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NON-EXTENDABILITY OF SEMILATTICE-VALUED MEASURES ON PARTIALLY ORDERED SETS

FRIEDRICH WEHRUNG

ABSTRACT. For a poset P and a distributive $\langle \vee, 0 \rangle$ -semilattice S , a S -valued poset measure on P is a map $\mu: P \times P \rightarrow S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$, and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$. In relation with congruence lattice representation problems, we consider the problem whether such a measure can be extended to a poset measure $\bar{\mu}: \bar{P} \times \bar{P} \rightarrow S$, for a larger poset \bar{P} , such that for all $\mathbf{a}, \mathbf{b} \in S$ and all $x \leq y$ in \bar{P} , $\bar{\mu}(y, x) = \mathbf{a} \vee \mathbf{b}$ implies that there are a positive integer n and a decomposition $x = z_0 \leq z_1 \leq \dots \leq z_n = y$ in \bar{P} such that either $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{a}$ or $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{b}$, for all $i < n$.

In this note we prove that this is not possible as a rule, even in case the poset P we start with is a *chain* and S has size \aleph_1 . The proof uses a “monotone refinement property” that holds in S provided S is either a lattice, or countable, or strongly distributive, but fails for our counterexample. This strongly contrasts with the analogue problem for *distances* on (discrete) sets, which is known to have a positive (and even *functorial*) solution.

1. INTRODUCTION

In the paper [5], the author proved that for any lattice K , any distributive lattice S with zero, and any $\langle \vee, 0 \rangle$ -homomorphism φ from the $\langle \vee, 0 \rangle$ -semilattice $\text{Con}_c K$ of all finitely generated congruences of K to S , there are a lattice L , a lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\alpha: \text{Con}_c L \rightarrow S$ such that $\varphi = \alpha \circ \text{Con}_c f$. In the paper [4], J. Tůma and the author proved that for a $\langle \vee, 0 \rangle$ -semilattice S , this statement characterizes S being a lattice. The proof of this negative result strongly uses the lattice structure of the hypothetical lattice L , see the proof of [4, Corollary 1.3].

In the present paper, we show that for a certain semilattice S of cardinality \aleph_1 , the poset structure alone is sufficient to get a related counterexample. More precisely, for a $\langle \vee, 0 \rangle$ -semilattice S , a S -valued poset measure on a poset P is a map $\mu: P \times P \rightarrow S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ (*triangular inequality*) and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$. We say that μ is a V -measure, if for all $x \leq y$ in P and all $\mathbf{a}, \mathbf{b} \in S$, if $\mu(y, x) \leq \mathbf{a} \vee \mathbf{b}$, then there are a positive integer n and a decomposition $x = z_0 \leq z_1 \leq \dots \leq z_n = y$ in P such that either $\mu(z_{i+1}, z_i) \leq \mathbf{a}$ or $\mu(z_{i+1}, z_i) \leq \mathbf{b}$ for all $i < n$. In particular, if P is a lattice and $S = \text{Con}_c P$, then the map μ defined by $\mu(x, y) = \Theta^+(x, y) = \Theta(y, x \vee y)$ is a $\text{Con}_c P$ -valued V -measure on P .

This yields the following poset analogue of the abovementioned lattice-theoretical problem.

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Problem. *Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice. Does any S -valued poset measure on a given poset extend to some S -valued poset V -measure on a larger poset?*

A version of this problem for so-called *distances* (instead of measures) on discrete sets (instead of posets) is stated in [3]. The answer to this related question turns out to be positive (and easy). More surprisingly, this positive solution can be made *functorial*.

Nevertheless, we prove in the present paper that the problem above has a *negative* solution. Unlike what is done in [4], we do not reach here a characterization of all lattices among distributive $\langle \vee, 0 \rangle$ -semilattices. Our counterexample, denoted by $\mathcal{F}(\omega_1)$ (see Corollary 4.9) is obtained as an application of a certain “free construction” used by M. Ploščica and J. Tůma in [2]. The semilattice D of [4, Section 2], which is the simplest example of a $\langle \vee, 0 \rangle$ -semilattice which is not a lattice, does not satisfy the negative property used here. This is because D is *countable*, while we prove in Proposition 4.10 that no countable distributive $\langle \vee, 0 \rangle$ -semilattice can have the required negative property. On the other hand, in relation to [4, Problem 4], the proof of our counterexample uses very little of the Axiom of Choice (namely, only the Axiom of *countable* choices), while the proof of the negative property of the abovementioned semilattice D established in [4, Corollary 2.4] uses the existence of an embedding from ω_1 into the reals.

2. BASIC CONCEPTS

For posets (i.e., partially ordered sets) P and Q , a map $f: P \rightarrow Q$ is *isotone*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$. In addition, we say that f is *join-preserving*, if for any subset X of P , whenever the join $\bigvee X$ of X exists in P , $\bigvee f[X]$ exists in Q , and $\bigvee f[X] = f(\bigvee X)$. For a subset X of a poset P , we shall put $\downarrow X = \{p \in P \mid \exists x \in X \text{ such that } p \leq x\}$, and then $\downarrow a = \downarrow \{a\}$, for all $a \in P$. We say that X is a *lower subset* of P , if $X = \downarrow X$.

