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Super-replication of European options with a derivative asset under constrained finite variation strategies

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Abstract

We consider a financial market, in which a first asset will be referred as the underlying and the second one as a derivative. In this market, the volatility on the underlying depends of the price of the derivative. Furthermore, the derivative is constrained to be traded with finite variation strategies. We study the super-replication problem of an European option on the underlying, and characterize its price as the unique viscosity solution of a partial differential equation with appropriate boundary conditions. We also give a dual representation of the price, as the supremum of the risk neutral expectation over a range of dynamics of the price of the derivative.

Key words : Gamma constraints, super-replication, viscosity solutions, double stochastic integrals

AMS classification (2000) : Primary 93E20, 91B28; Secondary 35K55, 60H30

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1 Introduction

It is commonly known that, under unbounded stochastic volatility, with no instrument to hedge oneself against this volatility, the super-replication price of an European option is the price of the cheapest buy and hold strategy involving the underlying. Hence this price is the concave envelope of the payoff of this option. This was treated, for example in [8]. Meanwhile, another problem gives the same result: the super replication price under constant volatility with fixed or proportional transaction costs. For example, see [2] or [7]. On the other hand, we know that in some stochastic volatility models, for example Heston's model [4] or Hull and White's model [5], one can perform a perfect hedge with the underlying and another instrument, a "volatility asset". This is the case, for example, if another European option is traded. But what happens in such a model when there are some constraints on the strategies involving the "volatility asset"? (for example, if the asset is constrained to be traded with finite variation, due typically to transaction costs). These type of constraints are studied over the underlying in [2], and the result is again the cheapest buy and hold super-replicating strategy. But the case of constraints over the "volatility asset" with no constraints over the underlying is not yet considered in the literature, although important in practice. In this paper we focus on that case, and prove that the super-replication strategy is not necessarily a "Buy and Hold" strategy. Indeed, the price is convex with respect to the volatility asset price, but not w.r.t. the underlying. We characterize this price as the unique solution to a PDE in the viscosity sense, and the terminal condition is found to be the payoff itself. Moreover, we prove a dual representation as in [2] and [14], in which the price is the supremum of the risk neutral prices over all possible dynamics of the "volatility asset". Here, we do not consider vanishing transaction costs, but we constraint the quantity of asset in the portfolio to be of finite and almost surely bounded variation, as a limit case of gamma constraints used in [11], when the authorized "gamma" with respect to one asset is zero. This is a new feature of this paper. Moreover, the gamma constraints considered here are not symmetric, which involve a new result about double stochastic integral as in [12], which is valid for non-symmetric integrands.

The structure of this paper is the following: In section 2, we define the model, the super-replication problem and the portfolio gamma constraints. We also state the main results. Then, in section 3 we show that the super-replication price is a solution of a partial differential equation with specific terminal and boundary conditions. The uniqueness of this solution is proved in section 4 with the help of a comparison principle. Finally, in section 5, we prove a dual representation of the solution, which can be interpreted as the supremum of the risk neutral prices of the option over a range of dynamics of the volatility asset.

2 Problem formulation and main results

2.1 Model

We consider a financial market with three different assets. The first one is a riskless bond which price is constant and equal to 1. The second one is a risky asset S , and the third one is an asset X whose price is linked to the instantaneous volatility $\sigma(X)$ of S . This asset X distributes an instantaneous cash flow $\mu(X)$. Our problem is to find a super-replication price, hence we are only interested in almost sure events. Therefore we can specify our market under a risk neutral probability measure. We assume that the prices of the considered assets evolve according to the dynamics:

$$\begin{cases} dS(t) = S(t) \sigma(t, X_t) dW_t^1 \\ dX(t) = -\mu(t, X_t) dt + \zeta(t, X_t) dW_t^1 + \xi(t, X_t) dW_t^2 \end{cases} \quad (2.1)$$

Here, uncertainty is due to a two dimensional standard Brownian motion (W_t) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote $\{\mathcal{F}_t, 0 \leq t \leq T\}$ the usual augmented filtration of $\{W_t, 0 \leq t \leq T\}$. Here we supposed that S and X are uncorrelated. X_t can be viewed as a pure volatility asset, for instance a variance swap. In order to obtain a unique strong solution, one needs to assume some properties of the functions σ , μ and ξ .

$$\sigma^2, \mu, \zeta, \xi \text{ are locally Lipschitz on } (0, T) \times (0, +\infty) \quad (2.2)$$

Furthermore, we will need two other assumptions, in order to find the boundary condition of the pricing PDE:

$$\xi(., 0) = 0 \text{ and } \xi(., x) > 0 \text{ for } x \neq 0 \quad (2.3)$$

$$\zeta(., 0) = 0 \text{ and } \zeta(., x) > 0 \text{ for } x \neq 0 \quad (2.4)$$

$$\text{There exists } 0 < C_1 < 1 \text{ such that } -\mu(t, x) \leq \frac{x C_1}{T-t} \quad (2.5)$$

$$\text{for all } (t, x) \in [0, T) \times \mathbb{R}_+$$

$$\text{There exists a constant } C_2 \text{ such that } \sigma^2(t, x) \leq C_2 x \text{ for all } (t, x) \in [0, T) \quad (2.6)$$

With these assumptions, one can prove that equation (2.1) has a unique strong solution (S, X) valued in $[0, +\infty)^2$ given:

$$S(t) = s, X(t) = x, (t, s, x) \in [0, T] \times [0, +\infty)^2$$

Remark 2.1. It would be more realistic to consider the asset X as a call option, for example. But conditions (2.3) and (2.6) would have to be modified (by arbitrage, the value of a call option can not be below its discounted payoff). Modifying (2.3) would change the domain of the pricing PDE, in the case of a call option from $(0, T) \times (0, +\infty)^2$ to $(0, T) \times (0, +\infty) \times ((s - K)_+, s)$. Hence, the proof of the comparison principle would be more complicated. Changing condition (2.6) may change the limit condition of the value function near the

boundary of the domain. It could make it more tedious to derive the equivalent of propositions 3.3 and 3.4. Hence we decide to study the simple case which embeds variance swaps, or futures on VIX index (futures contracts on the implied volatility level). Another issue is whether these results can be adapted to diffusion in dimension $n > 2$, with $k > 1$ constrained assets. If $k = 1$, there is not much work to adapt the case. Meanwhile, $k > 1$ would mean that every of the k boundary condition would be the solution of the same kind of problem with $k - 1$ constrained assets, which would introduce some new difficulties.

The issue of this paper will be to derive a hedging price for a contingent claim $g(S_T)$ under certain constraints described in the following. For the sake of simplicity, we will sometimes consider some regularity assumptions on the payoff function:

$$g \text{ is bounded by a constant } C^* \quad (2.7)$$

$$g \text{ is } \mathcal{C}^2 \text{ and } s \rightarrow s^2 g''(s) \text{ is bounded by a constant } C_3 \quad (2.8)$$

The first assumption could be omitted if we simply considered the primal problem (some linear growth conditions would suffice), but the dual problem characterization is much more easier with it. The second assumption could be relaxed with little efforts (considering a sequence of regular payoffs above the one of interest). Indeed, this assumption will only be used in the proofs of propositions 3.3 and 3.4, and one can see that for most common payoffs, these can be adapted.

2.2 The super-replication problem

2.2.1 Value function

The agent can trade assets on the market with self financing strategies, and its wealth process can be written as:

$$Y(t)_{t,s,x,y}^\pi(r) = y + \int_t^r \pi_{t,s,x}^S dS_{t,s,x}(u) + \int_t^r \pi_{t,s,x}^X (dX_{t,s,x}(u) + \mu(u, X(u))du)$$

Our problem is to find the super-replication price of a contingent claim $g(S_T)$ with a limited set of admissible strategies. One must find the minimum amount of money which enables to super-replicate the payoff of the option. Hence, the problem is to characterize the following value function:

$$v(t, s, x) = \inf_{y \in \mathbb{R}} \{y : Y_{t,s,x,y}^\pi \geq g(S_{t,s}(T)) \text{ a.s. for some } \pi \in \mathcal{A}_{t,s,x}\} \quad (2.9)$$

2.2.2 Gamma constraints

Here, we describe the set of admissible strategies $\mathcal{A}_{t,s,x}$. The specificity of this paper is the following: we can buy and sell the asset S freely, without transaction cost or waiting time, but the asset X is far less liquid, so we need some time to buy and sell it. Mathematically, this means that almost every adapted self-financed

strategies (excepted doubling ones) will be admissible for the asset S , but that the set of admissible strategies will be far more constrained for the asset X . A trading strategy is a vector $\pi(t) = (\pi_S(t), \pi_X(t))$, where $\pi_S(t)$ is the amount (in unity of assets) of assets S of the strategy at time t . π is in $\mathcal{A}_{t,s,x}$ if $\pi(t)$ is of the form:

$$\pi_S(r) = \sum_{n=0}^{N-1} y_s^n 1_{\tau_s^n > t} + \int_t^r \alpha_s(u) du + \int_t^r \gamma^1(u) dS_{t,s}(u) + \int_t^r \gamma^2(u) dX_{t,x}(u)$$

Where τ_s^n are stopping times for each n , y_s^n are $\mathcal{F}_{\tau_s^n}$ -measurable random variables, and α_s, γ_s^1 and γ_s^2 are almost surely bounded adapted processes. While π_X satisfies:

$$\pi_X(r) = \sum_{n=0}^{N-1} y_x^n 1_{\tau_x^n > t} + \int_t^r \alpha_x(u) du$$

With τ_s^n and α_s filling the same conditions as above. For technical reasons, $\gamma = (\gamma^1, \gamma^2)$ above must be of the form:

$$\gamma = \sum_{n=0}^{N-1} z^n 1_{\tau_n \leq t < \tau_{n+1}} + \int_t^r \psi_u^s du + \int_t^r \chi_u^s dS_{t,s}(u) + \int_t^r \kappa_u^s dX_{t,x}(u)$$

With ψ, χ, κ adapted and uniformly bounded. This is necessary to apply the result on double stochastic integrals proved in the appendix.

2.3 Main results

2.3.1 Operators

The operator used to to define the super-replication price equation will be:

$$F(t, s, x, Du, D^2u) = \tag{2.10}$$

$$\lambda^- \begin{pmatrix} -\frac{\partial u}{\partial t} + \mu(t, x) \frac{\partial u}{\partial X} - \frac{1}{2} s^2 \sigma^2(t, x) \frac{\partial^2 u}{\partial S^2} & -\frac{1}{2} s \sigma(t, X) \xi(t, X) \frac{\partial^2 u}{\partial S \partial X} \\ -\frac{1}{2} s \sigma(t, x) \xi(t, x) \frac{\partial^2 u}{\partial S \partial X} & -\frac{1}{2} (\xi(t, x))^2 \frac{\partial^2 u}{\partial X^2} \end{pmatrix} \\ = \lambda^- (J(t, s, x, Du, D^2u)) \tag{2.11}$$

Where λ^- represents the smallest eigenvalue of a matrix. One can easily check that this operator is parabolic. In the following, we will denote the matrix H^ξ as:

$$H^\xi(t, S, X, Du, D^2u) = \begin{pmatrix} -\frac{\partial u}{\partial t} + \mu(t, X) \frac{\partial u}{\partial X} - \frac{1}{2} S^2 \sigma^2(t, X) \frac{\partial^2 u}{\partial S^2} & -\frac{1}{2} S \sigma(t, X) \xi \frac{\partial^2 u}{\partial S \partial X} \\ -\frac{1}{2} S \sigma(t, X) \xi \frac{\partial^2 u}{\partial S \partial X} & -\frac{1}{2} \xi^2 \frac{\partial^2 u}{\partial X^2} \end{pmatrix} \tag{2.12}$$

And one can see that considering that a 2×2 matrix is positive if and only if its diagonal terms and its determinant are, operator F is positive if and only if matrix H^ξ is positive for a given $\xi \neq 0$. Furthermore, we see that ζ does not enter into account in these operators.

2.3.2 Equations

Now, we can state the two main results of this paper:

Proposition 2.2. *The solution v of the super-replication problem is the unique viscosity solution of the equation:*

$$F(t, s, x, Dv, D^2v) = 0 \text{ on } (0, T) \times (0, +\infty)^2$$

such that v is continuous on the boundaries $x = 0$ and $t = T$, with $v(t, s, 0) = v(T, s, x) = g(s)$ for all $(t, s, x) \in [0, T] \times [0, +\infty)^2$.

The second main result is a dual representation theorem of the value function.

