Pre-compact families of finite sets of integers and weakly null sequences in Banach spaces
Jordi Lopez Abad, S. Todorcevic

To cite this version:
Jordi Lopez Abad, S. Todorcevic. Pre-compact families of finite sets of integers and weakly null sequences in Banach spaces. 2007. hal-00012297v3

HAL Id: hal-00012297
https://hal.science/hal-00012297v3
Preprint submitted on 9 Jun 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
PRE-COMPACT FAMILIES OF FINITE SETS OF INTEGERS AND WEAKLY NULL SEQUENCES IN BANACH SPACES

J. LOPEZ-ABAD AND S. TODORCEVIC

1. Introduction

The affinities between the infinite-dimensional Ramsey theory and some problems of the Banach space theory and especially those dealing with Schauder basic sequences have been explored for quite some time, starting perhaps with Farahat’s proof of Rosenthal’s $\ell_1$-theorem (see [13] and [19]). The Nash-Williams’ theory though implicit in all this was not fully exploited in this context. In this paper we try to demonstrate the usefulness of this theory by applying it to the classical problem of finding unconditional basic-subsequence of a given normalized weakly null sequence in some Banach space $E$. Recall that Bessaga and Pelczynski [7] have shown that every normalized weakly null sequence in a Banach space contains a subsequence forming a Schauder basis for its closed linear span. However, as demonstrated by Maurey and Rosenthal [16] there exist weakly null sequences in Banach spaces without unconditional basic subsequences. So one is left with a task of finding additional conditions on a given weakly null sequence guaranteeing the existence of unconditional subsequences. One such condition, given by Rosenthal himself around the time of publication of [16] (see also [19]). When put in a proper context Rosenthal’s condition reveals the connection with the Nash-Williams theory. It says that if a weakly null sequence $(x_n)$ in some space of the form $\ell_\infty(\Gamma)$ is such that each $x_n$ takes only the values 0 or 1, then $(x_n)$ has an unconditional subsequence. To see the connection, consider the family

$$\mathcal{F} = \{\{n \in \mathbb{N} : x_n(\gamma) = 1 \} : \gamma \in \Gamma\}$$

and note that $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. As pointed out in [19], Rosenthal result is equivalent saying that there is an infinite subset $M$ of $\mathbb{N}$ such that the trace

$$\mathcal{F}[M] = \{t \cap M : t \in \mathcal{F}\}$$

is hereditary, i.e., it is downwards closed under inclusion. On the other hand, recall that the basic notion of the Nash-Williams’ theory is the notion of a barrier, which is simply a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ no two members of which are related under the inclusion which has the property that an arbitrary infinite subset of $\mathbb{N}$ contains an initial segment in $\mathcal{F}$. Thus, in particular, $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. Though the trace of an arbitrary pre-compact family might be hard to visualize, a trace $\mathcal{B}[M]$ of a barrier $\mathcal{B}$ is easily to compute as it is simply equal to the downwards closure of its restriction

$$\mathcal{B} \upharpoonright M = \{t \in \mathcal{B} : t \subseteq M\}.$$
A further examination of Rosenthal’s result shows that for every pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ there is an infinite set $M$ such that the trace $\mathcal{F}[M]$ is actually equal to the downward closure of a uniform barrier $\mathcal{B}$ on $M$, or in other words that the $\mathbb{C}$-maximal elements of $\mathcal{F}[M]$ form a uniform barrier on $M$. As it turns out, this fact holds considerably more information that the conclusion that $\mathcal{F}[M]$ is merely a hereditary family which is especially noticeable if one needs to perform further refinement of $M$ while keeping track on the original family $\mathcal{F}$. This observation was the motivating point for our research which helped us to realize that further extensions of Rosenthal’s result require analysis of not only pre-compact families of finite subsets of $\mathbb{N}$ but also maps from barriers into pre-compact families of finite subsets of $\mathbb{N}$, or, more generally, into weakly compact subsets of $c_0$. We have explained this point in our previous paper [14], where we have presented various results on partial unconditionality such as near-unconditionality or convex-unconditionality as consequences of the structure theory of this kind of mappings. This paper is as a continuation of this line of research. In Section 3 we show how the combinatorics on barriers can be used to prove the $c_0$-saturation for Banach spaces $C(K)$ when $K$ is a countable compactum. Recall that the $c_0$-saturation of Banach spaces $C(K)$ over countable compacta $K$ is a result originally due to Pełczyński and Semadeni [21] (see also [5] and [12] for recent accounts on this result.) More particularly, we show that if $(x_i) \subseteq C(K)$ is a normalized weakly-null sequence, then there is $C \geq 1$, some infinite set $M$, some uniform barrier $\mathcal{B}$ on $M$ of rank at most the Cantor-Bendixon rank of $K$ and some uniform assignment $\mu : B \to c_0^+$ with the property that $\text{supp} \mu(s) \subseteq s$ for every $s \in B$, and such that for every block sequence $(s_n)$ of elements of $B$, the corresponding sequence $(x(s_n))$ of linear combinations,

$$x(s_n) = \sum_{i \in s_n} (\mu(s_n))(i)x_i,$$

is a normalized block sequence $C$-equivalent to the standard basis of $c_0$.

The last section concerns the following natural measurement of unconditionality present in a given weakly null sequence $(x_n)$ in a general Banach space $E$. Given a family $\mathcal{F}$ of finite sets, we say that $(x_n)$ is $\mathcal{F}$-unconditional with constant at most $C \geq 1$ if for every sequence of scalars $(a_n)$,

$$\sup_{s \in \mathcal{F}} \|\sum_{n \in s} a_n x_n\| \leq C \|\sum_{n \in \mathbb{N}} a_n x_n\|.$$

Thus, if for some infinite subset $M$ of $\mathbb{N}$ the trace $\mathcal{F}[M]$ contains the family of all finite subsets of $M$, the corresponding subsequence $(x_n)_{n \in M}$ is unconditional. Typically, one will not be able to find such a trace, so one is naturally led to study this notion when the family $\mathcal{F}$ is pre-compact, or equivalently, when $\mathcal{F}$ is a barrier. Since for every pair $\mathcal{F}_0$ and $\mathcal{F}_1$ of barriers on $\mathbb{N}$ there is an infinite set $M$ such that $\mathcal{F}_0[M] \subseteq \mathcal{F}_1[M]$ or $\mathcal{F}_1[M] \subseteq \mathcal{F}_0[M]$ and since the two alternatives depend on the ranks of $\mathcal{F}_0$ and $\mathcal{F}_1$, one is also naturally led to the following measurement of unconditionality that refers only to a countable ordinal $\gamma$ rather than a particular barrier of rank $\gamma$. Thus, we say that a normalized basic sequence $(x_n)$ of a Banach space $X$ is $\gamma$-unconditionally saturated with constant at most $C \geq 1$ if there is a $\gamma$-uniform barrier $\mathcal{B}$ on $\mathbb{N}$ such that for every infinite $M \subseteq \mathbb{N}$ there is infinite $N \subseteq M$ such that the corresponding subsequence $(x_n)_{n \in N}$ of $(x_n)$ is $\mathcal{B} \upharpoonright N$-unconditional with constant at most $C$. (Here, $\mathcal{B} \upharpoonright N$ denotes the topological closure of the restriction $\mathcal{B} \upharpoonright N$ which in turn is equal to the trace $\mathcal{B}[N]$, a pleasant property of
any barrier. It turns out that only indecomposable countable ordinals \( \gamma \) matter for this notion. We shall see, extending the well-known example of Maurey-Rosenthal of a normalized weakly-null sequence without unconditional subsequences, that every normalized basic sequence has a subsequence which is \( \omega \)-unconditionally saturated, and that this cannot be extended further. For example, we show that for every indecomposable countable ordinal \( \gamma > \omega \) there is a compactum \( K \) of Cantor-Bendixson rank \( \gamma + 1 \) and a normalized 1-basic weakly-null sequence \( (x_n) \subseteq C(K) \) such that \( (x_n) \) is \( \beta \)-unconditionally saturated for all \( \beta < \gamma \) but not \( \gamma \)-unconditionally saturated. More precisely, the summing basis of \( c_0 \) is finitely block-representable in every subsequence of \( (x_n) \), and so in particular, no subsequence of \( (x_n) \) is unconditional.

2. Preliminaries

Let \( N \) denote the set of all non-negative integers and let \( \text{FIN} \) denote the family of all finite sets of \( N \). The topology on \( \text{FIN} \) is the one induced from the Cantor cube \( 2^N \) via the identification of subsets of \( N \) with their characteristic functions. Observe that this topology coincides with the one induced by \( c_0 \), the Banach space of sequences converging to zero, with the same identification of finite sets and corresponding characteristic functions. Thus, we say that a family \( \mathcal{F} \subseteq \text{FIN} \) is compact if it is a compact space under the induced topology. We say that \( \mathcal{F} \subseteq \text{FIN} \) is pre-compact if its topological closure \( \bar{\mathcal{F}} \) taken in the Cantor cube \( 2^N \) consists only of finite subsets of \( N \). Given \( X, Y \subseteq N \) we write

1. \( X < Y \) iff \( \max X < \min Y \). We will use the convention \( \emptyset < X \) and \( X < \emptyset \) for every \( X \).
2. \( X \subseteq Y \) iff \( X \subseteq Y \) and \( X < Y \setminus X \).

A sequence \( (s_i) \) of finite sets of integers is called a block sequence iff \( s_i < s_j \) for every \( i < j \), and it is called a \( \Delta \)-sequence iff there is some finite set \( s \) such that \( s \subseteq s_i \) \( (i \in \mathbb{N}) \) and \( (s_i \setminus s) \) is a block sequence. The set \( s \) is called the root of \( (s_i) \). Note that \( s_i \to s \) iff for every subsequence of \( (s_i) \) has a \( \Delta \)-subsequence with root \( s \). It follows that the topological closure \( \overline{\mathcal{F}} \) of a pre-compact family \( \mathcal{F} \) of finite subsets of \( N \) is included in its downwards closure

\[
\overline{\mathcal{F}} = \{s \subseteq t : t \in \mathcal{F}\}
\]

with respect to the inclusion relation and also included in its downwards closure

\[
\mathcal{F} = \{s \subseteq t : t \in \mathcal{F}\}
\]

with respect to the relation \( \subseteq \). We say that a family \( \mathcal{F} \subseteq \text{FIN} \) is \( \subseteq \)-hereditary if \( \mathcal{F} = \mathcal{F} \) and \( \subseteq \)-hereditary if \( \mathcal{F} = \overline{\mathcal{F}} \). The \( \subseteq \)-hereditary families will simply be called hereditary families. We shall consider the following two restrictions of a given family \( \mathcal{F} \) of subsets of \( N \) to a finite or infinite subset \( X \) of \( N \)

\[
\mathcal{F} \mid X = \{s \in \mathcal{F} : s \subseteq X\},
\]

\[
\mathcal{F}[X] = \{s \cap X : s \in \mathcal{F}\}.
\]

There are various ways to associate an ordinal index to a pre-compact family \( \mathcal{F} \) of finite subsets of \( N \). All these ordinal indices are based on the fact that for \( n \in \mathbb{N} \), the index of the family

\[
\mathcal{F}(n) = \{s \in \text{FIN} : n < s, \{n\} \cup s \in \mathcal{F}\}
\]
is smaller or equal from that of \( \mathcal{F} \). For example, one may consider the \textit{Cantor-Bendixson index} \( r(\mathcal{F}) \), the minimal ordinal \( \alpha \) for which the iterated Cantor-Bendixson derivative \( \partial^n(\mathcal{F}) \) is equal to \( 0 \), then clearly \( r(\mathcal{F}_{(n)}) \leq r(\mathcal{F}) \) for all \( n \in \mathbb{N} \). Recall that \( \partial \mathcal{F} \) is the set of all proper accumulation points of \( \mathcal{F} \) and that \( \partial^n(\mathcal{F}) = \bigcap_{k<n} \partial(\mathcal{F}) \). The rank is well defined since \( \mathcal{F} \) is countable and therefore a scattered compactum so the sequence \( \partial^k(\mathcal{F}) \) of iterated derivatives must vanish. Observe that if \( \mathcal{F} \) is a nonempty compact, then necessarily \( r(\mathcal{F}) \) is a successor ordinal.

