Evaluating convolutions of divisor sums with quasimodular forms

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EVALUATING CONVOLUTION SUMS OF THE DIVISOR FUNCTION BY QUASIMODULAR FORMS

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A . We compute many sums involving divisor functions using the theory of quasimodular forms. Some of our identities were not known, others were only partially known. The method is systematic.

1. I

1.1. **Results.** Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For n and j in $\mathbb{N}^*$ we set

$$\sigma_j(n) := \sum_{d|n} d^j$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_j(n) = 0$. For $N \in \mathbb{N}^*$ we define

$$W_N(n) := \sum_{m<n/N} \sigma_1(m)\sigma_1(n-Nm)$$

where m runs through the positive integers $< n/N$. We call W_N the convolution of level N (of the divisor function). We present a method (introduced in [LR05]) to compute some of these sums using quasimodular forms. In particular, we evaluate W_6, W_7, W_8 and W_{10} which were not known and recover W_N for $N \in \{1,2,3,4,5,9\}$ by a systematic method. The first evaluation of W_9 is due to Williams [Wil05, Theorem 1.1].

For $N \in \mathbb{N}^*$, we denote by $\Delta_{4,N}$ the unique cuspidal form spanning the cuspidal subspace of the modular forms of weight 4 on $\Gamma_0(N)$ with Fourier development[1] $\Delta_{4,N}(z) = e^{2\pi iz} + O\left(e^{4\pi iz}\right)$. We define

$$\Delta_{4,N}(z) = \sum_{n=1}^{+\infty} \tau_{4,N}(n) e^{2\pi inz}.$$

Theorem 1. *Let $n \in \mathbb{N}^*$, then*

$$W_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n),$$

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right),$$

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[1] In this paper, "Fourier development" always means "Fourier development at the cusp ∞.

$$W_3(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{12}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{3}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{3}\right),$$

$$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right),$$

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) - \frac{1}{20}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{5}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{5}\right) - \frac{1}{130}\tau_{4,5}(n),$$

$$W_6(n) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{24}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{6}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{6}\right) - \frac{1}{120}\tau_{4,6}(n),$$

$$W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{28}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{7}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{7}\right) - \frac{1}{70}\tau_{4,7}(n),$$

$$W_8(n) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) - \frac{1}{32}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{8}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{8}\right) - \frac{1}{64}\tau_{4,8}(n),$$

$$W_9(n) = \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{36}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{9}\right) - \frac{1}{54}\tau_{4,9}(n).$$

$$W_{10}(n) = \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{10}\right) - \frac{1}{40}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{10}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{10}\right) - \frac{1}{120}\tau_{4,10}(n) - \frac{3}{260}\tau_{4,5}(n) - \frac{3}{65}\tau_{4,5}\left(\frac{n}{2}\right).$$

For dimensional reasons, the forms $\Delta_{4,N}$ are primitive forms for $N \in \{5, \dots, 10\}$. It follows that the arithmetical functions $\tau_{4,N}$ are multiplicative and satisfy the relation (2) (see below). Moreover, if

$$\eta(z) := e^{i\pi z/12} \prod_{n=1}^{+\infty} \left[1 - e^{2\pi i n z}\right]$$

then, following [Koi84], one obtains

$$\begin{aligned}
\Delta_{4,5}(z) &= \left[\eta(z)\eta(5z)\right]^4, \\
\Delta_{4,6}(z) &= \left[\eta(z)\eta(2z)\eta(3z)\eta(6z)\right]^2, \\
\Delta_{4,8}(z) &= \left[\eta(2z)\eta(4z)\right]^4, \\
\Delta_{4,9}(z) &= \left[\eta(3z)\right]^8,
\end{aligned}$$

whereas $\Delta_{4,7}$ and $\Delta_{4,10}$ *are not* products of the η function. However, using M [BCP97] (see [Ste04] for the algorithms based on the computation of the spectrum of Hecke operators on modular symbols), one can compute their Fourier coefficients.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,7}(n)$	1	−1	−2	−7	16	2	−7	15	−23	−16	−8
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,7}(n)$	14	28	7	−32	41	54	23	−110	−112	14	8

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,10}(n)$	1	2	−8	4	5	−16	−4	8	37	10	12
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,10}(n)$	−32	−58	−8	−40	16	66	74	−100	20	32	24

An interesting historic of this topics may be found in [Wil05].

Our method, with the introduction of Dirichlet characters, allows also to recover a second result of Williams [Wil05, Theorem 1.2] and to extend a result of Melfi [Mel98, Theorem 2, (7)]. For $b \in \mathbb{N}^*$ and $a \in \{0, \ldots, b-1\}$, we define

$$S[a,b](n) \coloneqq \sum_{\substack{m=0 \\ m \equiv a \pmod{b}}}^{n} \sigma_1(m)\sigma_1(n-m).$$

We compute $S[i,3]$ for $i \in \{0,1,2\}$. Our result uses the primitive Dirichlet character χ_3 defined by

$$\chi_3(n) \coloneqq \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv -1 \pmod 3 \end{cases}$$

for all $n \in \mathbb{N}^*$.

Theorem 2. *Let $n \in \mathbb{N}^*$, then,*

$$\begin{aligned}
S[0,3](n) = {} & \\
& \frac{11}{72}\sigma_3(n) + \frac{25}{18}\sigma_3\left(\frac{n}{3}\right) - \frac{9}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{4}n\sigma_1(n) - n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{4}n\sigma_1\left(\frac{n}{9}\right) \\
& \qquad + \frac{1}{24}\left[1 + \delta(3 \mid n)\right]\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n),
\end{aligned}$$

$$S[1,3](n) = \frac{19}{144}\sigma_3(n) + \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\delta(3 \mid n-1)\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n).$$

and

$$S[2,3](n) = \frac{19}{144}\sigma_3(n) - \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{8}n\sigma_1(n) + \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\delta(3 \mid n-2)\sigma_1(n) - \frac{1}{9}\tau_{4,9}(n).$$

where $\delta(3 \mid n)$ *is* 1 *if* 3 *divides* n *and* 0 *otherwise.*

We next consider convolutions of different divisor sums and complete results of Melfi [Mel98, Theorem 2, (9), (10)], [HOSW02, Theorem 6]. We shall use the unique cuspidal form $\Delta_{8,2}$ spanning the cuspidal subspace of the modular forms of weight 8 on $\Gamma_0(2)$ with Fourier development $\Delta_{8,2}(z) = e^{2\pi iz} + O\left(e^{4\pi iz}\right)$. Using [Koi84], we have

$$\Delta_{8,2}(z) = [\eta(z)\eta(2z)]^8.$$

We define

$$\Delta_{8,2}(z) = \sum_{n=1}^{+\infty} \tau_{8,2}(n)e^{2\pi inz}.$$

This is again a primitive form, hence the arithmetical function $\tau_{8,2}$ is multiplicative and satisfies the relation (2) (see below).