A $\langle \vee, 0 \rangle$ -semilattice S is *distributive*, if $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$ in S implies that there are $\mathbf{x} \leq \mathbf{a}$ and $\mathbf{y} \leq \mathbf{b}$ in S such that $\mathbf{c} = \mathbf{x} \vee \mathbf{y}$. A distributive $\langle \vee, 0 \rangle$ -semilattice S is *strongly distributive*, if every element of S is the join of a finite set of join-irreducible elements of S ; equivalently, S is isomorphic to the semilattice of all finitely generated lower subsets of some poset.

We shall denote by $\text{otp } P$ the order-type of a well-ordered set P . Hence $\text{otp } P$ is an ordinal. We shall also use standard set-theoretical notation and terminology, referring the reader to [1] for further information. In particular, we shall denote by ω_1 the first uncountable ordinal. A subset C of ω_1 is *closed unbounded*, if C is unbounded in ω_1 and the join of any nonempty bounded subset of C belongs to C . It is well-known that the closed unbounded subsets form a countably complete filterbasis on ω_1 , see [1, Lemma 7.4]. Hence containing a closed unbounded set is a notion of “largeness” for subsets of ω_1 .

3. FREE DISTRIBUTIVE EXTENSION OF A $\langle \vee, 0 \rangle$ -SEMILATTICE

There are several non-equivalent definitions of what should be the “free distributive extension” of a given $\langle \vee, 0 \rangle$ -semilattice. The one that we shall use is introduced in [2, Section 2]. Let us first recall the construction.

For a $\langle \vee, 0 \rangle$ -semilattice S , we shall put $\mathcal{C}(S) = \{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in S^3 \mid \mathbf{w} \leq \mathbf{u} \vee \mathbf{v}\}$. A *finite* subset \mathbf{x} of $\mathcal{C}(S)$ is *reduced*, if it satisfies the following conditions:

- (1) \mathbf{x} contains exactly one diagonal triple, that is, a triple of the form $\langle \mathbf{u}, \mathbf{u}, \mathbf{u} \rangle$; we put $\mathbf{u} = \pi(\mathbf{x})$.
- (2) $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x}$ and $\langle \mathbf{v}, \mathbf{u}, \mathbf{w} \rangle \in \mathbf{x}$ implies that $\mathbf{u} = \mathbf{v} = \mathbf{w}$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S$.
- (3) $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x} \setminus \{\langle \pi(\mathbf{x}), \pi(\mathbf{x}), \pi(\mathbf{x}) \rangle\}$ implies that $\mathbf{u}, \mathbf{v}, \mathbf{w} \not\leq \pi(\mathbf{x})$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S$.

We denote by $\mathcal{R}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$, endowed with the partial ordering \leq defined by

$$\mathbf{x} \leq \mathbf{y} \iff \forall \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x} \setminus \mathbf{y}, \text{ either } \mathbf{u} \leq \pi(\mathbf{y}) \text{ or } \mathbf{w} \leq \pi(\mathbf{y}). \quad (3.1)$$

Furthermore, we shall identify \mathbf{x} with the element $\{\langle \mathbf{x}, \mathbf{x}, \mathbf{x} \rangle\}$ of $\mathcal{R}(S)$, for all $\mathbf{x} \in S$. For set-theoretical purists, this can for example be done by replacing $\mathcal{R}(S)$ by the disjoint union of S with the set of non-singletons in $\mathcal{R}(S)$. The disjointness can easily be achieved by a suitable modification of the standard definition of a triple. We shall use the symbol \bowtie to denote the canonical generators of $\mathcal{R}(S)$, so that

$$\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{cases} \mathbf{w}, & \text{if either } \mathbf{u} = \mathbf{v} \text{ or } \mathbf{v} = 0 \text{ or } \mathbf{w} = 0, \\ 0, & \text{if } \mathbf{u} = 0, \\ \{\langle 0, 0, 0 \rangle, \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle\}, & \text{otherwise,} \end{cases}$$

Observe that the canonical map $\pi: \mathcal{R}(S) \rightarrow S$ is *isotone* and that the restriction of π to S is the identity. Furthermore, $\pi(\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})) = 0$, for any non-diagonal $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$. The following is an easy consequence of (3.1).

$$\mathbf{x} \leq \mathbf{y} \iff \mathbf{x} \leq \pi(\mathbf{y}), \quad \text{for all } \langle \mathbf{x}, \mathbf{y} \rangle \in S \times \mathcal{R}(S). \quad (3.2)$$

We recall the standard facts established in [2] about this construction.

Proposition 3.1.