We are now ready to introduce the basic combinatorial concepts of this section. For this we need the following piece of notation, where \( X \) and \( Y \) are subsets of \( \mathbb{N} \)

\[
\cdot X = X \setminus \{ \min X \} \quad \text{and} \quad X/Y = \{ m \in X : \max Y < m \}
\]

The set \( \cdot X \) is called the \textit{shift} of \( X \). Given integer \( n \in \mathbb{N} \), we write \( X/n \) to denote \( X/\{n\} = \{ m \in X : m > n \} \). The following notions have been introduced by Nash-Williams.

**Definition 2.1.** ([15]) Let \( \mathcal{F} \subseteq \text{FIN} \).

1. \( \mathcal{F} \) is called \textit{thin} if \( s \not\subset t \) for every pair \( s, t \) of distinct members of \( \mathcal{F} \).
2. \( \mathcal{F} \) is called \textit{Sperner} if \( s \not\subset t \) for every pair \( s \neq t \in \mathcal{F} \).
3. \( \mathcal{F} \) is called \textit{Ramsey} if for every finite partition \( \mathcal{F} = \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_k \) there is an infinite set \( M \subseteq \mathbb{N} \) such that at most one of the restrictions \( \mathcal{F}_i \upharpoonright M \) is non-empty.
4. \( \mathcal{F} \) is called a \textit{front} on \( M \) if \( \mathcal{F} \subseteq \mathcal{P}(M) \), it is thin, and for every infinite \( N \subseteq M \) there is some \( s \in \mathcal{F} \) such that \( s \subseteq N \).
5. \( \mathcal{F} \) is called a \textit{barrier} on \( M \) if \( \mathcal{F} \subseteq \mathcal{P}(M) \), it is Sperner, and for every infinite \( N \subseteq M \) there is some \( s \in \mathcal{F} \) such that \( s \subseteq N \).

Clearly, every barrier is a front but not vice-versa. For example, the family \( \mathbb{N}^{[k]} \) of all \( k \)-element subsets of \( \mathbb{N} \) is a barrier. The basic result of Nash-Williams [15] says that every front (and therefore every barrier) is Ramsey. Since as we will see soon there are many more barriers than those of the form \( \mathbb{N}^{[k]} \) this is a far reaching generalization of the classical result of Ramsey.

To see a typical application, let \( \mathcal{F} \) be a front on some infinite set \( M \) and consider its partition \( \mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \), where \( \mathcal{F}_0 \) is the family of all \( \subseteq \)-minimal elements of \( \mathcal{F} \). Since \( \mathcal{F} \) is Ramsey there is an infinite \( N \subseteq M \) such that one of the restrictions \( \mathcal{F}_i \upharpoonright M \) is empty. Note that \( \mathcal{F}_1 \upharpoonright N \) must be empty. Since \( \mathcal{F}_0 \upharpoonright N \) is clearly a Sperner family, it is a barrier on \( N \). Thus we have shown that every front has a restriction that is a barrier. Since barrier are more pleasant to work with one might wonder why introducing the notion of front at all. The reason is that inductive constructions lead more naturally to fronts rather than barriers. To get an idea about this, it is instructive to consider the following notion introduced by Pudlák and Rödl.

**Definition 2.2.** ([22]) For a given countable ordinal \( \alpha \), a family \( \mathcal{F} \) of finite subsets of a given infinite set \( M \) is called \textit{\( \alpha \)-uniform on \( M \)} provided that:

(a) \( \alpha = 0 \) implies \( \mathcal{F} = \{ \emptyset \} \),
(b) \( \alpha = \beta + 1 \) implies that \( \mathcal{F}_{(n)} \) is \( \beta \)-uniform on \( M/n \),
(c) \( \alpha > 0 \) limit implies that there is an increasing sequence \( \{ \alpha_n \}_{n \in M} \) of ordinals converging to \( \alpha \) such that \( \mathcal{F}_{(n)} \) is \( \alpha_n \)-uniform on \( M/n \) for all \( n \in M \).

\( \mathcal{F} \) is called uniform on \( M \) if it is \( \alpha \)-uniform on \( M \) for some countable ordinal \( \alpha \).
Remark 2.3. (a) If $\mathcal{F}$ is a front on $M$, then $\mathcal{F} = \mathcal{F}^E$.
(b) If $\mathcal{F}$ is uniform on $M$, then it is a front (though not necessarily a barrier) on $M$.
(c) If $\mathcal{F}$ is $\alpha$-uniform (front, barrier) on $M$ and $\Theta : M \to N$ is the unique order-preserving onto mapping between $M$ and $N$, then $\Theta^* \mathcal{F} = \{ \Theta^* s : s \in \mathcal{F} \}$ is $\alpha$-uniform (front, barrier) on $M$.
(d) If $\mathcal{F}$ is $\alpha$-uniform (front, barrier) on $M$ then $\mathcal{F} \upharpoonright N$ is $\alpha$-uniform (front, barrier) on $N$ for every $N \subseteq M$.
(e) If $\mathcal{F}$ is uniform (front, barrier) on $M$, then for every $s \in \mathcal{F}^E$ the family
$$\mathcal{F}_s = \{ t : s < t \text{ and } s \cup t \in \mathcal{F} \}$$
is uniform (front, barrier) on $M/s$.
(f) If $\mathcal{F}$ is $\alpha$-uniform on $M$, then $\partial^\alpha (\mathcal{F}) = \emptyset$, hence $r(\mathcal{F}) = \alpha + 1$. (Hint: use that $\partial^\beta (\mathcal{F}_{[n]}) = (\partial^\beta (\mathcal{F}))_{[n]}$ for every $\beta$ and every compact family $\mathcal{F}$).
(g) An important example of a $\omega$-uniform barrier on $\mathbb{N}$ is the family $\mathcal{S} = \{ s : |s| = \min(s) + 1 \}$. We call $\mathcal{S}$ a Schreier barrier since its downwards closure is commonly called a Schreier family. Indeed, it can be proved a $\mathcal{B}$ is a $\omega$-uniform family on $M$ iff there is an unbounded mapping $f : M \to \omega$ such that $\mathcal{B} = \{ s \subseteq M : |s| = f(\min s) + 1 \}$.

The following result based on Nash-Williams’ extension of Ramsey’s theorem explains the relationship between the concepts introduced above (see [4] for proofs and fuller discussion).

Proposition 2.4. The following are equivalent for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$:
(a) $\mathcal{F}$ is Ramsey.
(b) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is Sperner.
(c) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or uniform on $M$.
(d) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a front on $M$.
(e) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a barrier on $M$.
(f) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is thin.
(g) There is an infinite $M \subseteq \mathbb{N}$ such that for every infinite $N \subseteq M$ the restriction $\mathcal{F} \upharpoonright N$ cannot be split into two disjoint families that are uniform on $N$. \hfill \Box

In this kind of Ramsey theory one frequently performs diagonalisation arguments that can be formalized using the following notion.

Definition 2.5. An infinite sequence $(M_k)_{k \in \mathbb{N}}$ of infinite subsets of $\mathbb{N}$ is called a fusion sequence of subsets of $M \subseteq \mathbb{N}$ if for all $k \in \mathbb{N}$:
(a) $M_{k+1} \subseteq M_k \subseteq M$,
(b) $m_k < m_{k+1}$, where $m_k = \min M_k$.

The infinite set $M_\infty = \{ m_k \}_{k \in \mathbb{N}}$ is called the fusion set (or limit) of the sequence $(M_k)_{k \in \mathbb{N}}$.

We have also the following simple facts connecting these combinatorial notions with the topological concepts considered at the beginning of this section.

Proposition 2.6. Fix a family $\mathcal{F} \subseteq \text{FIN}$.
(a) If $\mathcal{F}$ is a barrier on $M$ then $\mathcal{F}^\mathcal{E} = \mathcal{F}^\infty = \mathcal{F}$, and hence $\mathcal{F}^\mathcal{E}$ is a compact family.
(b) If $\mathcal{F}$ is a barrier on $M$ then for every $N \subseteq M$, $\mathcal{F} \cap N^\mathcal{E} = \mathcal{F} \cap N$.
(c) Suppose that $\mathcal{F}$ is a barrier on $M$. Then for every $N \subseteq M$ such that $M \setminus N$ is infinite we have that $\mathcal{F}[N] = \mathcal{F}[N]^\mathcal{E}$, and in particular $\mathcal{F}[N]$ is downwards closed.
(d) A family $\mathcal{F} \subseteq M^{<\infty}$ is the topological closure of a barrier on $M$ iff $\mathcal{F}^{\mathcal{E} - \max} = \mathcal{F}^{\mathcal{E} - \max}$ is a barrier on $M$.

Barriers describe small families of finite sets, as it is shown in the following.

**Theorem 2.7.** [14] Let $\mathcal{F} \subseteq \text{FIN}$ be an arbitrary family. Then there is an infinite set $M \subseteq \mathbb{N}$ such that either
(a) $\mathcal{F}[M]$ is the closure of a uniform barrier on $M$, or
(b) $M^{[\infty]} \subseteq \mathcal{F}^\mathcal{E}$.

Note that it follows that if $\mathcal{F}$ is pre-compact then condition (a) must hold.

We shall follow standard terminology and notation when dealing with sequences in Banach spaces (see [13]). We recall now few standard definitions we are going to use along this paper.

**Definition 2.8.** Let $(x_i)$ be a sequence in a Banach space $E$.
(a) $(x_i)$ is called **weakly-null** iff for every $x^* \in E^*$, the sequence of scalars $(x^*(x_i))_i$ tends to 0.
(b) $(x_i)$ is called a **Schauder basis** of $E$ iff for every $x \in E$ there is a unique sequence of scalars $(a_i)$ such that $x = \sum_i a_i x_i$. This is equivalent to say that $x_i \neq 0$ for every $i$, the closed linear span of $(x_i)$ is $X$, and there is a constant $\theta \geq 1$ such that for every sequence of scalars $(a_i)$, and every interval $I \subseteq \mathbb{N}$,
\[
\left\| \sum_{i \in I} a_i x_i \right\| \leq \theta \sum_{i \in \mathbb{N}} |a_i|.
\]
(c) $(x_i)$ is called a basic sequence iff it is a Schauder basis of its closed linear span, i.e., $x_i \neq 0$ for every $i$, and there is $\theta \geq 1$ such that for every sequence of scalars $(a_i)$, and every interval $I \subseteq \mathbb{N}$, $\left\| \sum_{i \in I} a_i x_i \right\| \leq \theta \sum_{i \in \mathbb{N}} |a_i|$. The infimum of those constants $\theta$ is called the basic constant of $(x_i)$.
(d) $(x_i)$ is called **$\theta$-unconditional** (\(\theta \geq 1\)) iff for every sequence of scalars $(a_i)$, and every subset $A \subseteq \mathbb{N}$,
\[
\left\| \sum_{i \in A} a_i x_i \right\| \leq \theta \sum_{i \in \mathbb{N}} |a_i|.
\]
$(x_i)$ is called unconditional if it is $\theta$-unconditional for some $\theta \geq 1$.