Theorem 3. *Let* $n \in \mathbb{N}^*$. *Then,*

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_3(n-k) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_3(k) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_3\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_3(n-2k) = \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{16}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1\left(\frac{n}{2}\right).$$

Moreover,

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_5(n-k) = \frac{5}{126}\sigma_7(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_5(n-2k) =$$
$$\frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{24}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}\tau_{8,2}(n),$$

and

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_5(k) =$$
$$\frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{12}n\sigma_5\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}\tau_{8,2}(n).$$

Our method allows also to evaluate sums of Lahiri's type

$$S[(a_1,\dots,a_r),(b_1,\dots,b_r),(N_1,\dots,N_r)](n) \coloneqq \sum_{\substack{(m_1,\dots,m_r)\in\mathbb{N}^r \\ m_1+\dots+m_r=n}} m_1^{a_1}\dots m_r^{a_r}\sigma_{b_1}\left(\frac{m_1}{N_1}\right)\dots\sigma_{b_r}\left(\frac{m_r}{N_r}\right) \tag{1}$$

[HOSW02, §3] where the a_i are nonnegative integers, the N_i are positive integers and the b_i are odd positive integers. To simplify the notations, we introduce

$$S[(a_1,\dots,a_r),(b_1,\dots,b_r)](n) \coloneqq S[(a_1,\dots,a_r),(b_1,\dots,b_r),(1,\dots,1)](n).$$

For example, we prove the following.

Theorem 4. *Let $n \in \mathbb{N}^*$. Then,*

$$S[(0,1,1),(1,1,1)](n) = \frac{1}{288}n^2\sigma_5(n) - \frac{1}{72}n^3\sigma_3(n) + \frac{1}{288}n^2\sigma_3(n) + \frac{1}{96}n^4\sigma_1(n) - \frac{1}{288}n^3\sigma_1(n).$$

and

$$\begin{aligned} -24^5 S[(0,0,0,1,1),(1,1,1,1,1)](n) = &-\frac{48}{5}n^2\sigma_9(n) + 128n^3\sigma_7(n) \\ &- 80n^2\sigma_7(n) - 600n^4\sigma_5(n) \\ &+ 648n^3\sigma_5(n) + \frac{8208}{7}n^5\sigma_3(n) \\ &- 144n^2\sigma_5(n) - \frac{11232}{7}n^4\sigma_3(n) - \frac{3456}{5}n^6\sigma_1(n) \\ &+ 576n^3\sigma_3(n) + \frac{5184}{5}n^5\sigma_1(n) \\ &- 432n^4\sigma_1(n) - 48n^2\sigma_3(n) \\ &+ 48n^3\sigma_1(n) + \frac{8}{35}n\tau(n) - \frac{8}{35}\tau(n). \end{aligned}$$

We continue our evaluations by the more complicated sum $S[(0,1),(1,1),(2,5)]$. The reason why it is more difficult is that the underlying space of new cuspidal modular forms has dimension 3.

The space of newforms of weight 6 on $\Gamma_0(10)$ has dimension 3. Let $\{\Delta_{6,10,i}\}_{1\leq i\leq 3}$ be the unique basis of primitive forms with

$$\Delta_{6,10,1}(z) = e^{2\pi iz} + 4e^{4\pi iz} + 6e^{6\pi iz} + O(e^{8\pi iz}),$$
$$\Delta_{6,10,2}(z) = e^{2\pi iz} - 4e^{4\pi iz} + 24e^{6\pi iz} + O(e^{8\pi iz}),$$
$$\Delta_{6,10,3}(z) = e^{2\pi iz} - 4e^{4\pi iz} - 26e^{6\pi iz} + O(e^{8\pi iz}).$$

Again, by [Koi84], we know that these functions *are not* products of the η function. We denote by $\tau_{6,10,i}(n)$ the nth Fourier coefficient of $\Delta_{6,10,i}$. Note that the sequences $\tau_{6,10,i}$ are multiplicative. Again, Stein's algorithms on M give the following tables.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,1}(n)$	1	4	6	16	−25	24	−118	64	−207	−100	192
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,1}(n)$	96	1106	−472	−150	256	762	−828	−2740	−400	−708	768

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,2}(n)$	1	−4	24	16	25	−96	−172	−64	333	−100	132
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,2}(n)$	384	−946	688	600	256	−222	−1332	500	400	−4128	−528

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,3}(n)$	1	−4	−26	16	−25	104	−22	−64	433	100	−768
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,3}(n)$	−416	−46	88	650	256	378	−1732	1100	−400	572	3072

We also need the unique primitive form

$$\Delta_{6,5}(z) = \sum_{n=1}^{+\infty} \tau_{6,5}(n) e^{2\pi inz}$$

of weight 6 on $\Gamma_0(5)$. It is not a product of the η function and, its first Fourier coefficients are given in the following table.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,5}(n)$	1	2	−4	−28	25	−8	192	−120	−227	50	−148
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,5}(n)$	112	286	384	−100	656	−1678	−454	1060	−700	−768	−296