- (1) For any $\langle \vee, 0 \rangle$ -semilattice S , $\mathcal{R}(S)$ is a $\langle \vee, 0 \rangle$ -semilattice, and the inclusion map from S into $\mathcal{R}(S)$ is a $\langle \vee, 0 \rangle$ -embedding.
- (2) For $\langle \vee, 0 \rangle$ -semilattices S and T , every $\langle \vee, 0 \rangle$ -homomorphism $f: S \rightarrow T$ extends to a unique $\langle \vee, 0 \rangle$ -homomorphism $\mathcal{R}(f): \mathcal{R}(S) \rightarrow \mathcal{R}(T)$ such that $\mathcal{R}(f)(\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})) = \bowtie(f(\mathbf{u}), f(\mathbf{v}), f(\mathbf{w}))$, for all $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$.
- (3) The assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is a functor.

The extension $\mathcal{R}(S)$ is defined in such a way that $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \mathbf{u}$ and $\mathbf{w} = \bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) \vee \bowtie(\mathbf{v}, \mathbf{u}, \mathbf{w})$, for all $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$. Hence, putting $\mathcal{R}^0(S) = S$ and $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$ for each n , we obtain that the increasing union $\mathcal{D}(S) = \bigcup (\mathcal{R}^n(S) \mid n < \omega)$ is a distributive $\langle \vee, 0 \rangle$ -semilattice, extending S . Furthermore, putting $\mathcal{D}(f) = \bigcup (\mathcal{D}^n(f) \mid n < \omega)$ for each $\langle \vee, 0 \rangle$ -homomorphism f , we obtain that \mathcal{D} is a functor. The proof of the following lemma is straightforward.

Lemma 3.2. *Let S be a $\langle \vee, 0 \rangle$ -semilattice and let $\langle S_i \mid i \in I \rangle$ be a family of $\langle \vee, 0 \rangle$ -subsemilattices of S . The following statements hold:*

- (1) $\mathcal{R}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{D}(S_i)$.
- (2) If I is a nonempty upward directed poset and $\langle S_i \mid i \in I \rangle$ is isotone, then $\mathcal{R}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{D}(S_i)$.

Definition 3.3. For a $\langle \vee, 0 \rangle$ -semilattice S and an element $\mathbf{x} \in \mathcal{D}(S)$, we define the *rank* of \mathbf{x} , denoted by $\text{rk } \mathbf{x}$, as the least natural number n such that $\mathbf{x} \in \mathcal{R}^n(S)$, and the *complexity* of \mathbf{x} , denoted by $\|\mathbf{x}\|$, by $\|\mathbf{x}\| = 0$ if $\mathbf{x} \in S$, and

$$\|\mathbf{x}\| = \sum (\|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\| + 1 \mid \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{D}(S) \setminus S.$$

4. THE SEMILATTICES $\mathfrak{S}(\Lambda)$ AND $\mathfrak{F}(\Lambda)$

For any *chain* Λ , we shall denote by $\mathfrak{S}(\Lambda)$ the $\langle \vee, 0 \rangle$ -semilattice defined by generators \mathbf{a} , \mathbf{b} , and \mathbf{c}_i , for $i \in \Lambda$, and relations $\mathbf{c}_i \leq \mathbf{a} \vee \mathbf{b}$ and $\mathbf{c}_i \leq \mathbf{c}_j$, for all $i \leq j$ in Λ . Hence the elements of $\mathfrak{S}(\Lambda)$ either belong to $\mathfrak{S}(\emptyset) = \{0, \mathbf{a}, \mathbf{b}, \mathbf{a} \vee \mathbf{b}\}$ or have the form \mathbf{c}_i , $\mathbf{a} \vee \mathbf{c}_i$, or $\mathbf{b} \vee \mathbf{c}_i$, for some $i \in \Lambda$. We shall identify $\mathfrak{S}(X)$ with the $\langle \vee, 0 \rangle$ -subsemilattice of $\mathfrak{S}(\Lambda)$ generated by $\mathfrak{S}(\emptyset) \cup \{\mathbf{c}_i \mid i \in X\}$, for any $X \subseteq \Lambda$.

For chains X and Y , any isotone map $f: X \rightarrow Y$ gives raise to a unique $\langle \vee, 0 \rangle$ -homomorphism $\mathfrak{S}(f): \mathfrak{S}(X) \rightarrow \mathfrak{S}(Y)$ fixing \mathbf{a} and \mathbf{b} and sending \mathbf{c}_i to $\mathbf{c}_{f(i)}$, for all $i \in X$. Of course, the assignment $\Lambda \mapsto \mathfrak{S}(\Lambda)$, $f \mapsto \mathfrak{S}(f)$ is a functor.

We denote by $\mathfrak{F} = \mathfrak{D} \circ \mathfrak{S}$ the composition of the two functors \mathfrak{D} and \mathfrak{S} .

The proof of the following lemma is straightforward.

Lemma 4.1. *Let Λ be a chain and let $\langle X_i \mid i \in I \rangle$ be a family of subsets of Λ . The following statements hold:*

- (1) $\mathfrak{S}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \mathfrak{S}(X_i)$.
- (2) *If I is a nonempty upward directed poset and $\langle X_i \mid i \in I \rangle$ is isotone, then $\mathfrak{S}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \mathfrak{S}(X_i)$.*

As an easy consequence of Lemmas 3.2 and 4.1, we get the following.