Given two basic sequences $(x_i)_{i \in M}$ and $(y_i)_{i \in N}$ of some Banach spaces $E$ and $F$, indexed by the infinite sets $M, N \subseteq \mathbb{N}$, we say that $(x_i)_{i \in M} \subseteq E$ and $(y_i)_{i \in N} \subseteq F$ are $\theta$-equivalent, denoted by $(x_i)_{i \in M} \sim_\theta (y_i)_{i \in N}$, if the order preserving bijection $\Phi$ between the two index-sets $M$ and $N$ lifts naturally to an isomorphism between the corresponding closed linear spans of these sequences sending $x_i$ to $y_{\Phi(i)}$.

The **sequence of evaluation functionals** of $c_0$ is the biorthogonal sequence $(p_i)$ of the natural basis $(e_i)$ of $c_0$, i.e. if $x = \sum_i a_i e_i \in c_0$, then $p_i(x) = a_i$. Note that weakly compact subsets $K$ of $c_0$ are characterized by the property that every sequence in $K$ has a pointwise converging subsequence to an element of $K$. It is clear that for every weakly-compact subset $K \subseteq c_0$ the
restrictions of evaluation mappings \((p_i)\) to \(K\) is weakly-null in \(C(K)\). The sequence of restrictions will also be denoted by \((p_i)\). Observe that \((p_i)\) as a sequence in the Banach space \(C(K)\) is a monotone basic sequence iff \(K\) is closed under restriction to initial intervals.

There are two particularly examples of weakly-compact subsets of \(c_0\) naturally associated to a normalized weakly null sequence \((x_i)_{i\in M}\) of a Banach space \(E\):

(a) the set

\[
R_E((x_i)_{i\in M}) = \{(x^*(x_i))_{i\in M} \in c_0 : x^* \in B_{E^*}\}
\]

is symmetric, 1-bounded and weakly-compact subset of \(c_0\).

(b) If \(E = C(K)\), \(K\) compactum, then the set

\[
R_K((x_i)_{i\in M}) = \{(x_i(e))_{i\in M} \in c_0 : e \in K\}
\]

is also 1-bounded and weakly-compact.

In both cases one has that \((x_i)_{i\in M}\) is 1-equivalent to the evaluation mapping sequences of \(C(R_E((x_i)_{i\in M}))\) and \(C(R_K((x_i)_{i\in M}))\).

We say that a subset \(X\) of \(c_0\) is weakly pre-compact if its closure relative to the weak topology of \(c_0\) is weakly compact. We have then the following, not difficult to prove.

**Proposition 2.9.** (a) \(\mathcal{F} \subseteq \text{FIN}\) is pre-compact iff the set \(\{\chi_s : s \in \text{FIN}\} \subseteq c_0\) of characteristic functions of sets in \(\mathcal{F}\) is weakly-pre-compact.

(b) For every weakly-pre-compact subset \(X\) of \(c_0\) and every \(\varepsilon > 0\) one has that

\[\text{supp}_\varepsilon X = \{n \in \mathbb{N} : |\xi(n)| \geq \varepsilon : \xi \in X\}\] is pre-compact.

Finally, we introduce few combinatorial notions concerning mappings from families of finite sets of integers into \(c_0\). For more details see [14].

**Definition 2.10.** ([14]) Let \(\mathcal{F} \subseteq \text{FIN}\) be an arbitrary family, and let \(f : \mathcal{F} \to c_0\).

(a) \(f\) is internal if for every \(s \in \mathcal{F}\) one has that \(\text{supp}\_f(s) \subseteq s\).

(b) \(f\) is uniform if for every \(t \in \text{FIN}\) one has that

\[|\{\varphi(s)\min(s/t)) \in t \subseteq s, s \in \mathcal{F}\}| = 1\]

(c) \(f\) is Lipschitz if for every \(t \in \text{FIN}\) one has that

\[|\{\varphi(s) : t \subseteq s, s \in \mathcal{F}\}| = 1\]

(d) \(f\) is called a \(U\)-mapping if \(\mathcal{F}\) if it is internal and uniform.

(e) \(f\) is called a \(L\)-mapping if \(\mathcal{F}\) if it is internal and Lipschitz.

**Remark 2.11.** (a) Every uniform mapping is Lipschitz, but the reciprocal is in general false. For example, the mapping \(f : \text{FIN} \to c_0\) defined by \(f(s)(i) = i\) if \(i \in s\) and \(f(s)(i) = 0\) is Lipschitz but not uniform.

(b) Every \(L\)-mapping \(f : \mathcal{F} \to c_0\) can be naturally extended to a continuous mapping \(f' : \mathcal{F}^\mathcal{E} \to c_0\) by setting \(f'(t) = f(s) \upharpoonright t\) for (any) \(s \in \mathcal{F}\) such that \(t \subseteq s\).

(c) The importance of internal mappings can be seen, for example, by the well-known result of Pudlak-Rödl [22] stating that if \(f : B \to X\) is a function defined on a barrier \(B\) on \(M\) then
there is $N \subseteq M$, a barrier $C$ on $N$, and an internal mapping $g : B \upharpoonright N \to C$ such that for every $s, t \in B \upharpoonright N$ one has that $f(s) = f(t)$ if $g(s) = g(t)$.

(d) $U$-mappings were used in [14] to produce some weakly-null sequences playing an important role in the better understanding of an abstract concept of unconditionality (see [14] for more details).

The main result on mappings defined on barriers is the following:

**Theorem 2.12.** [14] Suppose that $B$ is a barrier on $M$, $K \subseteq c_0$ is weakly-compact and suppose that $f : B \to K$. Then for every $\varepsilon > 0$ there is $N \subseteq M$ and there is a $U$-mapping $g : B \upharpoonright N \to c_0$ such that for every $s \in B \upharpoonright N$ one has that

$$\|f(s) \upharpoonright N - g(s)\|_{l_1} \leq \varepsilon.$$

**Corollary 2.13.** Suppose that $f : B \to c_0$ is an internal mapping defined on a barrier $B$. Suppose that in addition $f$ is bounded, i.e. there is $C$ such that for every $s \in B$ one has that $\|f(s)\|_{\infty} \leq C$. Then for every $\varepsilon > 0$ there exists a $U$-mapping $g : B \upharpoonright N \to c_0$ such that for every $s \in B \upharpoonright N$ one has that

$$\|f(s) - g(s)\|_{l_1} \leq \varepsilon.$$

**Proof.** Let us prove first that the image of $f$ is weakly-pre-compact: For suppose that $(f(s_n))_n$ is an arbitrary sequence. Let $M \subseteq \mathbb{N}$ be such that $(\text{supp } f(s_n))_{n \in M}$ converges to some $s \in B_{c_0}$. This is possible because $f$ is internal. Since $f$ is bounded, we can find $N \subseteq M$ such that $(f(s_n))_{n \in N}$ is weak-convergent in $c_0$.

Now the desired result follows from 2.12 by using that $f$ is in addition internal. \qed

3. $c_0$-saturation of $C(K)$ for a countable compactum $K$

Recall the result of Pełczyński and Semadeni [21] which says that every Banach space of the form $C(K)$ for a countable compactum is $c_0$-saturated in the sense that every of its closed infinite-dimensional subspaces contains an isomorphic copy of $c_0$. The purpose of this section is to examine the $c_0$-saturation using the theory of mappings on barriers developed above in Section 3. We start with a convenient reformulation of the problem. We start with a definition.

**Definition 3.1.** For a given subset $X$ of $c_0$, let $\text{supp } X = \{\{i \in \mathbb{N} : \xi(i) \neq 0\} : \xi \in X\}$ be the support set of $X$. We say that a weakly compact subset $K$ of $c_0$ is supported by a barrier on $M$ if its support set $\text{supp } K$ is the closure of a uniform barrier on $M$.

**Lemma 3.2.** Suppose that $K$ is a countable compactum. Suppose that $(x_i)_{i \in M}$ is a normalized weakly null sequence. Then for every $\varepsilon > 0$ there is a subsequence $(x_{i_1})_{i_1 \in M}$ and a weakly-compact subset $L \subseteq c_0$ supported by a barrier on $\mathbb{N}$ of rank not bigger than the Cantor-Bendixson rank of $K$ such that $(x_{i_1})_{i_1 \in M}$ and the evaluation mapping $\langle p_i \rangle_{i \in \mathbb{N}}$ of $C(L)$ are $(1 + \varepsilon)$-equivalent.

**Proof.** Fix $\varepsilon > 0$. Find first an strictly decreasing sequence $(\varepsilon_i)$ such that $\sum \varepsilon_i \leq \varepsilon$ and such that

$$\{\varepsilon_i : i \in \mathbb{N}\} \cap \{x_i(c) : c \in K\} = \emptyset. \quad (3)$$

This is possible because $K$ is countable. Now define $\varphi : K \to \mathcal{P}(\mathbb{N})$ by $\varphi(c) = \{i \in \mathbb{N} : |x_i(c)| \geq \varepsilon_i\}$. Note that (3) implies that $\varphi$ is a continuous function. Enumerate $K = \{c_k\}_{k \in \mathbb{N}}$. ...
Since \((x_i)\) is weakly-null we can find a fusion sequence \((M_k)\) such that for every \(k\) and every \(i \in M_k\) one has that \(|x_i(e_k)| < \varepsilon_k\). Now if we set \(N\) to be the corresponding fusion set then for every \(k\) one has that \(\{i \in M : x_i(e_k) \geq \varepsilon_i\} \subseteq \{n_0, \ldots, n_{k-1}\}\). This means that the mapping \(\psi = \xi_M \cdot \psi\) is continuous with image included in \(\text{FIN}\). Set \(N = \{M\}\) and denote the immediate predecessor of \(i \in N\) in \(M\) by \(i^-\). Since \(K\) is a zero-dimensional compactum, we can find clopen sets \(C_i \subseteq K\) (\(i \in N\)) such that

\[K \setminus x_i^{-1}((\varepsilon_i, -\varepsilon_i)) \subseteq C_i \subseteq K \setminus x_i^{-1}([-\varepsilon_i, \varepsilon_i])\ \text{for every} \ i \in N.\]

Set \(y_i = \chi_{C_i} x_i\) for each \(i \in N\). So one has

(i) \(\|x_i - y_i\|_K < \varepsilon_i\), so \((x_i)_{i \in N}\) and \((y_i)_{i \in N}\) are 1 + \(\varepsilon\)-equivalent, and

(ii) for every \(c \in K\) and every \(i \in N\), if \(|y_i(c)| < \varepsilon_i\), then \(y_i(c) = 0\).