Proposition 5. *Let $n \in \mathbb{N}^*$. Define*

$$A(n) = \frac{12}{13}n\sigma_3(n) + \frac{48}{13}n\sigma_3\left(\frac{n}{2}\right) + \frac{300}{13}n\sigma_3\left(\frac{n}{5}\right) + \frac{1200}{13}n\sigma_3\left(\frac{n}{10}\right)$$
$$B(n) = -\frac{48}{5}n^2\sigma_1\left(\frac{n}{2}\right) - 48n^2\sigma_1\left(\frac{n}{5}\right)$$
$$C(n) = 24n\sigma_1\left(\frac{n}{5}\right)$$
$$D(n) = \frac{12}{5}n\tau_{4,10}(n) - \frac{216}{65}n\tau_{4,5}(n) - \frac{864}{65}n\tau_{4,5}\left(\frac{n}{2}\right)$$
$$E(n) = \frac{108}{35}\tau_{6,5}(n) + \frac{864}{35}\tau_{6,5}\left(\frac{n}{2}\right)$$
$$F(n) = -\frac{24}{5}\tau_{6,10,1}(n) + \frac{12}{7}\tau_{6,10,2}(n).$$

Then,

$$5 \times 24^2 \sum_{\substack{(a,b)\in\mathbb{N}^{*2} \\ 2a+5b=n}} b\sigma_1(a)\sigma_1(b) = A(n) + B(n) + C(n) + D(n) + E(n) + F(n).$$

In each of our previous examples, we did not leave the field of rational numbers. This may not happen since, the primitive forms have no necessarily rational coefficients. However, every evaluation will make use of totally real algebraic numbers for coefficients since the extension of $\mathbb{Q}$ by the Fourier coefficients of a primitive form is finite and totally real [Shi72, Proposition 1.3]. We shall evaluate $S[(1,1),(1,1),(1,5)]$. Let t be one of the two roots of $X^2 - 20X + 24$. There exist three primitive forms of weight 8 on $\Gamma_0(5)$ determined by the beginning of their Fourier development:

$$\Delta_{8,5,1}(z) = e^{2\pi iz} - 14e^{4\pi iz} + O(e^{6\pi iz})$$
$$\Delta_{8,5,2}(z) = e^{2\pi iz} + (20-t)e^{4\pi iz} + O(e^{6\pi iz})$$
$$\Delta_{8,5,3}(z) = e^{2\pi iz} + te^{4\pi iz} + O(e^{6\pi iz}).$$

The function $\Delta_{8,5,3}$ is obtained from $\Delta_{8,5,2}$ by application of the conjugation of $\mathbb{Q}(t)$ (*i.e* $t \mapsto 20 - t$) on the Fourier coefficients. We denote by $\tau_{8,5,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{8,5,i}$.

n	1	2	3	4	5	6	7	8	9
$\tau_{8,5,1}(n)$	1	−14	−48	68	125	672	−1644	840	117
n	10	11	12	13	14	15	16	17	18
$\tau_{8,5,1}(n)$	−1750	172	−3264	3862	23016	−6000	−20464	−12254	−1638

n	1	2	3	4	5
$\tau_{8,5,2}(n)$	1	$-t+20$	$8t-70$	$-20t+248$	−125
n	6	7	8	9	10
$\tau_{8,5,2}(n)$	$70t-1208$	$-56t+510$	$-120t+1920$	$160t+1177$	$125t-2500$
n	11	12	13	14	15
$\tau_{8,5,2}(n)$	$-400t+6272$	$184t-13520$	$608t-4310$	$-510t+8856$	$-1000t+8750$

Proposition 6. *Let $n \in \mathbb{N}^*$. Define*

$$A(n) = \frac{24}{13} n^2 \sigma_3(n) + \frac{600}{13} n^2 \sigma_3\left(\frac{n}{5}\right)$$
$$B(n) = -\frac{24}{5} n^3 \sigma_1(n) - 24 n^3 \sigma_1\left(\frac{n}{5}\right)$$
$$C(n) = -\frac{288}{325} n^2 \tau_{4,5}(n)$$
$$D(n) = \frac{792 + 12t}{475} \tau_{8,5,2}(n) + \frac{1032 - 12t}{475} \tau_{8,5,3}(n).$$

Then

$$5 \times 24^2 \sum_{\substack{(a,b) \in \mathbb{N}^{*2} \\ a+5b=n}} ab\sigma_1(a)\sigma_1(b) = A(n) + B(n) + C(n) + D(n).$$

Remark 1. The two terms in the right hand side of the definition of $D(n)$ in proposition 6 being conjugate, we have

$$D(n) = \mathrm{tr}_{\mathbb{Q}(t)/\mathbb{Q}}\left[\frac{792 + 12t}{475} \tau_{8,5,2}(n)\right] \in \mathbb{Q}.$$

To stay in the field of rational numbers, we could have used the fundamental fact that, for every even $k > 0$ and every integer $N \geq 1$, the space of cuspidal forms of weight k on $\Gamma_0(N)$ has a basis whose elements have a Fourier development with integer coefficients [Shi94, Theorem 3.52]. However, the coefficients of these Fourier developments are often not multiplicative: this is a good reason to leave $\mathbb{Q}$.

Remark 2. If τ_* is one of our τ functions, its values are the Fourier coefficients of a primitive forms (of weight k on $\Gamma_0(N)$ say). It therefore satisfies the following multiplicativity relation

$$\tau_*(mn) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \mu(d) d^{k-1} \tau_*\left(\frac{m}{d}\right) \tau_*\left(\frac{n}{d}\right). \tag{2}$$

1.2. **Method.** Since our method is based on quasimodular forms (anticipated by Rankin [Ran56] and formally introduced by Kaneko & Zagier in [KZ95]), we briefly recall the basics on these functions, refering to [MR05] and [LR05] for the details.

