clark-Ocone formulas as in [Ané01] and [Wu00] for instance. We hope that some of these extensions will make the matter of

forthcoming articles. We have in mind the construction of a functional bridge between discrete space Markov processes and “curved” diffusion processes, which complements, by mean of quantitative functional inequalities, the approximation in law. The recent articles [Goe04], [BT03], [CP04, BCDPP05], and [JG05] may help for such a program.

Let us give an outline of the rest of the article. In the introduction, the definition of the $M/M/\infty$ queueing process is followed by the presentation of links and analogies with the Ornstein-Uhlenbeck process, and then by the introduction of the Φ -entropy together with the A, B, C transforms of Φ . Section one is a two points space approach to binomial-Poisson entropic inequalities. In Section two, we address the exponential decay of Φ -entropy functionals along the $M/M/\infty$ queue, we give various proofs of entropic inequalities by using semigroup interpolations, and we use a scaling limit to recover Gaussian inequalities for the Ornstein-Uhlenbeck process. The third and last section is devoted to key convexity properties related to the A, B, C transforms.

1.1 The $M/M/\infty$ queueing process

The $M/M/\infty$ queue with input rate $\lambda \geq 0$ and service rate $\mu \geq 0$ is a particular space-inhomogeneous and time-homogeneous birth and death process on \mathbb{N} . Let X_t be the number of customers in the queue – i.e. the length of the queue – at time t . The name “ $M/M/\infty$ ” comes from Kendall’s classification of simple queueing processes. It corresponds to an infinite number of servers with random memoryless inter-arrivals (first M) and service times (second M), see for instance [Rob03, p. xiii]. Since the number of servers is infinite, each client gets immediately his own dedicated server, and the length of the queue is exactly the number of busy servers. The infinitesimal Markov generator \mathbf{L} of the $M/M/\infty$ queue $(X_t)_{t \geq 0}$ is given for any $f : \mathbb{N} \rightarrow \mathbb{R}$ and any $n \in \mathbb{N}$ by

$$\mathbf{L}(f)(n) = n\mu D^*(f)(n) + \lambda D(f)(n), \quad (1)$$

where the discrete gradients D and D^* are defined respectively for any $f : \mathbb{Z} \rightarrow \mathbb{R}$ and any $n \in \mathbb{N}$ by

$$D(f)(n) := f(n+1) - f(n) \quad \text{and} \quad D^*(f)(n) := f(n-1) - f(n). \quad (2)$$

The operators D and D^* commute with translations, but \mathbf{L} does not. Notice that $f(-1)$ does not need to be defined in (1) since it is multiplied by 0. The stated notation for D^* comes from the fact that D^* is the adjoint of D with respect to the counting measure on \mathbb{Z} . The identity $DD^* = D^*D = -(D + D^*)$ leads to a polarised version of the infinitesimal generator (1),

$$\mathbf{L}(f)(n) = -\lambda(DD^*)(f)(n) + (n\mu - \lambda)D^*(f)(n),$$

for any $n \in \mathbb{N}$ and $f : \mathbb{N} \rightarrow \mathbb{R}$. The operator DD^* is the discrete Laplacian defined by $(\text{DD}^*)(f)(n) = 2f(n) - f(n-1) - f(n+1)$ for any $f : \mathbb{Z} \rightarrow \mathbb{R}$ and any $n \in \mathbb{Z}$.

Consider the process conditional to the event $\{X_s = n\}$. Define the waiting time before next jump $T := \min\{t > s : X_t \neq X_s\} - s$. Then T follows an exponential law $\mathcal{E}(\lambda + n\mu)$ of mean $1/(\lambda + n\mu)$. The transition matrix \mathbf{J} of the embedded discrete time jump Markov chain is given for any $m, n \in \mathbb{N}$ by

$$\mathbf{J}(n, m) := \frac{1}{\lambda + n\mu} \begin{cases} n\mu & \text{if } m = n - 1 \\ \lambda & \text{if } m = n + 1, \\ 0 & \text{otherwise} \end{cases}$$

where we assumed for simplicity that $\lambda + \mu > 0$. The embedded chain is recurrent irreducible as soon as $\lambda > 0$ and $\mu > 0$. The jump intensity function $n \mapsto \lambda + n\mu$ is not bounded when $\mu > 0$, however, the process is not explosive.

Defining a stochastic process corresponds to specify a law on paths space. Following [Rob03, Chap. 6], the stochastic process $(X_t)_{t \geq 0}$ with $X_0 = n$ can be constructed as follows :

$$X_t = n + \mathcal{N}_\lambda([0, t]) - \sum_{i=1}^{\infty} \int_{]0, t]} 1_{\{X_{s-} \geq i\}} \mathcal{N}_\mu^i(ds),$$

where \mathcal{N}_λ is a Poisson random measure on \mathbb{R}_+ of intensity λ and where $(\mathcal{N}_\mu^i)_{i \in \mathbb{N}}$ is an i.i.d. collection of Poisson random measures on \mathbb{R}_+ of intensity μ , independent of \mathcal{N}_λ . In other words, the process $(X_t)_{t \geq 0}$ solves the Stochastic Differential Equation

$$dX_t = \mathcal{N}_\lambda(dt) - \sum_{i=1}^{X_{t-}} \mathcal{N}_\mu^i(dt).$$

Let us consider the filtration $(\mathcal{F}_t)_{t \geq 0}$ defined for any $t \in \mathbb{R}_+$ by

$$\mathcal{F}_t := \sigma\{\mathcal{N}_\lambda([0, s]); s \in [0, t]\} \vee \sigma\{\mathcal{N}_\mu^i([0, s]); (s, i) \in [0, t] \times \mathbb{N}\}.$$

The process $(X_t - X_0 - \lambda t + \mu \int_0^t X_s ds)_{t \geq 0}$ is a square integrable martingale with increasing process given by $\lambda t + \mu \int_0^t X_s ds$. More generally, the process $(X_t)_{t \geq 0}$ is a solution of the martingale problem associated to the Markov generator \mathbf{L} defined by (1). Namely, for any $f : \mathbb{N} \rightarrow \mathbb{R}$, the process

$$\left(f(X_t) - f(X_0) - \int_0^t \mathbf{L}(f)(X_s) ds \right)_{t \geq 0}$$

is a local martingale. When $f(n) = n$ for any $n \in \mathbb{N}$, we get $\mathbf{L}(f)(n) = \lambda - \mu n$. The Markov semigroup $(\mathbf{P}_t)_{t \geq 0}$ of $(X_t)_{t \geq 0}$ is defined for any bounded $f : \mathbb{N} \rightarrow \mathbb{R}$ by

$$\mathbf{P}_t(f)(n) := \mathbf{E}(f(X_t) | X_0 = n), \quad (3)$$

in such a way that $\mathbf{P}_t(\mathbf{I}_A)(n) = \mathbb{P}(X_t \in A | X_0 = n)$ for any $A \subset \mathbb{N}$. One has $\mathbf{P}_t(\cdot)(n) = \mathcal{L}(X_t | X_0 = n)$ for any $n \in \mathbb{N}$. In particular, $\mathbf{P}_t \circ \mathbf{P}_s = \mathbf{P}_{t+s}$, and $\mathbf{P}_0 = \text{Id}$, and $\mathbf{L}f := \partial_{t=0} \mathbf{P}_t(f)$. The coefficient ρ of the M/M/ ∞ queue is defined by

$$\rho := \frac{\lambda}{\mu}.$$

In the sequel, we denote by $\mathbf{E}_Q(f)$ or by $\mathbf{E}_Q f$ the mean of function f with respect to the probability measure Q , and by $L^p(Q)$ the Lebesgue space of measurable real valued functions f such that $|f|^p$ is Q -integrable. For a Borel measure on \mathbb{N} , we also denote $Q(n) := Q(\{n\})$ for any $n \in \mathbb{N}$.

1.2 The Ornstein-Uhlenbeck as a fluid limit of M/M/ ∞

The Ornstein-Uhlenbeck process can be recovered from the M/M/ ∞ queue as a fluid limit, by using a Kelly scaling. See for instance [Kel86, Kel91] and the books [Rob03], [EK86], and [KL99]. Namely, for any $N \in \mathbb{N}$, let $(X_t^N)_{t \geq 0}$ be the M/M/ ∞ queue with input rate $N\lambda$ and service rate $\mu > 0$. Define the process $(Y_t^N)_{t \geq 0}$ by

$$Y_t^N := \frac{1}{N} X_t^N.$$

For any $x \in \mathbb{R}_+$, let $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ be defined by $m(t) := \rho + (x - \rho)p(t)$ for any $t \in \mathbb{R}_+$. Consider a sequence of initial states $(X_0^N)_{N \in \mathbb{N}}$ such that

$$\lim_{N \rightarrow \infty} Y_0^N = \lim_{N \rightarrow \infty} \frac{1}{N} X_0^N = x.$$

Then, for any $t \in \mathbb{R}_+$, the sequence of random variables

$$\left(\sup_{0 \leq s \leq t} |Y_s^N - m(s)| \right)_{N \in \mathbb{N}}$$

converges in L^1 towards 0 when $N \rightarrow \infty$, see for instance [Rob03, Section 6.5]. In particular, for any $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq s \leq t} |Y_s^N - m(s)| > \varepsilon \right) = 0.$$

Moreover, this Law of Large Numbers is complemented by a Central Limit Theorem, see for instance [Bor67] and [Rob03]. Namely, define the process $(Z_t^N)_{t \geq 0}$ by

$$Z_t^N := \sqrt{N}(Y_t^N - m(t)) = \frac{X_t^N - Nm(t)}{\sqrt{N}}.$$

Notice that $m(0) = x$. Let $y \in \mathbb{R}$ and assume that the sequence of initial states $(X_0^N)_{N \in \mathbb{N}}$ satisfies additionally that

$$\lim_{N \rightarrow \infty} Z_0^N = \lim_{N \rightarrow \infty} \sqrt{N}(Y_0^N - x) = y.$$

A basic example is given by $X_0^N = [Nx + \sqrt{N}y]$ where $[\cdot]$ denotes the integer part. Then, the sequence of processes $(Z_t^N)_{t \geq 0}$ converges in distribution, when $N \rightarrow \infty$, towards a process denoted $(Z_t^\infty)_{t \geq 0}$, with non-homogenous independent increments, given by

$$Z_t^\infty := yp(t) + \int_0^t p(t-s) \sqrt{2\lambda + \mu(x-\rho)} dB_s,$$

where $(B_s)_{s \geq 0}$ is a standard Brownian Motion on the real line.

In particular, when $x = \rho$, then $m(s) = \rho$ for any $s \in \mathbb{R}_+$, and $(U_t)_{t \geq 0}$ is in that case an Ornstein-Uhlenbeck process, solution of the Stochastic Differential Equation $Z_0^\infty = y$ and $dZ_t^\infty = \sqrt{2\lambda} dB_t - \mu Z_t^\infty dt$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion on the real line. Additionally, for any $t \in \mathbb{R}_+$,

$$\mathcal{L}(Z_t^\infty | Z_0^\infty = y) = \delta_{yp(t)} * \mathcal{N}(0, \rho(1-p(t)^2)),$$

where $\mathcal{N}(a, b)$ denotes the standard Gaussian law on \mathbb{R} of mean a and variance b . This Mehler formula is the continuous space analogue of (4). The Markov infinitesimal generator of $(Z_t^\infty)_{t \geq 0}$ is the linear differential operator which maps function $y \mapsto f(y)$ to function

$$y \mapsto \lambda f''(y) - \mu y f'(y).$$

The symmetric invariant measure of $(Z_t^\infty)_{t \geq 0}$ is the Gaussian law $\mathcal{N}(0, \rho)$. The μ parameter appears clearly here as a curvature, whereas the λ parameter appears as a diffusive coefficient.