Since for every \(c \in K\), by (ii) above, one has that

\[\{i \in N : y_i(c) \neq 0\} = \{i \in N : c \in C_i \text{ and } |x_i(c)| \geq \varepsilon_i\} = \psi(c),\]

it follows that the support set \(\mathcal{F}\) of \(R_K((y_i)_{i \in N})\) coincide with the image of \(\psi\), so it is a compact family of \(\mathbb{N}\). We use now Theorem 2.7 to find \(P \subseteq N\) such that \(\mathcal{F}[P]\) is the closure of a uniform barrier on \(P\). This implies that \(R_K((y_i)_{i \in P})\) is supported by a barrier \(\mathcal{B}\) on \(P\). Let \(\theta\) be the unique order preserving mapping from \(\mathbb{N}\) onto \(P\), and let \(\Theta : c_0 \to \{0\} \subseteq c_0 \to c_0\) be defined by \(\Theta(\xi)(n) = \xi(\theta(n))\). This is an homeomorphism between \(c_0 \to P\) and \(c_0\) both with the weak topology, so \(L = \Theta^* R_K((y_i)_{i \in P})\) is a weakly-compact subset of \(c_0\) and supported by the barrier \(\theta^{-1} B = \{\theta^{-1} s : s \in \mathcal{B}\}\) on \(\mathbb{N}\). Now it is easy to see that the evaluation mapping \((p_i)_{i \in N}\) of \(C(L)\) is a normalized weak-null sequence 1 + \(\varepsilon\)-equivalent to \((x_i)_{i \in P}\).

\section*{Theorem 3.3.} Suppose that \((x_i) \subseteq C(K)\) is a normalized weak-null sequence for a countable compactum \(K\). Then there is a constant \(C \geq 1\), an infinite set \(M\), a uniform barrier \(\mathcal{B}\) on \(M\) whose rank is at most the Cantor-Bendixson rank of \(K\), and some \(U\)-mapping \(\mu : \mathcal{B} \to c_0^+\) such that for every block sequence \((x_n) \subseteq \mathcal{B}\) the corresponding sequence of linear combinations \((\sum_{i \in s}(\mu(s_n))(i)x_{i,n})\) is a normalized block sequence \(C\)-equivalent to the unit vector basis of \(\mu\).

\textbf{Proof.} The proof is by induction on the Cantor-Bendixson rank of \(K\). First of all, by Lemma 3.2 we may assume that \(K\) is a weakly-compact subset of \(c_0\) supported by a barrier \(\mathcal{B}\) on \(\mathbb{N}\) and that the normalized weak null sequence \((x_i)\) is the corresponding evaluation mapping sequence \((p_i)_{i \in N}\). If \(\alpha = 1\), then \(\mathcal{B} = \mathbb{N}^{[1]}\) and clearly \((p_i)\) is equivalent to the unit vector basis of \(c_0\). So assume that \(\alpha > 1\). By going to a subsequence of \((p_i)\) if needed, we may also assume in this case that \(|s| \geq 2\) for every \(s \in \mathcal{B}\). For each integer \(n\) set \(\mathcal{F}_n = \bigcup_{m \leq n} \overline{B(m)}\). Since \(\mathcal{B}\) is a \(\alpha\)-uniform family, we have that for every \(n\), \(\partial^\alpha \mathcal{F}_n = \emptyset\), so its Cantor-Bendixson rank is strictly smaller than \(\alpha + 1\). For each \(n \in \mathbb{N}\), let

\[K_n = \{f : s \in \mathcal{F}_n\}.\]

This is a compactum whose support is \(\mathcal{F}_n\) and whose rank is strictly smaller than \(\alpha + 1\). So, the evaluation mapping sequence \((p_i)\) is a weakly-null sequence of \(C(K_n)\) for every \(n\). Observe that for every sequence of scalars \((a_i)\) we have that

\[\|\sum_i a_i p_i\|_n = \|\sum_i a_i p_i\|_{K_n} = \sup\{\|\sum_i a_i p_i\|_{K} : s \in \mathcal{F}_n\}.\]
Using the fact that the family $F_n$ is hereditary, we obtain that $(p_i)$ is 1-unconditional. Since we assume that all the singletons $\{i\}$ belong to $F_n$, it follows that $(p_i)_{i \geq 1}$ is indeed a 1-unconditional normalized weakly null sequence in $C(K_n)$.

Fix $\varepsilon > 0$, and let $(\varepsilon_n)$ be a summable sequence with $\sum_n \varepsilon_n < \varepsilon/2$. By the Ramsey property of the uniform barrier $B$, we can find a fusion sequence $(M_k)_k$ such that, setting $n_k = \min M_k$ for each $k \in \mathbb{N}$, we have that for every $k$ the following dichotomy holds:

(I) Either for every $s \in B \setminus M_k$ there is some $\mu_k(s) \in \ell_\infty$ with $\text{supp} \mu_k(s) \subseteq s$, $0 \leq \mu_k(s)(i) \leq 1$ for every $i \in s$, and such that for every such that $\| \sum_{i \in s} \mu_k(s)(i) p_i \|_{K} = 1$ while $\| \sum_{i \in s} \mu_k(s)(i) p_i \|_{n_k} < \varepsilon_k$, or else

(II) $\| \sum_{i \in s} a_i p_i \|_{K} \leq 2 \varepsilon_k^{-1} \| \sum_{i \in s} a_i p_i \|_{n_k}$ for every $s \in B \setminus M_k$ and every $(a_i)_{i \in \mathbb{N}}$.

Suppose first that (I) holds for every $k$. Let $M_\infty = \{n_k\}$ be the corresponding fusion set. Then let $C = B \setminus M_\infty$. For $s \in C$, define $\mu(s) = \mu_k(s)$, where $n_k = \min s$. This is well defined since $s \in B \setminus M_k$. For a given $s \in C$, let

$$x(s) = \sum_{i \in s} \mu(s)(i) p_i.$$ 

Our intention is to show that for every block sequence $(s_i)_i$ in $C$ one has that $(x(s_i))_i$ is $2 + \varepsilon$-equivalent to the $c_0$-basis. So fix such sequence $(s_i)_i$ and let $(b_i)_{i \in \mathbb{N}}$ be a sequence of scalars with $|b_i| \leq 1$ for every integer $i$. Since each $x(s_i)$ is normalized and since $(p_i)$ is monotone, we obtain that

$$\| \sum_i b_i x(s_i) \|_K \geq (1/2) \| \sum_i b_i e_i \|_{\infty}.$$ 

Suppose that $\xi \in K$, and let $i_0 = \min \{i : s_i \cap \text{supp} \xi \neq \emptyset\}$. Fix $i > i_0$, and let $k_i$ be such that $n_{k_i} = \min s_i$. Since $\text{supp} \xi \cap s_i \in F_{\max s_i}$, we have that

$$|x(s_i)(\xi)| \leq \| \sum_{j \in s_i \cap \text{supp} \xi} a_i^{(k_i)} p_i \|_{\max s_i} < \varepsilon_k.$$ 

It follows that

$$| \sum_i b_i x(s_i)(\xi) | \leq |b_{i_0}| + \sum_{i > i_0} |b_i| |x(s_i)(\xi)| \leq |b_{i_0}| + \frac{\varepsilon}{2}. \quad (6)$$

So, $\| \sum_i b_i x(s_i) \|_{K} \leq (1 + \varepsilon/2) \| \sum_i b_i e_i \|_{\infty}$. Finally use Corollary 2.13 to perturb $\mu$ and make it $U$-mapping.

Suppose now that $k_0$ is the first $k$ such that (II) holds for $k$. Set $M = M_{k_0}$. It readily follows that for every $x$ in the closed linear span of $(p_i)_{i \in M}$ one has that $\| x \|_K \leq \varepsilon_k^{-1} \| x \|_{n_{k_0}}$. By inductive hypothesis applied to $(p_i) \subseteq C(K_{n_{k_0}})$, there is some $C \geq 1$, some uniform barrier $C$ on some $N \subseteq M$ of rank not bigger than the one of $K_{n_{k_0}}$ and some $\mu$ fulfilling the conclusions of the Lemma. Fix $s \in C$. Then $\| \mu(s) \|_{n_{k_0}} = 1$, so we can find some $t_s \subseteq s$ such that $1 = \| \mu(s) \|_{n_{k_0}} = \| \mu(s) \|_{t_s \setminus K}$. Observe that, by 1-unconditionality of $\| \cdot \|_{n_{k_0}}$, $\| \mu(s) \|_{t_s \setminus K} = 1$. Define $\nu : C \to \ell_\infty$ by $\nu(s) = \mu(s) \upharpoonright t_s$. Finally, let us check that $(x(s_i))_i \subseteq C(K)$ is $C \varepsilon^{-1}_{n_k}$-equivalent to the $c_0$-basis for every block sequence $(s_i)_i$ in $C$. Fix scalars $(a_i)$, $|a_i| \leq 1$ ($i \in \mathbb{N}$).

We obtain the inequality $\| \sum_i a_i \nu(s_i) \|_{K} \geq (1/2) \| \sum_i a_i e_i \|_{\infty}$ by the monotonicity of the basic
sequence $(p_i)$. Now,
\[ \left\| \sum_i a_i \nu(s_i) \right\|_K \leq \frac{1}{\varepsilon_{n_{k_0}}} \left\| \sum_i a_i \mu(s_i) \right\|_{n_{k_0}} \leq \frac{1}{\varepsilon_{n_{k_0}}} \left\| \sum_i a_i \varepsilon_i \right\|_\infty \leq \frac{C}{\varepsilon_{n_{k_0}}} \left\| \sum_i a_i \varepsilon_i \right\|_\infty. \] (7)

4. Conditionality

We start with the following natural slightly variation on the notion of $S_\varepsilon$-unconditionality from [3], and which is a generalization of unconditionality (see Definition 2.8 (d)).

**Definition 4.1.** Let $\mathcal{F}$ be a family of finite sets of integers. A normalized basic sequence $(x_n)$ of a Banach space $E$ is called $\mathcal{F}$-unconditional with constant at most $C \geq 1$ iff for every sequence of scalars $(a_n)$,
\[ \sup_{s \in \mathcal{F}} \left\| \sum_{n \in s} a_n x_n \right\| \leq C \left\| \sum_{n \in \mathbb{N}} a_n x_n \right\|. \]

This generalizes the notion of unconditionality covered by the case of $\mathcal{F} = \text{FIN}$. The question is whether every normalized weakly-null sequence has a $\mathcal{F}$-unconditional subsequence. Observe that the subsequence $(x_n)_{n \in M}$ is $\mathcal{F}$-unconditional iff it is $\mathcal{F}[M]$-unconditional, so the existence of an $\mathcal{F}$-unconditional subsequence is closely related to the form of the traces $\mathcal{F}[M]$. If we assume that in addition the family $\mathcal{F}$ is hereditary, then, by the Theorem 2.7, two possibilities can occur: The first one is that some trace of $\mathcal{F}$ consists on all finite subsets of some infinite set $M$. In this case, for subsequences of $(x_n)_{n \in M}$ the $\mathcal{F}$-unconditionality coincides with the unconditionality. The second case is when some trace of $\mathcal{F}$ is the closure of a uniform barrier. So one is naturally led to examining the standard compact families of finite subsets of $\mathbb{N}$. We begin with the following positive result announced in [16] and first proved by E. Odell [20] concerning the Schreier family $\mathcal{S} = \{ s \subseteq \mathbb{N} : |s| \leq \min(s) + 1 \}$.