Define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4,\ ad - bc = 1,\ N \mid c \right\}$$

for all integers $N \geq 1$. In particular, $\Gamma_0(1)$ is $\mathrm{SL}(2,\mathbb{Z})$. Denote by $\mathscr{H}$ the Poincaré upper half plane:

$$\mathscr{H} = \{z \in \mathbb{C} :\ \Im m\, z > 0\}.$$

Definition 7. *Let $N \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $s \in \mathbb{N}^*$. A holomorphic function*

$$f : \mathscr{H} \to \mathbb{C}$$

is a quasimodular form of weight k, depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathscr{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z)\left(\frac{c}{cz+d}\right)^i \tag{3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

Here is what is meant by the requirement for f_s to be holomorphic at the cusps. One can show [MR05, Lemme 119] that if f satisfies the quasimodularity condition (3), then f_s satisfies the modularity condition

$$(cz+d)^{-(k-2s)} f_s\left(\frac{az+b}{cz+d}\right) = f_s(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Asking f_s to be holomorphic at the cusps is asking that, for all $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(1)$, the function

$$z \mapsto (\gamma z+\delta)^{-(k-2s)} f_s\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$$

has a Fourier expansion of the form

$$\sum_{n=0}^{+\infty} \widehat{f_{s,M}}(n) e\left(\frac{nz}{u_M}\right)$$

where

$$u_M = \inf\{u \in \mathbb{N}^* \colon T^u \in M^{-1}\Gamma_0(N)M\}.$$

In other words, f_s is automatically a modular function and is required to be more than that, a modular form of weight $k-2s$ on $\Gamma_0(N)$. It follows that if f is a quasimodular form of weight k and depth s, non identically vanishing, then k is even and $s \leq k/2$.

A fundamental quasimodular form is the Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24\sum_{n=1}^{+\infty} \sigma_1(n) e^{2\pi i n z}.$$

It is a quasimodular form of weight 2, depth 1 on $\Gamma_0(1)$ (see *e.g.* [Ser77, Chapter 7]).

We shall denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N)]$ the space of quasimodular forms of weight k, depth $\leq s$ on $\Gamma_0(N)$ and $M_k[\Gamma_0(N)] = \widetilde{M}_k^{\leq 0}[\Gamma_0(N)]$ the space of modular forms of weight k on $\Gamma_0(N)$.

Our method for theorem 1 is to remark that the function

$$\begin{aligned} H_N(z) &= E_2(z)E_2(Nz) \\ &= 1 - 24\sum_{n=1}^{+\infty}\left[\sigma_1(n) + \sigma_1\left(\frac{n}{N}\right)\right] e^{2\pi i n z} + 576\sum_{n=1}^{+\infty} W_N(n) e^{2\pi i n z} \end{aligned}$$

is a quasimodular form of weight 4, depth 2 on $\Gamma_0(N)$ that we linearize using the following lemma.

Lemma 8. *Let $k \geq 2$ even. Then,*

$$\widetilde{M}_k^{\leq k/2}[\Gamma_0(N)] = \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i}[\Gamma_0(N)] \oplus \mathbb{C} D^{k/2-1} E_2.$$

We have set

$$D := \frac{1}{2i\pi} \frac{\mathrm{d}}{\mathrm{d}z}.$$

Let $\{B_k\}_{k\in\mathbb{N}}$ be the sequence of rational numbers defined by its exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{+\infty} B_k \frac{t^k}{k!}.$$

We shall use the Eisenstein series to express the needed basis:

$$E_{k,N}(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_3(n) e^{2\pi i n N z} \in M_k[\Gamma_0(N)]$$

for all $k \in 2\mathbb{N}^* + 2$, $N \in \mathbb{N}^*$. If $N = 1$ we simplify by writing $E_k := E_{k,N}$. For weight 2 forms, we shall need

$$\Phi_{a,b}(z) = \frac{1}{b-a} \left[bE_2(bz) - aE_2(az) \right] \in M_2[\Gamma_0(b)]$$

for all $b > 1$ and $a \mid b$.

Let χ be a Dirichlet character. If f satisfies all of what is needed to be a quasimodular form except (3) being replaced by

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \chi(d) \sum_{i=0}^{n} f_i(z) \left(\frac{c}{cz+d}\right)^i,$$

then, one says that f is a quasimodular form of weight k, depth s and character χ on $\Gamma_0(N)$. (In particular, we ask f_s to be a modular form of character χ). We denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi]$ the vector space of quasimodular forms of weight k, depth $\leq s$ and character χ on $\Gamma_0(N)$. If $\chi = \chi_0$ is a principal character to a modulus dividing N, then $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi] = \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$.

If $f \in \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$, then f has a Fourier development with Fourier coefficients $\{\widehat{f(n)}\}_{n\in\mathbb{N}}$. We define the twist of f by the Dirichlet character χ by

$$f \otimes \chi(z) = \sum_{n=0}^{+\infty} \chi(n) \widehat{f(n)} e^{2\pi i n z}.$$

In [LR05, Proposition 9], we proved the following proposition.

Proposition 9. *Let χ be a primitive Dirichlet character of conductor m. Let f be a quasimodular form of weight k and depth s on $\Gamma_0(N)$. Then $f \otimes \chi$ is a quasimodular form of weight k, depth less than or equal s and character χ^2 on $\Gamma_0\left(\mathrm{lcm}(N, m^2)\right)$.*

Remark 3. The condition of primitivity of the character may be replaced by the condition of non annulation of its Gauss sum.

The proof of theorem 2 follows from the linearization of $E_2 \cdot E_2 \otimes \chi_3$.

Theorems 3 and 4 follow from the linearization of derivatives of forms of type $E_j E_{k,N}$.

