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Large-time behaviour of the entropy solution of a scalar conservation law with boundary conditions

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Abstract

We study the large-time behaviour of the entropy solution of a scalar conservation law with boundary conditions. Under structural hypotheses on the flux of the equation, we describe the stationary solutions and show the convergence of the entropy solution to a stationary one. Numerical tests illustrate the theoretical results.

Résumé

Nous étudions le comportement asymptotique en temps ($t \rightarrow +\infty$) de la solution entropique d'une loi de conservation scalaire avec conditions aux limites. Sous des hypothèses structurelles sur le flux (non-autonome) de l'équation, nous décrivons les états stationnaires et établissons la convergence de la solution entropique vers une solution stationnaire. Des tests numériques illustrent les résultats théoriques.

Key words: Scalar conservation laws; Large-time asymptotic analysis; Stationary solutions; Boundary conditions.

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1 Introduction

1.1 Statement of the problem

We study the large-time behaviour of the solution of the one-dimensional non-autonomous conservation law with boundary conditions

$$\begin{cases} u_t(x, t) + (A(x, u))_x(x, t) = 0, & (x, t) \in (0, 1) \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in (0, 1), \\ u(0, t) = \bar{u}_0(t), & t > 0, \\ u(1, t) = \bar{u}_1(t), & t > 0. \end{cases} \quad (1)$$

This study was initially motivated by the analysis of a problem which arises in lubrication theory. The behaviour of a thin film flow of two non-miscible fluids between two surfaces in relative motion has been derived by Paoli [Pao03], and further developed by Bayada, Martin and Vázquez [BMV05]. In these papers, it is stated that the saturation of the reference fluid satisfies a scalar conservation law as the one described in Problem (1) (the so-called generalized Buckley-Leverett equation). As it was already pointed out in [BMV05], this model is of great interest in lubrication theory, especially for the understanding of cavitation phenomena in devices such as journal bearings of infinite width. Cavitation, which is defined as the rupture of the continuous liquid lubricant film due to the formation of gaseous bubbles (see [Do63,EA75] for precise physical explanations) may be approached using the generalized Buckley-Leverett model, by considering the bifluid as a liquid-gas mixture. Finally, for practical convenience, it is also relevant to focus on steady-state regimes which are particularly studied in the field of tribology.

References on the large-time behaviour of the entropy solution of scalar conservation laws with boundary conditions are quite few, to our knowledge. See the work of Mascia and Terracina [MT99] and references therein in case the equation has a source term. On the contrary, numerous works are devoted to the study of the large-time behaviour of the entropy solution of a scalar conservation law posed on the whole space (Cauchy problem). See in particular the results of decay of the entropy solution to N-waves, as studied in [Lax57,IO60,DiP75,LP84,Daf85,Kim03]. Consider also the studies on the stability of profiles for scalar conservation laws on the line, in particular in [KM85,MN94,LN97,FS01,Ser04]. We emphasize in particular the works of Serre and co-workers, who developed a general strategy for the study of the stability of profiles of scalar conservation laws which inspired us for the present analysis.

To the specific framework of lubrication theory is associated precise hypotheses on the flux A and on the data. Among these hypotheses, we have selected the following ones:

Assumption 1 (Data) *The boundary and initial data satisfy the following:*

(i) *Initial conditions:*

$$u_0 \in L^\infty(0, 1) \quad 0 \leq u_0 \leq 1 \text{ a.e.}, \quad (2)$$

(ii) *Boundary conditions:*

$$\bar{u}_0 = \bar{u}_1 = \bar{u} \in [0, 1]. \quad (3)$$

Remark 1 *The unknown u , as a saturation, takes value in $[0, 1]$, and so do the data. Furthermore \bar{u}_0 and \bar{u}_1 are constant functions: this allows to deal with realistic boundary conditions in infinite journal bearings as it will be discussed further.*

Assumption 2 (Flux function) *The flux function $A \in \mathcal{C}^1([0, 1]^2)$ satisfies the following properties:*

$$\forall x \in [0, 1], \quad A(x, \cdot) = A(1 - x, \cdot), \quad A(x, 0) = 0, \quad A(x, 1) = Q \geq 0, \quad (4)$$

$$\forall x \in [0, 1], \quad \exists \alpha(x) \in (0, 1), \quad \begin{cases} A_u(x, \cdot) > 0 \text{ on } [0, \alpha(x)), \\ A_u(x, \cdot) < 0 \text{ on } (\alpha(x), 1], \end{cases} \quad (5)$$

$$\forall u \in [0, 1], \quad x \mapsto A(x, u) \text{ is non-increasing on } [0, 1/2]. \quad (6)$$

Remark 2 (Lubrication theory) *In lubrication, a plain cylindrical journal bearing is made of an inner rotating cylinder and an outer cylinder. The two cylinders are closely spaced and the annular gap between the two cylinders is filled with some lubricant. The radial clearance is very small, typically $\Delta r/r = 10^{-3}$ for oil lubricated bearings so that the smallness of this ratio allows for a Cartesian coordinate to be located on the bearing surface. When cavitation occurs, the two-phase flow is described by Problem (1), with a flux $A(x, u) = Q f(u) + v_0 H(x) u(1 - f(u))$, where u denotes the (unknown) saturation of the liquid phase, f is a \mathcal{S} -shaped function (with $f(0) = 0$, $f(1) = 1$), H is the normalized converging-diverging gap between the two close surfaces in relative motion, v_0 is the shear velocity of the lower surface (the upper one being fixed) and Q is the flow input. Hypothesis (3) (see Assumption 1) is natural in infinite journal bearings: indeed, the annular gap is filled with some lubricant at the supply groove which is located at the maximum gap [BMV05], the saturation of the liquid phase being imposed in this located area (corresponding to each extremity in the Cartesian coordinates). Let us emphasize that the non-autonomous property of the flux function comes from the shear effects and/or the converging-diverging profile of the normalized gap. This affects the*

mathematical analysis of the large-time behaviour of the solution (mainly because balance effects appear, owing to the dependence on x of the flux and the possibility of more stationary profiles than in the case of an autonomous flux). However the non-zero shear velocity of the lower surface and the specific profile of the gap are characteristic features of realistic lubrication regimes.

Remark 3 (Model case) *A model case of flux function is given by*

$$A(x, u) = Qu + H(x)u(1 - u)$$

where, according to the strength of $H(x)/Q$, $\alpha(x) = 1$ ($A(x, \cdot)$ increasing on $[0, 1]$) or $\alpha(x) \in (0, 1)$ ($A(x, \cdot)$ has one strict maximum at $\alpha(x)$).

Remark 4 *Hypothesis (3) can be relaxed ((2) still being satisfied). Notice also that, under Hypotheses (2) and (4)–(6), Problem (1) models two-phase flows in porous media under a gravity field (see, e.g., [Kaa99,EGV03]) or two-phase flow in a pipe [EGV03].*

To achieve the discussion on Hypotheses (2)–(6), let us emphasize the necessity to restrict the possible shapes of the data and of the flux. Indeed, consider the case where $A = 0$. The solution of (1) is then $u(x, t) = u_0(x)$, whose behaviour in large-time cannot be expected. Similarly, in the case where $A(x, u) = u$, the solution of (1) is, for $t > 1$, $u(x, t) = \bar{u}_0(t - x)$ and $u(\cdot, t)$ converges to a function of $L^\infty(0, 1; [0, 1])$ in the L^1 -norm at $t \rightarrow +\infty$ only if the means of \bar{u}_0 converge to a constant.

1.2 Structure of the paper and main results

The paper is organized as follows: in Section 2, we recall the notion of entropy solution (also sub- and super-solution) for Problem (1) and its properties. In Section 3, we analyse the stationary solutions of Problem (1) to give their description in Theorem 3. In Section 4, we study the large-time behaviour of the entropy solution of (1). Our main result, Theorem 4, states that the entropy solution to (1) converges to a stationary solution in L^1 . In Section 5, we present some numerical tests which illustrate the theoretical results.

2 Entropy Solutions

2.1 Definitions

Definition 1 Assume that (2)–(4) hold. A function $u \in L^\infty((0, 1) \times (0, +\infty))$ is said to be an entropy sub-solution (respectively super-solution) of Problem (1) if for all $\kappa \in [0, 1]$, for all $\varphi \in \mathcal{C}_c^\infty([0, 1] \times [0, +\infty))$, $\varphi \geq 0$,

$$\begin{aligned} & \int_0^1 \int_0^\infty (u - \kappa)^\pm \varphi_t + \Phi^\pm(x, u, \kappa) \varphi_x - \operatorname{sgn}_\pm(u - \kappa) A_x(x, \kappa) \varphi \, dx \, dt \\ & \quad + \int_0^1 (u_0 - \kappa)^\pm \varphi(0, x) \, dx \quad (7) \\ & \quad + \mathcal{L} \int_0^\infty ((\bar{u}_0 - \kappa)^\pm \varphi(t, 0) + (\bar{u}_1 - \kappa)^\pm \varphi(t, 1)) \, dt \geq 0. \end{aligned}$$

The functions $u \mapsto (u - \kappa)^\pm$ are the so-called “semi Kružkov entropies” (see [Car99, Ser96, Vov02]), defined by

$$(u - \kappa)^+ = \begin{cases} u - \kappa, & \text{if } u \geq \kappa, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad (u - \kappa)^- = (\kappa - u)^+.$$

The functions $\Phi^\pm(x, u, \kappa)$ are the corresponding “semi Kružkov fluxes” defined by

$$\Phi^\pm(x, u, \kappa) = \operatorname{sgn}_\pm(u - \kappa)(A(x, u) - A(x, \kappa)),$$

and \mathcal{L} is a Lipschitz constant of $A(0, \cdot)$.

Definition 2 Assume that (2)–(4) hold. A function $u \in L^\infty((0, 1) \times (0, +\infty))$ is said to be an entropy solution of Problem (1) if it is a sub-solution and a super-solution of Problem (1).

2.2 Properties of the entropy solution

Theorem 1 (Existence, L^∞ bound) Assume that (2)–(4) hold. The problem (1) admits at least one entropy solution $u \in L^\infty((0, 1) \times (0, +\infty))$. Moreover, any entropy solution satisfies: $0 \leq u \leq 1$ a.e.

PROOF. Existence of entropy solutions of Problem (1) is given by the convergence of approximations (parabolic or numeric ones), see the following references [BLN79, Ott93, Ott96, MNRR96, Vov02, Mar05]. The bounds are directly

deduced from the weak entropy formulation, using appropriate choices of entropy/flux pairs and values of κ .

Theorem 2 (Uniqueness) *[BLN79, Ott93, Mar05] Assume that (2) and (4) hold. Let $u, v \in L^\infty((0, 1) \times (0, +\infty))$ be, respectively, entropy sub- and super-solutions of Problem (1) associated to respective data $u_0, \bar{u}_0, \bar{u}_1$ and $v_0, \bar{v}_0, \bar{v}_1$. Then, for all $\varphi \in \mathcal{C}_c^\infty([0, 1] \times [0, +\infty))$, $\varphi \geq 0$, u and v satisfy the comparison inequality*

$$\int_0^1 \int_0^\infty (u - v)^+ \varphi_t + \Phi^+(x, u, v) \varphi_x dx dt + \int_0^1 (u_0 - v_0)^+ \varphi(0, x) dx \\ \mathcal{L} \int_0^\infty ((\bar{u}_0 - \bar{v}_0)^+ \varphi(t, 0) + (\bar{u}_1 - \bar{v}_1)^+ \varphi(t, 1)) dt \geq 0. \quad (8)$$

In particular, entropy solutions of Problem (1) are uniquely determined by the initial and boundary data.