Proposition 2.3. *The solution v of the super-replication problem satisfies the dual representation:*

$$v(t, s, x) = \sup_{(\rho, \xi) \in \mathcal{U}} \mathbf{E} \left[g \left(S_{t,s,x}^{\rho, \xi}(T) \right) \right] \text{ for all } (t, s, x) \in [0, T] \times [0, +\infty)^2$$

Where $S_{t,s,x}^{\rho, \xi}(T)$ is the solution of the equation:

$$\begin{aligned} dS_{t,s,x}^{\rho, \xi}(u) &= \sigma(t, X_{t,x}^{\rho, \xi}) S_{t,s,x}^{\rho, \xi}(u) dW^1(u) \\ dX_{t,x}^{\rho, \xi}(u) &= -\mu(t, X_{t,x}^{\rho, \xi}) dt + \xi(u) X_{t,x}^{\rho, \xi}(u) dW^2(u) \\ \langle dW^1(u), dW^2(u) \rangle &= \rho(u) \\ S(t) &= s, X(t) = x \end{aligned}$$

Where \mathcal{U} is the set of all almost-surely bounded progressively measurable processes taking values in $[-1; 1] \times [0; +\infty)$

3 Viscosity property

3.1 Sub and supersolution characterization

The proof of the viscosity property is very close to the proof in [11]. Though there are two noticeable differences. The first one is that, here, the space of gamma constraints is of empty interior, because the "gamma" with respect to the second asset is constrained to be zero. It has some impact on the proof of the subsolution property, but most of all on the uniqueness theorem. The second difference is that we did not suppose that the matrix of the gammas was symmetric (i.e. we did not suppose $\gamma_u^2 = 0$, while we constrained the gamma component of X to be equal to zero), so we will need to study the small time behavior of double stochastic integrals involving non-symmetric matrices, which is a new feature. The proof of the sub and supersolution properties involves respectively two auxiliary value functions $\bar{v} \geq v$ and $\underline{v} \leq v$, and are

quite long and technical. Furthermore, the characterization of v is found by the comparison theorem, that gives $\bar{v} \leq \underline{v}$ leading

$$\bar{v} = v = \underline{v}$$

But here, for the sake of simplicity we will only give the main arguments of the proof, without involving rigorous mathematics. But the same steps as in [11] could be used. We will act as if we manipulated the original value function. It is to notice that one could object that we prove the boundary properties in the next sections for v and that we should do it for \bar{v} and \underline{v} . But the proof would be exactly the same, as the differences between the definitions of the three value functions would not interfere.

3.1.1 Subsolution property

Let us begin by defining the upper bound \bar{v} for the value function v . First, we define a norm on the controls:

$$\|\nu\|_{t,s}^{\beta,\infty} := \max \left\{ \|N\|_{L^\infty} ; \|Y\|_{t,s}^{\beta,\infty} ; \|\alpha\|_{t,s}^{\beta,\infty} ; \|\gamma\|_{t,s}^{\beta,\infty} ; \|\gamma^1\|_{t,s}^{\beta,\infty} ; \|\gamma^2\|_{t,s}^{\beta,\infty} \right\}$$

We define another set of admissible controls with:

$$\mathcal{A}_{t,s,x}^M = \left\{ \nu \in \mathcal{A}_{t,s,x} : \|\nu\|_{t,s}^{\beta,\infty} \leq M \right\}$$

And the auxiliary value function:

$$v^M(t, s) := \inf \left\{ y \in \mathbb{R} : X_{t,s,x,y}^v(T) \geq g(S_{t,s,x}(T)) \text{ for some } \nu \in \mathcal{A}_{t,s,x}^M \right\}$$

And at last we get the upper bound of v taking:

$$\bar{v} := \inf_{M>0} (v^M)^*(t, s, x)$$

Now we can state the viscosity subsolution property:

Proposition 3.1. *The function \bar{v} is a viscosity subsolution of equation*

$$F(t, s, x, D\bar{v}, D^2\bar{v}) = 0 \text{ on } (0, t) \times (0, +\infty)$$

Proof. This proof is an adaptation of the proof in [11]. We omit some technical condition, which can be transcribed easily. There are some differences, thought, as the space of controls is of empty interior. As the payoff function is bounded, we know that the value function is finite. Let $\varphi \in C^\infty$ be a test function such that:

$$0 = (\bar{v} - \varphi)(t_0, s_0, x_0) > (\bar{v} - \varphi)(t, s, x) \text{ for all } (t, s, x) \neq (t_0, s_0, x_0)$$

Then assume that on the contrary

$$F(t_0, s_0, x_0, D\varphi, D^2\varphi) > 0 \tag{3.1}$$

We will obtain a contradiction. Denote:

$$\sigma(t_0, x_0) = \sigma_0, \mu(t_0, x_0) = \mu_0$$

First, remark that (3.1) leads to:

$$\frac{1}{2}S_0^2\sigma_0^2\frac{\partial^2\varphi}{\partial S^2}(t_0, S_0, X_0) - \mu_0\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) + \frac{\partial\varphi}{\partial t}(t_0, S_0, X_0) < 0 \quad (3.2)$$

For $\varepsilon > 0$ sufficiently small, we consider a compact neighborhood \mathcal{N} of (t_0, s_0, x_0) such that:

$$\begin{aligned} & -\frac{\sigma^2(t, X)S^2}{\sigma_0^2S_0^2} \left(\frac{\partial\varphi}{\partial t}(t_0, S_0, X_0) - \mu_0\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) + \varepsilon \right) \\ & - \frac{\partial\varphi}{\partial X}(t_0, S_0, X_0)\mu(t, X) + \max_{S, X \in \mathcal{N}} \frac{\partial\varphi}{\partial t}(t, S, X) \leq 0 \quad (3.3) \\ & \left(\begin{array}{c} \frac{1}{2}\sigma_0^2S_0^2\frac{\partial^2\varphi}{\partial S^2}(t, S, X) + \left(\frac{\partial\varphi}{\partial t} - \mu_0\frac{\partial\varphi}{\partial X} \right)(t_0, S_0, X_0) + \varepsilon \\ \frac{1}{2}S_0\sigma_0\xi_0\frac{\partial^2\varphi}{\partial S\partial X}(t, S, X) \end{array} \right. \\ & \left. \begin{array}{c} \frac{1}{2}S_0\sigma_0\xi_0\frac{\partial^2\varphi}{\partial S\partial X}(t, S, X) \\ \frac{1}{2}\xi_0^2\frac{\partial^2\varphi}{\partial X^2}(t, S, X) \end{array} \right) \leq 0 \quad (3.4) \end{aligned}$$

for all $(t, S, X) \in \mathcal{N}$

As φ is C^∞ and satisfies (3.1), \mathcal{N} is nonempty and $(t_0, S_0, X_0) \notin \partial\mathcal{N}$ for sufficiently small ε . As (t_0, s_0, x_0) is a strict maximizer of $\bar{v} - \varphi$, there exists $\eta > 0$ such that $(\bar{v} - \phi)(t, s, x) < 2\eta$ on $\partial\mathcal{N}$. Let θ be the stopping time:

$$\theta := \inf \{t \geq t_0 : (t, S_t, X_t) \notin \mathcal{N}\}$$

And consider the following decomposition of φ :

$$\begin{aligned} \varphi(t, S, X) &= \psi_1(t, S, X) + \psi_2(t, S, X) \\ \psi_1(t, S, X) &= \varphi(t_0, S_0, X_0) - \frac{(S - S_0)^2}{S_0^2\sigma_0^2} \left(\frac{\partial\varphi}{\partial t}(t_0, S_0, X_0) + \varepsilon - \mu_0\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) \right) \\ & \quad + (X - X_0)\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) + (S - S_0)\frac{\partial\varphi}{\partial S}(t_0, S_0, X_0) \\ & \quad + \int_{t_0}^t \max_{(u, s, x) \in \mathcal{N}} \frac{\partial\varphi}{\partial t}(u, s, x) du \\ \psi_2(t, S, X) &= \varphi(t, S, X) - \varphi(t_0, S_0, X_0) \\ & \quad + \frac{(S - S_0)^2}{S_0^2\sigma_0^2} \left(\frac{\partial\varphi}{\partial t}(t_0, S_0, X_0) + \varepsilon - \mu_0\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) \right) \\ & \quad - (S - S_0)\frac{\partial\varphi}{\partial S}(t_0, S_0, X_0) - (X - X_0)\frac{\partial\varphi}{\partial X}(t_0, S_0, X_0) \\ & \quad - \int_{t_0}^t \max_{(u, s, x) \in \mathcal{N}} \frac{\partial\varphi}{\partial t}(u, s, x) du \end{aligned}$$

First, let us prove that one can super-replicate the first part of this decomposition. Consider the initial capital:

$$y_0 := \bar{v}(t_0, s_0, x_0) - \eta$$

And the control:

$$\pi_0 = D\varphi(t_0, s_0, x_0), \alpha(t) := 0, \pi_S(t) = -\frac{2(S - S_0)}{S_0^2 \sigma_0^2} \left[\left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, s_0, x_0) + \varepsilon \right]$$

Denote $(Y, \pi) := (Y_{t_0, s_0, x_0, y_0}^\nu, \pi_{t_0, s_0, x_0}^\nu)$. Then, by Ito's formula, combined with conditions (3.2) and (3.3):

$$\begin{aligned} dY(t) &= \frac{\partial \varphi}{\partial X}(t_0, s_0, x_0) (dX + \mu(t, X)dt) - \frac{2(S - s_0)}{s_0^2 \sigma_0^2} \left[\left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, s_0, x_0) + \varepsilon \right] dS \\ d\Psi_1 &= -\frac{2(S - s_0)}{s_0^2 \sigma_0^2} \left[\left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, s_0, x_0) + \varepsilon \right] dS + \max_{\mathcal{N}} \frac{\partial \varphi}{\partial t} dt \\ &\quad + \frac{2S^2 \sigma^2(t, X)}{S_0^2 \sigma_0^2} \left[\left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, s_0, x_0) + \varepsilon \right] \\ d(Y(t) - \psi_1(t, S(t), X(t))) &\geq 0 \end{aligned}$$

Which shows that $Y(\theta) - \psi_1(\theta, S(\theta), X(\theta)) \geq -\eta$. Now, let us show that $\psi_2 \leq 0$ on \mathcal{N} . For $t = t_0$, we have, differentiating with respect to S and X :

$$D\psi_2(t_0, S_0, X_0) = 0$$

And the Hessian matrix of ψ_2 is for any $(t_0, S, X) \in \mathcal{N}$:

$$D^2\psi_2(t_0, S_0, X_0) = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial S^2}(t, S, X) + \frac{2}{S_0^2 \sigma_0^2} \left(\left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, S_0, X_0) + \varepsilon \right) & \frac{\partial^2 \varphi}{\partial S \partial X}(t, S, X) \\ \frac{\partial^2 \varphi}{\partial S \partial X}(t, S, X) & \frac{\partial^2 \varphi}{\partial X^2}(t, S, X) \end{pmatrix}$$

Let $\Sigma_0 = \begin{pmatrix} \sigma_0 S_0 & 0 \\ 0 & \xi_0 \end{pmatrix}$. We get, with assumption (3.3):

$$\begin{aligned} \frac{1}{2} \Sigma_0 D^2\psi_2(t_0, S_0, X_0) \Sigma_0 &= \\ \begin{pmatrix} \frac{1}{2} \sigma_0^2 S_0^2 \frac{\partial^2 \varphi}{\partial S^2}(t, S, X) + \left(\frac{\partial \varphi}{\partial t} - \mu_0 \frac{\partial \varphi}{\partial X} \right) (t_0, S_0, X_0) + \varepsilon & \frac{1}{2} S_0 \sigma_0 \xi_0 \frac{\partial^2 \varphi}{\partial S \partial X}(t, S, X) \\ \frac{1}{2} S_0 \sigma_0 \xi_0 \frac{\partial^2 \varphi}{\partial S \partial X}(t, S, X) & \frac{1}{2} \xi_0^2 \frac{\partial^2 \varphi}{\partial X^2}(t, S, X) \end{pmatrix} \\ &\leq 0 \end{aligned}$$

Hence the function ψ_2 is concave on $\mathcal{N} \cap \{t = t_0\}$, and its value and first order derivative are 0 on (t_0, S_0, X_0) . Hence, it is negative on \mathcal{N} for $t = t_0$. Now, remark that its time derivative is negative on \mathcal{N} , therefore, ψ_2 is negative on \mathcal{N} for $t > t_0$. So, we have:

$$Y(\theta) \geq \varphi(\theta, S_\theta, X_\theta) - \eta \geq v(\theta, S_\theta, X_\theta)$$

Therefore the dynamic programming principle is violated and this concludes the proof. \square

3.1.2 Supersolution property

To prove the supersolution property, one has to define the the relaxed stochastic control problem:

$$\underline{v}^M(t, s, x) := \inf \left\{ y \in \mathbb{R} : \tilde{Y}_{t,s,x,y}^\nu(T) \geq g(\tilde{S}_{t,s,x}(T)) \text{ for some } (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P}) \text{ and } \tilde{\nu} \in \mathcal{A}_{t,s}^M(\tilde{\Omega}) \right\}$$

For $M > 0$. Where \tilde{Y}, \tilde{S} and $\mathcal{A}_{t,s}^M(\tilde{\Omega})$ are defined as in the original control problem. Then, define \underline{v} as the lower semicontinuous envelope of the inferior bound of \underline{v}^M over all $M > 0$. The change of probability and filtration is due to technical reason in order to obtain existence of and optimal control, and lower semicontinuity of the value function. These, and the corresponding dynamic programming principle are obtained in lemmas 5.1 and 5.2 in [11].