1.3 The M/M/ ∞ as a discrete Ornstein-Uhlenbeck

Let $(X_t)_{t \geq 0}$ be an M/M/ ∞ with rates λ and μ . When μ vanishes, $(X_t)_{t \geq 0}$ reduces to a simple Poisson process of intensity λ , and admits the counting measure on \mathbb{N} as a symmetric measure. A contrario, when λ vanishes, $(X_t)_{t \geq 0}$ is a pure death process, and admits δ_0 as an invariant probability measure.

The M/M/ ∞ queue plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for standard Brownian Motion. The law of the M/M/ ∞ queue $(X_t)_{t \geq 0}$ is explicitly given for any $n \in \mathbb{N}$ by the following Mehler like formula

$$\mathcal{L}(X_t | X_0 = n) = \mathcal{B}(n, p(t)) * \mathcal{P}(\rho q(t)), \quad (4)$$

where

$$p(t) := e^{-\mu t} \quad \text{and} \quad q(t) := 1 - p(t).$$

When $\mu = 0$, we set $\rho q(t) = \lambda$, since $\lambda = \lim_{\mu \rightarrow 0^+} \rho q(t)$. Here and in the sequel, $\mathcal{B}(n, p)$ stands for the binomial distribution $\mathcal{B}(n, p) := (p\delta_1 + q\delta_0)^{*n}$ of size $n \in \mathbb{N}$ and parameter $p \in [0, 1]$, with the convention $\mathcal{B}(n, 0) := \delta_0$ and $\mathcal{B}(n, 1) := \delta_n$. The notation $\mathcal{P}(\sigma)$ stands for the Poisson measure on \mathbb{N} of intensity $\sigma > 0$, defined by $\mathcal{P}(\sigma) := e^{-\sigma} \sum_{k=0}^{\infty} \frac{1}{k!} \sigma^k \delta_k$. When $\mu > 0$, the process $(X_t)_{t \geq 0}$ is ergodic and admits $\mathcal{P}(\rho)$ as a reversible invariant measure. In other words, for any $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$,

$$\lim_{n \rightarrow \infty} \mathcal{L}(X_t | X_s = n) = \mathcal{P}(\rho).$$

Moreover, $\mathbf{E}_{\mathcal{P}(\rho)}(\mathbf{P}_t f) = \mathbf{E}_{\mathcal{P}(\rho)}(f)$ for any $f \in L^1(\mathcal{P}(\rho))$ and any $t \in \mathbb{R}_+$, or equivalently $\mathbf{E}_{\mathcal{P}(\rho)}(\mathbf{L}f) = 0$ for any $f \in L^1(\mathcal{P}(\rho))$. As for the Ornstein-Uhlenbeck process, this convergence is not uniform in n since for any $\alpha > 0$,

$$\lim_{n \rightarrow \infty} \mathcal{L}(X_{\mu^{-1} \log(n/\alpha)} | X_s = n) = \mathcal{P}(\alpha + \rho).$$

The mean and variance of $\mathcal{L}(X_t | X_s = n)$ with $t \geq s \geq 0$ are given respectively by

$$np(t-s) + \rho q(t-s) \quad \text{and} \quad (np(t-s) + \rho)q(t-s).$$

The semigroup $(\mathbf{P}_t)_{t \geq 0}$ of the M/M/ ∞ queue shares the nice ‘‘constant curvature’’ property with the Ornstein-Uhlenbeck semigroup. Namely, for any $t \in \mathbb{R}_+$, any $n \in \mathbb{N}$, and any bounded $f : \mathbb{N} \rightarrow \mathbb{R}$,

$$\mathbf{D}\mathbf{P}_t f = e^{-\mu t} \mathbf{P}_t \mathbf{D}f. \tag{5}$$

This is due to the fact that $[\mathbf{L}, \mathbf{D}] = \mu \mathbf{D}$. The commutation (5) can be deduced easily from (4). Namely, if X_1, \dots, X_{n+1}, Y are independent random variables with $X_i \sim \mathcal{B}(1, p(t))$ and $Y \sim \mathcal{P}(\rho q(t))$, we simply write

$$\begin{aligned} \mathbf{P}_t(f)(n+1) &= \mathbf{E}(f(X_1 + \dots + X_{n+1} + Y)) \\ &= \mathbf{E}(\mathbf{E}(f(X_1 + \dots + X_{n+1} + Y) | X_{n+1})) \\ &= p(t) \mathbf{P}_t(f(1 + \cdot))(n) + q(t) \mathbf{P}_t(f)(n) \\ &= p(t) \mathbf{P}_t(\mathbf{D}f)(n) + \mathbf{P}_t(f)(n). \end{aligned}$$

This fact and the properties of the A, B, C transforms introduced in the sequel give rise to various entropic inequalities, by using the semigroup $(\mathbf{P}_t)_{t \geq 0}$ as an interpolation flow.

Let $H_{n,p}(m) := \binom{n}{m} p^m q^{n-m}$ for any $p \in [0, 1]$ and any integers n and m with $0 \leq m \leq n$. One has then $m H_{n,p}(m) = np H_{n-1,p}(m-1)$ as soon as $0 < m \leq n$. As a consequence, for any function $f : \mathbb{N} \rightarrow \mathbb{R}$, any $n \in \mathbb{N}^*$ and any $p \in [0, 1]$

$$\mathbf{E}_{\mathcal{B}(n,p)}(hf) = np \mathbf{E}_{\mathcal{B}(n-1,p)}(f(1 + \cdot)), \tag{6}$$

where $h : \mathbb{N} \rightarrow \mathbb{R}$ is defined by $h(k) = k$ for any $k \in \mathbb{N}$. Similarly, $(n-m)H_{n,p}(m) = nqH_{n-1,p}(m)$ as soon as $0 \leq m < n$, which gives for any function $f : \mathbb{N} \rightarrow \mathbb{R}$, any $n \in \mathbb{N}^*$ and any $p \in [0, 1]$

$$\mathbf{E}_{\mathcal{B}(n,p)}((n-h)f) = nq\mathbf{E}_{\mathcal{B}(n-1,p)}(f). \quad (7)$$

For any $\rho > 0$, the binomial approximation of Poisson measure which lets np tend to ρ when n goes to ∞ gives from (6)

$$\mathbf{E}_{\mathcal{P}(\rho)}(hf) = \rho\mathbf{E}_{\mathcal{P}(\rho)}(f(1+\cdot)). \quad (8)$$

Some algebra provides by conditioning a mixed binomial-Poisson version

$$\mathbf{E}_{\mathcal{B}(n,p)*\mathcal{P}(\rho)}(hf) = np\mathbf{E}_{\mathcal{B}(n-1,p)*\mathcal{P}(\rho)}(f(1+\cdot)) + \rho\mathbf{E}_{\mathcal{B}(n,p)*\mathcal{P}(\rho)}(f(1+\cdot)). \quad (9)$$

In particular, the Mehler like formula (4) gives for any $n \in \mathbb{N}^*$ and $t \in \mathbb{R}_+$,

$$\mu\mathbf{P}_t(hf)(n) = \mu np(t)\mathbf{P}_t(f(1+\cdot))(n-1) + \lambda q(t)\mathbf{P}_t(f(1+\cdot))(n), \quad (10)$$

where $h : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $h(n) := n$ for any $n \in \mathbb{N}$.

1.4 Convex functionals

For any convex domain \mathcal{D} of \mathbb{R}^n , let us denote by $\mathcal{C}_{\mathcal{D}}$ the convex set of smooth convex functions from \mathcal{D} to \mathbb{R} . Let $\mathcal{I} \subset \mathbb{R}$ be an interval of \mathbb{R} and $\Phi \in \mathcal{C}_{\mathcal{I}}$. We denote by $L^{1,\Phi}(Q)$ the convex subset of functions $f \in L^1(Q)$ such that $\Phi(f) \in L^1(Q)$. We define the Φ -entropy $\mathbf{Ent}_Q^\Phi(f)$ of $f \in L^{1,\Phi}(Q)$ by

$$\mathbf{Ent}_Q^\Phi(f) := \mathbf{E}_Q(\Phi(f)) - \Phi(\mathbf{E}_Q f). \quad (11)$$

It is also known as ‘‘Jensen divergence’’ since Jensen inequality gives $\mathbf{Ent}_Q^\Phi(f) \geq 0$. Moreover, $\mathbf{Ent}_Q^\Phi(f) = 0$ if and only if $\Phi(f)$ is Q -a.s. constant. One can distinguish for function Φ the following three usual special cases.

(P1) $\Phi(u) = u \log(u)$ on $\mathcal{I} = \mathbb{R}_+$, and $\mathbf{Ent}_Q^\Phi(f) = \mathbf{Ent}_Q(f) := \mathbf{E}_Q(f \log f / \mathbf{E}_Q f)$;

(P2) $\Phi(u) = u^2$ on $\mathcal{I} = \mathbb{R}$, and $\mathbf{Ent}_Q^\Phi(f) = \mathbf{Var}_Q(f) := \mathbf{E}_Q((f - \mathbf{E}_Q f)^2)$;

(P3) $\Phi(u) = u^\alpha$ on $\mathcal{I} = \mathbb{R}_+$ with $\alpha \in (1, 2)$.

The \mathbf{Ent}_Q^Φ functional is linear in Φ and vanishes when Φ is affine. Let us define from the interval $\mathcal{I} \subset \mathbb{R}$ the convex subset $\mathcal{T}_{\mathcal{I}}$ of \mathbb{R}^2 by

$$\mathcal{T}_{\mathcal{I}} := \{(u, v) \in \mathbb{R}^2, (u, u+v) \in \mathcal{I} \times \mathcal{I}\}. \quad (12)$$

The A, B, C transforms of Φ are the functions $A^\Phi, B^\Phi, C^\Phi : \mathcal{T}_I \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} A^\Phi(u, v) &:= \Phi(u + v) - \Phi(u) - \Phi'(u)v; \\ B^\Phi(u, v) &:= v(\Phi'(u + v) - \Phi'(u)); \\ C^\Phi(u, v) &:= \Phi''(u)v^2. \end{aligned}$$

These three transforms are linear in Φ , and their kernel contains any affine function. Various additional properties of these three transforms are collected in Section 4. In particular, the convexity of Φ on \mathcal{I} is equivalent to the non negativity of its A, B, C transforms on \mathcal{T}_I . For **(P1-P2-P3)**,

- A^Φ, B^Φ, C^Φ are non negative and convex on \mathcal{T}_I ;
- $\mathbf{Ent}_Q^\Phi(\cdot)$ is non negative and convex on $L^{1,\Phi}(Q)$;
- $A^\Phi \leq B^\Phi$;
- $2A^\Phi = B^\Phi = C^\Phi$ for **(P2)** and $A^\Phi \leq C^\Phi$ for **(P1)**.

Table 1 gives explicit examples. On the two point space $\{0, 1\}$, the Φ -entropy gives rise naturally to A^Φ . Namely, for any $f : \{0, 1\} \rightarrow \mathbb{R}$ with $(a, b) := (f(0), f(1))$ and $(u, v) := (a, b - a)$,

$$\begin{aligned} \mathbf{Ent}_{B(1,p)}^\Phi(f) &= q\Phi(a) + p\Phi(b) - \Phi(qa + pb) \\ &= pA^\Phi(u, v) - A^\Phi(u, pv). \end{aligned} \tag{13}$$

The A, B, C transforms are the germs of discrete Dirichlet forms via the identities

$$\begin{aligned} A^\Phi(f, Df) &= D\Phi(f) - \Phi'(f)Df; \\ B^\Phi(f, Df) &= D(f)D(\Phi'(f)); \\ C^\Phi(f, Df) &= \Phi''(f)|Df|^2. \end{aligned}$$

The reader may find explicit examples in table 1. We used above the following identity valid for any $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and any $f : \mathbb{Z} \rightarrow \mathbb{R}$,

$$D(\varphi(f)) = \varphi(f(1 + \cdot)) - \varphi(f) = \varphi(f + Df) - \varphi(f),$$

where $f(1 + \cdot)$ stands for $\mathbb{Z} \ni n \mapsto f(1 + n)$. In particular, $f(1 + \cdot) = f + Df$. The usage of the A, B, C transforms allow, as presented in the sequel, to derive several entropic inequalities in the same time, including Poincaré inequalities and various modified logarithmic Sobolev inequalities. They reduce most of the proofs to convexity, and they provide various comparisons for discrete Dirichlet forms.