Remark 5 (Localisation of the comparison) *If $u, v \in L^\infty$ are entropy solutions in the interior of $(0, 1) \times (0, +\infty)$, in the sense that (7) holds for non-negative test-functions $\varphi \in \mathcal{C}_c^\infty(\omega \times (t_1, t_2))$ where ω is an open subset of $(0, 1)$ and $0 < t_1 < t_2$, then (8) remains true on $\omega \times (t_1, t_2)$, i.e. (8) holds for non-negative $\varphi \in \mathcal{C}_c^\infty(\omega \times (t_1, t_2))$.*

We have also the following properties:

Lemma 1 (Continuity w.r.t. time) *Assume that (2)–(4) hold. The entropy solution $u \in L^\infty((0, 1) \times (0, +\infty))$ of Problem (1) is continuous from $[0, +\infty)$ in L^1 :*

$$u \in \mathcal{C}([0, +\infty); L^1(0, 1)) \quad \text{and} \quad \lim_{t \rightarrow 0} u(t) = u_0 \text{ in } L^1(0, 1).$$

Remark-Definition 1 *In particular, and by Lemma 1, Problem (1) defines a semi-group $S(t) : (u_0, \bar{u}) \mapsto u(t)$ from $L^\infty(0, 1; [0, 1]) \times [0, 1]$ into $L^\infty(0, 1; [0, 1])$, which associates to (u_0, \bar{u}) the value at time t of the entropy solution of Problem (1) with initial datum u_0 and boundary data $\bar{u}_0 = \bar{u}_1 = \bar{u}$. We have*

$$\lim_{t \rightarrow 0} S(t)(u_0, \bar{u}) = u_0 \text{ in } L^1(0, 1).$$

A classical consequence of (8) is the fact that, for $\bar{u} \in [0, 1]$, $S(t)(\cdot, \bar{u})$ is non-expansive on L^1 : for every $u_0, v_0 \in L^\infty(0, 1; [0, 1])$, one has

$$\forall t > 0, \quad \|S(t)(u_0, \bar{u}) - S(t)(v_0, \bar{u})\|_{L^1(0,1)} \leq \|u_0 - v_0\|_{L^1(0,1)}. \quad (9)$$

In particular, $S(t)(\cdot, \bar{u})$ is continuous on $L^\infty(0, 1; [0, 1])$ endowed with the topology of the L^1 -norm.

We give the proof of Lemma 1 in Appendix A. To complete Remark-Definition 1, we give the following proposition:

Proposition 1 *Assume that (4) holds ((2) and (3) may be relaxed, the data being L^∞ functions). Let $v \in L^\infty(0, 1; [0, 1]) \cap \mathcal{C}([0, +\infty); L^1(0, 1))$ (respectively $w \in L^\infty(0, 1; [0, 1]) \cap \mathcal{C}([0, +\infty); L^1(0, 1))$) be an entropy sub- (resp. super-)solution of (1) with data (v_0, \bar{v}) (resp. (w_0, \bar{w})) and let*

$$(u_0, \bar{u}) \in L^\infty(0, 1; [0, 1]) \times [\bar{v}, \bar{w}].$$

Let $S(t)$ be the semi-group defined in Remark-Definition 1. Then

$$t \mapsto \|(v(t) - S(t)(u_0, \bar{u}))^+\|_{L^1(0,1)} \text{ and } t \mapsto \|(S(t)(u_0, \bar{u}) - w(t))^+\|_{L^1(0,1)}$$

are non-increasing. In particular, if $v_0 \leq u_0$ (resp. $u_0 \leq w_0$), then $v(t) \leq S(t)(u_0, \bar{u})$ (resp. $S(t)(u_0, \bar{u}) \leq w(t)$) a.e. on $(0, 1)$.

PROOF. Proposition 1 is a direct consequence of Theorem 2: choose a test function φ independent of x to get, in the weak sense

$$\rho'(t) \leq 0, \quad \rho(t) = \int_0^1 (v(t) - S(t)(u_0, \bar{u}))^+ dx \leq 0.$$

Besides, ρ is a continuous function, it is therefore non-increasing.

Lemma 2 (BV bound) [BLN79] *Assume that (4) holds. Then there exists a non-negative function C_{BV} from \mathbb{R}_+ to \mathbb{R}_+ such that, for $\bar{u} \in [0, 1]$,*

$$\|S(t)(u_0, \bar{u})\|_{BV(0,1)} \leq C_{BV}(\|u_0\|_{BV(0,1)}) \quad (10)$$

for all $u_0 \in BV(0, 1) \cap L^\infty(0, 1; [0, 1])$.

In the context of Lemma 2, $u(t) := S(t)(u_0, \bar{u}) \in BV(0, 1)$ for every t . In particular the traces $u(0, t)$ and $u(1, t)$ make sense and the boundary conditions are given by the so-called BLN inequalities [BLN79].

Lemma 3 (Boundary conditions) *Assume that (2)–(4) hold. Suppose $u_0 \in BV(0, 1) \cap L^\infty(0, 1; [0, 1])$ and $\bar{u} \in [0, 1]$. Then, a function*

$$u \in \mathcal{C}([0, +\infty); L^1(0, 1)) \cap L^\infty(0, +\infty, BV(0, 1))$$

is the entropy solution of (1) (i.e. $u(t) = S(t)(u_0, \bar{u})$) iff it satisfies the entropy inequalities inside $(0, 1)$: for all $\kappa \in [0, 1]$, for all $\varphi \in \mathcal{C}_c^\infty((0, 1) \times [0, +\infty))$, $\varphi \geq 0$,

$$\begin{aligned} \int_0^1 \int_0^\infty (u - \kappa)^\pm \varphi_t + \Phi^\pm(x, u, \kappa) \varphi_x - \text{sgn}_\pm(u - \kappa) A_x(x, \kappa) \varphi dx dt \\ + \int_0^1 (u_0 - \kappa)^\pm \varphi(0, x) dx \geq 0. \end{aligned} \quad (11)$$

and if, furthermore, it satisfies the BLN boundary conditions ($[a, b]$ denotes the interval of extremities a and b):

$$\begin{cases} \forall t > 0, \forall \kappa \in [u(0, t), \bar{u}_0], \operatorname{sgn}(u(0, t) - \bar{u})(A(0, u(0, t)) - A(0, \kappa)) \leq 0, \\ \forall t > 0, \forall \kappa \in [u(1, t), \bar{u}_1], \operatorname{sgn}(u(1, t) - \bar{u})(A(1, u(1, t)) - A(1, \kappa)) \geq 0. \end{cases}$$

Let us pause here to analyse the above BLN condition. Since $A(0, \cdot) = A(1, \cdot)$ is an increasing function on $(0, \alpha(0))$ and a decreasing function on $(\alpha(0), 1)$, we can associate to each $z \in [0, 1]$ its A -conjugate z^A defined by: $\alpha(0)^A = \alpha(0)$, $A(0, z^A) = A(0, z)$, $z^A \neq z$ if $A(0, z) \geq Q$ and $z \neq \alpha(0)$, $z^A = 1$ if $0 \leq z \leq \theta(0)$. Then, at $x = 0$, the BLN condition is equivalent to

$$\begin{cases} \text{either } 0 \leq \bar{u} \leq \alpha(0) \text{ and } u(0, t) = \bar{u} \text{ or } \alpha(0) \leq u(0, t) \leq 1, \\ \text{or } \alpha(0) \leq \bar{u} \leq 1 \text{ and } \bar{u}^A \leq u(0, t) \leq 1, \end{cases} \quad (12)$$

and, at $x = 1$, the BLN condition is equivalent to

$$\begin{cases} \text{either } 0 \leq \bar{u} \leq \alpha(0) \text{ and } 0 \leq u(1, t) \leq \alpha(0), \\ \text{or } \alpha(0) \leq \bar{u} \leq 1 \text{ and } u(1, t) = \bar{u} \text{ or } 0 \leq u(1, t) \leq \bar{u}^A. \end{cases} \quad (13)$$

3 Stationary solutions

3.1 Constant flux

Recall that $S(t)$ is defined in Remark-Definition 1.

Definition 3 Assume that (2) and (4) hold. A function $w \in L^\infty(0, 1; [0, 1])$ is said to be a stationary solution of Problem (1) if there exists a boundary datum $\bar{u} \in [0, 1]$ such that $w = S(t)(w, \bar{u})$ for all $t > 0$.

We begin the analysis of stationary solutions of (1) by the inversion of the equation $A(x, w) = c \in \mathbb{R}$.

Proposition 2 Let $w \in L^\infty(0, 1; [0, 1])$ be a stationary entropy solution of Problem (1). Then $A(x, w)$ is constant a.e.

PROOF. An entropy solution is a weak solution: $(A(x, w))_x = 0$ in $\mathcal{D}'((0, 1) \times (0, +\infty))$. Since $w = w(x)$, this remains true in $\mathcal{D}'(0, 1)$. This last fact implies $A(x, w) = c \in \mathbb{R}$ by the following Lemma 4.

Lemma 4 *Let $T \in \mathcal{D}'(0, 1)$ be such that $T' = 0$. Then $T = c \in \mathbb{R}$ (i.e. T is a function of $L^1(0, 1)$ which is constant a.e.).*

PROOF. The result is stated in two steps:

■ *1st step* - If $\psi \in \mathcal{D}(0, 1)$ with $\int_0^1 \psi = 0$, then $T\psi = 0$, using $\psi = \Psi'$ with

$$\Psi : x \mapsto \int_0^x \psi \in \mathcal{D}(0, 1).$$

■ *2nd step* - Let (K_n) be an increasing sequence of compact sets of $(0, 1)$ such that

$$(0, 1) = \bigcup_{n \geq 0} K_n,$$

and let ϕ_n be a sequence of function of $\mathcal{D}(0, 1)$ such that

$$\text{supp}(\phi_n) \subset K_{n+1}, \quad \int_0^1 \phi_n = 1.$$

For every $\psi \in \mathcal{D}(0, 1)$ such that $\text{supp}(\psi) \subset K_n$,

$$\psi_n := \psi - \left(\int_0^1 \psi \right) \phi_n$$

is such that $\int_0^1 \psi_n = 0$. Therefore, $T\psi_n = 0$, i.e.

$$T\psi = T\phi_n \int_0^1 \psi.$$

In particular, applying the result to $\psi = \phi_{n-1}$, we get $T\phi_{n-1} = T\phi_n$, which is therefore a constant c independant of n . We then deduce

$$T\psi = c \int_0^1 \psi$$

for every $\psi \in \mathcal{D}(0, 1)$ and, by injection of $L^1(0, 1)$ in $\mathcal{D}'(0, 1)$, $T = c$ in $L^1(0, 1)$.