Proposition 3.2. *For all M sufficiently large, \underline{v}^M is a viscosity supersolution of equation:*

$$F(t, s, x, D\underline{v}^M, D^2\underline{v}^M) = 0 \text{ on } [0, t[\times]0, +\infty[\quad (3.5)$$

Proof. This proof is exactly like Theorem 5.4 in [11], excepted for the limit result on double stochastic integrals. Indeed, in that paper, the integrand of the double integral is supposed to be symmetric, whereas here it is not. But anyway the result is the same, as the integrand turns out to be necessarily symmetric. For these reasons, we only give a sketch of the demonstration. Let $M \geq C^*$ (where C^* is the bound of g in assumption (2.7)) be fixed. By lemma 5.2, in [11] \underline{v}^M is finite and lower semicontinuous. Consider a $(t_0, s_0, x_0) \in [0, T[\times \mathbb{R}_+^2$ and a test function $\varphi \in C^\infty[0, T] \times \mathbb{R}_+^2$ such that:

$$0 = (\underline{v}^M - \varphi)(t_0, s_0, x_0) = \min_{(t,s,x) \in [0,T[\times \mathbb{R}_+^2} (\underline{v}^M - \varphi)(t, s, x)$$

Set $y_0 = \underline{v}^M(t_0, s_0, x_0)$. By Lemma 5.2 in [11], there exists a two dimensional Brownian motion \tilde{W} on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ satisfying the usual conditions and a control $\pi \in \mathcal{A}_{t_0, s_0, x_0}^M$ such that, for any stopping time $t_0 \leq \theta \leq T$:

$$\tilde{Y}_{t_0, s_0, x_0}^{\tilde{\pi}}(\theta) \geq \underline{v}^M(\theta, \tilde{S}(\theta), \tilde{X}(\theta)) \geq \varphi(\theta, \tilde{S}(\theta), \tilde{X}(\theta))$$

By twice applying Ito's lemma, one obtains, writing $\tilde{Z} = \begin{pmatrix} \tilde{S} \\ \tilde{X} \end{pmatrix}$:

$$\int_{t_0}^{\theta} l(r) dr + \int_{t_0}^{\theta} \left(c + \int_{t_0}^r a(u) du + \int_{t_0}^r b(u) d\tilde{Z}(u) \right)^T d\tilde{Z}(r) \geq 0 \quad (3.6)$$

where:

$$\begin{aligned} l(r) &:= -\mathcal{L}\varphi \left(r, \tilde{S}(r), \tilde{X}(r) \right) \\ a(r) &:= \tilde{\alpha}(r) - \mathcal{L}(D\varphi) \left(r, \tilde{S}(r), \tilde{X}(r) \right) \\ b(r) &:= \tilde{\gamma}(r) - \mathcal{L}(D^2\varphi) \left(r, \tilde{S}(r), \tilde{X}(r) \right) \\ c &:= \tilde{\pi}^0 - D\varphi(t_0, s_0, x_0) \end{aligned}$$

In [11] it is then proved that $c = 0$ and that considering $\theta^\eta = \min(\theta, \eta)$, then for any real number $\varepsilon > 0$:

$$\lim_{\eta \rightarrow 0^+} \eta^{\varepsilon-3/2} \int_{t_0}^{\theta^\eta} \left(\int_{t_0}^r a(u) du \right)^T d\tilde{Z}(r) = 0 \quad (3.7)$$

Then it follows from (3.6) that:

$$\liminf_{\eta \rightarrow 0^+} \frac{1}{\eta \log \log \frac{1}{\eta}} \int_{t_0}^{\theta^\eta} \left(\int_{t_0}^r b(u) d\tilde{Z}(u) \right)^T d\tilde{Z}(r) \geq 0 \quad (3.8)$$

Therefore, it follows from proposition A.1 that:

$$b(t_0) \text{ is symmetric and positive} \quad (3.9)$$

Hence, by definition of b , $\gamma_0^2 = 0$, and the following matrix is positive:

$$\Sigma_0 b(t_0) \Sigma_0 = \begin{pmatrix} -\sigma^2(t_0, X_0) S_0^2 \left(\frac{\partial^2 \varphi}{\partial S^2} - \gamma_0^1 \right) & -\sigma_0 S_0 \xi_0 \frac{\partial^2 \varphi}{\partial S \partial X} \\ -\sigma_0 S_0 \xi_0 \frac{\partial^2 \varphi}{\partial S \partial X} & -\xi_0^2 \frac{\partial^2 \varphi}{\partial X^2} \end{pmatrix} \geq 0 \quad (3.10)$$

Where all the derivatives are taken at point (t_0, S_0, X_0) . At last, theorem A.2 in [11] shows:

$$\limsup_{\eta \rightarrow 0^+} \frac{1}{\eta} \int_{t_0}^{\theta^\eta} \left(\int_{t_0}^r b(u) d\tilde{Z}(u) \right)^T d\tilde{Z}(r) = \frac{1}{2} \text{Tr} [\Sigma_0 b(t_0) \Sigma_0]$$

Dividing (3.6) by η , remembering (3.7) and that $c = 0$, then taking the limit for $\eta \rightarrow 0$ leads to:

$$-\mathcal{L}\varphi(t_0, s_0) - \frac{1}{2} \text{Tr} [\Sigma_0 b(t_0) \Sigma_0] \geq 0$$

That is:

$$-\frac{\partial \varphi}{\partial t} + \mu(t_0, X_0) \frac{\partial \varphi}{\partial X} - \frac{1}{2} \sigma^2(t_0, X_0) S_0^2 \gamma_0^1 \geq 0$$

Plugging it into (3.10) finishes the proof. \square

Then, one can prove that \underline{v} is supersolution of the same equation, using the same steps as in Corollary 5.5 in [11].

3.2 Boundary conditions

Following remark of example 4.1 in the last section, the viscosity property of the value function v in the interior of the domain is not enough to ensure the characterization of v . Indeed, one needs to derive the boundary behavior near the border $X = 0$ to obtain uniqueness of the solution of equation (2.10) with this additional constraint. This is why we need assumptions (2.3) and (2.5) to exhibit a superhedging strategy when $X \rightarrow 0$ which gives an upper bound on v .

3.2.1 Terminal condition

In many super-replication problems, the value function converges to a face-lifted payoff when time tends to maturity. This is not the case here. Let us demonstrate that the terminal condition of v corresponds to the payoff function g .

Proposition 3.3. *The terminal condition of the value function v is g . In other words: For any $(s, x) \in \mathbb{R}_+^2$*

$$\lim_{t \nearrow T, s' \rightarrow s, x' \rightarrow x} v(t, s', x') = g(s) \quad (3.11)$$

That is, the value function is continuous on $t = T$

Proof. Remember assumptions (2.5), (2.8) and (2.6). Consider an instant $t < T$ and a state of the market (s, x) . Consider the following portfolio for any time $t \leq u \leq T$, which will be the key of the demonstrations below:

$$\pi_u = \left(\begin{array}{c} g'(S_u) \\ \frac{1}{2} \frac{C_2 C_3 (T-u)}{1-C_1} \end{array} \right) \quad (3.12)$$

Where C_3 is the constant in assumption (2.8) and C_1 the constant in (2.5). On the other hand, by Ito's formula one has:

$$g(S_T) - g(S_t) = \int_t^T g'(S_u) dS_u + \frac{1}{2} \sigma^2(u, X_u) S_u^2 g''(S_u) du$$

Plugging conditions (2.6) and (2.8) one obtains:

$$g(S_T) - g(S_t) \leq \int_t^T g'(S_u) dS_u + \frac{1}{2} C_2 C_3 X_u du \quad (3.13)$$

Starting with the initial wealth $g(S_t) + X_t \frac{1}{2} \frac{C_2 C_3 (T-t)}{1-C_1}$, the continuous selling of X_u gives:

$$\int_t^T \frac{1}{2} \frac{C_2 C_3}{1-C_1} X_u du$$

While the (possibly negative) dividends μ gives:

$$\int_t^T \frac{1}{2} \frac{C_2 C_3 (T-u)}{1-C_1} \mu(u, X_u) du$$

So, with condition (2.5) on μ the profit and loss associated with the component in X (excluded the buying price at the beginning) of the portfolio dominates:

$$\int_t^T \frac{1}{2} C_2 C_3 X_u du$$

Compounded with (3.13), one gets that wealth $g(S_t) + \frac{1}{2}x \frac{C_2 C_3 (T-t)}{1-C_1}$ is enough to super-replicate the payoff. Hence:

$$v(t, s, x) \leq g(s) + \frac{1}{2}x \frac{C_2 C_3 (T-t)}{1-C_1}$$

The reverse inequality is more common, it comes from the fact that v is dominated by the replication price $u(t, s, x)$ without constraints, which is the expectation of the payoff. We will not prove this assertion here as it is classical. Once this is done, applying Fatou's lemma finishes the proof. As:

$$u(t, s, x) \leq v(t, s, x) \leq g(s) + \frac{1}{2}x \frac{C_2 C_3 (T-t)}{1-C_1}$$

Then as the LHS and the RHS converge to g , the value function does too. \square

3.2.2 Lateral condition

The next proposition deals with the same type of conditions near $x = 0$.

Proposition 3.4. *The boundary condition of the value function v near $x = 0$ is g : For any $(s, t) \in \mathbb{R}_+^2$*

$$\lim_{t' \rightarrow t, s' \rightarrow s, x' \rightarrow 0} v(t', s', x') = g(s) \tag{3.14}$$

Proof. The proof is essentially the same as above. \square

4 The comparison result

In this section, we prove that equation (2.10) has a unique solution, by establishing a comparison result. Our proof mostly relies on a strict supersolution argument, which has been introduced by Ishii and Lions in [6] and used by Soner and al. in [11]. The idea is to prove a comparison for perturbed sub and super-solutions, and then to take the limit of the resulting inequalities when the perturbation tends to zero. But first, we will see under which conditions does the comparison principle hold.

4.1 Boundary conditions

Interestingly, unlike in most similar parabolic problems, one will not only need a terminal condition to obtain uniqueness, but also some border conditions, when the spot price and the volatility asset tend to zero. Another boundary condition is hidden by the fact that we only consider bounded solutions, which is, intuitively, equivalent to a Neumann conditions near infinity. This is because equation (2.10) is not parabolic in the most common sense, due to a nonlinearity in front of the time derivative. Here is a simple example to illustrate this fact:

Example 4.1. This equation, defined for $u(t, S)$, $(t, S) \in [O, T] \times [0, +\infty[$:

$$\min \left\{ \frac{\partial v}{\partial t} + S^2 \frac{\partial^2 v}{\partial S^2}, \frac{\partial^2 v}{\partial S^2} \right\} = 0 \quad (4.1)$$

$$u(T) = 0 \quad (4.2)$$

has no unique solution. Indeed, let us consider two families of functions:

$$u(t, S) = (t - T)\lambda \text{ and } u(t, S) = (t - T)S\lambda$$

With $\lambda \geq 0$. Both are solutions of equation (4.1). In order to eliminate these solution we need to impose a condition like:

$$u(t, 0) = 0 \text{ for all } t \in [0, T]$$

To eliminate the first kind of solution, and a more common condition

$$u \text{ is bounded on } [0, T] \times [0, +\infty[$$

For the second one. Then, with these boundary conditions, using the following method, one can prove that $u = 0$ is now the only solution of equation (4.1) in the viscosity sense.

This is why one must use boundary conditions (3.11) and (3.14)

$$\lim_{t \rightarrow T^-} v(t, S, \cdot) = g(S), \quad \lim_{X \rightarrow 0^+} v(\cdot, S, X) = g(S),$$

and v is bounded by a constant C .

4.2 Equivalent equation

In order to establish the comparison result, we can reformulate the operator (2.10) with $\xi(t, X) = \max(1, X)$. We can easily see that changing the operator leaves the equation unchanged on the open domain, because of assumption (2.2). Indeed, changing $\xi(t, X) > 0$ into another positive function does not change the sign of the operator F for fixed (t, S, X, Du, D^2u) . So we can introduce a new assumption to prove the uniqueness theorem:

$$\xi(t, X) = \max(1, X) \quad (4.3)$$

4.3 Strict viscosity supersolutions

Let us now introduce the notion of strict supersolution, as in [11] and [6]. This strict supersolution property will be necessary to prove the comparison principle.

Definition 4.2. For a strictly positive constant η , a function w is an η -strict viscosity supersolution of equation (2.10) if:

$$F(t_0, y_0, D\varphi(t_0, y_0), D^2\varphi(t_0, y_0)) > \eta \quad (4.4)$$

for all $(t_0, y_0) \in [0, T] \times [0, +\infty]^2$ and $\varphi \in C^\infty([0, T] \times [0, +\infty]^2)$ such that

$$0 = (w_* - \varphi)(t_0, y_0) = \min_{(t, y) \in [0, T] \times [0, +\infty]^2} (w_* - \varphi)(t, y)$$

The next step is to find a function $w^1 \geq 0$ which one can add to any viscosity supersolution w of (2.10) to build a superior and arbitrary close strict supersolution $w + \varepsilon w^1$. As we will prove a comparison result for strict supersolutions, the next lemma will enable us to manage the comparison with any non-strict supersolution. Indeed, by perturbing the supersolution and taking the limit when the perturbation tends to zero, one can extend comparison. The main difficulty is that w^1 must always be superior to zero, and be concave enough to have $\frac{\partial^2 w^1}{\partial X^2}$ sufficiently negative to ensure property (4.4).