1.3. **Generalisation of the results.** For $N \geq 1$ and $k \geq 2$, let $A^*_{N,k}$ be the set of triples (ψ, ϕ, t) such that ψ is a primitive Dirichlet character of modulus L, ϕ is a primitive Dirichlet character of modulus M and t is an integer such that $tLM \mid N$ (and $tLM \neq 1$ if $k = 2$) with the extra condition

$$\psi\phi(n) = \begin{cases} 1 & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N}^*). \tag{4}$$

We write $\mathbf{1}$ for the primitive character of modulus 1 (the constant function $n \mapsto 1$). We extend the definition of σ_k: for k and n in $\mathbb{N}^*$ we set

$$\sigma_k^{\psi,\phi}(n) := \sum_{d|n} \psi\left(\frac{n}{d}\right)\phi(d)d^k$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_k^{\psi,\phi}(n) = 0$. If M is the modulus of the primitive character ϕ, we define the sequence $\{B_k^\phi\}_{k\in\mathbb{N}}$ by its exponential generating function

$$\sum_{c=0}^{M-1} \phi(c)\frac{te^{ct}}{e^{Mt}-1} = \sum_{k=0}^{+\infty} B_k^\phi \frac{t^k}{k!}.$$

For any $(\psi, \phi, t) \in A^*_{N,k}$, define

$$E_k^{\psi,\phi}(z) := \delta(\psi = \mathbf{1}) - \frac{2k}{B_k^\phi}\sum_{n=1}^{+\infty} \sigma_{k-1}^{\psi,\phi}(n)e^{2\pi inz}$$

and

$$E_{k,t}^{\psi,\phi}(z) := \begin{cases} E_k^{\psi,\phi}(tz) & \text{if } (k, \psi, \phi) \neq (2, \mathbf{1}, \mathbf{1}) \\ E_2^{\mathbf{1},\mathbf{1}}(z) - tE_2^{\mathbf{1},\mathbf{1}}(tz) & \text{otherwise} \end{cases}$$

where $\delta(\psi = \mathbf{1})$ is 1 if $\psi = \mathbf{1}$ and 0 otherwise.

For $N \geq 1$ and $k \geq 2$ even, the set

$$\left\{E_{k,t}^{\psi,\phi} : (\psi, \phi, t) \in A^*_{N,k}\right\}$$

is a basis for the orthogonal subspace (called Eisenstein subspace, the scalar product being the Petersson one) of the cuspidal subspace $S_k[\Gamma_0(N)]$ of $M_k[\Gamma_0(N)]$ [DS05, Chapter 4].

Moreover, by Atkin-Lehner-Li theory [DS05, Chapter 5], a basis for $S_k[\Gamma_0(N)]$ is

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \alpha_d\left(H_k^*[\Gamma_0(M)]\right)$$

where α_d is

$$\begin{array}{rccc} \alpha_d & : & M_k[\Gamma_0(M)] & \to & M_k[\Gamma_0(M)] \\ & & f & \mapsto & [z \mapsto f(dz)] \end{array}$$

and $H_k^*[\Gamma_0(M)]$ is the set of primitive forms of weight k on $\Gamma_0(M)$.

A corollary is the following generalisation of theorems 1 and 3. If f is a modular form, we denote by $\{\widehat{f(n)}\}_{n\in\mathbb{N}}$ the sequence of its Fourier coefficients.

Proposition 10. *Let $N \geq 1$. There exist scalars $a_{\psi,\phi,t}$, $a_{M,d,f}$ and a such that, for all $n \geq 1$, we have*

$$W_N(n) = \sum_{(\psi,\phi,t)\in A^*_{N,4}} a_{\psi,\phi,t}\sigma_3^{\psi,\phi}\left(\frac{n}{t}\right) + \sum_{(\psi,\phi,t)\in A^*_{N,2}} a_{\psi,\phi,t}n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right) + an\sigma_1(n)$$
$$+ \sum_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}} \sum_{f\in H_4^*[\Gamma_0(M)]} a_{M,d,f}\widehat{f}\left(\frac{n}{d}\right) + \sum_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}} \sum_{f\in H_2^*[\Gamma_0(M)]} a_{M,d,f}n\widehat{f}\left(\frac{n}{d}\right)$$
$$+ \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{N}\right).$$

More generally, for any $N \geq 1$ and any even $\ell \geq 4$, the arithmetical functions

$$n \mapsto \sum_{k<n/N} \sigma_1(n-kN)\sigma_{\ell-1}(k) - \frac{B_\ell}{2\ell}\sigma_1(n) - \frac{1}{24}\sigma_{\ell-1}\left(\frac{n}{N}\right)$$

and

$$n \mapsto \sum_{k<n/N} \sigma_1(k)\sigma_{\ell-1}(n-kN) - \frac{B_\ell}{2\ell}\sigma_1\left(\frac{n}{N}\right) - \frac{1}{24}\sigma_{\ell-1}(n)$$

are linear combinations of the sets of functions

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell+2}} \left\{n \mapsto \sigma_{\ell+1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell}} \left\{n \mapsto n\sigma_{\ell-1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}} \bigcup_{f\in H^*_{\ell+2}[\Gamma_0(M)]} \left\{n \mapsto \widehat{f}\left(\frac{n}{d}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}} \bigcup_{f\in H^*_{\ell}[\Gamma_0(M)]} \left\{n \mapsto n\widehat{f}\left(\frac{n}{d}\right)\right\}.$$

The same allows to generalize theorem 2. If $b \geq 1$ is an integer, denote by $X(b)$ the set of Dirichlet character of modulus b. By orthogonality, we have

$$S[a,b](n) = \frac{1}{\varphi(b)} \sum_{\chi\in X(b)} \overline{\chi(a)} \sum_{m=1}^{n-1} \chi(m)\sigma_1(m)\sigma_1(n-m).$$

It follows that the function to be considered is now

$$\frac{1}{\varphi(b)} \sum_{\chi\in X(b)} \overline{\chi(a)} E_2 \cdot E_2 \otimes \chi.$$

We restrict to b squarefree so that the Gauss sum associates to any character of modulus b is non vanishing. For $N \geq 1$, let $\chi_N^{(0)}$ be the principal character

of modulus N. For $\chi \in X(b)$, we define $A^*_{N,k,\chi}$ as $A^*_{N,k}$ except we replace condition (4) by

$$\psi\phi = \chi_N^{(0)}\chi.$$

Then, similary to the proposition 10, we have the following proposition.