Lemma 5 *Let $T \in \mathcal{D}'(0, 1)$ be such that $T' \geq 0$. Then $T \in L^1_{\text{loc}}(0, 1)$ and T is a monotone non-decreasing function. If, besides, T is a function with values in $\{0, 1\}$, then there exists $x_0 \in [0, 1]$ such that $T(x) = \text{sgn}_+(x - x_0)$.*

PROOF. The non-negative distribution T' is of order 0: for every compact subset K of $(0, 1)$, there exists a constant C_K such that for every $\varphi \in \mathcal{D}(0, 1)$ with $\text{supp}(\varphi) \subset K$, $|T'(\varphi)| \leq C_K \|\varphi\|_{L^\infty}$ (choose a non-negative $\psi \in \mathcal{D}(0, 1)$

such that $\psi = 1$ on K and write that $T'(\varphi \pm \|\varphi\|_{L^\infty}\psi)$ is non-negative (respectively non-positive) to get the result with $C_K = T'(\psi)$. By the Riesz representation theorem T' is represented by a Radon measure μ on $(0, 1)$. Let $x_0 \in (0, 1)$. By the green formula for BV functions, T and the BV function $\tilde{T} : x \mapsto \mu((x_0, 0, x))$ have the same derivative in $\mathcal{D}'(0, 1)$. By Lemma 4, we conclude that $T = \tilde{T} + \text{cte}$. Since $BV(0, 1) \subset L^\infty(0, 1) \subset L^1_{\text{loc}}(0, 1)$, T is a function, which is non-decreasing. If, furthermore, T takes values in $\{0, 1\}$, then, by monotony of T , $T^{-1}(0)$ and $T^{-1}(1)$ are connected, hence are subintervals of $[0, 1]$ and $T(x) = \text{sgn}_+(x - x_0)$ with $x_0 := \sup T^{-1}(0) = \inf T^{-1}(1)$.

Now we examine the inversion of the identity $A(x, w) = c \in \mathbb{R}$. Denote by q the value of $A(x, w)$. Under Hypotheses (4)–(5), the equation $A(x, w) = q$ may have zero, one or two branches of solutions according to the value of q (by branch of solution, we mean a continuous function w from $[0, 1]$ into itself such that $A(x, w(x)) = q$ for all x). Indeed, let us define

$$q_{\text{lim}} := \min\{A(x, \alpha(x)), x \in [0, 1]\} = A(1/2, \alpha(1/2)).$$

Thus, we have either $q_{\text{lim}} > Q$ or $q_{\text{lim}} = Q$.

Lemma 6 *Assume that (4)–(6) hold. We focus on the solutions of*

$$A(x, w(x)) = q \in \mathbb{R}, \quad x \in [0, 1], \quad w : [0, 1] \rightarrow [0, 1] \text{ continuous.} \quad (14)$$

We have successively, for increasing q :

- *If $q < 0$, there are no solutions.*
- *If $0 \leq q < Q$, there is one solution denoted λ .*

To continue the study of (14), we have to make the distinction between the following cases:

- *If $Q < q_{\text{lim}}$, we have the following subcases:*
 - *If $Q \leq q < q_{\text{lim}}$, there are two solutions, denoted λ and μ . We have $\lambda < \mu$.*
 - *If $q = q_{\text{lim}}$, there are two solutions, denoted λ and μ . We have $\lambda \leq \mu$ and λ and μ cross each other: $\lambda(1/2) = \mu(1/2)$.*
 - *If $q > q_{\text{lim}}$, there are no solutions to (14).*
- *If $Q = q_{\text{lim}}$, we have also a bifurcation at $q = q_{\text{lim}}$:*
 - *If $Q = q = q_{\text{lim}}$, there are two solutions denoted λ and μ . We have $\lambda \leq \mu$ and λ and μ cross each other: $\lambda(1/2) = \mu(1/2)$. Actually, μ is the constant function 1.*
 - *If $q > q_{\text{lim}}$, there are no solutions to (14).*

PROOF. The proof of Lemma 6 is a simple examination of cases. We use the fact that, by Hypotheses (4)–(5), and for all $x \in [0, 1]$, the function $A(x, \cdot)$ induces homeomorphisms from $[0, \alpha(x)]$ onto $[0, A(x, \alpha(x))]$ and from $[\alpha(x), 1]$ onto $[Q, A(x, \alpha(x))]$.

Remark 6 Notice that Lemma 6 can be further detailed in the case $q > q_{\text{lim}}$. Indeed, either $A(0, \alpha(0)) = q_{\text{lim}}$ in which case there is no solution to (14) for $q > q_{\text{lim}}$, or $A(0, \alpha(0)) > q_{\text{lim}}$ in which case the equation (14) for $A(0, \alpha(0)) > q > q_{\text{lim}}$ has solution on parts of the interval $[0, 1]$: if x_q is the greatest element x of $[0, 1/2]$ such that $A(x, \alpha(x)) = q$, then λ and μ are defined on $[0, x_q]$ and (by symmetry) on $[1 - x_q, 1]$:

$$\forall x \in [0, x_q] \cup [1 - x_q, 1], \quad \begin{cases} \lambda(x) = (A(x, \cdot)|_{[0, \alpha(x)]})^{-1}(q), \\ \mu(x) = (A(x, \cdot)|_{[\alpha(x), 1]})^{-1}(q), \end{cases} \quad (15)$$

Remark 7 In case the equation $A(x, w(x)) = q$ has more than one elementary solution, any function constructed by introducing jumps between two elementary solutions is also a solution, in $L^\infty((0, 1); [0, 1])$, of the equation. To specify the set of stationary solutions, we use the entropy condition and the boundary conditions.

3.2 Entropy and boundary conditions

Lemma 6 shows that, for $0 \leq q \leq q_{\text{lim}}$, the equation $A(x, w(x)) = q$ has one or two branches of solutions. We call them λ and μ , $\lambda \leq \mu$, with the convention $\lambda = \mu$ if there is actually only one branch.

Lemma 7 Assume that (4)–(6) hold. Let $q \in [0, q_{\text{lim}}]$. Let λ and μ be the branches of solutions to (14). Let us define

$$\begin{aligned} w_z &:= \lambda \mathbf{1}_{[0, z]} + \mu \mathbf{1}_{(z, 1]}, & \text{for } 0 \leq z \leq 1, \\ w_{z, z'} &:= w_z \mathbf{1}_{[0, 1/2]} + w_{z'} \mathbf{1}_{[1/2, 1]}, & \text{for } 0 \leq z \leq 1/2 \leq z' \leq 1. \end{aligned}$$

Then any stationary solution w to Problem (1) such that $A(x, w(x)) = q$ a.e. is one of the w_z , $z \in [0, 1]$ or one of the $w_{z, z'}$, $0 \leq z \leq 1/2 \leq z' \leq 1$.

PROOF. In the case where $\lambda = \mu$, the statement and its proof are obvious. We suppose therefore that λ and μ are distinct functions. Moreover, we have to distinguish two cases:

- Let us suppose that $Q \leq q < q_{\text{lim}}$. Let $w \in L^\infty(0, 1; [0, 1])$ be a stationary entropy solution of (1) with flux q . Choose $\tilde{q} \in (q, q_{\text{lim}})$. We denote by λ ,

$\tilde{\mu}$ the corresponding stationary solutions constructed as above. On the one hand, by comparison (see Remark 5 – we consider the case $\omega = (0, 1)$ here), we have

$$\int_0^1 \Phi^+(x, \tilde{\lambda}, w) \varphi_x dx \geq 0$$

for all non-negative $\varphi \in \mathcal{C}_c^\infty(0, 1)$. It follows that $\partial_x \Phi^+(x, \tilde{\lambda}, w) \leq 0$ in $\mathcal{D}'(0, 1)$. On the other hand, we have

$$\Phi^+(x, \tilde{\lambda}, w) = \operatorname{sgn}_+(\tilde{\lambda} - w)(A(x, \tilde{\lambda}) - A(x, w)) = \operatorname{sgn}_+(\tilde{\lambda} - w)(\tilde{q} - q)$$

and, therefore, $\partial_x \operatorname{sgn}_+(\tilde{\lambda} - w) \leq 0$ in $\mathcal{D}'(0, 1)$. Now, by Lemma 5, there exists some $x_0 \in [0, 1]$ such that

$$\operatorname{sgn}_+(\tilde{\lambda} - w)(x) = 1 - \operatorname{sgn}_+(x - x_0), \quad x \in (0, 1).$$

For every $x \in [0, 1]$, $u \mapsto \operatorname{sgn}_+(\tilde{\lambda}(x) - u)A(x, u)$ is an homeomorphism from $[0, \tilde{\lambda}(x))$ onto $[0, \tilde{q})$; therefore the identity

$$\operatorname{sgn}_+(\tilde{\lambda}(x) - w(x))A(x, w(x)) = q, \quad x \in (0, x_0)$$

implies that $w = \lambda$ on $(0, x_0)$. Similarly, we would prove that there exists some $x_1 \in [0, 1]$ such that $w = \mu$ on $(x_1, 1)$. Finally, since $\lambda < \mu$, we have $x_0 \leq x_1$. If $x_0 < x_1$, then $\tilde{\lambda} \leq w \leq \tilde{\mu}$ on (x_0, x_1) , which contradicts $A(x, w(x)) = q$ a.e. Therefore $x_0 = x_1 = z \in [0, 1]$ and $w = w_z$.

- Let us suppose that $q = q_{\lim}$. The proof of the result is similar, except that the solutions $\tilde{\lambda}$ and $\tilde{\mu}$ are not defined everywhere on $(0, 1)$: let $w \in L^\infty(0, 1; [0, 1])$ be a stationary entropy solution of (1) with flux q . Suppose that z_* is the smallest element of $[0, 1/2]$ such that $\lambda = \mu$ on $[z_*, 1 - z_*]$. Since λ and μ are distinct, we have $0 < z_* \leq 1/2$. By minimality of z_* , we have, for any $\varepsilon > 0$, $\lambda(z_* - \varepsilon) < \mu(z_* - \varepsilon)$. In particular,

$$A(z_* - \varepsilon, \alpha(z_* - \varepsilon)) > q_{\lim}.$$

Now, let us choose \tilde{q} such that

$$q_{\lim} < \tilde{q} < A(z_* - \varepsilon, \alpha(z_* - \varepsilon)).$$

Then $\tilde{\lambda}$ exists on $[0, z_* - \varepsilon]$ (see Equation (15)) and, as in the proof of the case $Q \leq q < q_{\lim}$ (using Remark 5 with $\omega = (0, z_* - \varepsilon)$), we can show that $\operatorname{sgn}_+(\tilde{\lambda} - w)$ is non-increasing on $(0, z_* - \varepsilon)$. Now, for every $x \in [0, z_* - \varepsilon]$, $u \mapsto \operatorname{sgn}_+(\tilde{\lambda}(x) - u)A(x, u)$ is an homeomorphism from $[0, \tilde{\lambda}(x))$ onto $[0, \tilde{q})$ so that there exists $x_\varepsilon \in [0, z_*]$ such that $w = \lambda$ on $(0, x_\varepsilon)$. Letting ε tend to 0, we infer that there exists $x_0 \in [0, z_*]$ such that $w = \lambda$ on $(0, x_0)$. Similarly, we show that there exists $x_1 \in [0, z_*]$ such that $w = \mu$ on (x_1, z_*) . If $x_0 < x_1$, we have $x_0 < x_1 - \varepsilon$ for some $\varepsilon > 0$. Then choosing again some $\tilde{q} \in (q_{\lim}, A(z_* - \varepsilon, \alpha(z_* - \varepsilon)))$, we see that, necessarily, $\tilde{\lambda} \leq w \leq \tilde{\mu}$

on (x_0, x_1) , which contradicts $A(x, w(x)) = q$ a.e. As a conclusion, there exists $z \in [0, z_*]$ such that $w = w_z$ on $[0, 1/2]$. Similarly, we can show that there exists $z' \in [1 - z_*, 1]$ such that $w = w_{z'}$ on $[1/2, 1]$. This shows that $w = w_{z, z'}$ and concludes the proof of the lemma.