Lemma 4.2. *Let Λ be a chain. Then for any $\mathbf{x} \in \mathfrak{F}(\Lambda)$, there exists a least (with respect to the inclusion) subset X of Λ such that $\mathbf{x} \in \mathfrak{F}(X)$; this subset is finite.*

We denote by $\text{supp}(\mathbf{x})$ the subset given by Lemma 4.2, and we call it the *support* of \mathbf{x} .

Notation 4.3. For a chain Λ , well-ordered subsets X and Y of Λ such that $\text{otp } X \leq \text{otp } Y$, and $\mathbf{x} \in \mathfrak{F}(X)$, we set $\mathbf{x}[Y/X] = \mathfrak{F}(e_{X,Y})(\mathbf{x})$, where $e_{X,Y}$ denotes the unique embedding from X into Y whose range is a lower subset of Y .

Hence $\mathbf{x}[Y/X]$ belongs to $\mathfrak{F}(Y)$, for all $\mathbf{x} \in \mathfrak{F}(X)$.

Lemma 4.4. *Let Λ be a chain and let X, Y be well-ordered subsets of Λ such that $\text{otp } X \leq \text{otp } Y$ and $X \cap Y$ is a lower subset of both X and Y . Then $\mathbf{x}[Y/X] = \mathbf{x}$, for all $\mathbf{x} \in \mathfrak{F}(X \cap Y)$.*

Proof. As the set $Z = X \cap Y$ is a lower subset of both X and Y , the homomorphism $\mathfrak{F}(e_{Z,X})$ (resp., $\mathfrak{F}(e_{Z,Y})$) is the inclusion map from $\mathfrak{F}(Z)$ into $\mathfrak{F}(X)$ (resp., $\mathfrak{F}(Y)$). In particular, $\mathbf{x} = \mathfrak{F}(e_{Z,X})(\mathbf{x}) = \mathfrak{F}(e_{Z,Y})(\mathbf{x})$. Therefore,

$$\mathbf{x}[Y/X] = \mathfrak{F}(e_{X,Y})(\mathbf{x}) = \mathfrak{F}(e_{X,Y}) \circ \mathfrak{F}(e_{Z,X})(\mathbf{x}) = \mathfrak{F}(e_{Z,Y})(\mathbf{x}) = \mathbf{x}. \quad \square$$

We are now reaching a crucial lemma.

Lemma 4.5 (Interpolation Lemma). *Let Λ be a chain, let X, Y be finite subsets of Λ , and let $(\mathbf{x}, \mathbf{y}) \in \mathfrak{F}(X) \times \mathfrak{F}(Y)$. If $\mathbf{x} \leq \mathbf{y}$, then either there exists $\mathbf{z} \in \mathfrak{F}(X \cap Y)$ such that $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$ or $(Y \not\subseteq X \text{ and } \mathbf{c}_{\min(Y \setminus X)} \leq \mathbf{y})$.*

Proof. We shall denote by π_k^l the canonical map from $\mathfrak{R}^l \mathfrak{S}(\Lambda)$ onto $\mathfrak{R}^k \mathfrak{S}(\Lambda)$, for all natural numbers $k \leq l$. Put $m = \text{rk } \mathbf{x}$ and $n = \text{rk } \mathbf{y}$. Observe that $\text{supp}(\mathbf{x}) \subseteq X$ and $\text{supp}(\mathbf{y}) \subseteq Y$. We argue by induction on $\|\mathbf{x}\| + \|\mathbf{y}\|$. If either $\text{supp}(\mathbf{x}) \subseteq Y$ or $\text{supp}(\mathbf{y}) \subseteq X$ then either $\mathbf{z} = \mathbf{x}$ or $\mathbf{z} = \mathbf{y}$ belongs to $\mathfrak{F}(X \cap Y)$ and satisfies the inequalities $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$, so we are done. So suppose that $\text{supp}(\mathbf{x}) \not\subseteq Y$ and $\text{supp}(\mathbf{y}) \not\subseteq X$. In particular, $X \not\subseteq Y$ and $Y \not\subseteq X$, and both $\text{supp}(\mathbf{x})$ and $\text{supp}(\mathbf{y})$ are nonempty. We put $\xi = \min(Y \setminus X)$.

Suppose that $m = n = 0$, that is, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(\Lambda)$. Pick $i \in \text{supp}(\mathbf{x})$. As $\mathbf{c}_i \leq \mathbf{x} \leq \mathbf{y}$, we obtain that either $\mathbf{y} = \mathbf{a} \vee \mathbf{b}$ (a contradiction, as then $\text{supp}(\mathbf{y}) = \emptyset$) or $\mathbf{y} \in \{\mathbf{c}_j, \mathbf{a} \vee \mathbf{c}_j, \mathbf{b} \vee \mathbf{c}_j\}$ for some $j \geq i$. If $i = j$, then $\text{supp}(\mathbf{x}) = \text{supp}(\mathbf{y}) = \{i\}$, a contradiction. If $i < j$, then $\xi = j$ and so $\mathbf{c}_\xi \leq \mathbf{y}$.