Lemma 4.3. *Assume (4.3). Then the function*

$$w^1(t, S, X) := (T - t) + \ln(1 + X) \geq 0$$

is a η -strict viscosity supersolution of (2.10) for some $\eta > 0$. Furthermore, if w is a supersolution of (2.10) with $w(T, \cdot) \geq g$, then, for any $\varepsilon > 0$, $w + \varepsilon w^1$ is a $\varepsilon^2 \eta$ -strict supersolution of (2.10) with $(w + \varepsilon w^1)(T, \cdot) \geq g$.

Proof. One can easily check that w^1 is a strict supersolution:

$$\begin{aligned} F(t, S, X, Dw^1, D^2w^1) &= \lambda^- \left(\begin{array}{cc} 1 + \frac{\mu(X)}{1+X} & 0 \\ 0 & \max(1, X^2) \frac{1}{(1+X)^2} \end{array} \right) \\ &\geq 1 \end{aligned}$$

Now, we check that $w + \varepsilon w^1$ is a ε -strict supersolution. Indeed, for any $(t_0, S_0, X_0) \in (0, T) \times (0, +\infty)^2$, and for any test function $\varphi \in C^\infty((0, T) \times (0, +\infty)^2 \rightarrow \mathbb{R})$ which satisfies:

$$\text{Min}(\varphi - w - \varepsilon w^1) = (\varphi - w - \varepsilon w^1)(t_0, S_0, X_0) = 0$$

Then, as $w^1 \in C^\infty((0, T) \times (0, +\infty)^2 \rightarrow \mathbb{R})$, $\psi = \varphi - \varepsilon w^1$ is a test function for w such that $\psi - w$ attains its strict minimum at (t_0, S_0, X_0) . Hence, we have, by the supersolution property of w :

$$F(t_0, S_0, X_0, D\psi, D^2\psi) \geq 0$$

Then, considering that for any symmetric matrices: $\lambda^-(A + B) \geq \lambda^-(A) + \lambda^-(B)$

$$\begin{aligned} F(t_0, S_0, X_0, D\varphi, D^2\varphi) &\geq F(t_0, S_0, X_0, D\psi, D^2\psi) + F(t_0, S_0, X_0, D(\varepsilon w^1), D^2(\varepsilon w^1)) \\ &\geq F(t_0, S_0, X_0, D\psi, D^2\psi) + \varepsilon F(t_0, S_0, X_0, Dw^1, D^2w^1) \\ &\geq \varepsilon \end{aligned}$$

By homogeneity of F in w . □

Before approaching the technical proof of the comparison principle involving strict-supersolutions, let us see how the preceding lemma allows us to extend that principle to any supersolution, thus proving the main theorem that follows:

Proposition 4.4. *If w and u are respectively super and subsolution of (2.10), and there exists a function h such that for any $(t, s) \in [0, T] \times \mathbb{R}_+$:*

$$\lim_{(t', s', x') \rightarrow (t, s, 0)} u \leq h(t, s) \leq \lim_{(t', s', x') \rightarrow (t, s, 0)} w$$

and a function g such that for any s in \mathbb{R}_+ :

$$\lim_{(t', s', x') \rightarrow (T, s, x)} u \leq g(t, x) \leq \lim_{(t', s', x') \rightarrow (T, s, x)} w$$

Then $u \leq w$ on $[0, T] \times \mathbb{R}_+^2$. In particular, the solution of equation (2.10) in the viscosity sense with boundary conditions is unique.

Proof. We use the same technique as in [11]. If w and u are respectively super and subsolutions of (2.10). Furthermore suppose that they both verify the limit conditions (3.11) and (3.14). Then, for any $\varepsilon > 0$, with lemma 4.3, $w + \varepsilon w^1$ satisfy the boundedness, strict supersolution and boundary limits assumptions of theorem 4.6. Applying it, one gets:

$$w + \varepsilon w^1 \geq u \text{ on } \mathbb{R}^2 \times [0; T]$$

Finally, letting ε converge to zero by positive values, we get the result:

$$w \geq u \text{ on } \mathbb{R}^2 \times [0; T]$$

□

4.4 Modulus of continuity of F

We now introduce some technical lemmas which are classical in the viscosity solutions theory. We need a modulus of continuity for the operator F . It is given in the next lemma:

Lemma 4.5. *Let A and $A' \in S^2(\mathbb{R})$ such that:*

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad (4.5)$$

For any (t, S, X, p) and (t', S', X', p') for which:

- There exists a constant C_1 such that:

$$\sigma^2(X) \leq \frac{C_1}{2\varepsilon} \text{ and } \sigma^2(X') \leq \frac{C_1}{2\varepsilon}$$

- There exists a constant C_2^K such that Σ and μ are Lipschitz of constant C_2^K on a convex domain K that contains (t, S, X) (t', S', X')
- There is a constant C_3 (possibly dependent of ε) such that:

$$\max(\sigma^2(X), \sigma^2(X'), X^2, X'^2, 1) \leq \frac{C_3}{3}$$

Then the following inequality holds:

$$\begin{aligned}
& F(t', S', X', p', q', A' - D^2 f^\varepsilon(S', X')) - F(t, S, X, p, q, A + D^2 f^\varepsilon(S, X)) \leq \\
& \quad 3C_2^K \alpha \|(t - t', S - S', X - X')\|^2 + C_2^K \|X - X'\|_{q_2} + C_1 \varepsilon + |p - p'| \\
& \quad + C_3 \|(X - X_\varepsilon, S - S_\varepsilon)\| + C_3 \|(X' - X_\varepsilon, S' - S_\varepsilon)\| \quad (4.6)
\end{aligned}$$

Proof. This proof is an adaptation of example 3.6 in [10]. First, by multiplying inequality (4.5) by $(\Sigma \quad \Sigma')$ on the left and $\begin{pmatrix} \Sigma \\ \Sigma' \end{pmatrix}$ on the right, one gets:

$$\Sigma A \Sigma - \Sigma' A' \Sigma' \leq 3\alpha (\Sigma - \Sigma')^2$$

then, we introduce the symmetric matrices:

$$\begin{aligned}
B &= \begin{pmatrix} p + \mu(X)q_2 & 0 \\ 0 & 0 \end{pmatrix} + \Sigma D^2 f^\varepsilon(S, X) \Sigma \\
&= \begin{pmatrix} p + \mu(X)q_2 + \sigma^2(X)(\varepsilon^2 + 3(S - S_\varepsilon)^2) & 0 \\ 0 & \max(1, X^2)(3(X - X_\varepsilon)^2) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
B' &= \begin{pmatrix} p' + \mu(X')q'_2 & 0 \\ 0 & 0 \end{pmatrix} - \Sigma D^2 f^\varepsilon(S', X') \Sigma \\
&= \begin{pmatrix} p' + \mu(X')q'_2 - \sigma^2(X)(\varepsilon^2 + 3(S' - S_\varepsilon)^2) & 0 \\ 0 & -\max(1, X'^2)(3(X' - X_\varepsilon)^2) \end{pmatrix}
\end{aligned}$$

And we add $B - B'$ on both sides:

$$\Sigma A \Sigma + B - \Sigma' A' \Sigma' - B' \leq 3\alpha (\Sigma - \Sigma')^2 + B - B'$$

and

$$\Sigma A \Sigma + B \leq \Sigma' A' \Sigma' + B' + 3\alpha (\Sigma - \Sigma')^2 + B - B'$$

Then, we use the fact that for two symmetric matrices X and Y one has:

$$\lambda^+(X + Y) \leq \lambda^+(X) + \lambda^+(Y)$$

Where λ^+ is the largest eigenvalue of a symmetric matrix. This gives:

$$\begin{aligned}
\lambda^+(\Sigma A \Sigma + B) &\leq \lambda^+(\Sigma' A' \Sigma' + B' + 3\alpha (\Sigma - \Sigma')^2 + B - B') \\
&\leq \lambda^+(\Sigma' A' \Sigma' + B') + \lambda^+(3\alpha (\Sigma - \Sigma')^2) + \lambda^+(B - B')
\end{aligned}$$

Thus, knowing that $\lambda^+(X) = -\lambda^-(-X)$ where λ^- is the smallest eigenvalue, we obtain:

$$\lambda^-(-\Sigma' A' \Sigma' - B') - \lambda^-(-\Sigma A \Sigma - B) \leq 3\alpha \lambda^+((\Sigma - \Sigma')^2) + \lambda^+(B - B')$$

By definition (2.10) of the operator F , this is equivalent to:

$$\begin{aligned} F(t', S', X', p, q, A' - D^2 f^\varepsilon(S', X')) - F(t, S, X, p, q, A + D^2 f^\varepsilon(S, X)) \leq \\ 3\alpha\lambda^+(\Sigma - \Sigma')^2 + \lambda^+(B - B') \end{aligned} \quad (4.7)$$

Now, we focus on the right hand side of (4.7) to obtain the result. First we have, as B and B' are diagonal matrices:

$$\begin{aligned} \lambda^+(B - B') &= \max[B_{11} - B'_{11}, B_{22} - B'_{22}] \\ &\leq \varepsilon^2(\sigma^2(X) + \sigma^2(X')) + |\mu(X) - \mu(X')| |q_2| \\ &\quad + C_3(\|(X - X_\varepsilon, S - S_\varepsilon)\| + \|(X' - X_\varepsilon, S' - S_\varepsilon)\|) \\ &\leq \varepsilon C_1 + C_2^K |X - X'| |q_2| + |p - p'| \\ &\quad + C_3(\|(X - X_\varepsilon, S - S_\varepsilon)\| + \|(X' - X_\varepsilon, S' - S_\varepsilon)\|) \end{aligned}$$

Finally, since Σ is Lipschitzian one gets:

$$3\alpha\lambda^+(\Sigma - \Sigma')^2 \leq 3\alpha C_2^K \|(t - t', S - S', X - X')\|^2$$

Plugging these two inequalities into 4.7, one obtains inequality 4.6, thus proving the lemma. \square

4.5 Proof of the comparison principle

Proposition 4.6. *Suppose u is an upper semicontinuous viscosity subsolution of 2.10, bounded from above and w a lower semicontinuous η -strict viscosity supersolution of (2.10) bounded from below. If, furthermore, for some bounded functions g and h :*

$$u(T, \cdot, \cdot) \leq g(\cdot) \leq w(T, \cdot, \cdot) \text{ and } u(\cdot, \cdot, 0) \leq h(\cdot, \cdot) \leq w(\cdot, \cdot, 0)$$

Then $u^*(t, S, X) \leq w_*(t, S, X)$ for all $(t, S, X) \in [0, T] \times [0, +\infty]^2$

Proof. This proof is inspired by [11] and [6]. For $\varepsilon, \alpha > 0$, let $\Phi^{\varepsilon, \alpha}$ be the upper semicontinuous function:

$$\begin{aligned} \Phi^{\varepsilon, \alpha}(t, t', S, S', X, X') &= u(t, S, X) - w(t', S', X') - f_1^\varepsilon(S, S', X, X') + \varepsilon \ln\left(\frac{t}{T}\right) \\ &\quad - \alpha(d(t - t', S - S', X - X')) \end{aligned}$$

where

$$d(a, b, c) = \frac{1}{2}(a^2 + b^2 + c^2)$$

and

$$f_1^\varepsilon(S, S', X, X') = (S + S' + X + X')\varepsilon - (\ln(S) + \ln(S'))\varepsilon^2$$

to simplify, denote

$$f_1^\varepsilon(S, X) = f_1^\varepsilon(S, S, X, X)$$

Next, set:

$$\Phi^\varepsilon(t, S, X) = \Phi^{\varepsilon, \alpha}(t, t, S, S, X, X)$$

As $(u - w)$ is bounded from above by a constant C , one can see from the form of $f(S, S, X, X)$ that the supremum of Φ^ε is attained in $[\exp(\frac{-\beta}{C}), T] \times [\exp(\frac{-\varepsilon}{C}), \frac{\varepsilon}{C}] \times [0, \frac{\varepsilon}{C}]$ which is a compact set. Therefore, Φ^ε is an upper semi-continuous function and attains its supremum in a compact set, this supremum is a maximum. It follows that we can find a point $(t_\varepsilon, S_\varepsilon, X_\varepsilon)$ such that:

$$\max_{[0, T] \times [0, +\infty]^2} \Phi^\varepsilon(t, S, X) = \Phi^\varepsilon(t_\varepsilon, S_\varepsilon, X_\varepsilon)$$

Now, there are three possible cases:

- there exist a sequence $\varepsilon_k > 0$ such that $\varepsilon_k \rightarrow 0$ and $t_{\varepsilon_k} = T$ for every k .
- there exist a sequence $\varepsilon_k > 0$ such that $\varepsilon_k \rightarrow 0$ and $X_{\varepsilon_k} = 0$ for every k .
- there exist a constant $\varepsilon^- > 0$ such that $X_\varepsilon > 0$ and $t_\varepsilon < T$ for all $0 < \varepsilon < \varepsilon^-$

Cases 1 and 2: One can prove easily that there is a contradiction if one of the two first cases apply. Indeed, in the first case, one can see that for all ε_k :

$$\begin{aligned} u(t, S, X) - w(t, S, X) &= \Phi^{\varepsilon_k}(t, S, X) + f^{\varepsilon_k}(S, S, X, X) + \varepsilon_k \ln\left(\frac{t}{T}\right) \\ &\leq \Phi^{\varepsilon_k}(T, S_k, X_k) + f^{\varepsilon_k}(S, S, X, X) - \varepsilon_k \ln\left(\frac{t}{T}\right) \\ &= u(T, S_k, X_k) - w(T, S_k, X_k) + f^{\varepsilon_k}(S, X) - \varepsilon_k \ln\left(\frac{t}{T}\right) \\ &\quad - f^{\varepsilon_k}(S_k, X_k) \\ &\leq u(T, S_k, X_k) - w(T, S_k, X_k) + f^{\varepsilon_k}(S, X) - \varepsilon_k \ln\left(\frac{t}{T}\right) \end{aligned}$$

because $f^{\varepsilon_k}(S_k, X_k) \geq 0$. Since $u(T, \dots) \leq g(\cdot) \leq w(T, \dots)$ this implies

$$u(t, S, X) - w(t, S, X) \leq f^{\varepsilon_k}(S, X) - \varepsilon_k \ln\left(\frac{t}{T}\right)$$

For all $(t, S, X) \in [0, T] \times [0, +\infty]^2$ hence the proposition is proved by taking $k \rightarrow +\infty$. The same kind of proof applies for the second case.