Remark 8 *In fact, as it will appear in the proof, Lemma 7 states that a stationary solution to (1) can have only increasing discontinuities. If $\lambda < \mu$ (case $Q \leq q < q_{\text{lim}}$), there can be only one discontinuity, and w is one of the functions w_z . If λ and μ cross each other (case $q = q_{\text{lim}}$), then w can exhibit two discontinuities and is one of the functions $w_{z, z'}$.*

Every function w_z or $w_{z, z'}$ is an entropy solution inside $(0, 1)$ and admits traces at $x = 0$ and $x = 1$. On the basis of Lemmas 3 and 7, the analysis of the boundary conditions leads to the characterisation of the stationary solutions:

Theorem 3 *Assume that (2)–(6) hold. Let $u \in L^\infty((0, 1); [0, 1])$ be a stationary solution of (1) with boundary datum \bar{u} (i.e. $S(t)(u, \bar{u}) = u$, $\forall t > 0$) and set $\bar{q} := A(0, \bar{u})$. Then we have the following results.*

1. *The flux $A(x, u(x))$ is equal a.e. to a constant q .*
2. *Let λ and μ be the elementary solutions to the equation $A(x, w(x)) = q$ defined in Lemma 6. According to the value of \bar{q} , we have:*
 - *if $\bar{q} < q_{\text{lim}}$, then either $\bar{u} = \lambda(0)$ in which case $u = \lambda$, or $\bar{u} = \mu(0)$ in which case $u = \mu$;*
 - *if $\bar{q} = q_{\text{lim}}$, then either $\bar{u} = \lambda(0)$ in which case $u = \lambda \mathbf{1}_{[0, z] \cup [1/2, 1]} + \mu \mathbf{1}_{(z, 1/2]} = w_{z, 1}$ for $0 \leq z \leq 1/2$, or $\bar{u} = \mu(0)$ in which case $u = \lambda \mathbf{1}_{[1/2, z']} + \mu \mathbf{1}_{[0, 1/2] \cup (z', 1]} = w_{0, z'}$ for $1/2 \leq z' \leq 1$;*
 - *if $\bar{q} > q_{\text{lim}}$, then $\lambda(0) < \bar{u} < \mu(0)$ and $u = w_{0, 1} = \lambda \mathbf{1}_{[1/2, 1]} + \mu \mathbf{1}_{[0, 1/2]}$.*

Notice that discontinuous stationary solutions exist only in the case $\bar{q} = q_{\text{lim}}$.

PROOF. To prove Theorem 3, we use the BLN conditions in the converse sense of the usual one, i.e. we study which set of boundary conditions is compatible with a given trace of the unknown; that is to say, we will use the following characterisation (deduced from (12), (13)): the couple $(u(0), \bar{u})$ satisfies the BLN condition at $x = 0$ iff

$$\begin{cases} \text{either } 0 \leq u(0) \leq \alpha(0) \text{ and } u(0) = \bar{u}, \\ \text{or } \alpha(0) \leq u(0) \leq 1 \text{ and } u(0)^A \leq \bar{u} \leq 1, \end{cases} \quad (16)$$

and the couple $(u(1), \bar{u})$ satisfies the BLN condition at $x = 1$ iff

$$\begin{cases} \text{either } 0 \leq u(1) \leq \alpha(0) \text{ and } 0 \leq \bar{u} \leq u(1)^A, \\ \text{or } \alpha(0) \leq u(1) \leq 1 \text{ and } u(1) = \bar{u}. \end{cases} \quad (17)$$

Item 1. is the content of Proposition 2.

Item 2. is proved as follows: Lemma 7 shows that $u = w_z$ or $u = w_{z,z'}$ for $z \in [0, 1]$ or $0 \leq z \leq 1/2 \leq z' \leq 1$. The analysis of the boundary conditions is the following one:

- *Case $z > 0$:* if $z > 0$, then $u(0) = \lambda(0) \in [0, \alpha(0)]$. From (16), it follows that $\bar{u} = u(0) = \lambda(0)$. We then have $q = \bar{q}$ and the BLN condition (13) at $x = 1$ shows that, between $\mu(1) = \mu(0)$ and $\lambda(1) = \lambda(0)$, $u(1)$ takes the value $\lambda(0)$. Therefore:
 - if $q < q_{\text{lim}}$, then $u = \lambda$,
 - if $q = q_{\text{lim}}$, then $u = w_{z,1}$, $z \in (0, 1/2)$.
- *Case $z = 0$.* In the case $z = 0$, we have either $u = \mu$ or $u = w_{0,z'}$, with $1/2 \leq z' \leq 1$. If $u = w_{0,1}$, the conditions (16) and (17) give $\lambda(0) \leq \bar{u} \leq \mu(0)$. In that case, $q = q_{\text{lim}}$ (i.e. λ and μ cross each other, otherwise $w_{0,1}$ has a discontinuity which is not admissible) and $\bar{q} \geq q = q_{\text{lim}}$. If $u = \mu$ or $u = w_{0,z'}$ with $1/2 \leq z' < 1$, then $u(1) = \mu(1) = \mu(0)$ and, from (17), it follows that $\bar{u} = \mu(0)$. We then have $q = \bar{q}$.

According to the value of \bar{q} and q_{lim} , the three possibilities described in the theorem may occur.

As a corollary of Theorem 3, we have:

Corollary 1 *Assume that (4)–(6) hold. Set $\bar{q} := A(0, \bar{u})$ and $q := \min(\bar{q}, q_{\text{lim}})$. Let λ and μ be the elementary solutions of $A(x, w(x)) = q$ defined in Lemma 6. Set $\hat{w}_z = \lambda \mathbf{1}_{[0,z)} + \hat{\mu} \mathbf{1}_{(z,1]}$ with*

$$\hat{\mu} := \begin{cases} 1, & \text{if } q < Q, \\ \mu, & \text{if } q \geq Q. \end{cases}$$

- *If $\bar{q} < q_{\text{lim}}$, then the functions \hat{w}_z , $0 \leq z \leq 1$, are either sub- or super-solutions of Problem (1) with boundary datum \bar{u} .*
- *If $\bar{q} = q_{\text{lim}}$, then the functions $w_{0,z'}$, $1/2 \leq z' \leq 1$, are super-solutions while the functions $w_{z,1}$, $0 \leq z \leq 1/2$ are sub-solutions of Problem (1) with boundary datum \bar{u} .*

PROOF. It is easy to see, from Definition 1, that a function $w \in BV(0, 1)$ is a (stationary) sub- (resp. super-) entropy solution of Problem (1) iff it satisfies (7) *inside* $(0, 1)$ (i.e. for test-functions φ which vanish at $x = 0$ and $x = 1$) together with the *boundary conditions*

$$-\Phi^\pm(0, w(0), k) + \mathcal{L}(\bar{u} - k)^\pm \geq 0, \quad 0 \leq k \leq 1, \quad (18)$$

$$\Phi^\pm(1, w(1), k) + \mathcal{L}(\bar{u} - k)^\pm \geq 0, \quad 0 \leq k \leq 1. \quad (19)$$

Notice that (18) (resp. (19)) is trivial unless $(\bar{u} - k)^\pm = 0$.

First notice, as a first step of the proof, that the functions \hat{w}_z are always super-solutions inside $(0, 1)$. If $\bar{q} \geq Q$, then $\hat{\mu} = \mu$ and they are actually entropy solutions in $(0, 1)$. In a second step, we examine the boundary conditions with the help of (18)–(19).

If $\bar{q} < Q$, then $\bar{q} = q$ and $\bar{u} = \lambda(0)$, $\hat{\mu} = 1$. If $1 > z > 0$, we have $\hat{w}_z(0) = \lambda(0) = \bar{u}$, whence (18). Besides $\hat{w}_z(1) = 1$ and, for $k \leq \bar{u} = \lambda(0)$, we have $(\bar{u} - k)^- = 0$ and

$$\Phi^-(1, \hat{w}_z(1), k) = \Phi^-(1, 1, k) = 0 \geq 0.$$

Inequality (19) with semi-entropies $(u - k)^-$ is satisfied. If $z = 0$ this is still true ($\hat{w}_z = \lambda$); if $z = 1$, then we have to examine the boundary condition at $x = 0$: for $k \leq \bar{u} = \lambda(0)$, we have

$$-\Phi^-(0, \hat{w}_1(1), k) = -\Phi^-(0, 1, k) = 0 \geq 0$$

and inequality (18) with semi-entropies $(u - k)^-$ is satisfied. We conclude that the functions \hat{w}_z , $0 \leq z \leq 1$ are super-solutions.

Similar considerations allow to prove the Corollary in the case $Q \leq \bar{q} \leq q_{\text{lim}}$.

4 Large time asymptotic behaviour

Fix $\bar{u} \in [0, 1]$. Let X_0 denote the set $X_0 := L^\infty(0, 1; [0, 1])$ endowed with the topology of the L^1 -norm (X_0 is a Banach space for it is closed in L^1 since any convergent sequence of L^1 has a subsequence which converges almost everywhere). Denote by $S_{\bar{u}}(t) := S(\cdot, \bar{u})$ the semi-group which, to any $u_0 \in X_0$, associates the value $u(t) \in X_0$ at time t of the solution u to Problem (1) with data $(u_0, \bar{u}_0 = \bar{u}_1 = \bar{u})$. We study the behaviour of the trajectories $S_{\bar{u}}(t)u_0$ as $t \rightarrow +\infty$.

4.1 Compactness

Proposition 3 *Assume that (4) holds. Let $u_0 \in X_0$. Then $(S_{\bar{u}}(t)u_0)_{t>0}$ is relatively compact in X_0 .*

PROOF. Suppose first that $u_0 \in X_0$ additionally satisfies $u_0 \in BV(0, 1)$. By (10), $(S_{\bar{u}}(t)u_0)_{t>0}$ is bounded in $L^1 \cap BV(0, 1)$ and therefore relatively compact in $L^1(0, 1)$. In the general case, fix $\varepsilon > 0$, choose $u_0^\varepsilon \in X_0 \cap BV(0, 1)$ and $\|u_0 - u_0^\varepsilon\|_{L^1(0,1)} \leq \varepsilon$ ($u_0^\varepsilon := \widetilde{u}_0 \star \rho_\varepsilon$ where \widetilde{u}_0 is the extension of u_0 by 0 outside $[0, 1]$ and ρ_ε a classical (non-negative) approximation of unity will do). Then $\|S_{\bar{u}}(t)u_0^\varepsilon - S_{\bar{u}}(t)u_0\| \leq \varepsilon$ for every t by (9). Therefore, for every $\varepsilon > 0$, the set $E := \{S_{\bar{u}}(t)u_0; t > 0\}$ is at distance less than ε of a set $E_\varepsilon (= \{S_{\bar{u}}(t)u_0^\varepsilon; t > 0\})$ which is relatively compact in X_0 . Being relatively compact, each E_ε is totally bounded, therefore so is E . Since X_0 is a Banach space, the set E is relatively compact in X_0 .

As a consequence of Proposition 3, each ω -limit set

$$\omega(u_0) := \bigcap_{t>0} \overline{\{S_{\bar{u}}(\tau)u_0; \tau \geq t\}}$$

is non-empty, for $u_0 \in X_0$. Let $\omega(X_0)$ be the union of the ω -limit sets:

$$\omega(X_0) := \bigcap_{t>0} \overline{\{S_{\bar{u}}(\tau)X_0; \tau \geq t\}}.$$

We now study $\omega(X_0)$.