**Theorem 4.2.** Suppose that $(x_n)$ is a normalized weakly-null sequence of a Banach space $E$. For every $\varepsilon > 0$ there is a $\mathcal{S}$-unconditional subsequence with constant $2 + \varepsilon$. \hfill \Box

Recall that if $\mathcal{F}$ is a barrier on some set $M$ then its trace $\mathcal{F}[N]$ on any co-infinite subset $N$ of $M$ is hereditary and that for every pair $\mathcal{F}_0$ and $\mathcal{F}_1$ of barriers on the same domain $M$ there is an infinite set $N \subseteq M$ such that $\mathcal{F}_0[N] \subseteq \mathcal{F}_1[N]$ or $\mathcal{F}_1[N] \subseteq \mathcal{F}_0[N]$. Since the two alternatives are dependent on the ranks of $\mathcal{F}_0$ and $\mathcal{F}_1$, one is naturally led to the following measurement of unconditionality.

**Definition 4.3.** Suppose that $\gamma$ is a countable ordinal. A normalized basic sequence $(x_n)$ of a Banach space $E$ is called $\gamma$-unconditionally saturated with constant at most $C \geq 1$ if for every $\gamma$-uniform barrier $\mathcal{B}$ on $\mathbb{N}$ and for every infinite $M$ there is infinite $N \subseteq M$ such that the corresponding subsequence $(x_n)_{n \in N}$ of $(x_n)$ is $\mathcal{B}$-unconditional with constant at most $C$.

We say that $(x_n)_n$ is $\gamma$-unconditionally saturated if it is $\gamma$-unconditionally saturated with constant $C$ for some $C \geq 1$.

**Remark 4.4.** (a) A sequence $(x_n)_n$ is $\gamma$-unconditionally saturated iff given a $\gamma$-uniform barrier $\mathcal{B}$ every subsequence of $(x_n)_n$ has a further $\mathcal{B}$-unconditional subsequence. The reason for this is
that given two $\gamma$-uniform barriers $B$ and $C$ on a set $M$ we have that there is $N \subseteq M$ such that either $B \upharpoonright N \subseteq C \upharpoonright N \subseteq B \upharpoonright N \oplus N[\leq 1]$ or the symmetric situation holds, where $F \oplus G = \{s \cup t : s \in G, t \in F \text{ and } s < t\}$ (see [4]).

(b) It follows from Theorem 4.2 that every normalized weakly null sequence is $\omega$-unconditionally saturated. Since the $\omega$-uniform barriers are of the form $\{s \in \text{FIN} : |s| = f(\min s + 1)\}$ for some unbounded mapping $f : M \to N$ one can easily modify the proof of Theorem 4.2 to prove that every normalized weakly null sequence is $\omega$-unconditionally saturated with constant at most $2 + \varepsilon$.

(c) If the normalized basic sequence $(x_n)$ is monotone, then it is $\overline{B}$-unconditional iff it is $B$-unconditional for every uniform barrier $B$ on $N$.

(d) An analysis of the Maurey-Rosenthal [16] example of a weakly-null sequence $(x_n)$ with no unconditional basic subsequence (see Example 4.5 below) reveals an $\omega^2$-uniform barrier $B_{\text{MR}}$ such that no infinite subsequence $(x_n)_{n \in M}$ is $B_{\text{MR}}$-unconditional with any finite constant $C$. So this is an example of a normalized weakly-null sequence with no $\omega^2$-unconditionally saturated subsequence.

(e) Recall that an ordinal $\gamma$ is called indecomposable if for every $\beta < \gamma$, $\beta \omega \leq \gamma$. Equivalently, $\gamma = \omega^\beta$ for some $\beta$. Suppose that $\gamma$ is the maximal indecomposable ordinal smaller than some fixed ordinal $\alpha$. Then a normalized basic sequence $(x_n)$ is $\alpha$-unconditionally saturated if and only it is $\gamma$-unconditionally saturated.

Example 4.5. First of all, for a fixed $0 < \varepsilon < 1$ choose a fast increasing sequence $(m_i)$ such that

$$\sum_{i=0}^{\infty} \sum_{j \neq i} \min \{(m_i/m_j)^{1/2}, (m_j/m_i)^{1/2}\} \leq \frac{\varepsilon}{2}.$$ \hspace{1cm} (8)

Let $\text{FIN}^{[<\infty]}$ be the collection of all finite block sequences $E_0 < E_1 < \cdots < E_k$ of nonempty finite subsets of $N$. Now choose a $1 \to 1$ function

$$\sigma : \text{FIN}^{[<\infty]} \to \{m_i\}$$ \hspace{1cm} (9)

such that $\varphi((s_i)_{i=0}^n) > s_n$ for all $(s_i) \in \text{FIN}^{[<\infty]}$. Now let $B_{\text{MR}}$ be the family of unions $s_0 \cup s_1 \cup \cdots \cup s_n$ of finite sets such that

(a) $(s_i)$ is block and $s_0 = \{n\}$,
(b) $|s_i| = \sigma(s_0, \ldots, s_{i-1})$ ($1 \leq i \leq n$).

It turns out that $B_{\text{MR}}$ is a $\omega^2$-uniform barrier on $N$ (see Proposition 4.11 below), hence $\overline{B_{\text{MR}}} = \overline{B_{\text{MR}}^{<\infty}}$ is a compact family with rank $\omega^2 + 1$. Observe that by definition, every $s \in B_{\text{MR}}$ has a unique decomposition $s = \{n\} \cup s_1 \cup \cdots \cup s_n$ satisfying (a) and (b) above. Now define the mapping $\Phi : B_{\text{MR}} \to c_0$,

$$\Phi(s) = e_n + \sum_{i=1}^{n} \frac{1}{|s_i|} \sum_{k \in s_i} e_k.$$ \hspace{1cm} (10)

It follows that $\Phi$ is a $U$-mapping defined on the barrier $B_{\text{MR}}$. Now we can define the Banach space $X_{\text{MR}}$ as the completion of $c_0$ under the norm

$$\|x\|_{\text{MR}} = \sup\{\langle \Phi(s), x \rangle : s \in B_{\text{MR}}\}.$$
The natural Hamel basis \((e_n)\) of \(c_0\) is now a normalized weakly-null monotone basis of \(X_{MR}\) without unconditional subsequences. Indeed, without \(\omega^2\)-unconditionally saturated subsequences. Moreover this weakly-null sequence has the property that the summing basis \((S_i)\) of \(c\), the Banach space of convergent sequences of reals, is finitely-block representable in the linear span of every subsequence of \((e_i)\) (and so the summing basis of \(c_0\)), more precisely, for every \(M\), every \(n \in \mathbb{N}\) and every \(\varepsilon > 0\) there is a normalized block subsequence \((x_i)_{i=0}^{n-1}\) of \((e_i)_{i \in M}\) such that for every sequence of scalars \((a_i)_{i=0}^{n-1}\),

\[
\max\{|\sum_{i=0}^{m} a_i| : m < n\} \leq \|\sum_{i=0}^{n-1} a_i x_i\|_{c(K)} \leq (1 + \varepsilon) \max\{|\sum_{i=0}^{m} a_i| : m < n\}.
\]

On the other hand, by Proposition 4.2 the sequence \((p_i)\) is \(\omega\)-unconditionally saturated with constant \(\sim 2\).

Another presentation of this space is the following: Since \(\Phi\) is uniform, it is Lipschitz, so there is a unique extension \(\Phi : B_{\mathbb{MR}} \to c_0\), naturally defined by \(\Phi(s) = \Phi(t)|_t\), where \(t \in B_{MR}\) is (any) such that \(s \subseteq t\). Now define \(K = \Phi(B_{MR}) \subseteq c_0\). This is a weakly-compact subset of \(c_0\) whose rank the same than \(B_{\mathbb{MR}}\), i.e., \(\omega^2 + 1\). Then the corresponding evaluation sequence \((p_i) \subseteq C(K)\) is 1-equivalent to the basis \((e_i)\) of \(X_{MR}\).

Building on the idea of Example 4.5, we are now going to find, for every countable indecomposable ordinal \(\gamma\), a \(U\)-sequence with no unconditional subsequences but \(\beta\)-unconditionally saturated for every \(\beta < \gamma\). Before embarking into the construction, we need to recall a localized version of Pták’s Lemma. For this we need the following notation: Given a family \(F\), and \(n \in \mathbb{N}\), let

\[
F \odot n = \{s_0 \cup \cdots \cup s_{n-1} : (s_i)_{i=0}^{n-1} \subseteq F\text{ is block}\}.
\]

It can be shown that \(F \odot n\) is a \(\alpha\)-uniform family if \(F\) is an \(\alpha\)-uniform family.

Given \(\xi \in c_0\) we will write \(\xi^{1/2}\) to denote \((\xi(i))^{1/2}\). Given \(\xi \in c_0\) and a finite set \(s\), let \(\langle \xi, s \rangle = \langle \xi, \lambda_s \rangle = \sum_{i \in s} \xi(i)\).

**Definition 4.6.** A mean is an element \(\mu \in c_0^+\) with the property that \(\sum_{i \in \mathbb{N}} \mu(i) = 1\). We say that \(\mu : B \to c_0^+\) is a \(U\)-mean-assignment if \(\mu\) is a \(U\)-mapping such that for every \(s \in B\) one has that \(\mu(s)\) is a mean.

**Lemma 4.7.** Suppose that \(B\) is an \(\alpha\)-uniform barrier on \(M\), \(\alpha \geq 1\). Let \(\gamma = \gamma(\alpha)\) be the maximal indecomposable ordinal not bigger than \(\alpha\), and let \(n = n(\alpha) \in \mathbb{N}\), \(n \geq 1\), be such that \(\gamma n < \alpha < \gamma(n + 1)\). Then for every \(k \in \mathbb{N}\), \(k > 1\), every \(\varepsilon > 0\), and every \(\beta\)-uniform barrier \(C\) on \(M\) with \(\beta > \alpha k\) there \(N \subseteq M\) and \(U\)-mean-assignment \(\mu : C \upharpoonright N \to c_0^+\) such that

\[
\sup \{\mu(s)^{\frac{\gamma}{\alpha} - 1}, t : t \in B\} \leq \frac{(1 + \varepsilon)(n + 1)}{(nk)^{\frac{1}{\alpha}}}
\]

for every \(s \in C \upharpoonright N\).

**Proof.** The proof is by induction on \(\alpha\). Fix \(\varepsilon > 0\) and \(k > 1\). Let \(C\) be an \(\beta\)-uniform family on \(M\) such that \(\beta > \alpha k\).
Notice that if we prove that for every $N \subseteq M$ there is one mean $\mu$ with support in $C \upharpoonright N$ such that (11) holds, then the Ramsey property of the uniform barrier $C$ gives the existence for some $N \subseteq M$ of a mean-assignment $\mu : C \upharpoonright N \to c_0$ such that $\mu(s)$ has the property (11) for every $s \in C \upharpoonright N$. Then Corollary 2.13 gives the desired $U$-mapping.