Case 3 This is the technical part. Consider the function:

$$\begin{aligned} \hat{\Phi}^{\varepsilon, \alpha}(t, t', S, S', X, X') &:= \Phi^{\varepsilon, \alpha}(t, t', S, S', X, X') - \frac{1}{2} [(t - t_\varepsilon)^2 + (t' - t_\varepsilon)^2] \\ &\quad - \frac{1}{4} [(S - S_\varepsilon)^4 + (S' - S_\varepsilon)^4 + (X - X_\varepsilon)^4 + (X' - X_\varepsilon)^4] \end{aligned}$$

In the the following, we denote $f_2^\varepsilon(S, X) = \frac{1}{4} [(S - S_\varepsilon)^4 + (X - X_\varepsilon)^4]$ and $\hat{\Phi}^\varepsilon(t, S, X) := \hat{\Phi}^{\varepsilon, \alpha}(t, t, S, S, X, X)$. It is clear that

$$\hat{\Phi}^{\varepsilon, \alpha}(t, S, X) := \Phi^{\varepsilon, \alpha}(t, S, X) - (t - t_\varepsilon)^2 - 2f_2^\varepsilon(S, X)$$

Then for every $\varepsilon > 0$, $(t_\varepsilon, S_\varepsilon, X_\varepsilon)$ is a strict maximizer of $\hat{\Phi}^\varepsilon$. Therefore, by lemma 3.1 in [10], for every $\varepsilon < \varepsilon^-$ there exist a sequence $\alpha_k \rightarrow +\infty$ and maximizers $(t_{\varepsilon, \alpha}, t'_{\varepsilon, \alpha}, S_{\varepsilon, \alpha}, S'_{\varepsilon, \alpha}, X_{\varepsilon, \alpha}, X'_{\varepsilon, \alpha})$ of $\hat{\Phi}^{\varepsilon, \alpha}$ such that:

$$(t_{\varepsilon, \alpha}, t'_{\varepsilon, \alpha}, S_{\varepsilon, \alpha}, S'_{\varepsilon, \alpha}, X_{\varepsilon, \alpha}, X'_{\varepsilon, \alpha}) \rightarrow (t_\varepsilon, t_\varepsilon, S_\varepsilon, S_\varepsilon, X_\varepsilon, X_\varepsilon)$$

and

$$\alpha_k \|(t_{\varepsilon, \alpha} - t'_{\varepsilon, \alpha}, S_{\varepsilon, \alpha} - S'_{\varepsilon, \alpha}, X_{\varepsilon, \alpha} - X'_{\varepsilon, \alpha})\|^2 \rightarrow 0$$

And, as $(t_\varepsilon, S_\varepsilon, X_\varepsilon)$ are in the interior of the domain for $\varepsilon < \varepsilon^-$, then the maximizers of $\hat{\Phi}^{\varepsilon, \alpha}$ are also in its interior for α_k sufficiently large. With this result, we can apply theorem 3.2 in [10], to the sequence of local maxima. We obtain that, for sufficiently large α_k , there exists two symmetric matrices $A_k, A'_k \in S^2$ such that:

(A_k, A'_k) satisfies 4.5,

$$\begin{aligned} (p_k, q_k + D(f_1^\varepsilon + f_2^\varepsilon)(S_k, X_k), A_k + D^2(f_1^\varepsilon + f_2^\varepsilon)(S_k, X_k)) &\in J^{2, -}u(t_k, S_k, X_k), \\ (p'_k, q_k - D(f_1^\varepsilon + f_2^\varepsilon)(S'_k, X'_k), A'_k - D^2(f_1^\varepsilon + f_2^\varepsilon)(S'_k, X'_k)) &\in J^{2, +}w(t'_k, S'_k, X'_k) \end{aligned}$$

where, by taking $f = f_1 + f_2$

$$\begin{aligned} p_k &= \alpha_k (t_k - t'_k) + (t_k - t_\varepsilon) \\ p'_k &= \alpha_k (t_k - t'_k) - (t'_k - t_\varepsilon) \\ q_k &= \alpha_k \begin{pmatrix} S_k - S'_k \\ X_k - X'_k \end{pmatrix} \\ D^2 f^\varepsilon(S_k, X_k) &= \begin{pmatrix} \frac{\varepsilon^2}{S_k^2} + 3(S_k - S_\varepsilon)^2 & 0 \\ 0 & 3(X_k - X_\varepsilon)^2 \end{pmatrix} \end{aligned}$$

and with $J^{2, +}w(t'_k, S'_k, X'_k)$ and $J^{2, -}u(t_k, S_k, X_k)$ are, as in [10], the closed inferior and superior semijets of w and u respectively. Then by the definition of viscosity subsolutions and strict supersolutions we obtain:

$$F(t_k, S_k, X_k, p_k, A_k + D^2 f^\varepsilon(S_k, X_k)) \leq 0$$

and

$$F(t'_k, S'_k, X'_k, p'_k, A'_k - D^2 f^\varepsilon(S'_k, X'_k)) > \eta$$

Combining these two inequalities, one gets:

$$F(t'_k, S'_k, X'_k, p'_k, q'_k, A'_k - D^2 f^\varepsilon(S'_k, X'_k)) - F(t_k, S_k, X_k, p_k, q_k, A_k + D^2 f^\varepsilon(S_k, X_k)) > \eta \quad (4.8)$$

On the other hand, since the maximum point $t_\varepsilon, S_\varepsilon, X_\varepsilon$ is attained in $[\exp(\frac{-\beta}{C}), T] \times [\exp(\frac{-\varepsilon}{C}), \frac{\varepsilon}{C}] \times]0, \frac{\varepsilon}{C}]$, and since (t_k, S_k, X_k) and (t'_k, S'_k, X'_k) converge to this point, the local Lipschitz condition (2.2) proves the existence of the constants $C_1, C_2^\varepsilon, C_3$, of lemma 4.5, independent of k provided it is sufficiently large, and with C_1 independent of ε provided it is sufficiently small. Now we can apply it to obtain gives the inequality:

$$\begin{aligned} F(t'_k, S'_k, X'_k, p_k, A'_k - \varepsilon Dl(S'_k, X'_k)) - F(t_k, S_k, X_k, p_k, A_k + \varepsilon Dl(S_k, X_k)) \leq \\ C_2^\varepsilon \alpha_k \|(t_{\varepsilon, \alpha} - t'_{\varepsilon, \alpha}, S_{\varepsilon, \alpha} - S'_{\varepsilon, \alpha}, X_{\varepsilon, \alpha} - X'_{\varepsilon, \alpha})\|^2 + C_1 \varepsilon \\ + C_3 (\|(X - X_\varepsilon, S - S_\varepsilon)\| + \|(X' - X_\varepsilon, S' - S_\varepsilon)\|) \end{aligned} \quad (4.9)$$

The right hand side of (4.9) tends to $C_1 \varepsilon$ when k tends to infinity, and by sending ε to zero, this contradicts (4.8), thus proving the comparison result. \square

5 Dual representation

In this section, we give a dual expectation representation of the super-replication problem. The dual maximization problem is done over all volatilities of X and all possible correlations between X and S . This kind of duality was first introduced in [14] for one-dimensional processes. First we define the value function \tilde{v} of the dual problem:

$$\tilde{v}(t, s, x) = \sup_{(\rho, \xi) \in \mathcal{U}} E \left[g \left(S_{t, s, x}^{\rho, \xi}(T) \right) \right] \quad (5.1)$$

Where \mathcal{U} is the set of all almost-surely bounded progressively measurable processes taking values in $[-1; 1] \times [0; +\infty)$:

$$\mathcal{U} = \left\{ (\rho, \xi) \text{ valued in } [-1, 1] \times [0, +\infty) \text{ and progressively measurable} \mid \int_0^T \xi_t^2 dt < +\infty \right\}$$

And the process $S_{t, s}^{\rho, \xi}$ is defined for $u \geq t$ by the dynamics:

$$\left\{ \begin{array}{l} S_{t, s, x}^{\rho, \xi}(t) = s \quad \text{and} \quad X_{tx}^{\rho, \xi}(t) = x \\ dS_{t, s, x}^{\rho, \xi}(u) = \sigma(t, X_x^{\rho, \xi}(u)) S_{t, s, x}^{\rho, \xi} dW^1(u) \\ dX_x^{\rho, \xi}(u) = -\mu(t, X_x^{\rho, \xi}(u)) du + \xi_u X_x^{\rho, \xi}(u) dW^2(u) \\ \langle dW^1(u), dW^2(u) \rangle = \rho_u \end{array} \right.$$

The main goal of this section is to prove that \tilde{v} is also solution of the primal super-replication problem. In other words that $\tilde{v} = v$. First, we have to prove that the two functions verify the same equation. Then, we prove that the two function have the same boundary conditions. We then conclude by the comparison theorem.

Proposition 5.1. \tilde{v} is a viscosity supersolution of equation (2.10) on $(0; T) \times (\mathbb{R}_+^*)^2$.

Proof. This is a classical proof in the optimal control theory, see [13], chapter 4, for details. That framework applies to one-dimensional problems, but there is no difficulty in extending them to the multidimensional case. Hence the viscosity sub and supersolution characterization in terms of Hamiltonian will be admitted, and we will focus on the equivalence between the classical characterization and equation (2.10). The Hamiltonian of the problem is:

$$H(S, X, v, Dv, D^2v) = \inf_{\xi, \rho} \left\{ \frac{\partial v}{\partial X} \mu(t, X) - \frac{1}{2} \xi^2 \frac{\partial^2 v}{\partial X^2} - \frac{1}{2} \sigma^2(t, X) S^2 \frac{\partial^2 v}{\partial S^2} - \rho \xi \sigma(t, X) S \frac{\partial v}{\partial S \partial X} \right\}$$

And by classical techniques one can show that:

$$-\frac{\partial \tilde{v}}{\partial t} + H(S, X, \tilde{v}, D\tilde{v}, D^2\tilde{v}) \geq 0 \quad (5.2)$$

in the viscosity sense. Indeed, the Hamiltonian H is smooth, unless it takes infinite negative values. In order to prove that the continuous operator $F(S, X, \tilde{v}, D\tilde{v}, D^2\tilde{v})$ of (2.10) is such that F and $H - \frac{\partial}{\partial t}$ always have the same sign, the next step is to explicitly solve the Hamiltonian H . Writing, for any vector b and any 2×2 symmetric matrix A :

$$H(S, X, v, b, A) = \inf_{\xi, \rho} \left\{ b_2 \mu(t, X) - \frac{1}{2} \xi^2 A_{22} - \frac{1}{2} \sigma^2(t, X) S^2 A_{11} - \rho \xi \sigma(t, X) S A_{12} \right\}$$

By elementary techniques, the minimization of H over ρ and ξ gives:

$$\begin{cases} H(S, X, v, b, A) = -\infty & \text{if } A_{22} > 0 \\ H(S, X, v, b, A) = -\infty & \text{if } A_{22} = 0 \text{ and } A_{12} \neq 0 \\ H(S, X, v, b, A) = b_2 \mu(t, X) - \frac{1}{2} \sigma^2(t, X) S^2 A_{11} & \text{if } A_{22} = 0 \text{ and } A_{12} = 0 \\ H(S, X, v, b, A) = \frac{1}{2} \sigma^2(t, X) S^2 \left(\frac{A_{12}^2}{A_{22}} - A_{11} \right) + b_2 \mu(t, X) & \text{otherwise} \end{cases} \quad (5.3)$$