4.2 Bounds on the adherence values

Lemma 8 *Assume that (4)–(6) hold. Let $\bar{q} = A(0, \bar{u})$, let $q = \min(\bar{q}, q_{\text{lim}})$ and let λ and μ be the elementary solutions of the equation $A(x, w(x)) = q$ (see Lemma 6). Then we have*

$$\omega(X_0) \subset \{w \in X_0, \lambda \leq w \text{ a.e.}\}. \quad (20)$$

If, furthermore, $Q \leq q$ (in which case μ is different from λ), we have

$$\omega(X_0) \subset \{w \in X_0, \lambda \leq w \leq \mu \text{ a.e.}\}. \quad (21)$$

PROOF. If $\bar{q} = 0$, then $q = 0$, $\lambda = 0$ and (20) is obvious. Therefore, suppose that $\bar{q} > 0$. We have $q > 0$ and $\lambda > 0$. For small $\varepsilon > 0$, let $q - \varepsilon < q_\varepsilon < q$ and

let λ_ε be the first continuous branch of the equation $A(x, w(x)) = q_\varepsilon$. Since $q_\varepsilon < q$, we have $\lambda_\varepsilon < \lambda$, and since $\lambda \leq \alpha$, we have also $\lambda_\varepsilon < \alpha$ on $[0, 1]$. From Hypothesis (5), it follows that

$$A_u(x, \sigma) > 0, \quad \forall (x, \sigma) \in E := \{(x, \sigma) \in [0, 1]^2; 0 \leq \sigma \leq \lambda_\varepsilon(x)\}.$$

By continuity of λ_ε , the set E is closed, hence compact in $[0, 1]^2$. By continuity of A_u then defining $\gamma := \min_{(x, \sigma) \in E} A_u(x, \sigma) > 0$, it follows that,

$$\Phi^+(x, \lambda_\varepsilon(x), w) = \operatorname{sgn}_+(\lambda_\varepsilon(x) - w) \int_w^{\lambda_\varepsilon(x)} A_u(x, \sigma) d\sigma \geq \gamma (\lambda_\varepsilon(x) - w)^+, \quad (22)$$

for all $x, w \in [0, 1]$. The function λ_ε is a solution, hence a sub-solution of Problem (1) with the boundary datum $\lambda_\varepsilon(0)$. From (8) we deduce, for $u_0 \in X_0$, φ non-negative test-function in $\mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \int_0^1 \int_0^\infty (\lambda_\varepsilon - S_{\bar{u}}(t)u_0)^+ \varphi_t + \Phi^+(x, \lambda_\varepsilon, S_{\bar{u}}(t)u_0) \varphi_x dx dt + \int_0^1 (\lambda_\varepsilon - u_0)^+ \varphi(0, x) dx \\ + 2\mathcal{L} \int_0^\infty (\lambda_\varepsilon(0) - \bar{u})^+ (\varphi(t, 0) + \varphi(t, 1)) dt \geq 0. \end{aligned} \quad (23)$$

Since $\bar{q} \geq q \geq q_\varepsilon$, and $0 \leq \lambda_\varepsilon(0) \leq \alpha(0)$, we have $(\lambda_\varepsilon(0) - \bar{u})^+ = 0$ and the last term in (23) vanishes.

Choosing $\varphi(x, t) = \alpha(t)e^{-x}$ where α is a non-negative function of $\mathcal{C}_c^\infty(0, +\infty)$, we have

$$\int_0^1 \int_0^{+\infty} ((\lambda_\varepsilon - S_{\bar{u}}(t)u_0)^+ \alpha_t - \Phi^+(x, \lambda_\varepsilon, S_{\bar{u}}(t)u_0) \alpha) e^{-x} \geq 0.$$

By use of the estimate (22), we get

$$\int_0^{+\infty} \left(\int_0^1 (\lambda_\varepsilon - S_{\bar{u}}(t)u_0)^+ e^{-x} \right) (\alpha_t - \gamma\alpha) \geq 0$$

which gives the exponential decrease

$$\int_0^1 (\lambda_\varepsilon - S_{\bar{u}}(t)u_0)^+ e^{-x} dx \leq \left(\int_0^1 (\lambda_\varepsilon - u_0)^+ e^{-x} dx \right) e^{-\gamma t}$$

and the inclusion $\omega(X_0) \subset \{w \in X_0, \lambda_\varepsilon \leq w \text{ a.e.}\}$. Since $\omega(X_0)$ is closed, and since $\lambda \rightarrow \lambda_\varepsilon$ (uniformly on $[0, 1]$) when $\varepsilon \rightarrow 0$, we get (20) at the limit $\varepsilon > 0$. If $\bar{q} = q_{\text{lim}} = Q$, then $\mu = 1$ and the inclusion (21) is obvious. If $q_{\text{lim}} \geq \bar{q} > Q$, then $q = \bar{q}$ and the inclusion $\omega(X_0) \subset \{w \in X_0, w \leq \mu \text{ a.e.}\}$ is obtained by similar methods as (20), on the basis of the estimate

$$\Phi^+(x, w, \mu_\varepsilon(x)) \leq -\tilde{\gamma}(w - \mu_\varepsilon(x))^+,$$

$(x, w) \in [0, 1]^2$, $\tilde{\gamma} > 0$, which holds as μ_ε is an upper branch of solution for the equation $A(x, w(x)) = q_\varepsilon$, $q - \varepsilon < q_\varepsilon < q$.

4.3 Convergence to stationary solutions

We first prove the following proposition.

Proposition 4 *Assume that (2)–(6) hold. The set $\omega(X_0)$ is a subset of the set of stationary solutions to Problem (1).*

PROOF. We use the result and notations of Lemma 8. Set $\hat{\mu} := 1$ if $q < Q$, $\hat{\mu} := \mu$ if $q \geq Q$, so that the conclusion of Lemma 8 reads $\omega(X_0) \subset X_1$, $X_1 := \{w \in X_0, \lambda \leq w \leq \hat{\mu} \text{ a.e.}\}$. Let $w \in \omega(X_0)$ and let \hat{w} be a stationary sub-solution of Problem (1) with boundary datum \bar{u} . By Proposition 1, the function $u \mapsto \|(\hat{w} - u)^+\|_{L^1}$ is non-increasing along the trajectories given by $S_{\bar{u}}$. Therefore, by LaSalle Principle, $t \mapsto \|(\hat{w} - S_{\bar{u}}(t)w)^+\|_{L^1}$ is constant.

Similarly, if \hat{w} is a stationary super-solution of Problem (1) with boundary datum \bar{u} , we see that $t \mapsto \|(\hat{w} - S_{\bar{u}}(t)w)^-\|_{L^1}$ is constant.

Now, we use Corollary 1 to show that $S_{\bar{u}}(t)w = w$. If $q < Q$, for example, the functions \hat{w}_z are stationary super-solutions and we have

$$\|(\hat{w}_z - S_{\bar{u}}(t)w)^-\|_{L^1} = \|(\hat{w}_z - w)^-\|_{L^1}, \quad z \in [0, 1], t > 0.$$

Besides $w \in X_1$, therefore

$$\|(\hat{w} - w)^-\|_{L^1} = \int_z^1 (1 - w) dx.$$

By Proposition 1, the set X_1 is invariant under $S_{\bar{u}}$ so that we also have

$$\|(\hat{w} - S_{\bar{u}}(t)w)^-\|_{L^1} = \int_z^1 (1 - S_{\bar{u}}(t)w) dx.$$

We deduce

$$\int_z^1 w = \int_z^1 S_{\bar{u}}(t)w,$$

for all $z \in [0, 1], t > 0$, i.e. $S_{\bar{u}}(t)w = w$.

Similarly, for all the possible values of q and \bar{u} , we use the fact that Corollary 1 provides enough sub- (or super-)solutions to deduce $S_{\bar{u}}(t)w = w$ from the fact that $t \mapsto \|(\hat{w} - S_{\bar{u}}(t)w)^\pm\|_{L^1}$ is constant: we therefore conclude that any $w \in \omega(X_0)$ is a stationary solution to (1) with boundary datum \bar{u} .

Theorem 4 *Assume that (2)–(6) hold. For every $u_0 \in X_0$, the trajectory $S_{\bar{u}}(t)u_0$ converges to a stationary state. Stationary states are described in Theorem 3.*

Remark 9 If $\bar{q} := A(0, \bar{u})$ is different from q_{lim} , then there is only one stationary solution (see Theorem 3) and Theorem 4 gives the asymptotic behaviour of $S_{\bar{u}}(t)u_0$: for large time, it is close to the unique stationary solution. In case $\bar{q} = q_{\text{lim}}$, the asymptotic behaviour of the trajectories depends on their starting point. It is however possible to precise the limit in some situations, see Proposition 5.

PROOF. Let $u_0 \in X_0$. By Proposition 3, there exists $w \in \omega(X_0)$. By Proposition 4, we know that w is a stationary state. Together with the fact that $S_{\bar{u}}$ is non-expansive (Equation (9)), this gives $S_{\bar{u}}(t)u_0 \rightarrow w$ in X_0 . Indeed, given $\varepsilon > 0$, there exists a time t_ε at which $\|S_{\bar{u}}(t_\varepsilon)u_0 - w\|_{L^1} < \varepsilon$. Since $S_{\bar{u}}$ is L^1 non-expansive and w a fixed point of $S_{\bar{u}}$, this remains true for every time $t \geq t_\varepsilon$. Therefore $S_{\bar{u}}(t)u_0 \rightarrow w$ in X_0 .

The following proposition completes the result stated in Theorem 4 in the case $\bar{q} = q_{\text{lim}}$ (see Remark 9).

Proposition 5 Assume that (2)–(6) hold. Furthermore, we suppose that

$$\bar{q} := A(0, \bar{u}) = q_{\text{lim}}.$$

Then, we have the following cases:

- if $u_0 \leq \lambda$, then $S_{\bar{u}}(t)u_0$ converges to λ ,
- if $u_0 \geq \mu$, then $S_{\bar{u}}(t)u_0$ converges to μ ,
- if $\lambda \leq u_0 \leq \mu$ and $\bar{u} = \lambda(0)$, then $S_{\bar{u}}(t)u_0$ converges to $w_{z,1}$ where z is the unique element of $[0, 1/2]$ such that

$$\int_0^{1/2} (u_0 - \lambda) dx = \int_0^{1/2} (w_{z,1} - \lambda) dx,$$

- if $\lambda \leq u_0 \leq \mu$ and $\bar{u} = \mu(0)$, then $S_{\bar{u}}(t)u_0$ converges to $w_{0,z'}$ where z' is the unique element of $[1/2, 1]$ such that

$$\int_{1/2}^1 (\mu - u_0) dx = \int_{1/2}^1 (\mu - w_{0,z'}) dx.$$

Notice that if u_0 is not ordered with respect to λ and/or μ , we cannot specify the asymptotic stationary state.

PROOF. In the first two cases, the convergence follows from Lemma 8 and the fact that $\{w \leq \lambda\}$ and $\{w \geq \mu\}$ are invariant under $S_{\bar{u}}$. In case $\lambda \leq u_0 \leq \mu$

and $\bar{u} = \lambda(0)$, the conclusion is a consequence of the fact that

$$\int_0^{1/2} S_{\bar{u}}(t)u_0 - \lambda$$

is constant. Similarly, in case $\lambda \leq u_0 \leq \mu$ and $\bar{u} = \mu(0)$,

$$\int_{1/2}^1 \mu - S_{\bar{u}}(t)u_0$$

is constant. Let us prove this result, for example in the case $\bar{u} = \lambda(0)$: considerations similar as those stated in the proof of Corollary 1 show that the functions

$$w_{1/2,1/2} = \lambda \mathbf{1}_{[0,1/2]} + \mu \mathbf{1}_{[1/2,1]}, \quad w_{0,1} = \lambda \mathbf{1}_{[1/2,1]} + \mu \mathbf{1}_{[0,1/2]}$$

are, respectively, super- and sub-solution of the problem (1) with boundary datum $\lambda(0)$ (they are entropy solution inside $(0, 1)$, and satisfy (18)–(19) with $\bar{u} = \lambda(0)$ and, respectively, entropies $(u - k)^-$ and entropies $(u - k)^+$). From Proposition 1, we deduce that the functions

$$t \mapsto \int_0^1 (S_{\bar{u}}(t)u_0 - w_{1/2,1/2})^+, \quad t \mapsto \int_0^1 (w_{0,1} - S_{\bar{u}}(t)u_0)^+$$

are non-increasing. Since $\{\lambda \leq w \leq \mu\}$ is invariant by $S_{\bar{u}}$, we have

$$\int_0^{1/2} S_{\bar{u}}(t)u_0 - \lambda = \int_0^1 (S_{\bar{u}}(t)u_0 - w_{1/2,1/2})^+ = \int_0^{1/2} (\mu - \lambda) - \int_0^1 (w_{0,1} - S_{\bar{u}}(t)u_0)^+.$$

Therefore

$$t \mapsto \int_0^{1/2} S_{\bar{u}}(t)u_0 - \lambda$$

is at the same time non-increasing and non-decreasing, i.e. is constant.