Suppose now that $m < n$. Then $\mathbf{x} \leq \mathbf{y}$ means that $\mathbf{x} \leq \pi_m^n(\mathbf{y})$ (use (3.2)). As $\pi_m^n(\mathbf{y})$ has support contained in Y and rank at most m , it follows from the induction hypothesis that either $\mathbf{c}_\xi \leq \pi_m^n(\mathbf{y})$ (thus, *a fortiori*, $\mathbf{c}_\xi \leq \mathbf{y}$) or there exists $\mathbf{z} \in \mathcal{F}(X \cap Y)$ such that $\mathbf{x} \leq \mathbf{z} \leq \pi_m^n(\mathbf{y})$ (thus, *a fortiori*, $\mathbf{z} \leq \mathbf{y}$).

So suppose from now on that $m > 0$ (i.e., $\mathbf{x} \notin \mathcal{S}(\Lambda)$) and $m \geq n$. If $\mathbf{x} = \bigvee_{i < k} \mathbf{x}_i$ where $k \geq 2$ and each \mathbf{x}_i has support contained in X and complexity less than $\|\mathbf{x}\|$, then we apply the induction hypothesis to each inequality $\mathbf{x}_i \leq \mathbf{y}$, for $i < k$. If $\mathbf{c}_\xi \not\leq \mathbf{y}$, then for all $i < k$, there exists $\mathbf{z}_i \in \mathcal{F}(X \cap Y)$ such that $\mathbf{x}_i \leq \mathbf{z}_i \leq \mathbf{y}$. Hence $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$, where $\mathbf{z} = \bigvee_{i < k} \mathbf{z}_i$ belongs to $\mathcal{F}(X \cap Y)$. This reduces the problem to the case where $\mathbf{x} = \bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})$, where $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ is a non-diagonal triple of elements of $\mathcal{R}^{m-1}\mathcal{S}(X)$ of complexity less than $\|\mathbf{x}\|$.

If $m > n$, then, as $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{x} \leq \mathbf{y}$ with $\text{supp}(\mathbf{x}) \not\subseteq Y$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{R}^{m-1}\mathcal{S}(X)$, and $\mathbf{y} \in \mathcal{R}^{m-1}\mathcal{S}(Y)$, it follows from (3.1) that either $\mathbf{u} \leq \mathbf{y}$ or $\mathbf{w} \leq \mathbf{y}$. If, for example, $\mathbf{u} \leq \mathbf{y}$, then, by the induction hypothesis, either $\mathbf{c}_\xi \leq \mathbf{y}$ (in which case we are done) or there exists $\mathbf{z} \in \mathcal{F}(X \cap Y)$ such that $\mathbf{u} \leq \mathbf{z} \leq \mathbf{y}$. In the second case, $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$. The argument is similar in case $\mathbf{w} \leq \mathbf{y}$.

The remaining case is $m = n > 0$. As $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{x} \leq \mathbf{y}$ with $\text{supp}(\mathbf{x}) \not\subseteq Y$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{R}^{m-1}\mathcal{S}(X)$, and $\mathbf{y} \in \mathcal{R}^m\mathcal{S}(Y)$, it follows from (3.1) that either $\mathbf{u} \leq \pi_{m-1}^m(\mathbf{y})$ or $\mathbf{w} \leq \pi_{m-1}^m(\mathbf{y})$. If $\mathbf{u} \leq \pi_{m-1}^m(\mathbf{y})$, then, by the induction hypothesis, either $\mathbf{c}_\xi \leq \pi_{m-1}^m(\mathbf{y})$ (thus, *a fortiori*, $\mathbf{c}_\xi \leq \mathbf{y}$) or there exists $\mathbf{z} \in \mathcal{F}(X \cap Y)$ such that $\mathbf{u} \leq \mathbf{z} \leq \pi_{m-1}^m(\mathbf{y})$ (in which case $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$). The case where $\mathbf{w} \leq \pi_{m-1}^m(\mathbf{y})$ is similar. \square

Lemma 4.6. *Let Λ be a chain and let X be a nonempty subset of Λ admitting a supremum, say, ξ , in Λ . Then \mathbf{c}_ξ is the supremum of $\{\mathbf{c}_i \mid i \in X\}$ in $\mathcal{F}(\Lambda)$.*

Proof. Let $\mathbf{x} \in \mathcal{F}(\Lambda)$ such that $\mathbf{c}_i \leq \mathbf{x}$ for all $i \in X$, we prove that $\mathbf{c}_\xi \leq \mathbf{x}$. Put $n = \text{rk } \mathbf{x}$ and $\mathbf{y} = \pi_0^n(\mathbf{x})$. Let $i \in X$. From $\mathbf{c}_i \leq \mathbf{x}$ and $\mathbf{c}_i \in \mathcal{S}(\Lambda)$ it follows that $\mathbf{c}_i \leq \mathbf{y}$. This holds for all $i \in X$, hence, as \mathbf{c}_ξ is clearly the supremum of $\{\mathbf{c}_i \mid i \in X\}$ in $\mathcal{S}(\Lambda)$, we obtain that $\mathbf{c}_\xi \leq \mathbf{y}$. Therefore, $\mathbf{c}_\xi \leq \mathbf{x}$. \square

Now we can state the main technical result of the paper. It says that $\langle \mathbf{c}_\xi \mid \xi < \omega_1 \rangle$ is the least non-eventually constant isotone ω_1 -sequence in $\mathcal{F}(\omega_1)$ modulo the closed unbounded filter on ω_1 .