In addition, the operator F is positive if and only if the matrix $J(t, s, x, Du, D^2u)$ defined in (2.10) is positive, that is, if and only if the two diagonal terms J_{11} and J_{22} and the determinant of J are positive. Clearly, by (5.3), F is positive if and only if H is positive. Hence \tilde{v} is a viscosity supersolution of (2.10). \square

Now, we concentrate on the subsolution property:

Proposition 5.2. *\tilde{v} is a viscosity subsolution of equation (2.10) on $(0; T) \times (\mathbb{R}_+^*)^2$.*

Proof. Let $\varphi \in C^2((0; T) \times (\mathbb{R}_+^*)^2)$ a test function such that

$$0 = (v^* - \varphi)(\bar{t}, \bar{S}, \bar{X}) = \max_{(t, S, X) \in (0; T) \times (\mathbb{R}_+^*)^2} (v^* - \varphi)(t, S, X)$$

for some $(\bar{t}, \bar{S}, \bar{X}) \in (0; T) \times (\mathbb{R}_+^*)^2$. Suppose that, on the contrary,

$$F(t, S, X, \varphi, D\varphi, D^2\varphi) > 0$$

Hence

$$-\frac{\partial \tilde{v}}{\partial t} + H(t, S, X, \varphi, D\varphi, D^2\varphi) > 0$$

considering (5.3), as H is continuous in the interior of the domain delimited by $F > 0$, one can find a contradiction with a classical dynamic programming argument which can be found in [13] for instance. \square

In order to apply the uniqueness proposition 4.6, it would remain to verify that the value function \tilde{v} of the dual problem has the same boundary conditions as v . There are two parts in this question: The study for $t \rightarrow T$ and for $X \rightarrow 0$. Ideally, we would prove the two following propositions directly. However this may be quite difficult, and they will be demonstrated indirectly along the lines of the proof of proposition 5.7. Let us begin by the first limit:

Proposition 5.3. *The value function \tilde{v} of problem (5.1) extends continuously to a function \hat{v} on $(0; T] \times (\mathbb{R}_+^*)^2$ satisfying the terminal condition:*

$$\hat{v}(T, S, X) = g(S)$$

Moreover, we need the condition near $X = 0$

Proposition 5.4. *The function \hat{v} extends continuously to a function \bar{v} on $(0; T] \times (\mathbb{R}_+)^2$ satisfying the boundary condition:*

$$\bar{v}(t, S, 0) = g(S)$$

The study of the behavior of \tilde{v} near $X = 0$ requires several steps. We define the auxiliary value function:

$$\tilde{v}_C = \sup_{(\xi, \rho) \in U_C} E \left[g \left(S_{t, T}^{x, \xi, \rho} \right) \right]$$

Where $U_C = \{(\xi, \rho) \in U \mid \xi \leq C\}$. The preliminary goal is to prove the following technical lemma :

Lemma 5.5. *For any $C \geq 0$ there exists two constants C_1 and C_2 independent of $(x, t, u) \in \mathbb{R} \times [0; T]^2$ with $t < u$ such that for any adapted processes $(\xi, \rho) \in [0; C] \times [-1, 1]$ one has:*

$$E \left[\left(X_{t, u}^{x, \xi, \rho} \right)^2 \right] \leq x^2 + C_1 x^2 \int_t^u e^{C_1(u-s)} ds \quad (5.4)$$

$$E \left[X_{t, u}^{x, \xi, \rho} \right] \leq x + C_2 x \int_t^u e^{C_2(u-s)} ds \quad (5.5)$$

Proof. We use a similar procedure as in [13]. By Itô's formula, one has, for any stopping time τ , and any $0 \leq t \leq s, x, (\xi, \rho) \in U_C$:

$$\begin{aligned} \left(X_{t, u \wedge \tau}^{x, \xi, \rho}\right)^2 &= x^2 + \int_t^{u \wedge \tau} \left[-2X_{t, s}^{x, \xi, \rho} \mu(s, X_{t, s}^{x, \xi, \rho}) + \left(\xi X_{t, s}^{x, \xi, \rho}\right)^2 \right] ds \\ &\quad + \int_t^{u \wedge \tau} \xi \left(X_{t, s}^{x, \xi, \rho}\right)^2 dW^2(s) \end{aligned}$$

choosing a sequence of stopping times: $\tau_n = \inf \left\{ s \geq t : \xi \left(X_{t, s}^{x, \xi, \rho}\right)^2 \geq n \right\}$, which tends a.s. to infinity when $n \rightarrow +\infty$, when have for fixed n :

$$E \left[\int_t^{u \wedge \tau_n} \xi \left(X_{t, s}^{x, \xi, \rho}\right)^2 dW^2(s) \right] = 0$$

Then, using the linear growth coefficient K of μ and the bound C of ξ :

$$\begin{aligned} E \left[\left(X_{t, u \wedge \tau_n}^{x, \xi, \rho}\right)^2 \right] &= x^2 + E \left(\int_t^{u \wedge \tau_n} \left[-2X_{t, s}^{x, \xi, \rho} \mu(s, X_{t, s}^{x, \xi, \rho}) + \left(\xi X_{t, s}^{x, \xi, \rho}\right)^2 \right] ds \right) \\ &\leq x^2 + E \left(\int_t^{u \wedge \tau_n} \left[2K \left(X_{t, s}^{x, \xi, \rho}\right)^2 + C \left(X_{t, s}^{x, \xi, \rho}\right)^2 \right] ds \right) \\ &\leq x^2 + (2K + C) E \left(\int_t^{u \wedge \tau_n} \left[\left(X_{t, s}^{x, \xi, \rho}\right)^2 \right] ds \right) \end{aligned}$$

By Gronwall's lemma, writing $C_1 = 2K + C$:

$$E \left[\left(X_{t, u \wedge \tau_n}^{x, \xi, \rho}\right)^2 \right] \leq x^2 + C_1 x^2 \int_t^{u \wedge \tau_n} e^{C_1(u \wedge \tau_n - s)} ds$$

Finally, using Fatou's lemma and letting $n \rightarrow +\infty$

$$E \left[\left(X_{t, u}^{x, \xi, \rho}\right)^2 \right] \leq x^2 + C_1 x^2 \int_t^u e^{C_1(u-s)} ds$$

Moreover, with the same arguments, one can prove that there exists a constant C_2 independent of x, ξ, ρ, t, u such that :

$$E \left[\left(X_{t, u}^{x, \xi, \rho}\right) \right] \leq x + C_2 x \int_t^u e^{C_2(u-s)} ds$$

□

Now we have the tools to prove the convergence results for v_C .

Lemma 5.6. *The terminal condition of \tilde{v}_C is:*

$$\lim_{(t, s', x') \rightarrow (T, s, x)} \tilde{v}_C(t, s, x) = g(s) \text{ for any } (s, x, C) \in \mathbb{R}_+^3 \quad (5.6)$$

Furthermore, the lateral condition of \tilde{v}_C is:

$$\lim_{(t', s', x) \rightarrow (t, s, 0)} \tilde{v}_C(t, s, x) = g(s) \text{ for any } (t, s, C) \in [0, T] \times \mathbb{R}_+^2 \quad (5.7)$$

Proof. For sake of conciseness we prove the two propositions at the same time. Since function g is bounded, function \tilde{v}_C has the same bounds. Hence, if for a given point (t, s, x) and for any sequence $(t_n, s_n, x_n) \rightarrow (t, s, x)$, $\tilde{v}_C(t_n, s_n, x_n)$ admits $g(s)$ as an accumulation point, then \tilde{v}_C is continuous at (t, s, x) and equal to $g(s)$. Therefore, we have to prove this claim at points of type (T, s, x) and $(t, s, 0)$. Choose a sequence $(t_n, s_n, x_n) \in [0, T] \times \mathbb{R}_+^2$ converging to a given (T, s, x) or $(t, s, 0)$. Then, by definition of the value function \tilde{v}_C there exists a sequence of controls $(\xi_n, \rho_n) \in U_C$ such that, denoting S^n as S^{ξ_n, ρ_n} :

$$E(g(S_{t_n, s_n, x_n}^n)) \leq \tilde{v}_C(t_n, s_n, x_n) \leq E(g(S_{t_n, s_n, x_n}^n)) + \frac{1}{n} \quad (5.8)$$

Now, we get to the convergence of both sides. We use the Doleans exponential formula:

$$S_{t_n, s_n, x_n}^n = s_n \exp \left(\int_{t_n}^T \sigma(u, X_{t_n, x_n}^n(u)) dW_u + \frac{1}{2} \int_{t_n}^T \sigma^2(u, X_{t_n, x_n}^n(u)) du \right)$$

One gets, taking the logarithm (will not work with $s = 0$ but then the proof is trivial):

$$\begin{aligned} \ln(S_{t_n, s_n, x_n}^n) - \ln(s) &= \ln(s_n) - \ln(s) + \int_{t_n}^T \sigma(u, X_{t_n, x_n}^n(u)) dW_u \\ &\quad + \frac{1}{2} \int_{t_n}^T \sigma^2(u, X_{t_n, x_n}^n(u)) du \end{aligned}$$

Taking the square of this equality one gets:

$$\begin{aligned} (\ln(S_{t_n, s_n, x_n}^n) - \ln(s))^2 &= (\ln(s_n) - \ln(s))^2 + \left(\int_{t_n}^T \sigma(u, X_{t_n, x_n}^n(u)) dW_u \right)^2 \\ &\quad + \left(\int_{t_n}^T \frac{1}{2} \int_{t_n}^T \sigma^2(u, X_{t_n, x_n}^n(u)) du \right)^2 \end{aligned}$$

Using (2.6) together with (5.4) gives:

$$\begin{aligned} E \left[\int_{t_n}^T \sigma^2(u, X_{t_n, x_n}^n(u)) du \right] &\leq (T - t_n)Cx \\ E \left[\int_{t_n}^T \sigma^2(u, X_{t_n, x_n}^n(u)) du \right]^2 &\leq (T - t_n)Cx^2 \end{aligned}$$

And where C is independent of n . Hence, in the context of lemma 5.6 we get:

$$\lim_{n \rightarrow +\infty} \ln(S_{t_n, s_n, x_n}^n) = \ln(s) \text{ in } L^2$$

Then, there exists a subsequence n_k satisfying:

$$\lim_{k \rightarrow +\infty} \ln(S_{t_{n_k}, s_{n_k}, x_{n_k}}^{n_k}) = \ln(s) \text{ almost surely}$$

And as g is continuous and bounded, we conclude, by the dominated convergence theorem:

$$\lim_{k \rightarrow +\infty} E \left(g(S_{t_{n_k}, s_{n_k}, x_{n_k}}^{n_k}) \right) = E \left(g \left(\lim_{k \rightarrow +\infty} S_{t_{n_k}, s_{n_k}, x_{n_k}}^{n_k} \right) \right) = g(s)$$

Remembering inequalities (5.8), we get that $g(s)$ is an accumulation point of $\tilde{v}_C(t_n, s_n, x_n)$ then the proof is complete. \square

Now we are in position to prove the main result of this section:

Proposition 5.7. *The value function of the primal problem and the dual problem are the same. In other words:*

$$\tilde{v} = v$$

Proof. We use the fact that \tilde{v}_C^* is a viscosity subsolution of the following equation (5.9), studied in [2]. We will give no proof of this claim, see that paper for detail

$$\inf_{\rho\xi \in [-C; C]} \{ -\mathcal{G}^{\rho\xi} \varphi \} \geq 0 \text{ where} \quad (5.9)$$

$$\mathcal{G}^{\rho\xi} \varphi = \frac{\partial \varphi}{\partial t} + \frac{1}{2} \begin{pmatrix} S\sigma(t, X) & \rho\xi \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \varphi}{\partial S^2} & \frac{\partial^2 \varphi}{\partial S \partial X} \\ \frac{\partial^2 \varphi}{\partial S \partial X} & \frac{\partial^2 \varphi}{\partial X^2} \end{pmatrix} \begin{pmatrix} S\sigma(t, X) \\ \rho\xi \end{pmatrix} \quad (5.10)$$

And, because the negativity of matrix H^1 follows trivially from the negativity of $\mathcal{G}^{\rho\xi} \varphi$ (which is an minimum of the quadratic form defined by H over vectors of form $\begin{pmatrix} 1 \\ \rho\xi \end{pmatrix}$), we obtain that \tilde{v}_C^* is a viscosity subsolution of equation (2.10) in the viscosity sense. Together with terminal conditions (5.6) and (5.7), we obtain:

$$\tilde{v}_C \leq v$$

Letting $C \rightarrow +\infty$ we get

$$\tilde{v} \leq v$$

Furthermore as

$$\tilde{v}_0 \leq \tilde{v}$$

and \tilde{v}_0 is a Black-Scholes price with deterministic volatility, we have, for any $(t, s) \in [0, T] \times \mathbb{R}$

$$\lim_{x \rightarrow 0} \tilde{v}_0(t, s, x) = g(s)$$

Knowing the boundary condition of v by propositions 3.3 and 3.4, and as $\tilde{v}_0 \leq \tilde{v} \leq v$ we have :

$$\lim_{x \rightarrow 0} \tilde{v}(t, s, x) = \lim_{x \rightarrow 0} v(t, s, x) = g(s)$$