Proposition 11. *Let $b \geq 1$ squarefree and $a \in [0, b-1]$ be integers. Then, the function*

$$n \mapsto S[a,b](n) - \frac{1}{24}[\delta(b \mid a) + \delta(b \mid n-a)]\sigma_1(n)$$

is a linear combination of the set of functions

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t) \in A^*_{N,4,\chi}} \left\{ n \mapsto \sigma_3^{\psi,\phi}\left(\frac{n}{t}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t) \in A^*_{N,2,\chi}} \left\{ n \mapsto n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M) \in \mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_4^*[\Gamma_0(M), \chi_N^{(0)}\chi]} \left\{ n \mapsto \widehat{f}\left(\frac{n}{d}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M) \in \mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_2^*[\Gamma_0(M), \chi_N^{(0)}\chi]} \left\{ n \mapsto n\widehat{f}\left(\frac{n}{d}\right) \right\}$$

$$\{ n \mapsto n\sigma_1(n) \}$$

where N is the least common multiple of 2 and b^2 and $\delta(b \mid n-a)$ is 1 if $n \equiv a \pmod{b}$ and 0 otherwise.

2. C		3, 5, 6, 7, 8, 9	10

2.1. **Level 3.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(3)] = M_4[\Gamma_0(3)] \oplus DM_2[\Gamma_0(3)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(3)]$ has dimension 2 and is spanned by the two linearily independant forms E_4 and $E_{4,3}$. The vector space $M_2[\Gamma_0(3)]$ has dimension 1 and is spanned by $\Phi_{1,3}$. By computation of the first Fourier coefficients, it follows that

$$H_3 = \frac{1}{10}E_4 + \frac{9}{10}E_{4,3} + 4D\Phi_{1,3} + 4DE_2. \tag{5}$$

The comparison of the Fourier development in (5) leads to the corresponding result in theorem 1.

2.2. **Level** 5**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(5)] = M_4[\Gamma_0(5)] \oplus DM_2[\Gamma_0(5)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(5)]$ has dimension 3 and is spanned by the linearily independant forms E_4, $E_{4,5}$ and $\Delta_{4,5}$. The vector space $M_2[\Gamma_0(5)]$ has dimension 1 and is spanned by $\Phi_{1,5}$. By computation of the first Fourier coefficients, it follows that

$$H_5 = \frac{1}{26}E_4 + \frac{25}{26}E_{4,5} - \frac{288}{65}\Delta_{4,5} + \frac{24}{5}D\Phi_{1,5} + \frac{12}{5}DE_2 \tag{6}$$

The comparison of the Fourier development in (6) leads to the corresponding result in theorem 1.

2.3. **Level** 6**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(6)] = M_4[\Gamma_0(6)] \oplus DM_2[\Gamma_0(6)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(6)]$ has dimension 5 and is spanned by the five linearily independant forms E_4, $E_{4,2}$,$E_{4,3}$ $E_{4,6}$ and $\Delta_{4,6}$. The vector space $M_2[\Gamma_0(6)]$ has dimension 3 and is spanned by the three linearily independant forms $\Phi_{1,2}$, $\Phi_{1,3}$ and $\Phi_{3,6}$. By computation of the first Fourier coefficients, it follows that

$$H_6 = \frac{1}{50}E_4 + \frac{2}{25}E_{4,2} + \frac{9}{50}E_{4,3} + \frac{18}{25}E_{4,6} - \frac{24}{5}\Delta_{4,6} + 2D\Phi_{1,3} + 3D\Phi_{3,6} + 2DE_2. \tag{7}$$

The comparison of the Fourier development in (7) leads to the corresponding result in theorem 1.

2.4. **Level** 7**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(7)] = M_4[\Gamma_0(7)] \oplus DM_2[\Gamma_0(7)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(7)]$ has dimension 3 and is spanned by the three linearily independant forms E_4, $E_{4,7}$ and $\Delta_{4,7}$. The vector space $M_2[\Gamma_0(7)]$ has dimension 1 and is spanned by the form $\Phi_{1,7}$. By computation of the first Fourier coefficients, it follows that

$$H_7 = \frac{1}{50}E_4 + \frac{49}{50}E_{4,7} - \frac{288}{35}\Delta_{4,7} + \frac{36}{7}D\Phi_{1,7} + \frac{12}{7}DE_2. \tag{8}$$

The comparison of the Fourier development in (8) leads to the corresponding result in theorem 1.

2.5. **Level** 8**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(8)] = M_4[\Gamma_0(8)] \oplus DM_2[\Gamma_0(8)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(8)]$ has dimension 5 and is spanned by the five linearily independant forms E_4, $E_{4,2}$, $E_{4,4}$, $E_{4,8}$ and $\Delta_{4,8}$. The vector space $M_2[\Gamma_0(8)]$ has dimension 3 and is spanned by the forms $\Phi_{1,4}$, $\Phi_{1,8}$ and

$$\Phi_{1,4,2} \coloneqq z \mapsto \Phi_{1,4}(2z).$$

By computation of the first Fourier coefficients, it follows that

$$H_8 = \frac{1}{80}E_4 + \frac{3}{80}E_{4,2} + \frac{3}{20}E_{4,4} + \frac{4}{5}E_{4,8} - 9\Delta_{4,8} + \frac{21}{4}D\Phi_{1,8} + \frac{3}{2}DE_2. \tag{9}$$

The comparison of the Fourier development in (9) leads to the corresponding result in theorem 1.

2.6. **Level** 9**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(9)] = M_4[\Gamma_0(9)] \oplus DM_2[\Gamma_0(9)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(9)]$ has dimension 5 and is spanned by the five linearily independant forms E_4, $E_4 \otimes \chi_3$, $E_{4,3}$, $E_{4,9}$ and $\Delta_{4,9}$. The vector space $M_2[\Gamma_0(9)]$ has dimension 3 and is spanned by the forms $\Phi_{1,3}$, $\Phi_{1,3} \otimes \chi_3$ and $\Phi_{1,9}$.

By computation of the first Fourier coefficients, it follows that

$$H_9 = \frac{1}{90}E_4 + \frac{4}{45}E_{4,3} + \frac{9}{10}E_{4,9} - \frac{32}{3}\Delta_{4,9} + \frac{16}{3}D\Phi_{1,9} + \frac{4}{3}DE_2. \tag{10}$$

The comparison of the Fourier development in (10) leads to the corresponding result in theorem 1.