Function Φ	\mathcal{I}	A^Φ	$A^\Phi(f, Df)$
(P1) $u \log(u)$	\mathbb{R}_+	$(u+v)(\log(u+v) - \log(u)) - v$	$(f + Df)D(\log f) - Df$
(P2) u^2	\mathbb{R}	v^2	$ Df ^2$
(P3) u^α	\mathbb{R}_+	$(u+v)^\alpha - u^\alpha - \alpha u^{\alpha-1}v$	$D(f^\alpha) - \alpha f^{\alpha-1}Df$
		B^Φ	$B^\Phi(f, Df)$
(P1) $u \log(u)$	\mathbb{R}_+	$v(\log(u+v) - \log(u))$	$D(f)D(\log f)$
(P2) u^2	\mathbb{R}	$2v^2$	$2 Df ^2$
(P3) u^α	\mathbb{R}_+	$\alpha v((u+v)^{\alpha-1} - u^{\alpha-1})$	$\alpha D(f)D(f^{\alpha-1})$
		C^Φ	$C^\Phi(f, Df)$
(P1) $u \log(u)$	\mathbb{R}_+	$v^2 u^{-1}$	$ Df ^2 f^{-1}$
(P2) u^2	\mathbb{R}	$2v^2$	$2 Df ^2$
(P3) u^α	\mathbb{R}_+	$\alpha(\alpha-1)v^2 u^{\alpha-2}$	$\alpha(\alpha-1) Df ^2 f^{\alpha-2}$

Table 1: Examples of A , B , C transforms. For **(P3)**, $\alpha \in (1, 2)$.

2 From two points space to binomial-Poisson

Let $p \in [0, 1]$ and let $\mathcal{B}(1, p)$ be the Bernoulli measure $q\delta_0 + p\delta_1$ on $\{0, 1\}$, where $q := 1 - p$. We identify the two points space $\{0, 1\}$ with $\mathbb{Z}/2\mathbb{Z}$, for which $1 + 1 = 0$. In particular, the “+” sign in the definition (2) of D is taken modulo 2. Then, for any $f : \{0, 1\} \rightarrow \mathcal{I}$, the following identity holds.

$$pq\mathbf{E}_{\mathcal{B}(1,p)}(B^\Phi(f, Df)) - \mathbf{Ent}_{\mathcal{B}(1,p)}^\Phi(f) = A^\Phi(\sigma_p(a, b - a)) + A^\Phi(\sigma_q(b, a - b)),$$

where $(a, b) := (f(0), f(1))$ and where σ_p is as in Lemma 4.3. Now, Lemma (4.1) gives $A^\Phi \geq 0$ as soon as Φ is convex. Thus, when Φ is convex,

$$\mathbf{Ent}_{\mathcal{B}(1,p)}^\Phi(f) \leq pq\mathbf{E}_{\mathcal{B}(1,p)}(B^\Phi(f, Df)). \quad (14)$$

Unfortunately, the inequality (14) is not optimal for **(P2)** since in that case,

$$\mathbf{Ent}_{\mathcal{B}(1,p)}^\Phi(f) = pq(f(1) - f(0))^2 \quad \text{whereas} \quad \mathbf{E}_{\mathcal{B}(1,p)}(B^\Phi(f, Df)) = 2(f(1) - f(0))^2.$$

This is due to the fact that $B^\Phi(f, Df) = 2|Df|^2$ for **(P2)**. We derive in the sequel the A transform version, which is stronger and optimal for **(P2)** since $A^\Phi(f, Df) = |Df|^2$ in that case. All the inequalities obtained in this section involve the A transform in their right hand side. They hold for example in the cases **(P1)**, **(P2)**, **(P3)**. The A transform can be bounded by the B or the C

transforms, by using the elementary bounds given by Lemma 4.2. We start with an entropic inequality for the Bernoulli law $\mathcal{B}(1, p)$. By convolution, we derive from this two points space inequality a new entropic inequality for the binomial law $\mathcal{B}(n, p) = \mathcal{B}(1, p)^{*n}$. An inequality for the Poisson law $\mathcal{P}(\rho)$ is then obtained by binomial approximation. The binomial-Poisson case is derived by tensorisation.

Lemma 2.1 (Two points Lemma). *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $\Phi'' \in \mathcal{C}_{\mathcal{I}}$. Consider the function $U : [0, 1] \rightarrow \mathbb{R}$ defined by*

$$U(p) := \mathbf{Ent}_{\mathcal{B}(1,p)}^{\Phi}(f) - pq\mathbf{E}_{\mathcal{B}(1,p)}(g),$$

where $f, g : \{0, 1\} \rightarrow \mathcal{I}$. Then, $U \leq 0$ on $[0, 1]$ if and only if

$$U'(0) \leq 0 \leq U'(1). \quad (15)$$

Proof. This calculus Lemma is a Φ version of [BL98, Lemma 2] by Bobkov and Ledoux. We denote $(a, b) := (f(0), f(1))$ and $(\alpha, \beta) := (g(0), g(1))$. We get then

$$U(p) = q\Phi(a) + p\Phi(b) - \Phi(qa + pb) - pq(q\alpha + p\beta). \quad (16)$$

The last term is a polynomial in p of degree three. Taking the fourth derivative in p gives

$$U''''(p) = -(b - a)^4\Phi''''(qa + pb).$$

Since Φ'' is convex, one has $U'''' \leq 0$ on $(0, 1)$ and thus U'' is concave. Consequently, there exists $0 \leq p_0 \leq p_1 \leq 1$ such that $U'' \leq 0$ on $[0, p_0] \cup [p_1, 1]$ and $U'' \geq 0$ on $[p_0, p_1]$. One has that U is concave on $[0, p_0]$. But $U(0) = 0$ and by assumption $U'(0) \leq 0$, thus $U \leq 0$ on $[0, p_0]$ by concavity. A consequence is that $U(p_0) \leq 0$. Similarly by symmetry one has that $U \leq 0$ on $[p_1, 1]$ and $U(p_1) \leq 0$. Now since U is convex on $[p_0, p_1]$ and non-positive on the boundaries, it is non-positive on the whole interval $[p_0, p_1]$. Therefore (15) implies $U \leq 0$ on $[0, 1]$. \square

One can show by similar arguments that if additionally $f(0) \geq f(1)$ and $g(0) \geq g(1)$ (respectively $f(0) \leq f(1)$ and $g(0) \leq g(1)$), then the condition (15) may be weakened into $U'(0) \leq 0$ (respectively $U'(1) \geq 0$). Notice that in terms of A transform,

$$U'(0) = A^{\Phi}(a, b - a) - \alpha \quad \text{and} \quad U'(1) = A^{\Phi}(a, b - a) + \beta. \quad (17)$$

Lemma 2.2 (Two points entropic inequalities). *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $\Phi'' \in \mathcal{C}_{\mathcal{I}}$. Then, for any $f : \{0, 1\} \rightarrow \mathcal{I}$,*

$$\mathbf{Ent}_{\mathcal{B}(1,p)}^{\Phi}(f) \leq pq\mathbf{E}_{\mathcal{B}(1,p)}(A^{\Phi}(f, Df)), \quad (18)$$

where the “+” in (2) of D is taken modulo 2. Moreover, the inequality becomes an equality for **(P2)**.

Proof. Let U be as in (16) with $g = A^\Phi(f, Df)$. From (17) we get

$$U'(0) = A^\Phi(a, b-a) - A^\Phi(a, b-a) = 0 \quad \text{and} \quad U'(1) = A^\Phi(a, b-a) + A^\Phi(b, a-b),$$

where $(a, b) := (f(0), f(1))$. By Lemma 4.1, function A^Φ is non negative and thus $U'(1) \geq 0$. Therefore (18) follows by virtue of Lemma 2.1. Notice that since $+1 = -1$ in $\mathbb{Z}/2\mathbb{Z}$, one has $D = D^*$, and function $B^\Phi(f, Df)$ is constant. \square

Lemma 4.5 gives another expression of (18). Entropic inequalities like (18) belong to the so called family of Φ -Sobolev inequalities, which are known to be stable by convolution, cf. [Cha04, Corollary 3.1 page 342]. This observation leads to Theorem 2.4 below. We need the following key lemma.

Lemma 2.3 (Variational formula and tensorisation of Φ -entropy). *Let μ be a probability measure on a measurable space E and let $\Phi \in \mathcal{C}_{\mathcal{I}}$ be such that $C^\Phi \in \mathcal{C}_{\mathcal{I}^*}$. Then \mathbf{Ent}_μ^Φ is convex on $L^{1,\Phi}(\mu)$ and for any $f \in L^{1,\Phi}(\mu)$,*

$$\mathbf{Ent}_\mu^\Phi(f) = \sup_{g \in L^{1,\Phi}(\mu)} \{ \mathbf{E}_\mu((\Phi'(g) - \Phi'(\mathbf{E}_\mu g))(f - g)) + \mathbf{Ent}_\mu^\Phi(g) \}. \quad (19)$$

The supremum is achieved for $g = f$. Assume that $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ is a product probability measure on a product measurable space $E_1 \times \cdots \times E_n$. Then, for any $f \in L^{1,\Phi}(\mu)$,

$$\mathbf{Ent}_\mu^\Phi(f) \leq \mathbf{E}_\mu(\mathbf{Ent}_{\mu_1}^\Phi(f)) + \cdots + \mathbf{E}_\mu(\mathbf{Ent}_{\mu_n}^\Phi(f)), \quad (20)$$

where the expectation with respect to μ_i in $\mathbf{Ent}_{\mu_i}^\Phi(g)$ concerns only the i^{th} coordinate.

Proof. This Lemma is taken from [Cha04, Prop. 1.2 page 331] and [Cha04, Prop. 3.1 page 341], which was inspired essentially by [LO00]. See also [BBLM05]. Let us show here how the result can be quickly deduced from the properties of the A and C transforms. Let f and g be two bounded functions taking their values in \mathcal{I} . When A^Φ is convex, Jensen inequality gives that the quantity

$$J(f, g) := \mathbf{E}_\mu(A^\Phi(g, f - g)) - A^\Phi(\mathbf{E}_\mu g, \mathbf{E}_\mu(f - g))$$

is non negative, and vanishes when $f = g$. Now, it turns out that

$$\mathbf{Ent}_\mu^\Phi(f) = J(f, g) + \mathbf{E}_\mu((\Phi'(g) - \Phi'(\mathbf{E}_\mu g))(f - g)) + \mathbf{Ent}_\mu^\Phi(g).$$

This gives (19). Let $\alpha : [0, 1] \rightarrow \mathbb{R}$ be the continuous function defined for any $t \in [0, 1]$ by $\alpha(t) := \mathbf{Ent}_\mu^\Phi(h_t)$ where $h_t := tf + (1 - t)g$. The convexity of C^Φ yields

$$\alpha''(t) = \mathbf{E}_\mu(C^\Phi(h_t, f - g)) - C^\Phi(\mathbf{E}_\mu h_t, \mathbf{E}_\mu(f - g)) \geq 0.$$

Therefore, α is convex and $\alpha(\lambda) \leq \lambda\alpha(1) + (1 - \lambda)\alpha(0)$ for $\lambda \in [0, 1]$ writes

$$\mathbf{Ent}_\mu^\Phi(\lambda f + (1 - \lambda)g) \leq \lambda \mathbf{Ent}_\mu^\Phi(f) + (1 - \lambda) \mathbf{Ent}_\mu^\Phi(g),$$

which is nothing else but the convexity of $f \in L^{1,\Phi}(\mu) \mapsto \mathbf{Ent}_\mu^\Phi(f)$. In the other hand, since every convex function is the envelope of its tangents, one gets

$$\mathbf{Ent}_\mu^\Phi(f) = \alpha(1) = \sup_{t \in [0,1]} \{\alpha(t) + \alpha'(t)(1 - t)\},$$

which gives (19). The tensorisation (20) follows easily from (19). Namely, by virtue of (19) for the triple (μ, f, g) , it is sufficient to show that for any $g \in L^{1,\Phi}(\mu)$,

$$\mathbf{Ent}_\mu^\Phi(g) + \mathbf{E}_\mu((\Phi'(g) - \Phi'(\mathbf{E}_\mu g))(f - g)) \leq \sum_{i=1}^n \mathbf{E}_\mu(\mathbf{Ent}_{\mu_i}^\Phi(f)). \quad (21)$$