Let $D$ be a $\gamma'$-uniform barrier on $M$ (if $n = 1$ we take $D = B$), and fix $N \subseteq M$. Find first $P \subseteq N$ be such that $(D \odot nk) \upharpoonright P \subseteq \overline{C}$ as well as $B \upharpoonright P \subseteq D \odot (n + 1)$. Consider $(\gamma_i)_i \in P$ such that $D_i \upharpoonright P$ is $\gamma_i$-uniform on $P/i$. Observe that for every $i \in P$ we have that $\gamma_i < \gamma$, so, since $\gamma$ is indecomposable, $\gamma_i \omega \leq \gamma$. Let $\mu_0$ be any mean such that $\text{supp} \mu_0 \subseteq B \upharpoonright P$. By inductive hypothesis applied to appropriate $\alpha_i$'s, we can find a block sequence $(\mu_j)_{j=0}^{nk-1}$ of means with support in $B \upharpoonright P$ such that for every $1 \leq j \leq nk - 1$,

$$\sup \{ \langle \mu_j^{1/2}, t \rangle : t \in D, \text{ and } \min t \leq \max \text{supp} \mu_{j-1} \} < \frac{\varepsilon}{2^{j+1}}. \tag{12}$$

Let $\nu = \left(1/(nk)\right) \sum_{j=0}^{nk-1} \mu_j$. Observe that $\text{supp} \nu \subseteq (D \odot (nk)) \upharpoonright P \subseteq \overline{C}$. Then, for every $t \in B$, by (12),

$$\langle \nu^{1/2}, t \rangle = \frac{1}{(nk)^{1/2}} \sum_{j=0}^{k-1} \sum_{i \in \mu_i} \mu_j(i) \frac{1}{2} \leq \frac{1 + \frac{\varepsilon}{(nk)^{1/2}}}{n^{1/2}}. \tag{13}$$

Let us point out that $\text{supp} \nu$ is, possibly, not a set in $C$. However it is easy to slightly perturb $\nu$ to a newer mean with support in $C$ and satisfying (13) for every $t \in B$: Let $s \in C$ be such that $\text{supp} \nu \subseteq s$, and set $u = s \setminus \text{supp} \nu$. Let $\delta > 0$ be such that

$$(1 + \frac{\varepsilon}{2})(1 - \delta)^{1/2} + \frac{n k \delta |u|^{1/2}}{1 + \varepsilon} \leq 1 + \varepsilon. \tag{14}$$

Now set

$$\mu = (1 - \delta)\nu + \frac{\delta}{|u|} \chi_u. \tag{15}$$

$\mu$ is a mean whose support is $s \in C$. It can be shown now that for every $t \in B$,

$$\sum_{i \in \mu} \mu(i) \frac{1}{2} \leq \frac{1 + \frac{\varepsilon}{(nk)^{1/2}}}{n^{1/2}}. \tag{16}$$

by the choice of $\delta$. Finally, let $t \in B$ and let us compute $\sum_{i \in \mu} (\mu(i))^{1/2}$: First of all we have that $\sum_{i \in \mu} (\mu(i))^{1/2} = \sum_{i \in \mu} (\mu(i))^{1/2}$, where $u = t \cap P$. Now, since $u \in B \upharpoonright P \subseteq D \odot (n + 1)$, we can find $t_0 < \cdots < t_n$ in $D$ such that $u \subseteq t_0 \cup \cdots \cup t_n$, and hence

$$\langle \mu^{1/2}, t \rangle = \sum_{j=0}^{n} \langle \mu^{1/2}, t_j \rangle \leq \frac{(n + 1)(1 + \varepsilon)}{(nk)^{1/2}}, \tag{17}$$

as promised. \hfill \Box

**Corollary 4.8.** Suppose that $B$ is an $\alpha$-uniform barrier on $M$, $\alpha \geq 1$. Then for every $\varepsilon > 0$ there is some $k = k(\alpha, \varepsilon)$ such that for every $\beta$-uniform barrier on $M$ with $\beta > ak$ there $N \subseteq M$ and some $U$-mean-assignment $\mu : C \upharpoonright N \to c_0$ such that,

$$\sup \{ \langle \mu(s)^{1/2}, t \rangle : t \in B \} \leq \varepsilon \tag{18}$$

for every $s \in B \upharpoonright N$. \hfill \Box
Lemma 4.9. Fix an indecomposable countable $\alpha$ and a sequence $(\varepsilon_n)$ of positive reals. Then:

(a) there is a collection $(B_n)$ of $\alpha_n$-uniform barriers on $\mathbb{N}/n$ and a corresponding sequence of $U$-mean-assignments $\mu_n : B_n \rightarrow \epsilon_{\infty}$ with the following properties:

(a.1) $\alpha_n > 0$, $\sup_n \alpha_n = \alpha$,

(a.2) for every $m < n$ and every $s \in B_n$

$$\sup \{ (\mu_n(s)^{1/2}, t) : t \in B_m \} < \varepsilon_n.$$  \hspace{1cm} (19)

(b) Suppose that in addition $\alpha = \omega^\gamma$ with $\gamma$ limit. Let $\alpha_n \uparrow \alpha$ be any sequence such that $\alpha_n \omega \leq \alpha_{n+1}$ ($n \in \mathbb{N}$). Then there is a double sequence $(B^n)$ such that for every integers $n$ and $i$

(b.1) $B^n$ is an $\alpha^{(n)}$-uniform barrier on $\mathbb{N}/(n+i)$, with $\alpha^{(n)} > 0$ and $\alpha^{(n)} \uparrow; \alpha_n$.

(b.2) There are $U$-mean-assignments $\mu_{n,i} : B^n_i \rightarrow \epsilon_{\infty}$ such that for every $s \in B^n_i$, and every $(m, j) <_{\text{lex}} (n, i)$

$$\sup \{ (\mu_{n,i}(s)^{1/2}, t) : t \in B^m_j \} < \varepsilon_{n+i},$$  \hspace{1cm} (20)

where we recall that $<_{\text{lex}}$ denotes the lexicographical order on $\mathbb{N}^2$ defined by $(m, i) <_{\text{lex}} (n, j)$ iff $m < n$, or $m = n$ and $i < j$.

Proof. (a): Choose $\alpha_n \uparrow \alpha$ such that for every $n \in \mathbb{N}$, $\alpha_{n+1} > \alpha_n k(\alpha_n, \varepsilon_n)$, that is possible since $\alpha$ is indecomposable. Let $C_n$ be an $\alpha_n$-uniform family on $\mathbb{N} (n \in \mathbb{N})$. By Corollary 4.8 we can find a fusion sequence $(M_n)$ such that

(c) $C_n \uparrow M_n \subseteq C_n$ if $m \leq n$, and

(d) for every $n \in \mathbb{N}$ there is a $U$-mean-assignment $\nu_n : C_n \rightarrow \epsilon_{\infty}$ such that

$$\sup \{ (\nu_n(s)^{1/2}, t) : t \in \bigcup_{i < n} C_i \} < \varepsilon_n.$$  \hspace{1cm} (21)

for every $s \in C_n \uparrow M_n$. Let $M = \{ m_n \}$ be the fusion set of $(M_n)$, and $\Theta : M \rightarrow \mathbb{N}$ be the corresponding order preserving onto mapping. It is not difficult to see that $C_n \uparrow (\Theta^n B^n) \uparrow (\mathbb{N}/n)$, and $\mu_n : C_n \rightarrow \epsilon_{\infty}$ defined naturally out of $\nu_n \Theta$ fulfills all the requirements.

(b): Suppose that $\alpha = \omega^\gamma$ with $\gamma$ limit. Let $\alpha_n \uparrow \alpha$ be any sequence such that $\alpha_n \omega \leq \alpha_{n+1}$ ($n \in \mathbb{N}$).

Claim. There is a fusion sequence $(M_n)$, $M_n = \{ m^{(n)}_n \}$, a double sequence $(B^n_i)$ of $\alpha^{(n)}$-uniform barriers on $M_n/m^{(n)}_n$ and $U$-mean-assignments $\mu_{n,i} : B^n_i \rightarrow \epsilon_{\infty}$ such that

(e) $\alpha^{(n)} \uparrow; \alpha_n$ ($n \in \mathbb{N}$), and

(f) for every $(m, j) <_{\text{lex}} (n, i)$, every $s \in B^n_i$ and every $t \in B^m_j$, $(\mu_{n,i}(s)^{1/2}, t) < \varepsilon_{n+i}.$

Proof of Claim: First, use Corollary 4.8 applied to $\alpha_0$ to produce an infinite set $M_0 = \{ m^{(0)}_i \}$ and a sequence $(B^n_0)$ of $\alpha^{(0)}$-uniform barriers on $M_0/m^{(0)}_1$ with $\alpha^{(0)} \uparrow \alpha_0$ and $U$-mean-assignments $\mu_{0,i} : B^n_0 \rightarrow \epsilon_{\infty}$ such that for every $i$ and every $s \in B^n_0$, $(\mu_{0,i}(s)^{1/2}, t) \leq \varepsilon_i$ for every $t \in B^n_0$ with $j < i$. In general, suppose we have found for every $k \leq n M_k = \{ m^{(k)}_i \} \subseteq M_{k-1}$, $(B^n_k)$ $\alpha^{(k)}$-uniform barriers on $M_k/m^{(k)}_1$ and $U$-mean-assignments $\mu_{k,i} : B^n_k \rightarrow \epsilon_{\infty}$ such that for every $(k, j) <_{\text{lex}} (m, i)$ every $s \in B^n_m$ and every $t \in B^n_j$, $\mu_{m,i}(s)^{1/2}, t) \leq \varepsilon_{m+i}.$ Then for each $k \leq n$ define the following families

$$B_k = \{ s \subseteq M_k : s \in B^n_{\min s} \}.$$  \hspace{1cm} (22)
This is clearly an $\alpha_i$-uniform family on $M_i$. Since $\alpha_n \omega \leq \alpha_{n+1}$, we can use again Corollary 4.8 and find an infinite subset $M_{n+1} = \{m_{i}^{(n+1)}\} \subseteq M_n$ and a sequence $(B_{i}^{n+1})$ of $\alpha_i^{(n+1)}$-uniform barriers on $M_{n+1}/m_i^{(n+1)}$ and $U$-mean-assignments $\mu_{n+i} : B_{i}^{n+1} \rightarrow \mathcal{O}_0$ such that for every $s \in B_{i}^{n+1}$,
\[
\sup \{(\mu_{n+i}(s))^{\frac{1}{2}}, t) : t \in \bigcup_{k \leq n} B_{k} \cup \bigcup_{j < i} B_{j}^{n+1} \} \leq \varepsilon_{n+i+1},
\]
so, in particular for every $k \leq n$ and every $t \in B_{k}^{i}$, $(\mu_{n+i}(s))^{\frac{1}{2}}, t) \leq \varepsilon_{n+i+1}$. \hfill \Box

Let $M$ be the fusion set of $(M_n)$, i.e. $M = \{m_0^{(n)}\}$. Observe that $m_0^{(n+i)} \geq m_0^{(n)}$ for every $n$ and $i$, so $M/m_0^{(n)} \subseteq M/m_i^{(n)}$. Set $C_i^n = B_i^n \uparrow (M/m_0^{(n+i)})$. This is an $\alpha_i^{(n)}$-uniform barrier on $M/m_0^{(n+i)}$. Consider $\nu_{n,i} = \mu_{n,i} \mid C_i^n : C_i^n \rightarrow \mathcal{O}_0$ has the property that for every $(m, j) <_{lex} (n, i)$, every every $s \in C_i^n$ and every $t \in C_j^n$, $(\nu_{n,i}(s))^{1/2}, t) \leq \varepsilon_{n+i}$. Now use $\Theta : M \rightarrow \mathbb{N}$, $\Theta(m_0^{(n)}) = 0$, to define the desired mean-assignments and families. \hfill \Box

\textbf{Remark 4.10.} Observe that if $B$ is $\alpha$-uniform on $M$ with $\alpha > 0$, then $M^{[1]} \subseteq B$. It readily follows that the mean-assignments $\mu_n$ and $\mu_{n,i}$ obtained in Lemma 4.9 have the property that $||\mu_n(s)||^{1/2} \leq \varepsilon_n$ and $||\mu_{n,i}(s)||^{1/2} \leq \varepsilon_{n+i}$ for every $s$ in the corresponding domains.