5 Numerical results

In this section, we focus on the numerical approximation of Problem (1) with flux A given by the formula

$$A(x, u) = Qu + H(x)u(1 - u).$$

5.1 Finite volume method for scalar conservation laws

We present the numerical method that has been used for the simulation of the model case. Let us consider a uniform mesh in space (with $N + 1$ elements).

We classically denote by $h = 1/N$ the mesh size, k the time step, and $\mathcal{T} = \{K_i\}_{1 \leq i \leq N}$ the family of the N control volumes. Let us define the following relationship:

$$u_i^{n+1} = u_i^n - \frac{k}{h} \left(F_{i+1/2}(u_{i+1}^n, u_i^n) - F_{i-1/2}(u_i^n, u_{i-1}^n) \right), \quad 1 \leq i \leq N-1, \quad (24)$$

under the boundary constraints $u_0^n = \bar{u}_0$ and $u_N^n = \bar{u}_1$, for all $n \in \mathbb{N}$ and under the CFL condition $\mathcal{L}k/h < 1$, \mathcal{L} being the Lipschitz constant of A w.r.t. u . The numerical solution is then defined by:

$$\forall K_i \in \mathcal{T}, \forall n \in \mathbb{N}, u_{\mathcal{T},k}(t, x) = u_i^n, \quad \forall (t, x) \in [n\Delta t, (n+1)\Delta t] \times K_i. \quad (25)$$

The choice of the scheme now relies on the numerical fluxes $F_{i\pm 1/2}$, which should satisfy properties of regularity, consistency, conservativity and monotonicity. Then, the numerical solution converges to the entropy solution of (1) [Vov02]. For the numerical simulations, the following scheme has been used:

Definition 4 (ENO scheme) Let $\tilde{u}(x) \in (0, 1]$ be the maximum of $u \mapsto A(x, u)$ on $[0, 1]$:

$$\tilde{u}(x) = \min \left(1, \frac{1}{2} \left(1 + \frac{Q}{H(x)} \right) \right)$$

and let us define the partial fluxes:

$$A^-(x, u) = \begin{cases} A(x, u) - A(x, \tilde{u}(x)), & \text{if } u \geq \tilde{u}(x), \\ 0, & \text{else.} \end{cases}$$

$$A^+(x, u) = \begin{cases} A(x, \tilde{u}(x)), & \text{if } u \geq \tilde{u}(x), \\ A(x, u), & \text{else.} \end{cases}$$

The ENO flux is defined by the following formula:

$$F_{i+1/2}(u, v) = A^-(x_{i+1}, u) + A^+(x_i, v). \quad (26)$$

5.2 Numerical tests

In the whole section, simulations are performed with the ENO scheme. We also apply the boundary condition $\bar{u}_0 = \bar{u}_1 = \bar{u} = 0.35$ and we use a normalized gap which can be expressed as:

$$H(x) = (H_{\max} - H_{\min})(2x - 1)^2 + H_{\min}.$$

As it was pointed out in Theorem 3, different cases can be obtained according to the chosen values of Q , H_{\min} and H_{\max} :

- case I. : $\bar{q} < q_{\text{lim}}$,
- case II. : $Q < \bar{q} = q_{\text{lim}}$ (II(a).) or $Q = \bar{q} = q_{\text{lim}}$ (II(b).),
- case III. : $\bar{q} > q_{\text{lim}}$.

In TABLE 1, we present some data corresponding to each case, in order to illustrate the behaviour of the weak entropy solution.

	I.	II(a).	II(b).	III.
Q	0.3500	0.2935	0.3500	0.2500
H_{\min}	0.7000	0.5871	0.1750	0.1250
H_{\max}	1.0000	1.0000	1.0000	1.0000
\bar{q}	0.3500	0.3302	0.3500	0.3150
q_{lim}	0.3937	0.3302	0.3500	0.2500

Table 1
Numerical data

In order to illustrate the structural differences of the flux, FIG.1-2 provides the graphs of $u \mapsto A(x, u)$, at different fixed x , for a given set of data (H_{\min}, H_{\max}, Q) . Actually, FIG.1 (resp. FIG.2) corresponds to case I. (resp. case III.).

Now, let us discuss the numerical results:

- In case I., we observe that the weak entropy solution converges to the stationary one, as described in Theorem 3. Moreover, the stationary solution is continuous (therefore, numerical illustrations are omitted).
- In case II(a)., the stationary solution depends on the initial one. Computations illustrate the theoretical results described in Theorem 3. In particular, according to the choice of the initial solution, a discontinuous stationary solution may appear:
 - FIG.3-4 - The initial condition is defined by $u_0 = u_0^{(1)} \equiv \bar{u}$. Then, we observe that $S(t)(u_0^{(1)}, \bar{u})$ obviously converges to λ , since $u_0^{(1)} \leq \lambda$ and $\lambda(0) = \bar{u}$.
 - FIG.5-6 - The initial condition is defined by $u_0 = u_0^{(2)}$ with

$$u_0^{(2)}(x) = \left(1 - \frac{1}{10} \sin(2\pi x)\right) \lambda \left(\frac{1}{2}\right), \quad \text{i.e.} \quad \lambda \leq u_0 \leq \mu.$$

As $\lambda(0) = \bar{u}$, $S(t)(u_0^{(2)}, \bar{u})$ converges to some $w_{z,1} = \lambda \mathbf{1}_{[0,z) \cup [1/2,1]} + \mu \mathbf{1}_{(z,1/2]}$

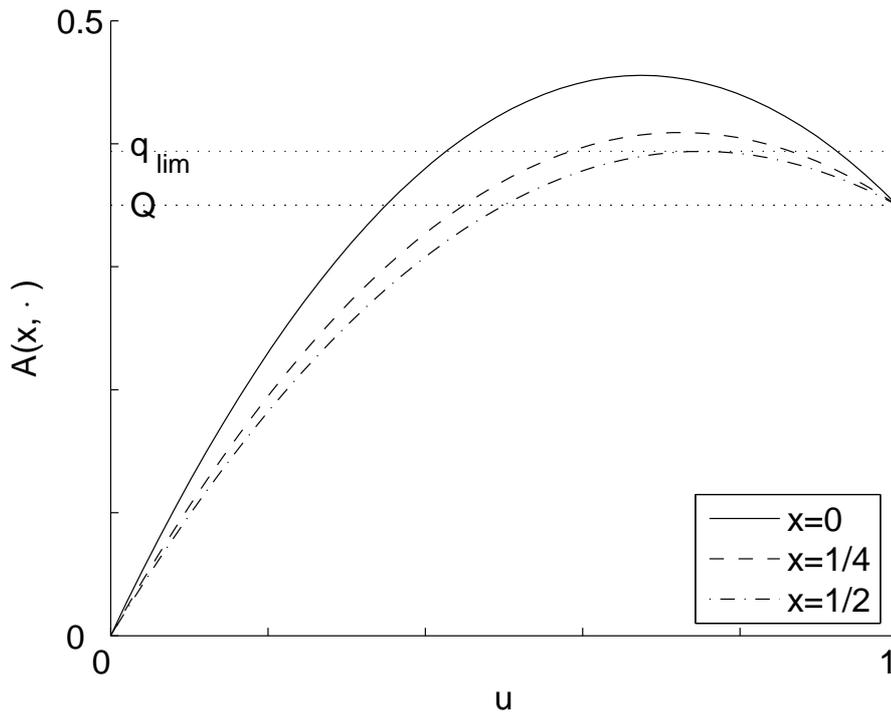


Fig. 1. Graph of $u \mapsto A(x, u)$ in case $Q = q_{lim}$.

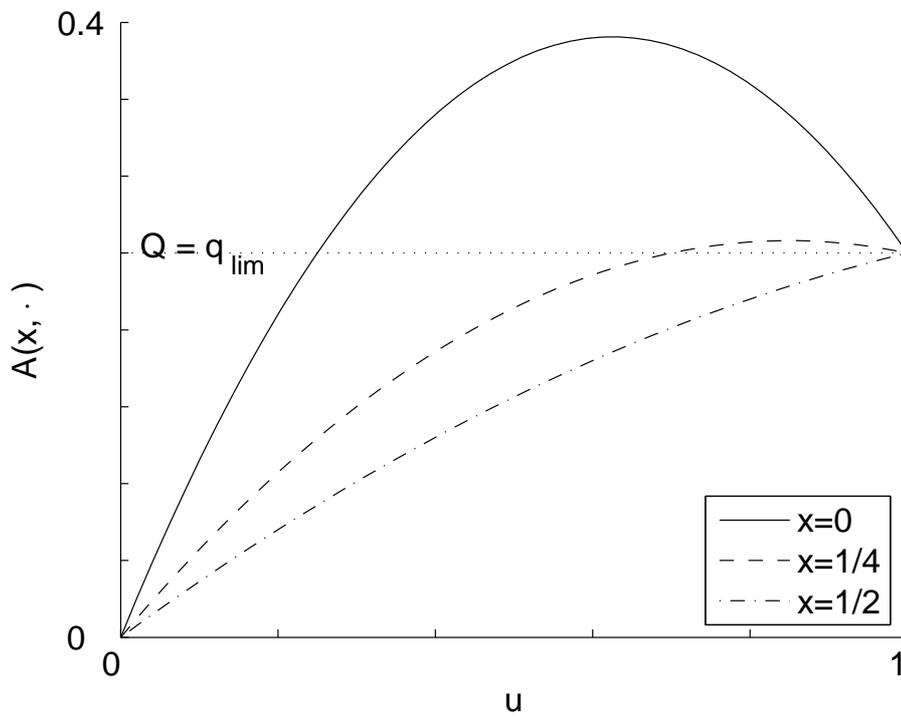


Fig. 2. Graph of $u \mapsto A(x, u)$ in case $Q < q_{lim}$.

with $0 \leq z \leq 1/2$. The position of the stationary shock satisfies:

$$\int_0^{1/2} w_{z,1} = \int_0^{1/2} u_0^{(2)}.$$

i.e. we obtain the conservativity of the mass on $[0, 1/2]$ (see Proposition 5).

- In case II(b)., FIG.7-8 is a simulation in the critical case. The initial condition is defined by $u_0 = u_0^{(3)}$ with

$$u_0^{(3)}(x) = \begin{cases} \lambda(x) + 3x(\mu(x) - \lambda(x))(1 + \sin(6\pi x)), & \text{if } 0 \leq x < \frac{1}{3}, \\ \mu(x), & \text{if } \frac{1}{3} \leq x < \frac{9}{10}, \\ \lambda(x), & \text{if } \frac{9}{10} \leq x \leq 1. \end{cases}$$

On $[0, 1/2]$, the initial solution lies between λ and μ . As mentioned, the mass is conserved on $[0, 1/2]$:

$$\int_0^{1/2} w_{z,1} = \int_0^{1/2} u_0^{(4)}.$$

On $[1/2, 1]$, the initial solution is a (non-entropy) stationary solution, the discontinuity being decreasing (and, therefore, non-admissible). As a consequence, this profile is not preserved as t tends to $+\infty$.