Theorem 4.7. *Let $\sigma = \langle \mathbf{x}_\xi \mid \xi < \omega_1 \rangle$ be an isotone ω_1 -sequence of elements of $\mathcal{F}(\omega_1)$. Then either σ is eventually constant or there exists a closed unbounded subset C of ω_1 such that $\mathbf{c}_\xi \leq \mathbf{x}_\xi$ for all $\xi \in C$.*

Proof. Assume that σ is not eventually constant. We put $X_\xi = \text{supp}(\mathbf{x}_\xi)$ and $n_\xi = |X_\xi|$, for all $\xi < \omega_1$. So $\mathbf{x}_\xi = \mathbf{x}'_\xi[X_\xi/n_\xi]$, for some $\mathbf{x}'_\xi \in \mathcal{F}(n_\xi)$. As all sets X_ξ are finite, it follows from the Δ -Lemma (see [1, Lemma 22.6]) that there are an uncountable subset I of ω_1 and a finite subset X of ω_1 such that $X_\xi \cap X_\eta = X$ for all distinct $\xi, \eta \in I$. We may further assume without loss of generality that there are $n < \omega$ and $\mathbf{x} \in \mathcal{F}(n)$ such that $n_\xi = n$ and $\mathbf{x}'_\xi = \mathbf{x}$, for all $\xi \in I$. Hence $\mathbf{x}_\xi = \mathbf{x}[X_\xi/n]$, for all $\xi \in I$. As σ is isotone but not eventually constant, it follows

that X is a proper subset of X_ξ , for all $\xi \in I$. Put $Y_\xi = X_\xi \setminus X$. Define $\rho(\xi)$ as the least element of Y_ξ .

For subsets U and V of ω_1 , let $U < V$ hold, if $u < v$ for all $\langle u, v \rangle \in U \times V$. By further shrinking I , we might assume that $X < Y_\xi$, for all $\xi \in I$. In particular, observe that $X = X_\xi \cap X_\eta$ is a lower subset of both X_ξ and X_η , for all $\xi \neq \eta$ in I .

Let $\xi < \eta$ in I and suppose that there exists $\mathbf{z} \in \mathcal{F}(X)$ such that $\mathbf{x}_\xi \leq \mathbf{z} \leq \mathbf{x}_\eta$. Applying the embedding $\mathcal{F}(e_{X_\xi, X_\eta})$ to the inequality $\mathbf{x}[X_\xi/n] \leq \mathbf{z}$ and using Lemma 4.4, we obtain the inequality $\mathbf{x}[X_\eta/n] \leq \mathbf{z}$, so $\mathbf{x}_\eta = \mathbf{x}[X_\eta/n] = \mathbf{z}$, a contradiction since the left hand side has support X_η while the right hand side has the smaller support X . Therefore, as $\mathbf{x}_\xi \leq \mathbf{x}_\eta$ and by Lemma 4.5, we obtain the inequality $\mathbf{c}_{\rho(\eta)} \leq \mathbf{x}_\eta$.

Hence, we may assume that $\mathbf{c}_{\rho(\xi)} \leq \mathbf{x}_\xi$ for all $\xi \in I$. It follows that

$$\mathbf{c}_{\bar{\rho}(\xi)} \leq \mathbf{x}_\xi, \quad \text{for all } \xi < \omega_1, \quad (4.1)$$

where we put

$$\bar{\rho}(\xi) = \bigvee (\rho(\eta) \mid \eta \in I, \eta < \xi), \quad \text{for all } \xi < \omega_1.$$

As the range of ρ is unbounded, so is the range of $\bar{\rho}$. Hence, as $\bar{\rho}$ is a complete join-homomorphism from ω_1 to ω_1 , the set $C = \{\xi < \omega_1 \mid \bar{\rho}(\xi) = \xi\}$ is a closed unbounded subset of ω_1 . It follows from (4.1) that the inequality $\mathbf{c}_\xi \leq \mathbf{x}_\xi$ holds for all $\xi \in C$. \square

The following corollary expresses that $\mathcal{F}(\omega_1)$ fails a certain ‘‘monotone refinement property’’.

Corollary 4.8. *There are no positive integer n and no finite collection of isotone ω_1 -sequences $\langle \mathbf{x}_{i,\xi} \mid \xi < \omega_1 \rangle$ of elements of $\mathcal{F}(\omega_1)$, for $0 \leq i \leq n$, such that*

- (1) $\mathbf{x}_{0,\xi} = 0$ and $\mathbf{x}_{n,\xi} = \mathbf{c}_\xi$, for all large enough $\xi < \omega_1$.
- (2) $\mathbf{x}_{i,\xi} \leq \mathbf{c}_\xi$, for all $i \leq n$ and all large enough $\xi < \omega_1$.
- (3) Either $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{x}_{i,\xi}$ or $\mathbf{x}_{i+1,\xi} \leq \mathbf{b} \vee \mathbf{x}_{i,\xi}$, for all $i < n$ and all $\xi < \omega_1$.