\textbf{Proposition 4.11.} (a) Suppose that $C$ and $B_i$ are $\beta$ and $\alpha_i$-uniform families on $M$ ($i \in \mathbb{N}$) with $\alpha_i \uparrow \alpha$, $\alpha_i, \beta \geq 1$. Let $\sigma : \text{FIN}^{[\infty]} \rightarrow \mathbb{N}$ be 1-1. Then for every $n \in \mathbb{N}$ the family
\[
\mathcal{D} = \{s_0 \cup \cdots \cup s_n : (s_i) \text{ is block, } s_0 \in C \text{ and } s_i \in B_{\sigma((s_0, \ldots, s_{i-1}))} \text{ for every } 1 \leq i \leq n - 1\}
\]
is $\gamma$-uniform on $M$, where $\gamma = \alpha n + \beta^-$ if $1 \leq \beta < \omega$ and $n > 0$, and $\gamma = \alpha n + \beta$ if $\beta \geq \omega$ or $n = 0$.

(b) Suppose that $B_i$ is $\alpha_i$-uniform on $M$ ($i \in \mathbb{N}$) with $\alpha_i \uparrow \alpha$. Let $\sigma : \text{FIN}^{[\infty]} \rightarrow \mathbb{N}$ be 1-1. Then the family
\[
\mathcal{C} = \{\{n\} \cup s_0 \cup \cdots \cup s_{n-1} : (\{n\}, s_0, \ldots, s_{n-1}) \text{ is block, and } s_i \in B_{\sigma((\{n\}, s_0, \ldots, s_{i-1}))} \text{ for every } 0 \leq i \leq n - 1\}
\]
is $\alpha \omega$-uniform on $M$.

\textbf{Proof.} (a): The proof is by induction on $n$. If $n = 0$, the result is clear. So suppose that $n > 0$. Now the proof is by induction on $\beta$. Suppose first that $\beta = 1$. Then $C = M^{[1]}$, and so, for every $m \in M$
\[
\mathcal{D}_{(m)} = \{s_1 \cup \cdots \cup s_n : (s_1, s_2, \ldots, s_n) \text{ is block, } s_1 \in B_{\sigma((m_1))} \text{ and } s_i \in B_{\sigma((m_1, s_{i-1}, \ldots, s_{i-1}))} \text{ for every } 2 \leq i \leq n - 1\},
\]
so, by inductive hypothesis, $\mathcal{D}_{(m)}$ is $\alpha(n-1) + \gamma_m$-uniform on $M/m$, depending whether $\alpha_m$ is finite or infinite, but in any case with $\gamma_m \uparrow \alpha$. Hence $\mathcal{D}$ is $\alpha \uparrow \alpha$-uniform on $M$. The general case for $1 \leq \beta < \omega$ is shown in the same way.

Suppose now that $\beta \geq \omega$. Then for every $m \in M$
\[
\mathcal{D}_{(m)} = \{t \cup s_1 \cup \cdots \cup s_n : (t, s_1, \ldots, s_n) \text{ is block, } t \in C_{(m)} \text{ and } s_i \in B_{\sigma((m \cup t, s_{i-1}, \ldots, s_{i-1}))} \text{ for every } 1 \leq i \leq n - 1\},
\]
By inductive hypothesis, $D_{\{m\}}$ is $\alpha n + \gamma_m$-uniform on $M/m$, with $\gamma_m \uparrow \beta$, so $D$ is $\alpha n + \beta$-uniform on $M$, as desired.

(b) follows easily from (a).

The following is a generalization of Maurey-Rosenthal example for arbitrary countable indecomposable ordinal $\alpha$.

**Theorem 4.12.** For every countable indecomposable ordinal $\alpha$ there is a normalized weakly-null sequence which is $\beta$-unconditionally saturated for every $\beta < \alpha$ but without unconditional subsequences.

**Proof.** Our example is a slightly modification of a $U$-sequence introduced in [14]. So, we are going to define a $\alpha$-uniform barrier $B$ on $N$, a $U$-mean-assignment $\varphi : B \to c_0$ and some $G \subseteq FIN \times FIN$ and then define the norm on $c_0$ by

$$||\xi|| = \max \{||\xi||_\infty, \sup \{|\langle \varphi(s) | t, \xi \rangle| : (s, t) \in G\} \}$$

(24)

where $G \subseteq FIN \times FIN$ is such that its first projection is $B$. Notice that some sort of restrictions have to be needed in the formula (24), since it is not difficult to see that for a compact and hereditary family $F$, a normalized weakly-null sequence $(x_i)_i$ is $F$-unconditional iff it is equivalent to the evaluation mapping sequence $(p_i)$ of a weakly-compact subset $K \subseteq c_0$ that is $F$-closed, i.e. closed under restriction on elements of $F$.

Fix $\varepsilon > 0$, and let $\varepsilon_n = \varepsilon / 2^{n+3}$. Suppose that $\alpha = \omega^\gamma$. There are two cases to consider. Suppose first that $\gamma = \beta + 1$. We apply Lemma 4.9 (a) to the indecomposable ordinal $\omega^\beta$ and $(\varepsilon_n)$ to produce the corresponding sequences of barriers $(C_n)$ and $U$-mean-assignments $\nu_n : C_n \to c_0$ ($n \in N$) satisfying the conclusions (a.1) and (a.2) of the Lemma. If $\gamma$ is limit, then we use the part (b) of that lemma to produce a double sequence $(B^n_i)$ and $U$-mean-assignments $\nu_{n, i} : C^n_i \to c_0$ satisfying (b.1) and (b.2). In order to unify the two cases we set for $n, i$$B^n_i = \begin{cases} C_i & \text{if } \gamma \text{ is successor ordinal} \\ C^n_i & \text{if } \gamma \text{ is limit ordinal} \end{cases}$

and

$\mu_{n, i} = \begin{cases} \nu_i & \text{if } \gamma \text{ is successor ordinal} \\ \nu_{n, i} & \text{if } \gamma \text{ is limit ordinal} \end{cases}$

Let $\sigma : FIN^{< \omega} \to N$ be 1-1 mapping such that $\sigma((s_0, \ldots, s_n)) = \max s_n$ for every block sequence $(s_0, \ldots, s_n)$ of finite sets. For each $n$ define

$C_n = \{s_0 \cup \cdots \cup s_{n-1} : (s_i)$ is block and $s_i \in B^n_{\sigma_i((n), \ldots, n-1))}$ for every $0 \leq i \leq n \}$

So, by Proposition 4.11, if $\alpha = \omega^{\beta + 1}$, then $C_n$ is a $\omega^\beta(n - 1) + \zeta$-uniform family on $N$, where $\zeta$ is such that $B^n_{\sigma_i((n))}$ is $\zeta$-uniform; while if $\alpha = \omega^\beta$ with $\gamma$ limit, then it is $\alpha n(n - 1) + \zeta$ where $\zeta$ is such that $B^n_{\sigma_i((n))}$ is $\zeta$-uniform. Now let

$C = \{s \in \text{FIN} : s \in C_{\min, n}\}$

(25)

It turns out that $C$ is an $\alpha$-uniform family on $N$ (so it is a front), not necessarily a barrier. Observe that every $s \in C$ has a unique decomposition $s = \{n\} \cup s(0) \cup \cdots \cup s(n - 1)$ with
Let $n = \min s$ and $s(i) \in B_{s(i)}$, and where $s[i] = \{n, s_0, \ldots, s_{i-1}\} (0 \leq i \leq n - 1)$. For every $s \in C$ and every $i \leq s$, set

$$
\Xi(s, i) = (\mu_{\min s, \sigma(s[i])}(s[i]))^{1/2}.
$$

Define now $\Phi : C \to c_00$ for every $s \in C$ by

$$
\Phi(s) = \epsilon_{\min s} + \sum_{i=0}^{n-1} \Xi(s, i),
$$

(26)

It is not difficult to see that $\Phi : C \to c_00$ is a $U$-mapping. Now define on $c_00$ the norm

$$
\|\Xi\| = \sup \{\|\Phi(s) \upharpoonright (s \setminus t), \Xi\| : s \in C, t \subseteq s(i), \text{ for some } i < \min s\} = \sup \{\|\Phi(s) \upharpoonright (u \setminus t), \Xi\| : u \subseteq s \in C, t \subseteq s(i), \text{ for some } i < \min s\},
$$

(27)

the last equality because $\Phi$ is Lipschitz and supported by a front. Let $X$ the completion of $c_00$ under this norm. Then the Hamel basis $(e_n)_n$ of $c_00$ is a normalized basis of $X$, moreover monotone (since $\Phi$ is Lipschitz with domain a front) and weakly-null: To prove this, it is enough to see that the set

$$
L = \{\Phi(s) \upharpoonright (u \setminus t) : s \in B, u \subseteq s, \text{ and } t \subseteq s(i) \text{ for some } i < \min s\}
$$

is weakly-compact. So, let $(\Phi(s_n) \upharpoonright (u_n \setminus t_n))_n$ a typical sequence in $L$. Since $C$ is a front, we can find an infinite set $M$ and $u \in \FIN$ such that $(u_n)_{n \in M}$ converges to $u$ and such that $(s_n)$ is a $\Delta$-system with root $u \subseteq r$. Since $\Phi$ is Lipschitz $\sigma$-de, we obtain that $(\Phi(s_n) \upharpoonright t_n)_{n \in M}$ converges to $\Phi(s_m) \upharpoonright t$ for (any) $m \in M$. If $u = \emptyset$, then $(\Phi(s_n) \upharpoonright (t_n \cup u))_{n \in M}$ converges to 0. Otherwise, let $N \subseteq M$ and $j < \min u$ be such that $t_n \subseteq s_n(j)$ for every $n \in N$. Now $(t_n)_{n \in N}$ is a sequence in the closure of $B^\sigma\sigma(s[i])$, hence, we can find $P \subseteq N$ such that $(t_n)_{n \in P}$ is convergent with limit $t$. It follows that $(\Phi(s_n) \upharpoonright (u_n \setminus t_n))_{n \in P}$ has limit $\Phi(s_n) \upharpoonright (u \setminus t) \in L$, where $n$ is (any) integer in $P$.

The next is a crucial computation.

**Claim.** For every $s, t \in C$ and every $i \leq \min s$ and $j \leq \min s$, we have that

$$
0 \leq \langle \xi(t, j), \xi(s, i) \rangle \leq \begin{cases} 
\varepsilon_{\max (\min s, \min t)} & \text{if } t[j] \neq s[i] \\
1 & \text{if } t[j] = s[i].
\end{cases}
$$

**Proof of Claim:** Set $n = \min s$, $m = \min t$, and assume that $t[j] \neq s[i]$. Suppose first that $\alpha = \omega^{\beta+1}$. Then, by definition of the mean assignments, $\langle \xi(t, j), \xi(s, i) \rangle \leq \varepsilon_{\max (\sigma(t[j]), \sigma(s[i]))}$, but $\sigma(u_0, \ldots, u_k) \geq \max u_k$ for every block sequence $(u_i)$, which derives into the desired inequality. Assume now that $\alpha = \omega^\gamma$ with limit ordinal. If $\min s = \min t$, then $\langle \xi(t, j), \xi(s, i) \rangle \leq \varepsilon_{\min s + \max (\sigma(t[j]), \sigma(s[i]))} \leq \varepsilon_{\min s}$. While if $t \neq s$, say $t < \min s$, then $\langle \xi(t, j), \xi(s, i) \rangle \leq \varepsilon_{\min s + \sigma(s[i])} \leq \varepsilon_{\min s}$.