Hence, we can conclude by the comparison principle of proposition 4.6 that:

$$\tilde{v} = v$$

\square

A Law of the iterated logarithm for some double stochastic integrals

Here, we prove the lemma which we use in the demonstration of the super-resolution property. In particular, we have to show that if the matrix $\Gamma - D^2\phi$ was constant, it would have to be symmetric and positive in order to satisfy relation (3.8). Then, corollary 3.8 in [12], proves, under some regularity assumptions, that if $\Gamma - D^2\phi$ changes over time, then a necessary and sufficient condition for (3.8) to hold is that $(\Gamma - D^2\phi)(0)$ is symmetric positive. As the symmetric case is studied in [3] we will deal with the non-symmetric case. The key issue is to estimate the limit of processes written as:

$$\liminf_{t \rightarrow 0} \frac{\int_0^t \left(\int_0^u \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} dW_v \right)^T dW_u}{t \log \log \frac{1}{t}}$$

Integrating the diagonal part, one obtains:

$$\frac{a}{2} (W_{1t}^2 + W_{2t}^2) - at + \int_0^t W_{1u} dW_{2u} - W_{2u} dW_{1u}$$

Hence, it is sufficient to study the limit of:

$$Z_t = \frac{a}{2} (W_{1t}^2 + W_{2t}^2) + \int_0^t W_{1u} dW_{2u} - W_{2u} dW_{1u}$$

The process L_t is called the Levy area:

$$L_t = \int_0^t W_{1u} dW_{2u} - W_{2u} dW_{1u}$$

First, let us see how this study will enable us to solve the consider problem. That is, we must find a result like the main theorem of [12] that embeds the case of nonsymmetric matrices:

Proposition A.1. *Let $M(t)$ be an \mathbb{R}^2 -valued martingale defined for any $t > 0$ by:*

$$M(t) = \int_0^t m(r) dW_r$$

where $m(t)$ is a \mathcal{M}^2 valued, \mathbb{F} progressively measurable process such that, for any $t > 0$:

$$\int_0^t |m(r)|^2 dr < +\infty$$

Let $b(t)$ be a bounded, \mathcal{M}^2 valued, \mathbb{F} progressively measurable process, and assume there exist a random variable $\varepsilon > 0$ such that almost surely:

$$\int_0^t |m(r) - m(0)|^2 dr = O(t^{1+\varepsilon}) \text{ and } \int_0^t |b(r) - b(0)|^2 dr = O(t^{1+\varepsilon}) \quad (\text{A.1})$$

For $t \rightarrow 0^+$. Then:

$$\liminf_{t \rightarrow 0^+} \frac{1}{t \log \log \frac{1}{t}} \int_0^t \left(\int_0^r b(u) dM_u \right)^T dM_r \geq 0 \text{ if and only if } b(0) \text{ is symmetric positive}$$

Proof. The proof is an extension of proof of theorem 3.3 in [12]. If $b(0)$ is symmetric, the proof is already done in corollary 3.7 of that paper. So, suppose that $b(0)$ is not symmetric. One can decompose the integral into:

$$\int_0^t \left(\int_0^r b(u) dM_u \right)^T dM_r = \int_0^t \left(\int_0^r m(0)^T b(0) m(0) dW_u \right)^T dW_r + R_1(t) + R_2(t)$$

where:

$$\begin{aligned} R_1(t) &= \int_0^t \left(\int_0^r b(u) [m(u) - m(0)] dW_u \right)^T m(0) dW_r \\ R_2(t) &= \int_0^t \left(\int_0^r b(u) m(u) dW_u \right)^T [m(r) - m(0)] dW_r \\ R_3(t) &= \int_0^t \left(\int_0^r [b(u) - b(0)] m(0) dW_u \right)^T m(0) dW_r \end{aligned}$$

in [12] it is shown that assumption (A.1) gives:

$$\lim_{t \rightarrow 0} \frac{R_1(t)}{t \log \log \frac{1}{t}} = \lim_{t \rightarrow 0} \frac{R_2(t)}{t \log \log \frac{1}{t}} = \lim_{t \rightarrow 0} \frac{R_3(t)}{t \log \log \frac{1}{t}} = 0$$

Hence, one only has to study the behavior of $\int_0^t \left(\int_0^r m(0)^T b(0) m(0) dW_u \right)^T dW_r$. We denote $m(0)^T b(0) m(0) = c$. Next we decompose c into a symmetric part c_1 and a skew-symmetric part c_2 . We use a base \tilde{W} for W where c_1 is diagonal. In this base:

$$c_1 = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \text{ and } c_2 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

Where $\lambda^+ > \lambda^-$. We define $c_3 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^- - \lambda^+ \end{pmatrix}$. Then we get:

$$\begin{aligned} \int_0^t \left(\int_0^r c dW_u \right)^T dW_r &= \int_0^t \left(\int_0^r (c_1 + c_2) dW_u \right)^T dW_r \\ &= \int_0^t \left(\int_0^r (c_2 + \lambda^+ I_2 + c_3) dW_u \right)^T dW_r \end{aligned}$$

Now, as $a \neq 0$ using lemma A.7 we get that:

$$\liminf_{t \rightarrow 0} \frac{1}{t \log \log \frac{1}{t}} \int_0^t \left(\int_0^r (c_2 + \lambda^+ I_2) dW_u \right)^T dW_r < 0$$

And as

$$\int_0^t \left(\int_0^r (c_3) dW_u \right)^T dW_r = (\lambda^- - \lambda^+) \frac{(\widetilde{W}_t^2)^2 - t}{2}$$

and

$$\liminf_{t \rightarrow 0} \frac{1}{t \log \log \frac{1}{t}} (\lambda^- - \lambda^+) \frac{(\widetilde{W}_t^2)^2 - t}{2} < 0$$

Then the proof is over. \square

A.1 Density of the considered process

Now let us study the probability density of the process Z . It is given in this lemma:

Lemma A.2. *For every $t > 0$, the probability density of the random variable Z_t is:*

$$\varphi(z_t) = \frac{\exp\left(\frac{z}{t} \arctan(a)\right)}{2\rho t \left[\text{ch}\left(\frac{\pi z}{2t}\right) \right]}$$

Proof. To begin, we use the Levy formula (see [9]):

$$E \left[e^{i\lambda L_t} \left| \sqrt{W_{1t}^2 + W_{2t}^2} = x \right. \right] = \frac{t\lambda}{\text{sh}(t\lambda)} \exp \left[\frac{|x|^2}{2t} (1 - t\lambda \coth(t\lambda)) \right]$$

As the expectation is conditional to $x^2 = W_{1t}^2 + W_{2t}^2$, multiplying it by a function of the conditioner, one obtains:

$$\begin{aligned} E \left[e^{i\lambda(L_t + \frac{a}{2}x^2)} \left| \sqrt{W_{1t}^2 + W_{2t}^2} = x \right. \right] &= \frac{t\lambda}{\text{sh}(t\lambda)} \exp \left[\frac{|x|^2}{2t} (1 - t\lambda \coth(t\lambda)) + i\lambda \frac{a}{2} x^2 \right] \\ &= \frac{t\lambda}{\text{sh}(t\lambda)} \exp \left[\frac{|x|^2}{2t} (1 - t\lambda \coth(t\lambda) + t\lambda ai) \right] \end{aligned}$$

As x^2 is the sum of two squared gaussian variables, it is distributed accordingly a two dimensional chi-squared law. Therefore, its probability density is:

$$d\mathbb{P}(x^2) = \frac{\exp\left(-\frac{x^2}{2t}\right)}{2t} d(x^2)$$

Thus, integrating expression (A.2) with respect to x^2 :

$$\begin{aligned} E \left[e^{i\lambda(L_t + \frac{a}{2}x^2)} \right] &= \int_0^{+\infty} E \left[e^{i\lambda(L_t + \frac{a}{2}x^2)} \left| \sqrt{W_{1t}^2 + W_{2t}^2} = x \right. \right] d\mathbb{P}(x^2) \\ &= \int_0^{+\infty} \frac{\lambda}{2\text{sh}(t\lambda)} \exp \left[\frac{x^2}{2t} (-t\lambda \coth(t\lambda) + t\lambda ai) \right] d(x^2) \\ &= \left[\frac{1}{\text{sh}(t\lambda) (-\coth(t\lambda) + ai)} \exp \left[\frac{x^2}{2t} (-t\lambda \coth(t\lambda) + t\lambda ai) \right] \right]_0^{+\infty} \\ &= \frac{1}{\text{ch}(t\lambda) - ai [\text{sh}(t\lambda)]} \end{aligned}$$

Then, manipulating the expression to obtain a canonical form, one derives, defining $\rho = \sqrt{1+a^2}$ and $\theta = \arg(1-ai) = \arctan(-a)$:

$$\begin{aligned} E \left[e^{i\lambda(L_t + \frac{a}{2}x^2)} \right] &= \frac{2}{e^{t\lambda} + e^{-t\lambda} - aie^{t\lambda} + aie^{-t\lambda}} \\ &= \frac{2}{(1-ai)e^{t\lambda} + (1+ai)e^{-t\lambda}} \\ &= \frac{2}{\rho[e^{t\lambda+i\theta} + e^{-t\lambda-i\theta}]} \\ &= \left(\frac{2}{\rho e^{i\theta}} \right) \frac{e^{t\lambda}}{e^{-2i\theta} + e^{2t\lambda}} \end{aligned}$$

To derive the probability density φ of the random variable $z_t = L_t + \frac{a}{2}x^2$, we must calculate the inverse Fourier transform of this function. Defining $\lambda' = -2t\lambda$ and $y = \frac{z}{2t}$ a change of variables gives:

$$\begin{aligned} \varphi(z) &= \left(\frac{2}{2\pi\rho e^{i\theta}} \right) \int_{-\infty}^{+\infty} \frac{e^{t\lambda} e^{-i\lambda z}}{e^{-2i\theta} + e^{2t\lambda}} d\lambda \\ \varphi(y) &= \frac{2e^{-i\theta}}{4\rho t\pi} \int_{-\infty}^{+\infty} \frac{e^{-\frac{\lambda'}{2}} e^{iy\lambda'}}{e^{-2i\theta} + e^{-\lambda'}} d\lambda' \end{aligned}$$

This Fourier transform can be found in [1]. Its inverse is:

$$\begin{aligned} \varphi(y) &= \frac{2e^{i\theta}}{4\rho t} \frac{e^{-i\theta-2y\theta}}{\sin\left(\frac{\pi}{2} - i\pi y\right)} \\ &= \frac{e^{-2y\theta}}{2\rho t \sin\left(\frac{\pi}{2} - i\pi y\right)} \\ &= \frac{e^{-2y\theta}}{2\rho t [\operatorname{ch}(\pi y)]} \end{aligned}$$

Hence we found the density of $z_t = L_t + \frac{a}{2}x^2$

$$\begin{aligned} \varphi(z_t) &= \frac{\exp\left(-\frac{z\theta}{t}\right)}{2\rho t \left[\operatorname{ch}\left(\frac{\pi z}{2t}\right)\right]} \\ \varphi(z_t) &= \frac{\exp\left(\frac{z}{t} \arctan(a)\right)}{2\rho t \left[\operatorname{ch}\left(\frac{\pi z}{2t}\right)\right]} \end{aligned}$$

□

This density function enables us to derive an upper bound for the repartition function of Z_t :

Lemma A.3. *For given $t > 0$ and $z > 0$, the probability $\mathbb{P}(Z_t > z)$ is majored by:*

$$\mathbb{P}(Z_t > z) \geq 1 - \frac{1}{2\rho} \frac{\exp\left(\left(\arctan(a) - \frac{\pi}{2}\right) \frac{Z}{t}\right)}{\frac{\pi}{2} - \arctan(a)}$$

Proof. Integration of φ gives the probability of Z to be above a given z :

$$\begin{aligned}
F(Z) &= \int_Z^{+\infty} \frac{\exp\left(\frac{z}{t} \arctan(a)\right)}{2\rho t \left[\operatorname{ch}\left(\frac{\pi z}{2t}\right) \right]} dz \\
&= \frac{1}{\rho t} \int_Z^{+\infty} \frac{\exp\left(\frac{z}{t} \arctan(a)\right)}{e^{\frac{\pi z}{2t}} + e^{-\frac{\pi z}{2t}}} dz \\
&\geq \frac{1}{2\rho t} \int_Z^{+\infty} \frac{\exp\left(\frac{z}{t} \arctan(a)\right)}{e^{\frac{\pi z}{2t}}} dz \\
&\geq \frac{1}{2\rho t} \int_Z^{+\infty} \exp\left(\frac{z}{t} \arctan(a) - \frac{\pi z}{2t}\right) dz \\
&\geq \frac{1}{2\rho} \frac{\exp\left(\left(\arctan(a) - \frac{\pi}{2}\right) \frac{Z}{t}\right)}{\frac{\pi}{2} - \arctan(a)}
\end{aligned}$$