We proceed by bounding below each term of the right-hand side of (21). Let $g_0 := g$ and $g_i := \mathbf{E}_{\mu_1 \otimes \dots \otimes \mu_i}(g)$ for any $i \in \{1, \dots, n\}$. Now, the variational formula (19) for the triple (μ_i, f, g_{i-1}) and the fact $\mathbf{E}_{\mu_i} g_{i-1} = g_i$ yield

$$\mathbf{Ent}_{\mu_i}^\Phi(f) \geq \mathbf{E}_{\mu_i}((\Phi'(g_{i-1}) - \Phi'(g_i))(f - g_{i-1})) + \mathbf{Ent}_{\mu_i}^\Phi(g_{i-1}).$$

The integration of both sides for μ and the usage of the identity $\mathbf{E}_\mu \mathbf{E}_{\mu_i} = \mathbf{E}_\mu$ gives

$$\mathbf{E}_\mu(\mathbf{Ent}_{\mu_i}^\Phi(f)) \geq \mathbf{E}_\mu((\Phi'(g_{i-1}) - \Phi'(g_i))(f - g_{i-1})) + \mathbf{E}_\mu(\mathbf{Ent}_{\mu_i}^\Phi(g_{i-1})).$$

We have by using the identities $\mathbf{E}_\mu \mathbf{E}_{\mu_i} = \mathbf{E}_\mu$ and $g_n = \mathbf{E}_\mu g$ that,

$$\sum_{i=1}^n \mathbf{E}_\mu(\mathbf{Ent}_{\mu_i}^\Phi(g_{i-1})) = \sum_{i=1}^n [\mathbf{E}_\mu(\Phi(g_{i-1})) - \mathbf{E}_\mu(\Phi(g_i))] = \mathbf{Ent}_\mu^\Phi(g).$$

By the same way, we obtain via a telescopic simplification,

$$\sum_{i=1}^n \mathbf{E}_\mu((\Phi'(g_{i-1}) - \Phi'(g_i))f) = \mathbf{E}_\mu((\Phi'(g) - \Phi'(\mathbf{E}_\mu g))f).$$

By using the fact that g_i and g_{i-1} do not depend on the coordinates with index less than of equal to $i - 1$, we get the last required identity

$$-\mathbf{E}_\mu((\Phi'(g_{i-1}) - \Phi'(g_i))g_{i-1}) = -\mathbf{E}_\mu((\Phi'(g_{i-1}) - \Phi'(g_i))g).$$

□

Theorem 2.4 (Bernoulli entropic inequalities). *Let $Q := \mathcal{B}(1, p_1) * \cdots * \mathcal{B}(1, p_n)$ and $C_Q := \max\{p_1 q_1, \dots, p_n q_n\}$ where $p_1, \dots, p_n \in [0, 1]$. Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $C^\Phi \in \mathcal{C}_{\mathcal{I}}$. Then, for any $f : \mathbb{N} \rightarrow \mathcal{I}$,*

$$\mathbf{Ent}_Q^\Phi(f) \leq C_Q \mathbf{E}_Q((n-h)A^\Phi(f, Df) + hA^\Phi(f, D^*f)), \quad (22)$$

where $h : \mathbb{N} \rightarrow \mathbb{R}$ is defined by $h(k) = k$ for any $k \in \mathbb{N}$. In particular, for any $n \in \mathbb{N}^*$, any $p \in [0, 1]$, and any $f : \mathbb{N} \rightarrow \mathbb{R}$,

$$\mathbf{Ent}_{\mathcal{B}(n,p)}^\Phi(f) \leq pq \mathbf{E}_{\mathcal{B}(n,p)}((n-h)A^\Phi(f, Df) + hA^\Phi(f, D^*f)). \quad (23)$$

Moreover, if τ is as in Lemma 4.1,

$$\mathbf{Ent}_{\mathcal{B}(n,p)}^\Phi(f) \leq npq \mathbf{E}_{\mathcal{B}(n-1,p)}(qA^\Phi(f, Df) + pA^\Phi(\tau(f, Df))). \quad (24)$$

The optimality of these inequalities in the case **(P2)** can be checked for linear f , say $f(k) = k$ for any $k \in \{0, \dots, n\}$.

Proof. First of all, by virtue of Lemma 4.2, the convexity of C^Φ on \mathcal{I} implies the convexity of Φ'' on \mathcal{I} . Let $(E_i, \mu_i) = (\{0, 1\}, \mathcal{B}(1, p_i))$ for any $i \in \{1, \dots, n\}$. Let $f : \mathbb{N} \rightarrow \mathcal{I}$ and consider the symmetric function $g : E_1 \times \cdots \times E_n \rightarrow \mathcal{I}$ defined by $g(x_1, \dots, x_n) := f(x_1 + \cdots + x_n)$. The tensorisation formula (20) of Lemma 2.3 together with the two points entropic inequality (18) of Lemma 2.2 gives

$$\mathbf{Ent}_{\mu_1 \otimes \cdots \otimes \mu_n}^\Phi(g) \leq C_Q \mathbf{E}_{\mu_1 \otimes \cdots \otimes \mu_n} \left(\sum_{i=1}^n A^\Phi(g, D^{(i)}g) \right),$$

where $D^{(i)}$ denotes the operator D acting on the i^{th} coordinate with modulo 2 as in Lemma 2.2. At this step, we notice by denoting $s_n := x_1 + \cdots + x_n$ that

$$\sum_{i=1}^n A^\Phi(g, D^{(i)}g)(x) = (n - s_n)A^\Phi(f, Df)(s_n) + s_n A^\Phi(f, D^*f)(s_n).$$

The values of f taken outside $\{0, \dots, n\}$ come with a null coefficient in the right hand side. The desired result follows since Q is the law of s_n under $\mu_1 \otimes \cdots \otimes \mu_n$. Inequality (22) reduces to (23) when $p_1 = \cdots = p_n = p$. It remains to establish (24). By virtue of (6) and (7), the right hand side of (23) equals

$$n \mathbf{E}_{\mathcal{B}(n-1,p)}(qA^\Phi(f, Df) + pA^\Phi(f, D^*f)(1 + \cdot)).$$

Inequality (24) follows then from the simple identity

$$(f, D^*f)(1 + \cdot) = \tau(f, Df). \quad (25)$$

When $n = 1$, then $Q = \mathcal{B}(1, p)$ and (22) reduces to (18), thanks to (6) and (7). \square

Corollary 2.5 (Poisson entropic inequality). *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $C^{\Phi} \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$. Then, for any $\rho \in \mathbb{R}_+$ and any f in $L^{1,\Phi}(\mathcal{P}(\rho))$,*

$$\mathbf{Ent}_{\mathcal{P}(\rho)}^{\Phi}(f) \leq \rho \mathbf{E}_{\mathcal{P}(\rho)}(A^{\Phi}(f, Df)). \quad (26)$$

Proof. Consider (24). Let p depend on n is such a way that $\lim_{n \rightarrow \infty} np_n = \rho$. Since $\lim_{n \rightarrow \infty} p_n = 0$, one has $q_n \rightarrow 1$. Moreover, $\mathcal{B}(n, p_n) \rightarrow \mathcal{P}(\rho)$ and $\mathcal{B}(n-1, p_n) \rightarrow \mathcal{P}(\rho)$. \square

To the author's knowledge, inequality (26) appeared for the first time in [Wu00] for **(P1)** and **(P2)**, and in [Cha04] for the general case. See also [CP04] and [BCDPP05]. The B and C transforms versions of (26), which are weaker, appeared in particular in [AL00] and [BL98].

Corollary 2.6 (Binomial-Poisson entropic inequality). *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $C^{\Phi} \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$. Let $Q_n = \mathcal{B}(n, p) * \mathcal{P}(\rho)$ where $p \in [0, 1]$, $\rho \in \mathbb{R}_+$, and $n \in \mathbb{N}$. Then, for any $f \in L^{1,\Phi}(\mathcal{P}(\rho))$, we have*

$$\mathbf{Ent}_{Q_n}^{\Phi}(f) \leq \rho \mathbf{E}_{Q_n}(A^{\Phi}(f, Df)) + npq \mathbf{E}_{Q_{n-1}}(qA^{\Phi}(f, Df) + pA^{\Phi}(\tau(f, Df))). \quad (27)$$

Proof. If $n = 0$, then (27) reduces to (26). Assume that $n > 0$. Let

$$(E_1, \mu_1) = (\mathbb{N}, \mathcal{B}(n, p)) \quad \text{and} \quad (E_2, \mu_2) = (\mathbb{N}, \mathcal{P}(\rho)).$$

Let $f : \mathbb{N} \rightarrow \mathcal{I}$ be some bounded function, and let $g : E_1 \times E_2 \rightarrow \mathcal{I}$ be defined by $g(x_1, x_2) = f(x_1 + x_2)$. Let us denote by $D^{(1)}$ and $D^{(2)}$ the D operator which acts on x_1 and x_2 respectively. Inequalities (20), (24), (26) yields that $\mathbf{Ent}_{\mu_1 \otimes \mu_2}^{\Phi}(g)$ is bounded above by

$$npq \mathbf{E}_{\mu_2}(\mathbf{E}_{\mu_0}(qA^{\Phi}(g, D^{(1)}g) + pA^{\Phi}(\tau(g, D^{(2)}g)))) + \rho \mathbf{E}_{\mu_1}(\mathbf{E}_{\mu_2}(A^{\Phi}(g, D^{(2)}g))),$$

where $\mu_0 := \mathcal{B}(n-1, p)$. Since D commutes with translations, we get

$$(g, D^{(i)}g)(x_1, x_2) = (f, Df)(x_1 + x_2)$$

for $i = 1, 2$. Inequality (27) follows since Q_n , respectively Q_{n-1} , is the law of $x_1 + x_2$ under $\mu_1 \otimes \mu_2$, respectively $\mu_0 \otimes \mu_2$. \square

The expectation with respect to Q_{n-1} in the right hand side of (27) may be rewritten as an expectation with respect to Q_n by using (9).

3 Entropies along the M/M/∞ queue

We start with the decay of the Φ -entropy functional along the queue.

Theorem 3.1 (Φ -entropies dissipation). *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$. Let $(\mathbf{P}_t)_{t \geq 0}$ be the M/M/∞ semigroup with input rate $\lambda > 0$ and service rate $\mu > 0$. Then for any $f \in L^{1,\Phi}(\mathcal{P}(\rho))$, the real function $t \in \mathbb{R}_+ \mapsto \mathbf{Ent}_{\mathcal{P}(\rho)}^{\Phi}(\mathbf{P}_t f)$ is non-increasing. Moreover, when $A^{\Phi} \in \mathcal{C}_{\mathcal{I}}$,*

$$\mathbf{Ent}_{\mathcal{P}(\rho)}^{\Phi}(\mathbf{P}_t f) \leq e^{-ct} \mathbf{Ent}_{\mathcal{P}(\rho)}^{\Phi}(f), \quad (28)$$

where c is the best (i.e. smallest) constant in the inequality

$$\forall f \in L^{1,\Phi}(\mathcal{P}(\rho)), \quad \lambda \mathbf{Ent}_{\mathcal{P}(\rho)}^{\Phi}(f) \leq c \mathbf{E}_{\mathcal{P}(\rho)}(B^{\Phi}(f, Df)).$$

It holds with $c = \mu$ in general, and with $c = 2\mu$ for **(P2)**.