2.7. **Level** 10**.** By lemma 8, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(10)] = M_4[\Gamma_0(10)] \oplus DM_2[\Gamma_0(10)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(10)]$ has dimension 7 and is spanned by the seven linearily independant forms E_4, $E_{4,2}$, $E_{4,5}$, $E_{4,10}$, $\Delta_{4,10}$, $\Delta_{4,5}$ and

$$\Delta_{4,5,2} \coloneqq z \mapsto \Delta_{4,5}(2z).$$

The vector space $M_2[\Gamma_0(10)]$ has dimension 3 and is spanned by the forms $\Phi_{1,10}$, $\Phi_{1,5}$ and

$$\Phi_{1,5,2} \coloneqq z \mapsto \Phi_{1,5}(2z).$$

By computation of the first Fourier coefficients, it follows that

$$\begin{aligned} H_{10} = \frac{1}{130}E_4 + \frac{2}{65}E_{4,2} + \frac{5}{26}E_{4,5} + \frac{10}{13}E_{4,10} - \frac{24}{5}\Delta_{4,10} - \frac{432}{65}\Delta_{4,5} - \frac{1728}{65}\Delta_{4,5,2} \\ + \frac{27}{5}D\Phi_{1,10} + \frac{6}{5}DE_2. \end{aligned} \tag{11}$$

The comparison of the Fourier development in (11) leads to the corresponding result in theorem 1.

3. C 1, 2, 4

The convolution of level dividing 4 have been evaluated in [LR05, Proposition 7]. We obtained

$$W_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n)$$

from the equality

$$E_2^2 = E_4 + 12DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(1)]$;

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right)$$

from the equality

$$H_2 = \frac{1}{5}E_4 + \frac{4}{5}E_{4,2} + 3D\Phi_{1,2} + 6DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(2)]$; and

$$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right).$$

from the equality

$$H_4 = \frac{1}{20}E_4 + \frac{3}{20}E_{4,2} + \frac{4}{5}E_{4,4} + \frac{9}{2}D\Phi_{1,4} + 3DE_2.$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(4)]$.

4. T

Let $\chi_3^{(0)}$ be the principal character of modulus 3. Noticing that

$$S[0,3](n) = \sum_{a+b=n} \sigma_1(a)\sigma_1(b) - \sum_{a+b=n} \chi_3^{(0)}(a)\sigma_1(a)\sigma_1(b),$$

we consider $E_2^2 - E_2(E_2 \otimes \chi_3^{(0)})$. Since $E_2 \otimes \chi_3^{(0)} \in \widetilde{M}_2^{\leq 1}[\Gamma_0(9), \chi_3^{(0)}] = \widetilde{M}_2^{\leq 1}[\Gamma_0(9)]$, we have

$$E_2^2 - E_2(E_2 \otimes \chi_3^{(0)}) \in \widetilde{M}_4^{\leq 2}[\Gamma_0(9)].$$

We use the same method and notations as in §2.6. We compute

$$E_2^2 - E_2(E_2 \otimes \chi_3^{(0)}) = \frac{11}{30}E_4 + \frac{10}{3}E_{4,3} - \frac{27}{10}E_{4,9} + 32\Delta_{4,9} + 16D\Phi_{1,3} - 16D\Phi_{1,9} + 12DE_2.$$

The evaluation of $S[1,3]$ given in theorem 2 follows by comparison of the Fourier developments.

We compute $S[1,3]$ after having remarked that

$$\frac{\chi_3^{(0)}(n) + \chi_3(n)}{2} = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the function to be linearized here is

$$\frac{1}{2}E_2[E_2 \otimes \chi_3^{(0)} + E_2 \otimes \chi_3]$$

whose nth Fourier coefficient ($n \in \mathbb{N}^*$) is

$$-24\delta(3 \mid n-1)\sigma_1(n) + 576S[1,3](n).$$

This is again a quasimodular form in $\widetilde{M}_4^{\leq 2}[\Gamma_0(9)]$ and, as in §2.6, we linearize it as

$$\frac{19}{60}E_4 + \frac{1}{20}E_4 \otimes \chi_3 - \frac{5}{3}E_{4,3} + \frac{27}{20}E_{4,9} + 32\Delta_{4,9} - 8D\Phi_{1,3} - 6D(\Phi_{1,3} \otimes \chi_3) + 8D\Phi_{1,9}.$$

The evaluation of $S[1,3]$ given in theorem 2 follows by comparison of the Fourier developments.

The evaluation of $S[2,3]$ follows immediately from

$$S[0,3](n) + S[1,3](n) + S[2,3](n) = W_1(n)$$

and theorem 1.

5. O M

The first three identities of theorem 3 are a direct consequence of the following ones:

$$E_2E_4 \in \widetilde{M}_6^{\leq 1}[\Gamma_0(1)] = M_6[\Gamma_0(1)] \oplus DM_4[\Gamma_0(1)] = \mathbb{C}E_6 \oplus \mathbb{C}DE_4,$$

and

$$\begin{aligned} E_2E_{4,2},\, E_4E_{2,2} \in \widetilde{M}_6^{\leq 1}[\Gamma_0(2)] &= M_6[\Gamma_0(2)] \oplus DM_4[\Gamma_0(2)] \\ &= \mathbb{C}E_6 \oplus \mathbb{C}E_{6,2} \oplus \mathbb{C}DE_4 \oplus \mathbb{C}DE_{4,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_4 = E_6 + 3DE_4,$$

$$E_2E_{4,2} = \frac{1}{21}E_6 + \frac{20}{21}E_{6,2} + 3DE_{4,2}$$

and

$$E_4E_{2,2} = \frac{5}{21}E_6 + \frac{16}{21}E_{6,2} + \frac{3}{2}DE_4.$$

The last three identities of theorem 3 are a direct consequence of the following ones:

$$E_2E_6 \in \widetilde{M}_8^{\leq 1}[\Gamma_0(1)] = M_8[\Gamma_0(1)] \oplus DM_6[\Gamma_0(1)] = \mathbb{C}E_8 \oplus \mathbb{C}DE_6,$$

and

$$\begin{aligned} E_{2,2}E_6,\, E_2E_{6,2} \in \widetilde{M}_8^{\leq 1}[\Gamma_0(2)] &= M_8[\Gamma_0(2)] \oplus DM_6[\Gamma_0(2)] \\ &= \mathbb{C}E_8 \oplus \mathbb{C}E_{8,2} \oplus \mathbb{C}\Delta_{8,2} \oplus \mathbb{C}DE_6 \oplus \mathbb{C}DE_{6,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_6 = E_8 + 2DE_6,$$

$$E_{2,2}E_6 = \frac{21}{85}E_8 + \frac{64}{85}E_{8,2} - \frac{2016}{17}\Delta_{8,2} + DE_6$$

and

$$E_2E_{6,2} = \frac{1}{85}E_8 + \frac{84}{85}E_{8,2} - \frac{504}{17}\Delta_{8,2} + 2DE_{6,2}.$$

6. O L ’

For $\mathbf{a} \coloneqq (a_1, \dots, a_r) \in \mathbb{N}^r$, $\mathbf{b} \coloneqq (b_1, \dots, b_r) \in (2\mathbb{N}+1)^r$ and $\mathbf{N} \coloneqq (N_1, \dots, N_r) \in \mathbb{N}^{*r}$ the sum $S[\mathbf{a}, \mathbf{b}, \mathbf{N}]$ defined in (1) is relied to the quasimodular forms *via* the function

$$D^{a_1}E_{b_1+1,N_1} \cdots D^{a_r}E_{b_r+1,N_r} \in \widetilde{M}^{\leq a_1+\dots+a_r+t(\mathbf{b})}_{b_1+\dots+b_r+r+2(a_1+\dots+a_r)}[\Gamma_0(\operatorname{lcm}(N_1, \dots, N_r))] \tag{12}$$

where

$$t(\mathbf{b}) = \#\{i \in \{1, \dots, r\} \colon b_i = 1\}.$$

Since we always can consider that the coordinates of $\mathbf{a}$ are given in increasing order, let ℓ be the nonnegative integer such that $a_1 = \cdots = a_\ell = 0$ and $a_{\ell+1} \neq 0$ (we take $\ell = 0$ if $\mathbf{a}$ has all its coordinates positive). We consider the function

$$\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}} \coloneqq \prod_{j=1}^{\ell}(E_{b_j+1,N_1} - 1) \prod_{j=\ell+1}^{r} D^{a_j} E_{b_j+1,N_j}$$
$$\in \bigoplus_{k=b_{j+1}+\cdots+b_r+r+2(a_{j+1}+\ldots+a_r)-j}^{b_1+\cdots+b_r+r+2(a_{j+1}+\ldots+a_r)} \widetilde{M}_k^{\leq a_1+\ldots+a_r+t(\mathbf{b})}\left[\Gamma_0\left(\operatorname{lcm}(N_1,\ldots,N_r)\right)\right]$$

We have

$$\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}}(z) = (-2)^r \left[\prod_{j=1}^{r} \frac{b_j+1}{B_{b_j+1}}\right] \sum_{n=1}^{+\infty} S[\mathbf{a},\mathbf{b},\mathbf{N}] e^{2\pi i n z}.$$

The evaluation of $S[(0,1,1),(1,1,1)]$ is a consequence (by lemma 8) of

$$(E_1 - 1)(DE_2)^2 \in$$
$$\mathbb{C}E_8 \oplus \mathbb{C}DE_6 \oplus \mathbb{C}D^2E_4 \oplus \mathbb{C}D^3E_2 \oplus \mathbb{C}E_{10} \oplus \mathbb{C}DE_8 \oplus \mathbb{C}D^2E_6 \oplus \mathbb{C}D^3E_4 \oplus \mathbb{C}D^4E_2.$$

The comparison of the first Fourier coefficients leads to

$$(E_2 - 1)(DE_2)^2 = -\frac{1}{5}D^2E_4 - 2D^3E_2 + \frac{2}{21}D^2E_6 + \frac{4}{5}D^3E_4 + 6D^4E_2.$$

Hence the evaluation of $S[(0,1,1),(1,1,1)]$ given in theorem 4.

The evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ is a consequence (by lemma 8) of

$$(E_2 - 1)^3(DE_2)^2 \in \mathbb{C}\Delta \oplus \mathbb{C}D\Delta \bigoplus_{i=1}^{7} \bigoplus_{j=0}^{7-i} \mathbb{C}D^j E_{2i}.$$

The comparison of the first Fourier coefficients leads to

$$\begin{aligned}(E_2 - 1)^3(DE_2)^2 = &-\frac{8}{35}\Delta + \frac{8}{35}D\Delta - 2D^3E_2 + 18D^4E_2 - \frac{216}{5}D^5E_2 + \frac{144}{5}D^6E_2\\ &- \frac{1}{5}D^2E_4 + \frac{12}{5}D^3E_4 - \frac{234}{35}D^4E_4 + \frac{171}{35}D^5E_4\\ &+ \frac{2}{7}D^2E_6 - \frac{9}{7}D^3E_6 + \frac{25}{21}D^4E_6 - \frac{1}{6}D^2E_8 + \frac{4}{15}D^3E_8 + \frac{2}{55}D^2E_{10}.\end{aligned}$$

Hence the evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ given in theorem 4.

We leave the proofs of propositions 5 and 6 to the reader. They are obtained from the linearizations of

$$(E_{2,2} - 1)DE_{2,5} \in \widetilde{M}_4^{\leq 2}[\Gamma_0(5)] \oplus \widetilde{M}_6^{\leq 3}[\Gamma_0(10)]$$

and

$$DE_2DE_{2,5} \in \widetilde{M}_8^{\leq 5}[\Gamma_0(5)].$$

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