Where we use the fact that $z > 0$, hence that $e^{\frac{\pi z}{2t}} + e^{-\frac{\pi z}{2t}} \leq 2e^{\frac{\pi z}{2t}}$. This is the only approximation in this formula, and one can see that the error is less than a factor 2. \square

A.2 Approximating the Laplace transform

The Laplace transform of the considered process $Z_t = L_t + \frac{a}{2} (W_{1t}^2 + W_{2t}^2)$ is defined as follows:

$$\Psi(c) = E(\exp(cz_t))$$

As one knows the density of Z_t , the following formula is straightforward:

$$\Psi(c) = \int_{-\infty}^{+\infty} \frac{\exp\left(\left(c - \frac{\theta}{t}\right)z\right)}{2\rho t \left[\operatorname{ch}\left(\frac{\pi z}{2t}\right) \right]} dz$$

One can see that $\Psi(c) < +\infty$ iff $-\frac{\pi}{2t} + \frac{\theta}{t} < c < \frac{\pi}{2t} + \frac{\theta}{t}$. If this holds true, then one can dominate the Laplace transform with:

$$\begin{aligned}
\Psi(c) &\leq \frac{1}{\rho t} \left[\int_0^{+\infty} \exp\left(\left(c - \frac{\theta}{t} - \frac{\pi}{2t}\right)z\right) dz + \int_{-\infty}^0 \exp\left(\left(c - \frac{\theta}{t} + \frac{\pi}{2t}\right)z\right) dz \right] \\
&\leq \frac{1}{\rho t} \left[\frac{1}{\frac{\theta}{t} + \frac{\pi}{2t} - c} + \frac{1}{c - \frac{\theta}{t} + \frac{\pi}{2t}} \right] \tag{A.2}
\end{aligned}$$

A.3 Proof of the law of the iterated logarithm

We study the superior limit of the process

$$Z_t = \frac{a}{2} (W_{1t}^2 + W_{2t}^2) + \int_0^t W_{1u} dW_{2u} - W_{2u} dW_{1u}$$

In the almost sure sense, when t goes to 0. The Laplace transform of Z_t can be used to show the first estimate:

Lemma A.4. *The process Z is such that*

$$\lim_{t \rightarrow 0^+} \sup \frac{Z_t}{t \log \log \frac{1}{t}} \leq \frac{1}{\frac{\pi}{2} - \arctan(a)}$$

In the almost sure sense.

Proof. Define $h(t) = \log \log \left(\frac{1}{t}\right)$, $g(t) = \frac{t}{\frac{\pi}{2} - \arctan(a)} h(t)$ consider two real numbers $0 < \beta < 1$ and $0 < \delta < 1$, take $t = \beta^{n-1}$ and $c = \frac{\frac{\pi}{2} + \theta}{t\sqrt{1+\delta}}$, then, Doob's maximal inequality shows that:

$$\begin{aligned} \mathbb{P} \left[\max_{0 \leq s \leq t} \{cZ_s\} > \sqrt{1+\delta}h(t) \right] &= \mathbb{P} \left[\max_{0 \leq s \leq t} \{\exp(cZ_s)\} > \exp\left(\sqrt{1+\delta}h(t)\right) \right] \\ &\leq \exp\left(-\sqrt{1+\delta}h(t)\right) E(\exp(cZ_t)) \end{aligned}$$

Using the upper bound (A.2) on the Laplace transform of Z , one gets:

$$\begin{aligned} \mathbb{P} \left[\max_{0 \leq s \leq t} \{cZ_s\} > \sqrt{1+\delta}h(t) \right] &\leq e^{-\sqrt{1+\delta}h(t)} \frac{1}{\rho t} \left[\frac{1}{\frac{\theta}{t} + \frac{\pi}{2t} - c} + \frac{1}{c - \frac{\theta}{t} + \frac{\pi}{2t}} \right] \\ &= e^{-\sqrt{1+\delta}h(t)} \left[\frac{1}{\theta + \frac{\pi}{2} - \frac{\frac{\pi}{2} + \theta}{\sqrt{1+\delta}}} + \frac{1}{\frac{\frac{\pi}{2} + \theta}{\sqrt{1+\delta}} - \theta + \frac{\pi}{2}} \right] \\ &= C(n-1)^{-\sqrt{1+\delta}} \end{aligned}$$

With C a constant independent of n . We obtain converging series, and we can apply the Borel-Cantelli lemma:

$$\begin{aligned} \mathbb{P} \left[\max_{0 \leq s \leq \beta^{n-1}} \{Z_s\} \leq (1+\delta) \frac{1}{\frac{\pi}{2} - \arctan(a)} \beta^{n-1} h(\beta^{n-1}), n \rightarrow +\infty \right] &= 1 \\ Z_t \leq \max_{0 \leq s \leq \beta^{n-1}} \{Z_s\} &\leq (1+\delta) \frac{1}{\beta} g(t) \frac{h(\beta^{n-1})}{h(\beta^n)} \text{ a.s.} \end{aligned} \tag{A.3}$$

Afterwards, taking $\beta \rightarrow 1$ and $\delta \rightarrow 0$ completes the proof. \square

The other inequality is slightly more complicated to derive. We begin with the preliminary lemma:

Lemma A.5. $\mathbb{P}(Z_t - Z_h > z | \mathcal{F}_h) \geq \frac{1}{4\rho} \frac{\exp\left(\left(\arctan(a) - \frac{\pi}{2}\right) \frac{z}{t-h}\right)}{\frac{\pi}{2} - \arctan(a)}$ for any $t > h > 0$

Proof. First, remark that for any $t > 0$ and $0 < h < t$:

$$\begin{aligned} Z_t &= Z_h + \tilde{L}_{t-h} + \frac{a}{2} (\tilde{W}_{1,t-h}^2 + \tilde{W}_{2,t-h}^2) + \left(W_{1,h} \tilde{W}_{2,t-h} - W_{2,h} \tilde{W}_{1,t-h} \right) \\ &\quad + a \left(W_{1,h} \tilde{W}_{1,t-h} - W_{2,h} \tilde{W}_{2,t-h} \right) \\ Z_t &= Z_h + \tilde{Z}_{t-h} + \left\langle \gamma_h, \tilde{W}_{t-h} \right\rangle \end{aligned}$$

Denote $\tilde{W}_{t-h} = W_t - W_h$, which is independent of W_h . Define $\tilde{Z}_{t-h}, \tilde{L}_{t-h}$, as before, but with \tilde{W} instead of W . Remark that γ_h is a vector which is a linear transform of W_h . Therefore, γ_h is independent of \tilde{L}, \tilde{Z} and \tilde{W} . This independence is the key property to use the Borel-Cantelli lemma. Next, one shows that for any $z \in \mathbb{R}$:

$$\begin{aligned} \mathbb{P}(Z_t - Z_h > z | \mathcal{F}_h) &= \mathbb{P}\left(\tilde{Z}_{t-h} + \langle \gamma_h, \tilde{W}_{t-h} \rangle > z \mid \mathcal{F}_h\right) \\ &\geq \mathbb{P}\left(\tilde{Z}_{t-h} > z, \langle \gamma_h, \tilde{W}_{t-h} \rangle > 0 \mid \mathcal{F}_h\right) \\ &\geq \frac{1}{2} \mathbb{P}\left(\tilde{Z}_{t-h} > z\right) \end{aligned}$$

The last inequality is due to the Brownian motion symmetry, and the independence property between γ_h and \tilde{Z} . Indeed taking $-\tilde{W}$ instead of \tilde{W} , does not change \tilde{Z} , and changes the sign of $\langle \gamma_h, \tilde{W}_{t-h} \rangle$. \square

Now, we must prove that the Borel-Cantelli lemma holds for the sequence of events $\{Z_{t_n} - Z_{t_{n+1}} > z\}$ with t_n a decreasing sequence of times. We cannot use Borel-Cantelli directly as these events are not independent.

Lemma A.6. (*Borel-Cantelli extension*) *Let $t_n, n \in \mathbb{N}$ be a decreasing sequence of positive numbers. And z_n a sequence of real numbers. If there exists a deterministic sequence B_n such that for any $n \in \mathbb{N}$*

$$\mathbb{P}\left(\{Z_{t_n} - Z_{t_{n+1}} > z_n\} \mid \mathcal{F}_{t_{n+1}}\right) \geq B_n$$

almost surely, an if:

$$\sum_{n=0}^{+\infty} B_n = +\infty$$

Then, $Z_{t_n} - Z_{t_{n+1}} > z_n$ infinitely often almost surely.

Proof. The proof is much the same as the original Borel-Cantelli lemma's proof. Denoting A_n as the event $\{Z_{t_n} - Z_{t_{n+1}} \leq z_n\}$, and

$$B_n = \mathbb{P}\left(\{Z_{t_n} - Z_{t_{n+1}} > z_n\} \mid \mathcal{F}_{t_{n+1}}\right)$$

One must prove that, for any $k \in \mathbb{N}$:

$$\mathbb{P}\left(\bigcap_{i=k}^{+\infty} A_i\right) = 0$$

As we have:

$$\begin{aligned}
\mathbb{P} \left(\bigcap_{i=k}^{+\infty} A_i \right) &= \mathbb{E} \left(\prod_{i=k}^{+\infty} \mathbb{1}_{A_i} \right) \\
&= \mathbb{E} \left[\mathbb{E} \left(\prod_{i=k}^{+\infty} \mathbb{1}_{A_i} \middle| \mathcal{F}_{t_{k+1}} \right) \right] \\
&= \mathbb{E} \left[\mathbb{E} (A_k | \mathcal{F}_{t_{k+1}}) \prod_{i=k+1}^{+\infty} \mathbb{1}_{A_i} \right] \text{ as } t_n \text{ is decreasing} \\
&\leq (1 - B_k) \mathbb{E} \left[\prod_{i=k+1}^{+\infty} \mathbb{1}_{A_i} \right]
\end{aligned}$$

Then, by a recursive argument, one obtains:

$$\mathbb{P} \left(\bigcap_{i=k}^{+\infty} A_i \right) \leq \prod_{i=k}^{+\infty} (1 - B_i)$$

At last, we use the inequality $1 - a \leq e^{-a}$ for any real number a . We obtain:

$$\mathbb{P} \left(\bigcap_{i=k}^{+\infty} A_i \right) \leq \exp - \sum_{i=k}^{+\infty} B_i \leq 0$$

As the series diverge by hypothesis. Then the proof is complete. \square

With these considerations, one can prove the reverse limit inequality, that is to say:

Lemma A.7. *The process Z is such that*

$$\limsup_{t \rightarrow 0^+} \frac{Z_t}{t \log \log \frac{1}{t}} \geq \frac{1}{\frac{\pi}{2} - \arctan(\alpha)} \tag{A.4}$$

In the almost sure sense.

Proof. Denoting the event:

$$A_n = \left\{ [Z(\beta^n) - Z(\beta^{n+1})] > (1 - \beta)^2 g(\beta^n) \right\}$$

Using lemma A.5, this event occurs with probability:

$$\begin{aligned}
\mathbb{P}(A_n | \mathcal{F}_h) &\geq \frac{1}{4\rho} \frac{\exp \left(\left(\arctan(\alpha) - \frac{\pi}{2} \right) \frac{(1-\beta)^2 g(\beta^n)}{\beta^n - \beta^{n+1}} \right)}{\frac{\pi}{2} - \arctan(\alpha)} \\
&\geq C.n^{-(1-\beta)}
\end{aligned}$$

Which is the general term of some diverging series. Hence, the Borel-Cantelli lemma A.6 leads to:

$$Z(\beta^n) > (1 - \beta)^2 g(\beta^n) + Z(\beta^{n+1})$$

infinitely often as $n \rightarrow +\infty$. Using the proof of lemma A.4 considering inequality (A.3) leads to:

$$Z(\beta^{n+1}) < (1 + \beta) g(\beta^{n+1})$$

almost surely for n sufficiently large. Therefore one gets:

$$Z(\beta^n) > (1 - \beta)^2 g(\beta^n) - (1 + \beta) g(\beta^{n+1}) > (1 - 4\beta) g(\beta^n)$$

Infinitely often when $n \rightarrow +\infty$. So:

$$\limsup_{t \rightarrow 0} \frac{\int_0^t \left(\int_0^u \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} dW_v \right)^T dW_u}{t \log \log \frac{1}{t}} \geq \frac{1 - 4\beta}{\frac{\pi}{2} - \arctan(a)}$$

For any $\beta > 0$. Taking $\beta \rightarrow 0$ finishes the proof. \square

Remark A.8. At last, it is remarkable that the same kind of result could easily be derived in dimension n instead of 2. One would have to work a little bit on the matrices to obtain boundaries for the original n dimensional quadratic form. These boundaries would be quadratic forms expressed as block-diagonal matrices with 2×2 and scalar blocks. Then the same type of reasoning would give some estimates on the lower and upper limits, and would enable to treat the gamma constraints problem in n dimensions with non symmetric matrices.

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