Proof. Let us denote $Q = \mathcal{P}(\rho)$. Since $\mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f) = \mathbf{E}_Q(\mathbf{P}_t \Phi(f)) - \Phi(\mathbf{E}_Q(f))$, the invariance of Q gives,

$$\partial_t \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f) = \mathbf{E}_Q(\Phi'(\mathbf{P}_t f) \mathbf{L} \mathbf{P}_t f).$$

Jensen inequality yields $\mathbf{P}_t(\Phi(f)) \geq \Phi(\mathbf{P}_t(f))$ as soon as Φ is convex. In particular $\mathbf{L}\Phi(f) \geq \Phi'(f) \mathbf{L}f$, which gives $\mathbf{E}_Q(\Phi'(f) \mathbf{L}f) \leq 0$ as soon as $\Phi(f)$ is Q -integrable. Hence, $t \mapsto \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f)$ is non-increasing, and we used only the convexity of Φ , the Markovian nature of $(\mathbf{P}_t)_{t \geq 0}$, and the invariance of Q . The Poisson integration by parts (8) – which is this time specific to our settings – yields for any g

$$\mathbf{E}_Q(\Phi'(g) \mathbf{L}g) = -\lambda \mathbf{E}_Q(D(g)D(\Phi'(g))) = -\lambda \mathbf{E}_Q(B^{\Phi}(g, Dg)). \quad (29)$$

In particular, for $g = \mathbf{P}_t(f)$, we get,

$$\partial_t \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f) = -\lambda \mathbf{E}_Q(B^{\Phi}(\mathbf{P}_t f, D\mathbf{P}_t f)).$$

Notice that since Φ is convex, one has $B^{\Phi} \geq 0$ by virtue of Lemma 4.1. When A^{Φ} is convex, the Poisson entropic inequality (26) together with the bound $A^{\Phi} \leq B^{\Phi}$ of Lemma 4.2 gives

$$-\lambda \mathbf{E}_Q(B^{\Phi}(\mathbf{P}_t f, D\mathbf{P}_t f)) \leq -\mu \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f).$$

Therefore, we get (28) since $\partial_t \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f) \leq -\mu \mathbf{Ent}_Q^{\Phi}(\mathbf{P}_t f)$. Finally, the correct constant for **(P2)** comes from the fact that $2A^{\Phi} = B^{\Phi}$ in that case. \square

For any probability measure γ on \mathbb{N} , and any $t \in \mathbb{R}_+$, we denote by $\gamma \mathbf{P}_t$ the probability measure on \mathbb{N} defined for any function $g : \mathbb{N} \rightarrow \mathbb{R}$ by $\mathbf{E}_{\gamma \mathbf{P}_t}(g) := \mathbf{E}_{\gamma}(\mathbf{P}_t(g))$. In particular, when $\gamma = \delta_n$ for some fixed $n \in \mathbb{N}$, we get $\delta_n \mathbf{P}_t = \mathbf{P}_t(\cdot)(n)$.

One has $\gamma \ll \mathcal{P}(\rho)$ for any probability measure γ on \mathbb{N} , as soon as $\rho > 0$. Let us define $f_\gamma := d\gamma/d\mathcal{P}(\rho)$. Since $\mathcal{P}(\rho)$ is symmetric for \mathbf{L} , one has that \mathbf{L} and $\mathbf{P}_t(\cdot)$ are self-adjoint in $L^2(\mathcal{P}(\rho))$. Therefore, one can write for any $g \in L^2(\mathcal{P}(\rho))$

$$\int_{\mathbb{N}} g \mathbf{P}_t(f_\gamma) d\mathcal{P}(\rho) = \int_{\mathbb{N}} \mathbf{P}_t(g) f_\gamma d\mathcal{P}(\rho) = \int_{\mathbb{N}} \mathbf{P}_t(g) d\gamma = \int_{\mathbb{N}} g d(\gamma \mathbf{P}_t).$$

Recall that the total variation norm $\|\alpha\|_{\text{TV}}$ of a Borel measure α on an at most countable set \mathcal{S} is defined by $\|\alpha\|_{\text{TV}} = \frac{1}{2} \|\alpha\|_1 = \frac{1}{2} \sum_{x \in \mathcal{S}} |\alpha(x)|$. If α and β are two probability measures on \mathcal{S} , the distance $\|\alpha - \beta\|_{\text{TV}}$ is given by

$$\|\alpha - \beta\|_{\text{TV}} = \sup_{A \subset \mathcal{S}} |\alpha(A) - \beta(A)| = \frac{1}{2} \sup_{\|f\|_\infty \leq 1} \left| \int f d\alpha - \int f d\beta \right|.$$

Recall the well known bound for any $a, b \in \mathbb{R}_+$, $\|\mathcal{P}(a) - \mathcal{P}(b)\|_{\text{TV}} \leq 1 - e^{-(b-a)}$, cf. [Rob03, Prop. 6.1 page 143], which gives from (4) for any $t \in \mathbb{R}_+$

$$\|\mathbf{P}_t(\cdot)(0) - \mathcal{P}(\rho)\|_{\text{TV}} \leq 1 - e^{-\rho e^{-\mu t}}.$$

Theorem 3.1 for **(P1)** produces in particular a bound for $\|\mathbf{P}_t(\cdot)(n) - \mathcal{P}(\rho)\|_{\text{TV}}$, as stated in the following Corollary.

Corollary 3.2. *Let $(\mathbf{P}_t)_{t \geq 0}$ be the semigroup of the $M/M/\infty$ queue with input rate $\lambda > 0$ and service rate $\mu > 0$. For any $n \in \mathbb{N}$ and any $t \in \mathbb{R}_+$,*

$$2\|\mathbf{P}_t(\cdot)(n) - \mathcal{P}(\rho)\|_{\text{TV}}^2 \leq e^{-\mu t} \log(e^\rho \rho^{-n} n!). \quad (30)$$

Proof of Corollary 3.2. We proceed like Diaconis and Saloff-Coste, cf. for example [DSC96, SC97]. Since $Q := \mathcal{P}(\rho)$ is symmetric for \mathbf{L} ,

$$\frac{d(\gamma \mathbf{P}_t)}{dQ} = \mathbf{P}_t(f_\gamma), \quad (31)$$

where $f_\gamma := d\gamma/dQ$. The Pinsker-Csiszár-Kullback inequality states that for any couple (α, β) of probability measures on the same measured space

$$2\|\alpha - \beta\|_{\text{TV}}^2 \leq \mathbf{Ent}(\alpha | \beta) = \mathbf{Ent}_\beta\left(\frac{d\alpha}{d\beta}\right), \quad (32)$$

where \mathbf{Ent}_β is the Φ -entropy in the case **(P1)**. Let $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. For $(\alpha, \beta) = (\mathbf{P}_t(\cdot)(n), Q)$ one can write by the successive use of (32) and (31) and (28)

$$2\|\gamma \mathbf{P}_t - Q\|_{\text{TV}}^2 \leq \mathbf{Ent}_Q(\mathbf{P}_t(f_\gamma)) \leq e^{-\mu t} \mathbf{Ent}_Q(f_\gamma).$$

For $\gamma = \delta_n$ for some fixed $n \in \mathbb{N}$, we get $\gamma \mathbf{P}_t = \mathbf{P}_t(\cdot)(n)$ and $f_{\delta_n} = \mathbf{I}_{\{n\}}/Q(n)$. As a consequence $\mathbf{Ent}_Q(f_{\delta_n}) = -\log Q(n) = \log(e^\rho \rho^{-n} n!)$. \square

3.1 Local inequalities and semigroup interpolation

Standard Brownian motion on \mathbb{R} starting from x interpolates on the time interval $[0, t]$ between the Dirac measure δ_x and the Gaussian measure $\mathcal{N}(0, t)$. It is known that this interpolation provides the optimal Gaussian logarithmic Sobolev inequality. Similarly, the simple Poisson process of intensity λ starting from n interpolates on the time interval $[0, t]$ between the Dirac measure δ_n and the Poisson measure $\mathcal{P}(\lambda t)$. By analogy, let us give a proof of the Poisson entropic inequality (26) by using the simple Poisson process, which corresponds to an M/M/ ∞ queue with $\mu = 0$. In that case, $\mathbf{L} = \lambda \mathbf{D}$ and $\mathbf{P}_t(\cdot)(0) = \mathcal{P}(\lambda t)$. Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $A^\Phi \in \mathcal{C}_{\mathcal{I}_X}$, cf. Lemma 4.2. One can write by abridging $\mathbf{P}_t(\cdot)(0)$ in $\mathbf{P}_t(\cdot)$ and denoting $F = \mathbf{P}_{t-s}(f)$,

$$\begin{aligned} \mathbf{Ent}_{\mathcal{P}(\lambda)}^\Phi(f) &= \mathbf{P}_1(\Phi(f)) - \Phi(\mathbf{P}_1(f)) \\ &= \int_0^t \partial_s \mathbf{P}_s(\Phi(\mathbf{P}_{t-s}f)) \, ds \\ &= \int_0^t \mathbf{P}_s(\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F) \, ds. \end{aligned}$$

Now, it turns out that $\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = A^\Phi(F, \mathbf{D}F)$. The commutation (5) with $\mu = 0$ gives $\mathbf{D}F = \mathbf{D}\mathbf{P}_{t-s}(f) = \mathbf{P}_{t-s}(\mathbf{D}f)$, and thus,

$$\begin{aligned} \mathbf{Ent}_{\mathcal{P}(\lambda)}^\Phi(f) &= \lambda \int_0^t \mathbf{P}_s(A^\Phi(F, \mathbf{D}F)) \, ds \\ &= \lambda \int_0^t \mathbf{P}_s(A^\Phi(\mathbf{P}_{t-s}f, \mathbf{P}_{t-s}\mathbf{D}f)) \, ds. \end{aligned}$$

Finally, Jensen inequality for convex function A^Φ gives then the desired result,

$$\begin{aligned} \mathbf{Ent}_{\mathcal{P}(\lambda)}^\Phi(f) &\leq \lambda \int_0^t \mathbf{P}_s(\mathbf{P}_{t-s}(A^\Phi(f, \mathbf{D}f))) \, ds \\ &= \lambda t \mathbf{P}_t(A^\Phi(f, \mathbf{D}f)). \end{aligned}$$

M/M/ ∞ semigroup interpolation on time interval $[0, +\infty]$

The standard Ornstein-Uhlenbeck process on \mathbb{R} starting from x interpolates on the time interval $[0, +\infty]$ by ergodicity between the Dirac measure δ_x and the standard Gaussian measure $\mathcal{N}(0, 1)$. It is known that this interpolation provides the Gaussian logarithmic Sobolev inequality. Similarly, the M/M/ ∞ queue with intensities (λ, μ) starting from n interpolates on the time interval $[0, +\infty]$ between the Dirac measure δ_n and the Poisson measure $\mathcal{P}(\rho)$ where $\rho = \lambda/\mu$. By analogy,

let us give a proof of the Poisson entropic inequality (26) by using the M/M/ ∞ queue.