If $\sigma(s[i]) = \sigma(t[j]) = i$, then $\min s = \min t = n$, and

$$
\langle \xi(s, i), \xi(t, j) \rangle \leq \|\langle \mu_{n,i}(s[i]) \rangle\|^{1/2}\|\langle \mu_{n,i}(t[j]) \rangle\|^{1/2} \leq 1,
$$

(28)

since both are means.

**Claim.** The summing basis $(S_n)$ of $c$ is finitely block represented in any subsequence of $(e_n)_n$. 

\[ \Box \]
Proof of Claim: Fix an infinite set \( M \) of integers, and \( i \in \mathbb{N} \). Let \( v \in \mathcal{B} \upharpoonright M/l, v = \{n\} \cup v(0) \cup \cdots \cup v(n - 1) \) its canonical decomposition, and set
\[
x_i = \sum_{j \in \Theta(n)} \xi(v, i)(j)\epsilon_j.
\] (29)

Observe that \( \langle \Phi(v), x(v, i) \rangle = \langle \xi(v, i), \xi(v, i) \rangle = 1 \), so from the previous claim we obtain that \( \|x_i\| = 1 \). Now consider scalars \( (a_i)_{i \leq n-1} \) with \( \|\sum_{i \leq n-1} a_iS_i\|_\infty = 1 \). Observe that this implies that \( \max_{i \leq n-1} |a_i| \leq 2 \). We are going to show that
\[
1 \leq \|\sum_{0 \leq i \leq n-1} a_ix_i\| \leq 3 + \epsilon.
\] (30)

To get the left hand inequality, suppose that \( 1 = \|\sum_{0 \leq i \leq n-1} a_iS_i\|_\infty = \|\sum_{i \leq m} a_i\| \), where \( m \leq n - 1 \). Let \( t = \{n\} \cup s(0) \cup \cdots \cup s(m) \). By (27) it follows that
\[
\|\sum_{i \leq n-1} a_ix_i\| \geq \langle \Phi(v) \upharpoonright t, \sum_{i \leq n-1} a_ix_i \rangle = \|\sum_{i \leq m} a_i\| = 1.
\] (31)

Next, fix \( s \in \mathcal{C} \) and \( t \subseteq s(k) \) for some \( k < \min s \). Suppose first that \( \min v = \min s \). Let \( i_0 = \max \{i \leq n - 1 : v(i) = s(i)\} \). If \( k > i_0 \) then by the previous claim we obtain
\[
\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_ix_i \rangle \leq \|\sum_{i \leq i_0} a_i\| + \sum_{i_0 < i, j \leq n-1} 2\|\xi(s, i), \xi(t, j)\| \leq \|\sum_{i \leq n-1} a_iS_i\|_\infty + 2n^2\epsilon_n \leq (1 + \epsilon)\|\sum_{i \leq n-1} a_iS_i\|_\infty.
\] (32)

Suppose that \( k \leq i_0 \). Then
\[
\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_ix_i \rangle \leq \|\sum_{i \leq i_0} a_i + a_k\xi(v, k), \xi(v, k) \upharpoonright (s(k) \setminus t)\| + \sum_{i_0 < i, j \leq n-1} 2\|\xi(s, i), \xi(t, j)\| \leq 3\|\sum_{i \leq n-1} a_iS_i\|_\infty + 2n^2\epsilon_n \leq (3 + \epsilon)\|\sum_{i \leq n-1} a_iS_i\|_\infty.
\] (33)

Suppose now that \( n = \min v \neq \min s \), say \( \min s < \min v \). Let \( i_0 < n \), if possible, be such that \( \min s \in v(i_0) \). Then,
\[
\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_ix_i \rangle \leq |a_{i_0}|\|\xi(v, i_0)\|_\infty + 2\sum_{i_0 \leq i < n} \sum_{0 \leq j < \min t} \|\xi(t, j), \xi(s, i)\| \leq 2\epsilon_n + 2n^2\epsilon_n \leq \epsilon.
\] (34)

Finally, it rests to show that the sequence \((\epsilon_n)\) is \( \beta \)-unconditionally saturated for every \( \beta < \alpha \).

We consider the two obvious cases:

Case 1. \( \alpha = \omega^{\beta+1} \). Let
\[
D = \{s \subseteq \mathbb{N} : s \in B_{\min s}^0\}.
\]

This is a \( \omega^\beta \)-uniform family on \( \mathbb{N} \) since each family \( B_m^0 \) is \( \alpha_m \)-uniform and \( \sup_m \alpha_m = \omega^\beta \). Therefore, the next claim gives that \((\epsilon_n)\) is \( \beta \)-unconditionally saturated for every \( \beta < \alpha \).
Claim. \((e_n)_n\) is \(\mathcal{D}\)-unconditional with constant at most \(2 + \varepsilon\).

Proof of Claim: Fix \(t \in \mathcal{D}\), and let \((a_i)_i \in \mathbb{N}\) be scalars such that \(\|\sum_{i \in \mathbb{N}} a_i e_i\| = 1\). Fix also \(s \in \mathcal{C}\).
Suppose first that \(\min s \in t\). Then since \(\sigma(s[i]) > \min s \geq \min t\) and \(s \in \mathcal{B}_0^{\min t}\), we obtain that
\[
\|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle \| \leq \|a_{\min s} + \varepsilon\| \sum_{i} a_i e_i\|
\]
(35)

Now suppose that \(\min s \notin t\), but \(s \cap t \neq \emptyset\) (otherwise \(\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle = 0\)). Let
\[
i_0 = \min \{i < \min s : s(i) \cap t \neq \emptyset\}.
\]
Then for every \(i_0 < i < \min s\) we have that \(\sigma(s[i]) > \max s_{i_0} \geq \min t\), so
\[
|\sum_{j \in t} a_j \xi(s, i)(j)| < \varepsilon \sigma(s[i]),
\]
(36)

hence
\[
|\langle \Phi(s) | u, \sum_{i \in t} a_i e_i \rangle | \leq |\sum_{j \in t \cap s(i_0)} a_j \xi(s, i_0)(j)| + \sum_{i_0 < i < \min s} |\sum_{j \in t} a_j \xi(s, i)(j)| =
\]
\[
= |\langle \Phi(s) | (\{n\} \cup s(0) \cup \cdots \cup s(i_0) \cap t)\rangle, \sum_{i \geq \min t} a_i e_i \rangle | +
\]
\[
+ \sum_{i_0 < i < \min s} |\sum_{j \in t} a_j \xi(s, i)(j)| \leq \|\sum_{i \geq \min t} a_i e_i \| + \varepsilon \|\sum_{i \in \mathbb{N}} a_i e_i\| \leq
\]
\[
\leq (2 + \varepsilon)\|\sum_{i} a_i e_i\|.
\]
(37)

the last inequality because \((e_i)\) is monotone. \(\square\)

Case 2. \(\alpha = \omega^\gamma\), \(\gamma\) a countable limit ordinal. The desired result follows from the following fact.

Claim. For every \(n \in \mathbb{N}\), the sequence \((e_i)_i\) is \(\mathcal{B}_0^n\)-unconditional with constant at most \(2n + 1\).

Proof of Claim: Fix \(n \in \mathbb{N}\) and \(t \in \mathcal{B}_0^n\). Let \((a_i)_i \in \mathbb{N}\) be scalars such that \(\|\sum_{i \in \mathbb{N}} a_i e_i\| = 1\). Fix \(s \in \mathcal{C}\). Suppose first that \(n \leq \min s\). Then in a similar manner that in Case 1 one can show that
\[
|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle | \leq \|a_{\min s} + \varepsilon\| \sum_{i} a_i e_i\|
\]
(38)

Suppose that \(m = \min s < n\), then
\[
|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle | \leq \|a_{\min s} + \sum_{i = 0}^{m-1} \sum_{j \in s(i) \cap t} a_j \xi(s, i)(j) \|
\]
\[
= \|a_{\min s} + \sum_{i = 0}^{m-1} \langle \Phi(s) | u_i, \sum_{j \geq \min s(i) \cap t} a_j \rangle | \leq
\]
\[
\leq (2m + 1)\|\sum_{i} a_i e_i\|
\]
(39)

where \(u_i = s(0) \cup \cdots \cup s(i) \cap t\). \(\square\)
Corollary 4.13. For every indecomposable ordinal $\alpha$ there is a weakly-compact $K \subseteq c_0$ such that
(a) $K \subseteq B_0$ is point-finite (i.e. $\{\xi(n) : \xi \in K\}$ is finite for every integer $n$) supported by a $\alpha$-uniform barrier on $\mathbb{N}$,
(b) the evaluation mapping sequence $(p_i)_i$ of $C(K)$ is a normalized weakly-null monotone basic sequence, and
(c) The summing basis of $e$ is 4-finitely representable in every subsequence of $(p_i)_i$; hence no subsequence of $(p_n)_n$ is unconditional, but
(d) $(p_i)_i$ is $\beta$-unconditionally saturated for every $\beta < \alpha$.

Proof. Let $C$ be the $\alpha$-uniform family on $\mathbb{N}$ and let $\Phi : C \to c_0$ be the $U$-mapping given in proof of Theorem 4.12. Let $M \subseteq \mathbb{N}$ be such that $C \restriction M$ is a $\alpha$-uniform barrier on $\mathbb{N}$. Let $\theta$ be the order-preserving mapping from $M$ onto $\mathbb{N}$. Let $B = \theta''C = \{\theta''(s) : s \in C\}$ and let $\varphi : B \to c_0$ be naturally defined by $\varphi(s) = \Phi(\theta^{-1}(s))$. $B$ is a uniform barrier on $\mathbb{N}$ and $\varphi$ is a $U$-mapping. Observe that every $s \in B$ has a unique decomposition, given by the one of $\theta^{-1}s$. Let

$$K = \{\varphi(s) \restriction (u \setminus t) : u \subseteq s \in B, t \subseteq s(i) \text{ for some } i\}.$$ 

This is a weakly-compact subset of $c_0$, and the corresponding evaluation mapping sequence $(p_i)_i$ is $1$-equivalent to the subsequence $(e_n)_{n \in M}$ of the weakly-null sequence $(e_i)_i$; given in the proof of Theorem 4.12. So $K$ fulfills all the requirements. \hfill \Box

References


Équipe de Logique Mathématique, Université Paris 7, 2 place Jussieu, 75251 Paris Cedex 05, France.

E-mail address: abade@logique.jussieu.fr

C.N.R.S., U.M.R. 7056 et Université Paris 7, U.F.R. de Mathématiques, Case 7012, 2 Place Jussieu, 75251 Paris, Cedex 05, France.

Department of Mathematics, University of Toronto, Toronto, Canada, M5S 3G3

E-mail address: stevo@math.toronto.edu