Proof. We prove that for all $i \leq n$, there exists $\eta_i < \omega_1$ such that $\mathbf{x}_{i,\xi} \leq \mathbf{c}_{\eta_i}$ for all $\xi < \omega_1$. We argue by induction on i . For $i = 0$ it holds by assumption, with $\eta_0 = 0$. Suppose that $\mathbf{x}_{i,\xi} \leq \mathbf{c}_{\eta_i}$, for all $\xi < \omega_1$. Let $\xi > \eta_i$. Assume, for example, that $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{x}_{i,\xi}$; so $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{c}_{\eta_i}$. Observing that $\mathbf{c}_\xi \not\leq \mathbf{a} \vee \mathbf{c}_{\eta_i}$, we get that $\mathbf{c}_\xi \not\leq \mathbf{x}_{i+1,\xi}$. As this holds for all $\xi > \eta_i$ and by Theorem 4.7, we obtain that $\langle \mathbf{x}_{i+1,\xi} \mid \xi < \omega_1 \rangle$ is eventually constant, and hence, by (2), below some $\mathbf{c}_{\eta_{i+1}}$, therefore completing the induction step.

In particular, for $i = n$, we obtain that the ω_1 -sequence $\langle \mathbf{c}_\xi \mid \xi < \omega_1 \rangle$ is eventually dominated by the constant \mathbf{c}_{η_n} , a contradiction. \square

Hence we get a negative extension property for posets.

Corollary 4.9. *There are a poset measure $\mu: (\omega_1 + 1) \times (\omega_1 + 1) \rightarrow \mathcal{F}(\omega_1)$ such that $\mu(\omega_1, 0) = \mathbf{a} \vee \mathbf{b}$ but there are no poset P containing $\omega_1 + 1$, no poset measure $\bar{\mu}: P \times P \rightarrow \mathcal{F}(\omega_1)$ extending μ , no positive integer n , and no decomposition $0 = z_0 \leq z_1 \leq \dots \leq z_n = \omega_1$ in P such that either $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{a}$ or $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{b}$ for all $i < n$.*

Proof. Define $\mu: (\omega_1 + 1) \times (\omega_1 + 1) \rightarrow \mathcal{F}(\omega_1)$ by

$$\mu(\xi, \eta) = \begin{cases} 0, & \text{if } \xi \leq \eta, \\ \mathbf{c}_\xi, & \text{if } \eta < \xi < \omega_1, \\ \mathbf{a} \vee \mathbf{b}, & \text{if } \eta < \xi = \omega_1. \end{cases} \quad \text{for all } \xi, \eta \leq \omega_1.$$

It is straightforward to verify that μ is a poset measure on $\omega_1 + 1$. Suppose that $P, \bar{\mu}, n, z_0, \dots, z_n$ satisfy the given conditions. We put $\mathbf{x}_{i,\xi} = \bar{\mu}(\xi, z_{n-i})$, for all $\xi < \omega_1$. It is not hard, using the triangular inequality, to verify that the elements $\mathbf{x}_{i,\xi}$ satisfy the assumptions (1)–(3) of Corollary 4.8, a contradiction. \square

As the following result shows, more “amenable” semilattices do satisfy a certain “monotone refinement property”.

Proposition 4.10. *Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice. If S is either a lattice, or strongly distributive, or countable, then for all $\mathbf{a}, \mathbf{b} \in S$, every chain Λ , and every isotone Λ -sequence $\langle \mathbf{c}_i \mid i \in \Lambda \rangle$ of elements of S such that $\mathbf{c}_i \leq \mathbf{a} \vee \mathbf{b}$ for all $i \in \Lambda$, then there are isotone Λ -sequences $\langle \mathbf{a}_i \mid i \in \Lambda \rangle$ and $\langle \mathbf{b}_i \mid i \in \Lambda \rangle$ of elements of S such that $\mathbf{a}_i \leq \mathbf{a}$, $\mathbf{b}_i \leq \mathbf{b}$, and $\mathbf{c}_i = \mathbf{a}_i \vee \mathbf{b}_i$, for all $i \in \Lambda$.*

Proof. If S is a lattice the conclusion is trivial: put $\mathbf{a}_i = \mathbf{a} \wedge \mathbf{c}_i$ and $\mathbf{b}_i = \mathbf{b} \wedge \mathbf{c}_i$, for all $i \in \Lambda$.

Now assume that S is strongly distributive. Denote by C_i the (finite) set of all maximal join-irreducible elements of S below \mathbf{c}_i , for all $i \in \Lambda$. Observe that $C_i \subseteq \downarrow C_j$, for all $i \leq j$ in Λ . For every finite subset I of Λ , denote by X_I the set of all families $\langle \langle A_i, B_i \rangle \mid i \in I \rangle$ such that

- (1) $A_i \subseteq \downarrow \mathbf{a}$, $B_i \subseteq \downarrow \mathbf{b}$, and $A_i \cup B_i = C_i$, for all $i \in I$.
- (2) For all $i \leq j$ in I , $A_i \subseteq \downarrow A_j$ and $B_i \subseteq \downarrow B_j$.