Let $(\mathbf{P}_t)_{t \geq 0}$ be the M/M/ ∞ queue semigroup with input rate λ and service rate μ . Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $B^\Phi \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$, cf. Lemma 4.2. We denote by Q the Poisson measure $\mathcal{P}(\rho)$. For bounded $f : \mathbb{N} \rightarrow \mathcal{I}$, one can write by ergodicity

$$\begin{aligned} \mathbf{Ent}_Q^\Phi(f) &= + \int_{\mathbb{N}} (\Phi(\mathbf{P}_0 f) - \Phi(\mathbf{P}_\infty f)) dQ \\ &= - \int_{\mathbb{N}} \int_0^\infty \partial_t \Phi(\mathbf{P}_t f) dt dQ \\ &= - \int_0^\infty \int_{\mathbb{N}} \Phi'(\mathbf{P}_t f) \mathbf{L} \mathbf{P}_t f dQ dt \\ &= \lambda \int_0^\infty \int_{\mathbb{N}} B^\Phi(\mathbf{P}_t f, \mathbf{D} \mathbf{P}_t f) dQ dt, \end{aligned}$$

where we used (29) for the last equality. Now, the commutation (5) yields

$$\mathbf{Ent}_Q^\Phi(f) = \lambda \int_0^\infty \int_{\mathbb{N}} B^\Phi(\mathbf{P}_t f, e^{-\mu t} \mathbf{P}_t \mathbf{D} f) dQ dt.$$

Jensen inequality for B^Φ and $\mathbf{P}_t(\cdot)$ followed by the invariance of Q give

$$\mathbf{Ent}_Q^\Phi(f) \leq \lambda \int_0^\infty \int_{\mathbb{N}} B^\Phi(f, e^{-\mu t} \mathbf{D} f) dQ dt.$$

But by Lemma 4.3, $B^\Phi(u, e^{-\mu t} v) \leq e^{-\mu t} B^\Phi(u, v)$ for any $(u, v) \in \mathcal{I}\mathcal{I}$, and thus

$$\begin{aligned} \mathbf{Ent}_Q^\Phi(f) &\leq \lambda \int_0^\infty e^{-\mu t} dt \int_{\mathbb{N}} B^\Phi(f, \mathbf{D} f) dQ \\ &= \rho \mathbf{E}_Q(B^\Phi(f, \mathbf{D} f)), \end{aligned}$$

which is exactly the B transform version of the Poisson entropic inequality (26).

Remark 3.3 (A, B, C transforms and discrete space). The interpolation on $[0, t]$ gives rise to the A^Φ transform whereas the interpolation on $[0, +\infty]$ leads to the B^Φ transform. In continuous space settings, the diffusion property permits to write $\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = \Phi''(F)\mathbf{\Gamma}(F, F)$ which is close to C^Φ and not to A^Φ in that case.

Local inequality and semigroup interpolation on $[0, t]$

Consider the semigroup $(\mathbf{P}_t)_{t \geq 0}$ of the M/M/ ∞ queue with input rate λ and service rate μ . The family $(\mathbf{P}_t(\cdot)(n))_{t \geq 0}$ interpolates between δ_n and $\mathcal{P}(\rho)$. In particular,

when $\lambda = 0$, the interpolation holds between δ_n and δ_0 . Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $A^\Phi \in \mathcal{C}_{\mathcal{I}}$, cf. Lemma 4.2. Inequality (27) together with (4) gives for any $n \in \mathbb{N}$, any $t \in \mathbb{R}_+$, and any bounded function $f : \mathbb{N} \rightarrow \mathcal{I}$,

$$\mathbf{Ent}_{\mathbf{P}_{(\cdot)}(n)}^\Phi(f) \leq \rho q(t) \mathbf{P}_t(A^\Phi(f, Df))(n) + np(t)q(t) \mathbf{P}_t(K_t(f))(n-1), \quad (33)$$

where $K_t(f) := q(t)A^\Phi(f, Df) + p(t)A^\Phi(\tau(f, Df))$. Now, we write

$$\begin{aligned} \mathbf{Ent}_{\mathbf{P}_{(\cdot)}(n)}^\Phi(f) &= \mathbf{P}_t(\Phi(f))(n) - \Phi(\mathbf{P}_t(f))(n) \\ &= \int_0^t \partial_s \mathbf{P}_s(\Phi(\mathbf{P}_{t-s}f))(n) ds \\ &= \int_0^t \mathbf{P}_s(\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F)(n) ds. \end{aligned}$$

where $F := \mathbf{P}_{t-s}(f)$. At this step, we notice that

$$\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = \lambda A^\Phi(F, DF) + \mu h A^\Phi(F, D^*F),$$

where $h : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $h(n) = n$ for any $n \in \mathbb{N}$. Thus, we get

$$\mathbf{Ent}_{\mathbf{P}_{(\cdot)}(n)}^\Phi(f) = \lambda \int_0^t \mathbf{P}_s(A^\Phi(F, DF))(n) ds + \mu \int_0^t \mathbf{P}_s(h A^\Phi(F, D^*F))(n) ds. \quad (34)$$

For **(P2)**, one has $A^\Phi(f, Df) = A^\Phi(\tau(f, Df)) = |Df|^2$, and thus (33) writes

$$\mathbf{Var}_{\mathbf{P}_{(\cdot)}(n)}(f) \leq \rho q(t) \mathbf{P}_t(|Df|^2)(n) + np(t)q(t) \mathbf{P}_t(|Df|^2)(n-1).$$

The identity (34) writes here

$$\mathbf{Var}_{\mathbf{P}_{(\cdot)}(n)}(f) = \lambda \int_0^t \mathbf{P}_s(|DF|^2)(n) ds + \mu \int_0^t \mathbf{P}_s(h|D^*F|^2)(n) ds.$$

The commutation formula (5) gives $|DF|^2 \leq p(t-s)^2 \mathbf{P}_{t-s}(|Df|^2)$, and thus

$$\mathbf{P}_s(|DF|^2) \leq p(t-s)^2 \mathbf{P}_t(|Df|^2).$$

In the other hand, the identities (10) and $|D^*F|^2(1+\cdot) = |DF|^2$, and the commutation formula (5), give

$$\mu \mathbf{P}_s(hG)(n) \leq \mu np(s)p(t-s)^2 \mathbf{P}_t(|Df|^2)(n-1) + \lambda q(s)p(t-s)^2 \mathbf{P}_t(|Df|^2)(n).$$

Putting all together, we obtain the optimal local Poincaré inequality

$$\mathbf{Var}_{\mathbf{P}_{(\cdot)}(f)} \leq n\alpha(t) \mathbf{P}_t(|Df|^2)(n-1) + \beta(t) \mathbf{P}_t(|Df|^2)(n),$$

where

$$\alpha(t) := \mu \int_0^t p(s)p(t-s)^2 ds = p(t)q(t)$$

and where

$$\beta(t) := \lambda \int_0^t (1+q(s))p(t-s)^2 ds = \rho q(t).$$

The author ignores if the same method leads to (33) for more general Φ functions.

3.2 Scaling limit of the entropic inequalities

Let us consider the Poisson distribution $\mathcal{P}(\rho)$ with parameter $\rho > 0$. For any $N \in \mathbb{N}^*$, let $\kappa_N : \mathbb{N} \rightarrow \mathbb{R}$ be the function defined by $\kappa_N(n) := N^{-1/2}(n - \rho N)$ for any $n \in \mathbb{N}$. By virtue of the Central Limit Theorem, the image measure of $\mathcal{P}(N\rho) = \mathcal{P}(\rho)^{*N}$ by κ_N converges weakly towards the Gaussian measure $\mathcal{N}(0, \rho)$ of mean 0 and variance ρ . Let $g : \mathbb{R} \rightarrow \text{int}(\mathcal{I})$ be smooth, bounded, with bounded derivatives, and set $f_N := g \circ \kappa_N$. We get in one hand,

$$\lim_{N \rightarrow \infty} \mathbf{Ent}_{\mathcal{P}(N\rho)}^\Phi(f_N) = \mathbf{Ent}_{\mathcal{N}(0, \rho)}^\Phi(g).$$

In the other hand, by a Taylor formula,

$$D(g \circ \kappa_N) = N^{-1/2}g'(\kappa_N) + o(N^{-1}).$$

Therefore, by virtue of (37), we get

$$A^\Phi(f_N, Df_N) = \frac{1}{2N}(g' \circ \kappa_N)^2 \Phi''(f_N) + o(N^{-1}).$$

This yields that

$$\lim_{N \rightarrow \infty} \rho N \mathbf{E}_{\mathcal{P}(\rho)}(A^\Phi(f_N, Df_N)) = \frac{1}{2} \rho \mathbf{E}_{\mathcal{N}(0, \rho)}(C^\Phi(g, g')).$$

Now, the A -transform based Poisson entropic inequality (26) gives finally that

$$\mathbf{Ent}_{\mathcal{N}(0, \rho)}^\Phi(g) \leq \frac{1}{2} \rho \mathbf{E}_{\mathcal{N}(0, 1)}(C^\Phi(g, g')). \quad (35)$$

Recall that the Poincaré inequality corresponds to **(P2)**. In that case,

$$\mathbf{Ent}_{\mathcal{N}(0, \rho)}^\Phi(g) = \mathbf{Var}_{\mathcal{N}(0, \rho)}(g) \quad \text{and} \quad C^\Phi(g, g') = 2|g'|^2.$$

The logarithmic Sobolev inequality corresponds to **(P1)**. In that case,

$$\mathbf{Ent}_{\mathcal{N}(0, \rho)}^\Phi(g) = \mathbf{Ent}_{\mathcal{N}(0, \rho)}(g) \quad \text{and} \quad C^\Phi(g, g') = \frac{|g'|^2}{g}.$$

The constant ρ in (35) is optimal, see for instance [Cha04]. It gives in particular the optimal Poincaré inequality for the Gaussian measure in the case **(P2)**, and the optimal logarithmic Sobolev inequality for the Gaussian measure in the case **(P1)**. The method was used in the case **(P1)** in [Wu00, Remark 1.6]. In some sense, the A transform is the right Dirichlet form to consider since it allows the derivation of optimal Gaussian entropic inequalities from their A -transform based Poisson versions. In contrast, it is shown in [BL98, pages 356-357] that the optimal B transform version for the Poisson measure does not lead to the optimal constant in the logarithmic Sobolev inequality for the Gaussian measure (lack of a multiplicative factor 2). The deep reason for this difference between A and B transforms consequences is the fact that the comparison $A^\Phi \leq B^\Phi$ improves by a factor 2 when v goes to 0, as stated in (37). This phenomenon does not hold for the Poincaré inequality, since $2A^\Phi = B^\Phi$ for **(P2)**.

As presented in Section 1.2, the M/M/ ∞ queueing process gives rise to an Ornstein-Uhlenbeck process via a fluid limit procedure. It is quite natural to ask about the behaviour of the binomial-Poisson entropic inequalities under this scaling limit.

Let $(X_t^N)_{t \geq 0}$ be an M/M/ ∞ queueing process with input rate $N\lambda > 0$ and service rate $\mu > 0$, where $N \in \mathbb{N}^*$. Let $(U_t)_{t \geq 0}$ be an Ornstein-Uhlenbeck process, solution of the Stochastic Differential Equation $dU_t = \lambda dB_t - \mu U_t dt$. Let $g : \mathbb{R} \rightarrow \text{int}(\mathcal{I})$ be smooth, bounded, with bounded derivatives. For any $x, y \in \mathbb{R}$, we define $z_N := \lfloor N\rho + N^{1/2}y \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part. According to Section 1.2, the image measure of $\mathcal{L}(X_t^N | X_0 = z_N)$ by function κ_N converges weakly towards $\mathcal{L}(U_t | U_0 = y)$ when N goes to ∞ . Notice that

$$\mathcal{L}(X_t^N | X_0 = z_N) = \mathcal{B}(z_N, p(t)) * \mathcal{P}(N\rho q(t))$$

and that

$$\mathcal{L}(U_t | U_0 = y) = \mathcal{N}(yp(t), \rho(1 - p(t)^2)).$$

In particular, if $f_N := g \circ \kappa_N$, then

$$\lim_{N \rightarrow \infty} \mathbf{Ent}_{\mathcal{L}(X_t^N | X_0^N = z_N)}^\Phi(f_N) = \mathbf{Ent}_{\mathcal{L}(U_t | U_0 = y)}^\Phi(g).$$

In the other hand, as for the pure Poisson measure case, we have

$$\lim_{N \rightarrow \infty} N \mathbf{E}_{\mathcal{L}(X_t^N | X_0^N = z_N)}(A^\Phi(f_N, Df_N)) = \frac{1}{2} \mathbf{E}_{\mathcal{L}(U_t | U_0 = y)}(C^\Phi(g, g')).$$

Similarly, we obtain

$$\lim_{N \rightarrow \infty} N \mathbf{E}_{\mathcal{L}(X_t^N | X_0^N = z_N)}(A^\Phi(\tau(f_N, Df_N))) = \frac{1}{2} \mathbf{E}_{\mathcal{L}(U_t | U_0 = y)}(C^\Phi(g, g')).$$

Putting all together, we deduce from (27) that

$$\mathbf{Ent}_{\mathcal{L}(U_t|U_0=y)}(g) \leq K(t) \mathbf{E}_{\mathcal{L}(U_t|U_0=y)}(C^\Phi(g, g')), \quad (36)$$

where $K(t) := \frac{1}{2}\rho q(t)(1 + 2p(t))$. It is well known that the best constant in (36) is $K^*(t) := \frac{1}{2}\rho(1 - p(t)^2)$, see for example [Cha04]. Let us define $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\theta(t) := \frac{K(t)}{K^*(t)} = \frac{q(t)(1 + 2p(t))}{1 - p(t)^2} = 1 + \frac{1}{1 + \frac{1}{p(t)}}.$$

This function is non-increasing, with $\theta(0) = \frac{3}{2}$ and $\lim_{t \rightarrow +\infty} \theta(t) = 1$. Consequently, the constant $K(t)$ in the inequality (36) improves when t increases. It is asymptotically optimal, when t goes to $+\infty$. The non-optimality of the constant in (36) at fixed time can be seen as a consequence of the presence of the binomial part in the law of X_t^N . This binomial part vanishes when t goes to $+\infty$.