- In case III., FIG.9-10 evidence the behaviour of the entropy solution, which converges to $w_{0,1}$. Indeed, $\lambda(0) \leq \bar{u} \leq \mu(0)$, starting from the initial solution $u_0 \equiv \bar{u}$, we may observe on $(0, 1/2)$ the formation of an unstationary shock which goes out from the domain: the discontinuity has been stabilized on the boundary $x = 0$.

Finally, by TABLE 2, we illustrate Proposition 5. Notice that TABLE 2 includes information on an additional numerical simulation (for which figures have been omitted) falling into the scope of case II(a), with the initial solution $u_0 = u_0^{(4)}$ defined by

$$u_0^{(4)}(x) = \begin{cases} \frac{1}{2}(\lambda(x) + \mu(x)), & \text{if } 0 \leq x < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

More precisely, denoting

$$m_i = \int_0^{1/2} u_0 - \lambda, \quad m_f = \int_0^{1/2} w_{z,1} - \lambda,$$

we obtain the following results:

Among the chosen set of numerical simulations, we observe that

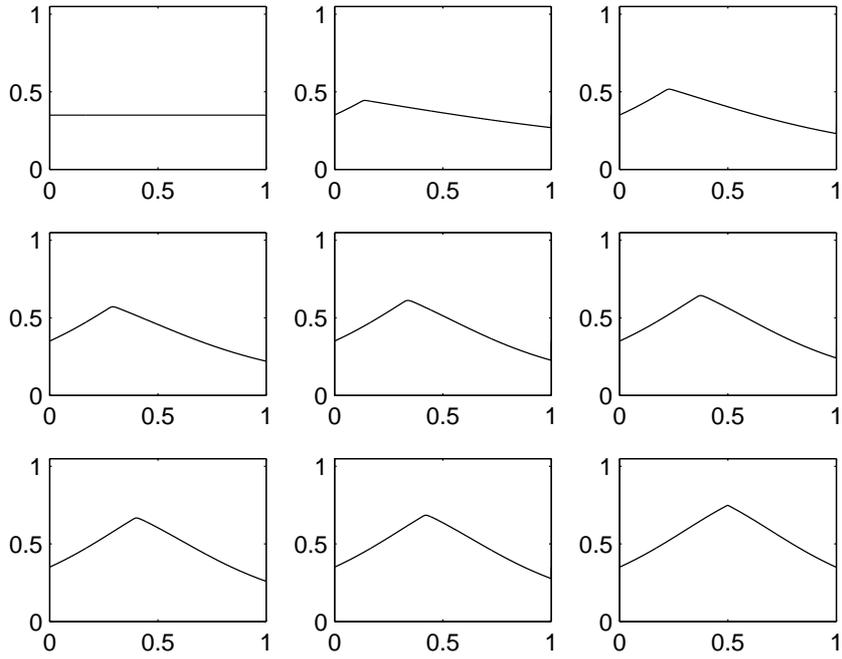


Fig. 3. II(a). Entropy solution at different time steps (initial condition $u_0^{(1)}$)

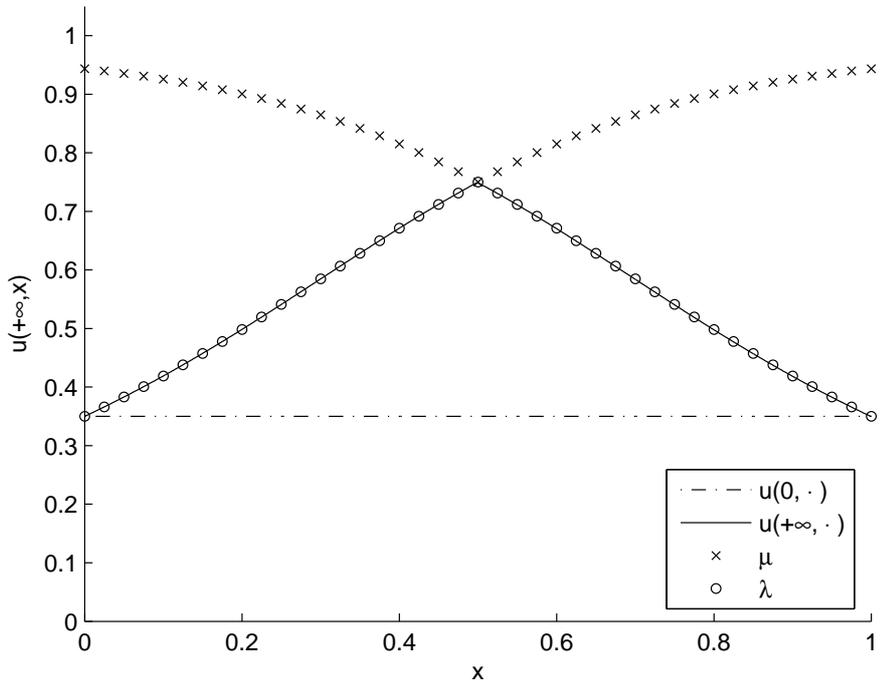


Fig. 4. II(a). Stationary entropy solution (initial condition $u_0^{(1)}$)

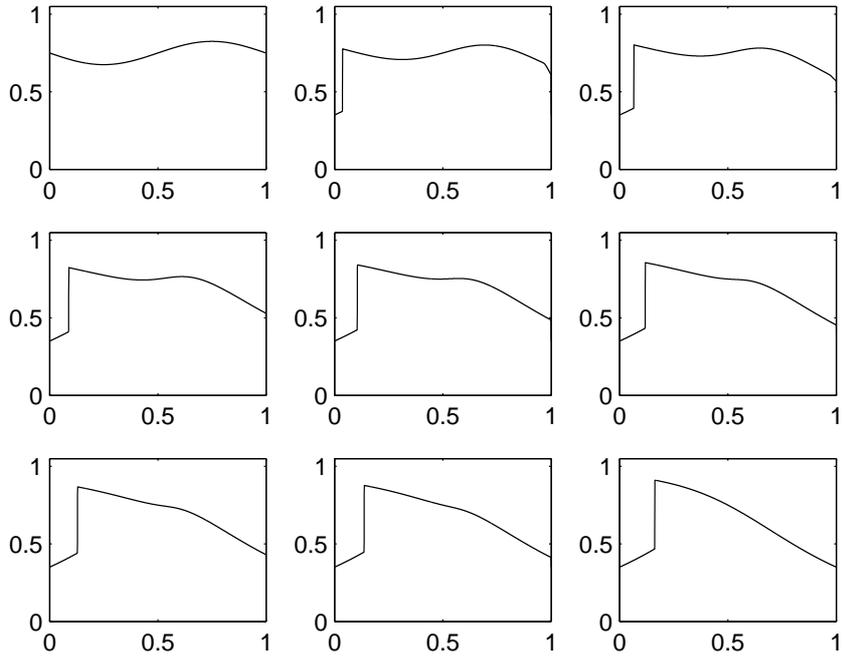


Fig. 5. II(a). Entropy solution at different time steps (initial condition $u_0^{(2)}$)

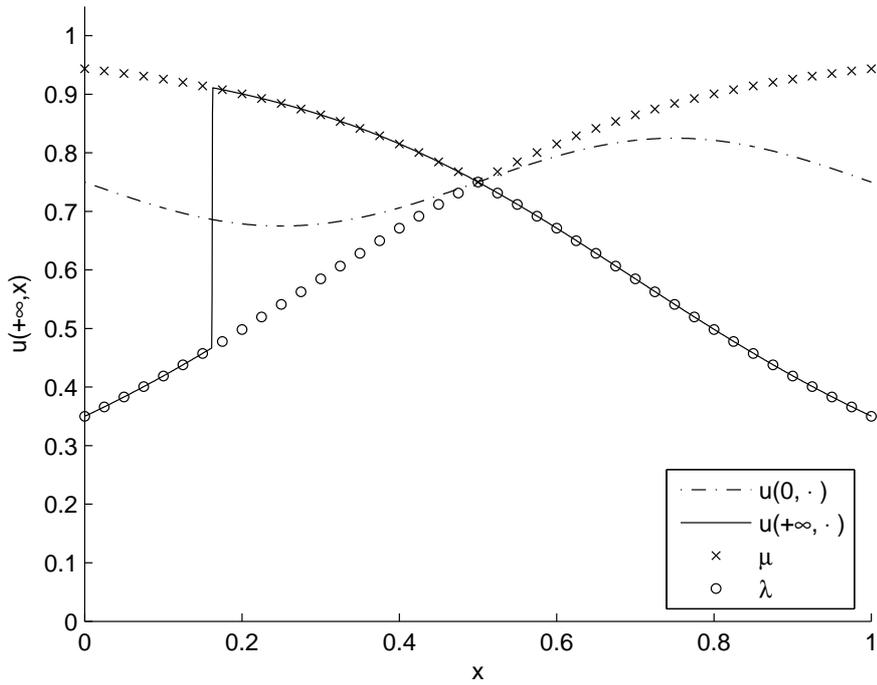


Fig. 6. II(a). Stationary entropy solution (initial condition $u_0^{(3)}$)

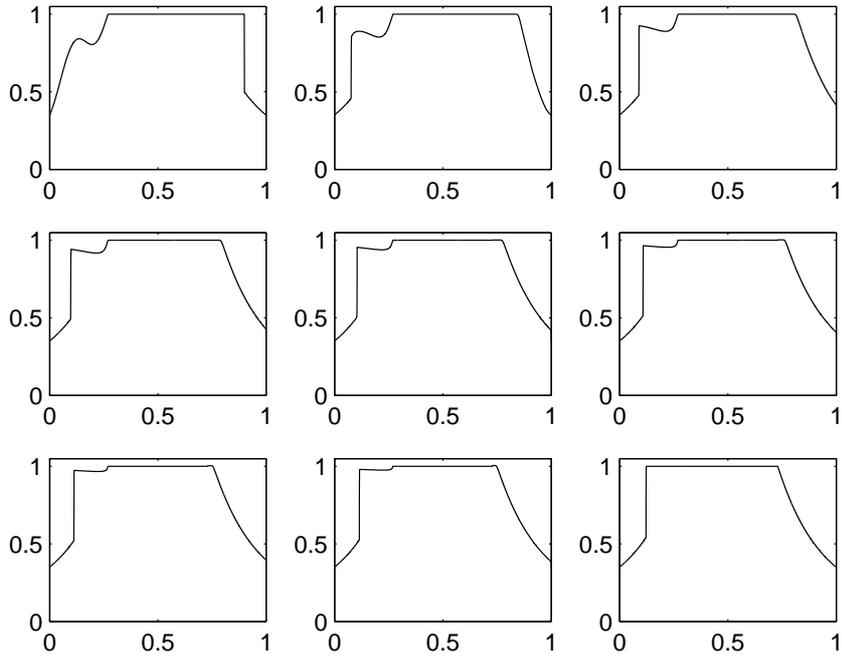


Fig. 7. II(b). Entropy solution at different time steps (initial condition $u_0^{(3)}$)