Remark 3.4 (The M/M/1 case). The M/M/1 queue with input rate λ and service rate μ is the birth and death process on \mathbb{N} with generator $\mathbf{L} = \mu\mathbf{D}^* + \mu\mathbf{D}$. One has $[\mathbf{L}, \mathbf{D}] = 0$ and the ‘‘curvature’’ is identically zero. When $\lambda > 0$ and $\mu > 0$, the symmetric invariant measure Q is given by $Q(n) = \rho^n$ for any $n \in \mathbb{N}$, with $\rho := \lambda/\mu$. The associated Markov semigroup $(\mathbf{P}_t)_{t \geq 0}$ satisfies to the exact commutation formula $\mathbf{D}\mathbf{P}_t = \mathbf{P}_t(\mathbf{D})$. Measure Q is finite if and only if $\rho \leq 1$, and Q is in that case the geometric measure $\mathcal{G}(1 - \rho)$ of mean $\lambda/(\mu - \lambda)$. This leads to Poisson like entropic inequalities for $\mathbf{P}_t(\cdot)$. The M/M/1 is a discrete space analog of the continuous process $(E_t)_{t \geq 0}$ solution of the Stochastic Differential Equation $dE_t = dB_t - \text{sign}(E_t)dt$.

Remark 3.5 (Spectrum). A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is an eigenvector associated to the eigenvalue $\alpha \in \mathbb{R}$ for the M/M/ ∞ infinitesimal generator \mathbf{L} defined by (1) if and only if for any $n \in \mathbb{N}$, $\lambda f(n+1) = (\lambda + \alpha + n\mu)f(n) - n\mu f(n-1)$. Obviously, for any $\alpha \in \mathbb{R}$ and any starting value $f(0) \neq 0$, the equation above has a unique non null solution. As a consequence, the spectrum of \mathbf{L} is \mathbb{R} . We will denote by f_α the unique solution such that $f_\alpha(0) = 1$. By the equation $\mathbf{L}f_\alpha = \alpha f_\alpha$ and the invariance of $Q := \mathcal{P}(\rho)$ we get that $\mathbf{E}_Q(f_\alpha) = 0$ as soon as $f_\alpha \in L^1(\mathbb{N}, Q, \mathbb{R})$. Suppose that $f_\alpha \in L^2(Q)$, then $0 \leq \mathbf{E}_Q(\Gamma(f, f)) = -\mathbf{E}_Q(f\mathbf{L}f) = -\alpha\mathbf{E}_Q(f^2)$ and thus $\alpha \leq 0$. Moreover, Theorem 3.1 for **(P2)** gives $\inf \{-\alpha \in \mathbb{R}, f_\alpha \in L^2(\mathbb{N}, Q, \mathbb{R})\} = \mu^{-1}$, cf. [Bak94, Prop. 2.3].

Remark 3.6 (Bakry \mathbf{I}_2 calculus). Define the Markovian functional quadratic forms Γ and \mathbf{I}_2 by $2\Gamma(f, f) := \mathbf{L}(f^2) - 2f\mathbf{L}f$ and $2\mathbf{I}_2(f, f) := \mathbf{L}\Gamma(f, f) - 2\Gamma(f, \mathbf{L}f)$. After some algebra based on (1), we get

$$2\Gamma(f, f)(n) = n\mu|\mathbf{D}^*f|^2(n) + \lambda|\mathbf{D}f|^2(n)$$

for any $f : \mathbb{N} \rightarrow \mathbb{R}$ and any $n \in \mathbb{N}$, and

$$2\mathbf{I}_2(f, f)(n) = \frac{3}{2}\lambda\mu|Df|^2(n) + \frac{n}{2}\mu^2|D^*f|^2(n) + \mathbf{R}(f, f)(n),$$

where $2\mathbf{R}(f, f)(n) := n(n-1)\mu^2|D^*D^*f|^2 + 2n\lambda\mu|DD^*f|^2 + \lambda^2|DDf|^2$. Notice that for the linear function $f(n) = n$, we get

$$2\mathbf{I}(f, f)(n) = \lambda + n\mu \quad \text{and} \quad 4\mathbf{I}_2(f, f)(n) = 3\lambda\mu + n\mu^2.$$

Since $\mathbf{R}(f, f) \geq 0$ for any f , we obtain immediately the bound $\mathbf{I}_2 \geq \frac{\mu}{2}\mathbf{I}$, which is the infinitesimal version of the commutation $\mathbf{I}\mathbf{P}_t \leq \exp(-t\frac{\mu}{2})\mathbf{P}_t\mathbf{I}$. Moreover, an integration by parts similar to (8) gives the integrated bound $\mathbf{E}_Q(\mathbf{I}_2 f) \geq \mu\mathbf{E}_Q(\mathbf{I}f)$, where $Q := \mathcal{P}(\rho)$. Such a bound gives, via integration by parts, the Poincaré inequality $\mathbf{Var}_Q(f) \leq \rho\mathbf{E}_Q(|Df|^2)$, which is exactly (26) for **(P2)**. However, the \mathbf{I}_2 bound above suggests that \mathbf{I}_2 is not the right tool in order to derive Φ -entropic inequalities beyond the **(P2)** case. Bakry-Émery type approaches are designed for diffusion. In discrete space settings, the lack of chain rule reduces their strength for the derivation of entropic inequalities beyond the **(P2)** case.

4 Convexity and Φ -calculus on A, B, C transforms

The following convexity Lemmas given here extend the results of [Cha04]. Unless otherwise stated, the statements hold for **(P1)**, **(P2)**, **(P3)**.

Lemma 4.1. *Let $\Phi : \mathcal{I} \rightarrow \mathbb{R}$ be a smooth function defined on an interval $\mathcal{I} \subset \mathbb{R}$. Let $\tau : \mathcal{T}_{\mathcal{I}} \rightarrow \mathcal{T}_{\mathcal{I}}$ be the bijective linear map defined for any $(u, v) \in \mathcal{T}_{\mathcal{I}}$ by $\tau(u, v) := (u + v, -v)$. Then the following statements hold.*

1. $A^\Phi + A^\Phi(\tau) = B^\Phi$ and $B^\Phi(\tau) = B^\Phi$;
2. Each of A^Φ, B^Φ, C^Φ is non-negative if and only if $\Phi \in \mathcal{C}_{\mathcal{I}}$.

Notice that τ is well defined since $(u, v) \in \mathcal{T}_{\mathcal{I}}$ implies that $(u+v, u+v-v) \in \mathcal{I} \times \mathcal{I}$ and thus $(u+v, -v) \in \mathcal{T}_{\mathcal{I}}$. If one writes $(a, b) := (u, u+v)$, the map τ transposes a and b and τ^2 is the identity map. One has also the expressions

$$\begin{aligned} A^\Phi(u, v) &= \Phi(b) - \Phi(a) - \Phi'(a)(b - a), \\ B^\Phi(u, v) &= (b - a)(\Phi'(b) - \Phi'(a)), \\ C^\Phi(u, v) &= \Phi''(a)(b - a)^2. \end{aligned}$$

Proof. The first statement of the Lemma is immediate. Let us establish the second one. Recall that for a smooth function, convexity is equivalent to non-negativity

of second derivative. Thus, C^Φ is non-negative on \mathcal{I}_τ if and only if Φ is convex on \mathcal{I} . B^Φ is non-negative on \mathcal{I}_τ if and only if Φ' is non-decreasing on \mathcal{I} , which is equivalent to the convexity of Φ on \mathcal{I} . For A^Φ , one can write when $v \geq 0$ the following Taylor formula

$$A^\Phi(u, v) = \int_u^{u+v} (u+v-w)\Phi''(w) dw,$$

and when $v \leq 0$ by considering $\Phi(u+v-\cdot)$ on $[u+v, u]$

$$A^\Phi(u, v) = \int_{u+v}^u (u-w)\Phi''(w) dw.$$

The result follows by non-negativity of Φ'' when Φ is convex on \mathcal{I} . The necessity of the convexity of Φ follows from $\Phi''(u) = 2v^{-2} \lim_{v \rightarrow 0^+} A^\Phi(u, v)$. \square

Lemma 4.1 tells that the transforms A, B, C map the set of convex functions on \mathcal{I} into the set of non-negative functions on \mathcal{I}_τ . Moreover, their null space contains any real valued affine functions on \mathcal{I} .

Lemma 4.2. *The following statements hold.*

1. In the cases **(P1)**, **(P2)**, **(P3)**, functions $\Phi, -\Phi', \Phi'', -1/\Phi''$ belong to $\mathcal{C}_\mathcal{I}$;
2. If $A^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$ then $B^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$. If $A^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$ or $C^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$ then $\Phi'' \in \mathcal{C}_\mathcal{I}$. Moreover when $\Phi'' > 0$ on \mathcal{I} , there is an equivalence between
 - (a) $A^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$;
 - (b) $C^\Phi \in \mathcal{C}_{\mathcal{I}_\tau}$;
 - (c) Φ'' and $-1/\Phi''$ are in $\mathcal{C}_\mathcal{I}$;
 - (d) $\Phi''''\Phi'' \geq 2\Phi''''^2$ on \mathcal{I} ;
3. If $\Phi \in \mathcal{C}_\mathcal{I}$ then $A^\Phi \leq B^\Phi$;
4. $2A^\Phi = B^\Phi = C^\Phi$ for **(P2)** and $A^\Phi \leq C^\Phi$ for **(P1)**.

Proof. The first statement is immediate. For the second one, we notice that the map τ defined in Lemma 4.1 is linear, and the expression of B^Φ in terms of A^Φ given by Lemma 4.1 induces that the convexity of A^Φ implies the convexity of B^Φ . Let us establish the equivalences. The assumption $\Phi'' > 0$ on \mathcal{I} implies that $1/\Phi''$ is well defined on \mathcal{I} , and also that Φ is strictly convex on \mathcal{I} .

- $2c \Rightarrow 2d$. Since $-1/\Phi''$ is convex, one has $(-1/\Phi'')' = (\Phi''''\Phi'' - 2\Phi''''^2)/\Phi''^4 \geq 0$ and thus $\Phi''''\Phi'' - 2\Phi''''^2 \geq 0$;

- 2d \Rightarrow 2c. From $\Phi''''\Phi'' - 2\Phi''^2 \geq 0$ and $\Phi'' > 0$ we get $\Phi'''' \geq 0$ and $(-1/\Phi'')'' \geq 0$ and thus Φ'' and $-1/\Phi''$ are convex on \mathcal{I} ;
- 2b \Leftrightarrow 2d. The Hessian matrix of C^Φ writes

$$\nabla^2 C^\Phi(u, v) = \begin{pmatrix} \Phi''''(u)v^2 & 2\Phi'''(u)v \\ 2\Phi'''(u)v & 2\Phi''(u) \end{pmatrix}.$$

The diagonal elements are non-negative on \mathcal{I} if and only if Φ and Φ'' are convex on \mathcal{I} . The determinant writes

$$\det(\nabla^2 C^\Phi(u, v)) = 2v^2(\Phi''''(u)\Phi''(u) - 2\Phi''^2(u)),$$

and is non-negative on \mathcal{I} if and only if $\Phi''''(u)\Phi''(u) - 2\Phi''^2(u) \geq 0$;

- 2a \Leftrightarrow 2c. The Hessian matrix of A^Φ writes

$$\nabla^2 A^\Phi(u, v) = \begin{pmatrix} A^{\Phi''}(u, v) & \Phi''(u+v) - \Phi''(u) \\ \Phi''(u+v) - \Phi''(u) & \Phi''(u+v) \end{pmatrix}.$$

The diagonal elements are non-negative on \mathcal{I} if and only if Φ and Φ'' are convex on \mathcal{I} . The determinant writes

$$\begin{aligned} \det(\nabla^2 A^\Phi(u, v)) &= \Phi''(u)(\Phi''(u+v) - \Phi''(u)) - \Phi''(u+v)\Phi'''(u)v \\ &= \Phi''(u+v)\Phi''^2(u)A^{-1/\Phi''}(u, v). \end{aligned}$$

The result follows from the non negativity of A^Φ given by Lemma 4.1.

The third statement follows from Lemma 4.1. Alternatively, one can proceed directly by using Taylor formulas as in the proof of Lemma 4.1. The last statement is elementary. \square

Optimality of A-B-C comparisons. The bound $A^\Phi \leq B^\Phi$ is optimal in the sense that $B^\Phi(u, v) \sim A^\Phi(u, v)$ at $v = +\infty$ for any $u \in \mathcal{I}$ for **(P1)**. However, $B^\Phi = 2A^\Phi = C^\Phi$ for **(P2)**; and in general

$$\lim_{v \rightarrow 0} v^{-2} 2A^\Phi(u, v) = \lim_{v \rightarrow 0} v^{-2} B^\Phi(u, v) = v^{-2} C^\Phi(u, v) = \Phi''(u). \quad (37)$$

Convexity of the A transform and nullity of Φ'' . When condition $\Phi'' > 0$ on \mathcal{I} is weakened in $\Phi'' \geq 0$ on \mathcal{I} , the equivalences $2a \Leftrightarrow 2b \Leftrightarrow 2c \Leftrightarrow 2d$ of Lemma 4.2 still hold on each connected component of $\{u \in \mathcal{I}; \Phi''(u) > 0\}$, with the corresponding \mathcal{I} set. The subset \mathcal{J}_Φ of \mathcal{I} defined by $\mathcal{J}_\Phi := \{u \in \mathcal{I}; \Phi''(u) = 0\}$ is convex when Φ and Φ'' are convex. On \mathcal{J}_Φ , function Φ is affine. However, if $u \notin \mathcal{J}_\Phi$ and $u + v \in \mathcal{J}_\Phi$, one has $\det(\nabla^2 A^\Phi(u, v)) = -\Phi''(u)^2 < 0$ and A^Φ is

thus not convex. The only case where A^Φ is convex on $\mathcal{I}_\mathcal{I}$ with a non empty \mathcal{J}_Φ corresponds to $\mathcal{J}_\Phi = \mathcal{I}$. It means that Φ is affine on \mathcal{I} , and thus that $A^\Phi \equiv 0$ on $\mathcal{I}_\mathcal{I}$.

Convexity of the B transform. One can ask if the convexity of B^Φ can be added to the list of equivalent items in Lemma 4.2. The answer is unfortunately negative. Consider for example the case where $\Phi(u) = -\log(u)$ on $\mathcal{I} = (0, 1)$. Then $B^\Phi(a, b-a) = (b-a)(a^{-1} - b^{-1})$ and B^Φ is convex since for any $(a, b) \in \mathcal{I} \times \mathcal{I}$,

$$\nabla^2 B^\Phi(a, b-a) = a^{-2}b^{-2} \begin{pmatrix} 2b^3a^{-1} & -(b^2 + a^2) \\ -(b^2 + a^2) & 2a^3b^{-1} \end{pmatrix}.$$

However, A^Φ is not convex since $-1/\Phi''$ is not convex ($1/\Phi''(u) = u^2$). More generally, the Hessian matrix of B^Φ writes for any $(u, v) \in \mathcal{I}_\mathcal{I}$,

$$\nabla^2 B^\Phi(u, v) = \begin{pmatrix} B^{\Phi''}(u, v) & D^\Phi(u, v) \\ D^\Phi(u, v) & 2\Phi''(u+v) + \Phi'''(u+v)v \end{pmatrix},$$

where $D^\Phi(u, v) := \Phi''(u+v) - \Phi''(u) + \Phi'''(u+v)v$. The convexity of B^Φ on $\mathcal{I}_\mathcal{I}$ implies in particular that $B^{\Phi''}$ is non-negative on $\mathcal{I}_\mathcal{I}$ which, by virtue of Lemma 4.2, implies the convexity of Φ'' on \mathcal{I} . However, there no way to deduce the convexity of $-1/\Phi''$.

The (P1-P2-P3) special cases again. One can notice that in **(P1)**, **(P2)**, **(P3)**, both Φ , $-\Phi'$, and Φ'' are convex on \mathcal{I} . One can ask if these three convexity conditions on \mathcal{I} are equivalent to the convexity of A^Φ on $\mathcal{I}_\mathcal{I}$. The answer is negative. Consider for example $\Phi(u) := -u \log(-u)$ on $\mathcal{I} = \mathbb{R}_-$, which makes A^Φ convex on $\mathcal{I}_\mathcal{I}$ according to Lemma 4.2. However, $-\Phi'$ is concave on \mathcal{I} . This lack of equivalence is due to the fact that the convexity of A^Φ is stable by symmetry on (Φ, \mathcal{I}) . Namely $A^{\Phi(\varsigma)} = A^\Phi(\varsigma)$ on $\mathcal{I}_{\varsigma(\mathcal{I})}$ where $\varsigma(u) := -u$ and $\varsigma(u, v) := (-u, -v)$ for any $u, v \in \mathbb{R}$. In the other hand, the convexity of $-\Phi'$ is not stable by symmetry. The case **(P2)** is exceptional since $-\Phi'$ is affine on $\mathcal{I} = \mathbb{R}$ and thus both convex and concave. Conversely, let us consider the case $\Phi(u) = -\log(u)$ on $\mathcal{I} = \mathbb{R}_+^*$. One has that Φ , $-\Phi'$, Φ'' are convex on \mathcal{I} , and that $\Phi'' > 0$ on \mathcal{I} . However, $-1/\Phi''$ is concave and not convex and thus A^Φ is not convex on $\mathcal{I}_\mathcal{I}$.

Lemma 4.3. *Let $\Phi \in \mathcal{C}_\mathcal{I}$. Let τ as in Lemma (4.1). For $p \in [0, 1]$, let $\sigma_p : \mathcal{I}_\mathcal{I} \rightarrow \mathcal{I}_\mathcal{I}$ be the linear map defined for any $(u, v) \in \mathcal{I}_\mathcal{I}$ by $\sigma_p(u, v) := (u, pv)$. Then, on $\mathcal{I}_\mathcal{I}$, and for any $p \in [0, 1]$,*

$$A^\Phi(\sigma_p) \leq pA^\Phi \quad \text{and} \quad B^\Phi(\sigma_p) \leq pB^\Phi.$$

Moreover, $A^\Phi(\sigma_p) = p^2A^\Phi$ and $B^\Phi(\sigma_p) = p^2B^\Phi$ for **(P2)**.

Notice that for any $(u, v) \in \mathcal{I}_\mathcal{I}$ and any $p \in [0, 1]$, $u + pv \in [u, u+v] \subset \mathcal{I}$ since \mathcal{I} is convex. Thus $(u, u + pv) \in \mathcal{I}_\mathcal{I}$ and the map σ_p is well defined.

Proof. For A^Φ , one has by convexity of Φ , and by denoting $q := 1 - p$,

$$pA^\Phi(u, v) - A^\Phi(u, pv) = p\Phi(u + v) + q\Phi(u) - \Phi(p(u + v) + qu) \geq 0.$$

For B^Φ , Φ' is non-decreasing since Φ is convex, and thus $pv\Phi'(u + pv) \leq pv\Phi'(u + v)$ regardless of the sign of v , which gives $B^\Phi(u, pv) \leq pB^\Phi(u, v)$. \square

The bound $B^\Phi(\sigma_p) \leq pB^\Phi$ is optimal in the sense that $B^\Phi(u, pv) \sim pB^\Phi(u, v)$ at $v = +\infty$ for any $(p, u) \in (0, 1) \times \mathcal{I}$ for **(P1)**. However, $B^\Phi(\sigma_p) = p^2B^\Phi$ for **(P2)**; and in general

$$\lim_{v \rightarrow 0} v^{-2}B^\Phi(u, pv) = p^2 \lim_{v \rightarrow 0} B^\Phi(u, v) = p^2\Phi''(u).$$

The same remark holds for A^Φ (up to a factor 2 in the case **(P2)**).

Remark 4.4. The inequality $A^\Phi(\sigma_p) \leq pA^\Phi$ can be derived directly from the convexity of A^Φ . Namely, for any $p \in [0, 1]$ and any $(u, v) \in \mathcal{T}_{\mathcal{I}}$, the convex combination $(u, pv) = p(u, v) + q(u, 0)$ yields

$$\begin{aligned} A^\Phi(\sigma_p(u, v)) &= A^\Phi(u, pv) \\ &\leq pA^\Phi(u, v) + qA^\Phi(u, 0) \\ &= pA^\Phi(u, v), \end{aligned}$$

where we used the fact $A^\Phi(\cdot, 0) \equiv 0$. Similarly, we get

$$A^\Phi(\tau(\sigma_p)) \leq pA^\Phi(\tau).$$

Lemma 4.5. *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$ such that $\Phi'' \in \mathcal{C}_{\mathcal{I}}$. Let τ and σ_p as in Lemmas 4.1 and 4.3. Then, for any $p \in [0, 1]$, one has on $\mathcal{T}_{\mathcal{I}}$ with $q := 1 - p$,*

$$pA^\Phi - A^\Phi(\sigma_p) \leq pq(pA^\Phi(\tau) + qA^\Phi); \quad (38)$$

Proof. Inequality (38) is a rewriting of (18). Namely, for any $(u, v) \in \mathcal{T}_{\mathcal{I}}$,

$$pA^\Phi(u, v) - A^\Phi(\sigma_p(u, v)) = \mathbf{Ent}_{\mathcal{B}(1,p)}^\Phi(f)$$

where $(f(0), f(1)) := (u, u + v)$, and also

$$\mathbf{E}_{\mathcal{B}(1,p)}(A^\Phi(f, Df)) = pA^\Phi(\tau(u, v)) + qA^\Phi(u, v).$$

\square

Some of the Lemmas above correct mistakes discovered by the author in [Cha04] after publication. In contrary to what appears in [Cha04, Page 330], $\mathcal{H}(\mathbf{2}')$ does not imply $\mathcal{H}(\mathbf{2})$. Actually, $\mathcal{H}(\mathbf{1})$ and $\mathcal{H}(\mathbf{2})$ are equivalent and $\mathcal{H}(\mathbf{2}')$ should be removed from [Cha04]. In particular, [Cha04, Remarks 8 and 11 pages 354-356] should be replaced by Lemma (4.1). These corrections are minor and simplifying and do not impact the results of [Cha04] at all.

Acknowledgement

The author would like to sincerely thank his friend and colleague Florent Malrieu for numerous stimulating discussions in Rennes, Saint-Malon sur Mel, and Toulouse.

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