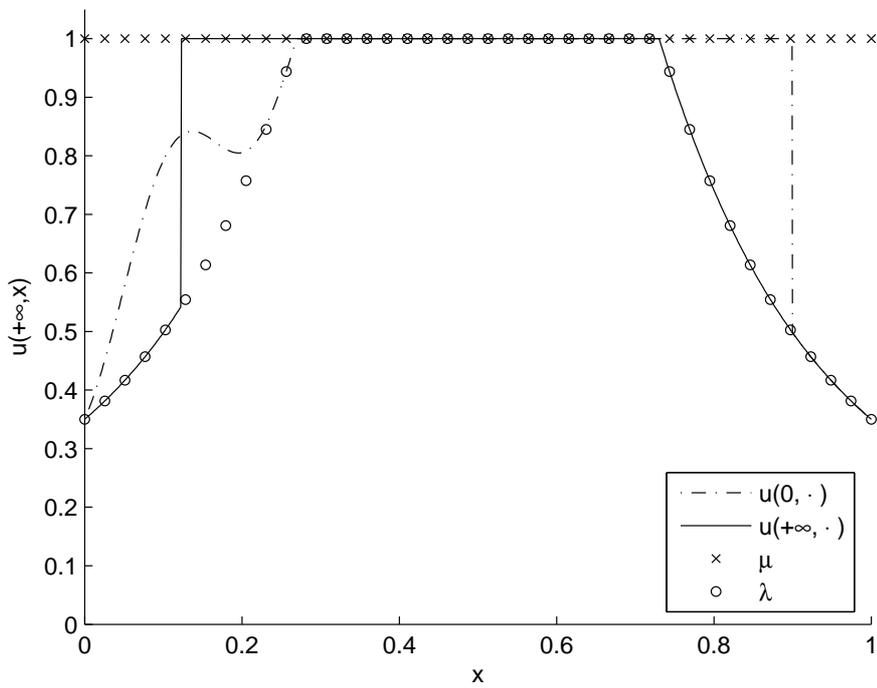


Fig. 8. II(b). Stationary entropy solution (initial condition $u_0^{(4)}$)

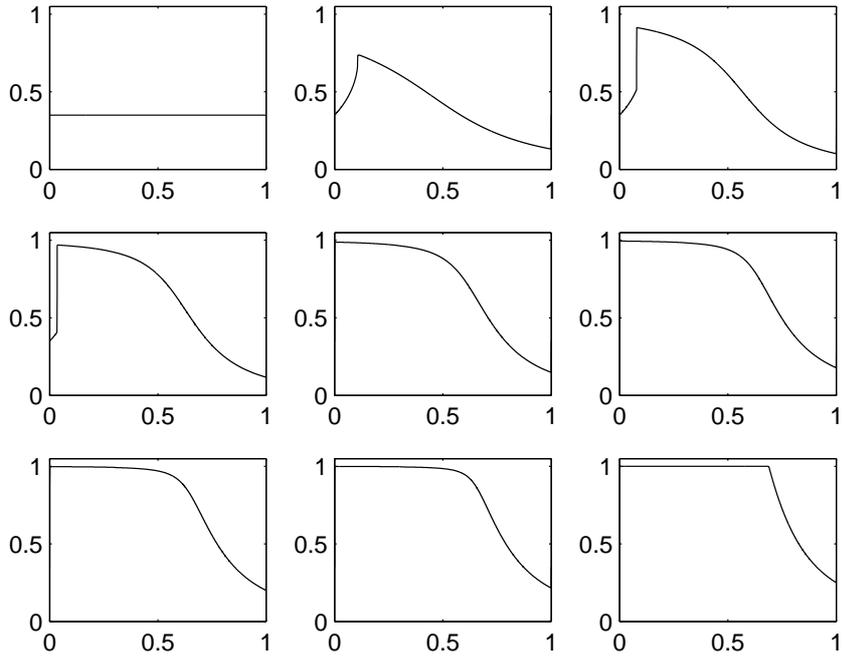


Fig. 9. III. Entropy solution at different time steps (initial condition $u_0^{(1)}$)

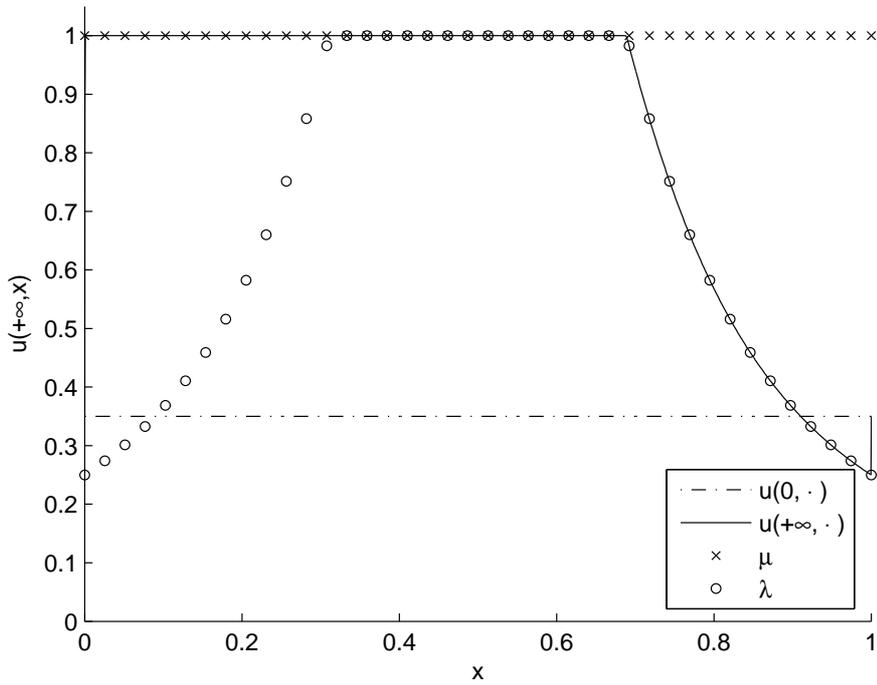


Fig. 10. III. Stationary entropy solution (initial condition $u_0^{(1)}$)

	II(a). $u_0^{(1)}$	II(a). $u_0^{(2)}$	II(b). $u_0^{(3)}$	II(a). $u_0^{(4)}$
m_i	$-9.0791 \cdot 10^{-2}$	$8.5147 \cdot 10^{-2}$	$3.4918 \cdot 10^{-2}$	$8.8099 \cdot 10^{-2}$
m_f	$6.3105 \cdot 10^{-3}$	$8.5060 \cdot 10^{-2}$	$3.4785 \cdot 10^{-2}$	$1.5205 \cdot 10^{-1}$
$(m_f - m_i)/m_i$	-1.0695	$-1.0259 \cdot 10^{-3}$	$-3.8318 \cdot 10^{-3}$	$7.2594 \cdot 10^{-1}$

Table 2

Position of the stationary shock in case II.

- in the cases II(a). with $u_0 = u_0^{(2)}$ and II(b). with $u_0 = u_0^{(3)}$, the integral equality $m_i = m_f$ is numerically satisfied with a relative error which is less than 0.5%. Indeed, the initial solution satisfies the assumption $\lambda \leq u_0 \leq \mu$,
- in the cases II(a). with $u_0 = u_0^{(1)}$ or $u_0 = u_0^{(2)}$, the integral equality $m_i = m_f$ does not hold, as the condition $\lambda \leq u_0 \leq \mu$ is not satisfied.

A Proof of Lemma 1

PROOF. Let T be positive. For $\sigma > 0$, $u^\sigma : (x, t) \mapsto u(x, t + \sigma)$ satisfies (7) for all non-negative $\varphi \in \mathcal{C}_c^\infty([0, 1] \times (0, +\infty))$ (use the test function $(x, t) \mapsto \varphi(x, t - \sigma)$ in (7)). By (8) and Remark 5, we have

$$\int_0^1 \int_0^\infty |u - u^\sigma| \varphi_t + \Phi(x, u, u^\sigma) \varphi_x dx dt \geq 0$$

for all non-negative $\varphi \in \mathcal{C}_c^\infty([0, 1] \times (0, +\infty))$. For $\varepsilon > 0, t_* \in (0, T)$, set

$$\varphi(x, t) = t\chi_{[0, t_*]}(t) + \frac{t_* + \varepsilon - t}{\varepsilon} \chi_{[t_*, t_* + \varepsilon]}(t)$$

in this last inequality to get

$$\frac{1}{\varepsilon} \int_{t_*}^{t_* + \varepsilon} \|u(t) - u^\sigma(t)\|_{L^1(0,1)} dt \leq \int_0^{t_*} \|u(t) - u^\sigma(t)\|_{L^1(0,1)} dt. \quad (\text{A.1})$$

Denote by v^ε the function

$$(x, t) \mapsto \frac{1}{\varepsilon} \int_t^{t+\varepsilon} u(x, s) ds.$$

The function v^ε is in $\mathcal{C}([0, T]; L^1(0, 1))$ and from (A.1), Chasles equality and the triangular inequality, we deduce

$$\|v^\varepsilon(t_*) - v^\varepsilon(t_* + \sigma)\|_{L^1(0,1)} \leq \eta(\sigma) := \int_0^{T+\sigma} \|u(t) - u^\sigma(t)\|_{L^1(0,1)} dt.$$

By continuity, this remains true for $t_* = 0$ or T . By continuity of the translations in $L^1([0, 1] \times (0, T))$, we have

$$\lim_{\sigma \rightarrow 0} \eta(\sigma) = 0,$$

and since η is independant of ε , the family v^ε is equicontinuous in the space $\mathcal{C}([0, T]; L^1(0, 1))$. Besides, since $u \in L^\infty((0, 1) \times (0, +\infty))$, it is pointwise bounded, independently of ε . By Ascoli's theorem, the sequence $v^{1/n}$ admits a subsequence which converges uniformly on $[0, T]$ to a function $v \in \mathcal{C}([0, T]; L^1(0, 1))$. Futhermore, we have $u = v$ at every Lebesgue point of the function $u : (0, T) \rightarrow L^1(0, 1)$, hence almost everywhere: this proves the first part of the lemma, u is equal almost everywhere to a function $v \in \mathcal{C}(0, T; L^1(0, 1))$. To end the proof, we just have to prove that $v(0) = u_0$, i.e. that

$$v^\varepsilon(0) \rightarrow u_0$$

in L^1 . This is a consequence of the fact that u is an entropy hence a weak solution of Problem (1): for all $\varphi \in \mathcal{C}_c^\infty((0, 1) \times [0, +\infty))$,

$$\int_0^1 \int_0^\infty u \varphi_t + A(x, u) \varphi_x dx dt + \int_0^1 u_0 \varphi(x, 0) dx = 0.$$

Taking

$$\varphi(x, t) = \frac{\varepsilon - t}{\varepsilon} \chi_{[0, \varepsilon]}(t) \psi(x),$$

where $\psi \in \mathcal{C}_c^\infty(0, 1)$, we have, by the dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 v^\varepsilon(x, 0) \psi(x) dx = \int_0^1 u_0(x) \psi(x) dx,$$

that is to say $v^\varepsilon(\cdot, 0)$ converges to u_0 in $\mathcal{D}'(0, 1)$. On the other hand, we know that $v^\varepsilon(\cdot, 0)$ converges to $v(0)$ in $L^1(0, 1)$. By injection of $L^1(0, 1)$ in $\mathcal{D}'(0, 1)$, we have $v(0) = u_0$.

Remark 10 Notice that we use the entropy conditions on $[0, 1] \times (0, +\infty)$ only, i.e. with test functions which vanish at $t = 0$. The initial condition is recovered only by mean of the weak formulation on the one hand, and by the bound on u in L^∞ in the second hand. The precedent proof thus answers the question on the continuity in time of entropy solutions adressed in [EGH00].

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