We claim that X_I is nonempty, for every finite subset I of Λ . We argue by induction on $|I|$. The conclusion is obvious for $I = \emptyset$. For $I = \{i\}$, put $A_i = \downarrow \mathbf{a} \cap C_i$ and $B_i = \downarrow \mathbf{b} \cap C_i$. Now suppose that $I = \{i\} \cup J$, where $i < j$ for all $j \in J$ and J is nonempty. By induction hypothesis, there exists an element $\langle \langle A_k, B_k \rangle \mid k \in J \rangle$ in X_J . Put $j = \min J$, $A_i = \downarrow A_j \cap C_i$, and $B_i = \downarrow B_j \cap C_i$. It is straightforward to verify that $\langle \langle A_k, B_k \rangle \mid i \in I \rangle$ belongs to X_I . This completes the induction step.

It follows that the set Ω_I of all families $\langle \langle A_i, B_i \rangle \mid i \in \Lambda \rangle$ of elements of the Cartesian product $\Omega = \prod (\mathfrak{P}(C_i) \times \mathfrak{P}(C_i) \mid i \in \Lambda)$ (where $\mathfrak{P}(X)$ denotes the powerset of a set X) whose restriction to I belongs to X_I is nonempty, for every finite subset I of Λ . Endow Ω with the product topology of the discrete topologies on all (finite) sets $\mathfrak{P}(C_i) \times \mathfrak{P}(C_i)$. By Tychonoff’s Theorem, Ω is compact. Hence the intersection of all Ω_I , for I a finite subset of Λ , is nonempty. Let $\langle \langle A_i, B_i \rangle \mid i \in \Lambda \rangle$ be an element of that intersection. Then the collection of all elements $\mathbf{a}_i = \bigvee A_i$ and $\mathbf{b}_i = \bigvee B_i$, for $i \in \Lambda$, satisfies the required conditions.

Assume, finally, that S is countable. Define an equivalence relation \equiv on Λ by $i \equiv j$ iff $\mathbf{c}_i = \mathbf{c}_j$, for all $i, j \in \Lambda$, and denote by $[i]$ the \equiv -equivalence class of i , for any $i \in \Lambda$. Putting $\mathbf{c}_{[i]} = \mathbf{c}_i$ makes it possible to replace Λ by Λ/\equiv . In particular, as S is countable, Λ becomes countable as well. Now write $\Lambda = \bigcup (\Lambda_n \mid n < \omega)$, where $\langle \Lambda_n \mid n < \omega \rangle$ is an increasing sequence of finite subsets of Λ with $|\Lambda_n| = n$, for all $n < \omega$. Denote by Y_n the set of all families $\langle \langle \mathbf{a}_l, \mathbf{b}_l \rangle \mid l \in \Lambda_n \rangle$ such that $\mathbf{a}_i \leq \mathbf{a}$, $\mathbf{b}_i \leq \mathbf{b}$, $\mathbf{c}_i = \mathbf{a}_i \vee \mathbf{b}_i$, and $i \leq j$ implies that $\mathbf{a}_i \leq \mathbf{a}_j$ and $\mathbf{b}_i \leq \mathbf{b}_j$, for all $i \leq j$ in Λ_n . Suppose that we are given an element of Y_n as above, and denote by k the unique

element of $Y_{n+1} \setminus Y_n$. Suppose, for example, that $\min \Lambda_n < k < \max \Lambda_n$, and denote by i (resp., j) the largest (resp., least) element of Λ_n below k (resp., above k). As $\mathbf{c}_k \leq \mathbf{c}_j = \mathbf{a}_j \vee \mathbf{b}_j$, there are $\mathbf{a}' \leq \mathbf{a}_j$ and $\mathbf{b}' \leq \mathbf{b}_j$ such that $\mathbf{c}_k = \mathbf{a}' \vee \mathbf{b}'$. Put $\mathbf{a}_k = \mathbf{a}_i \vee \mathbf{a}'$ and $\mathbf{b}_k = \mathbf{b}_i \vee \mathbf{b}'$. Then $\langle \langle \mathbf{a}_l, \mathbf{b}_l \rangle \mid l \in \Lambda_{n+1} \rangle$ is an element of Y_{n+1} . So every element of Y_n extends to an element of Y_{n+1} . The proof is even easier in case either $k < \min \Lambda_n$ or $k > \max \Lambda_n$. Hence we have constructed inductively a family $\langle \langle \mathbf{a}_i, \mathbf{b}_i \rangle \mid i \in \Lambda \rangle$ whose restriction to Λ_n belongs to Y_n , for all $n < \omega$. Therefore, the elements \mathbf{a}_i and \mathbf{b}_i , for $i \in \Lambda$, are as required. \square

The “monotone refinement property” described above fails in $\mathcal{F}(\omega_1)$. Indeed, consider the isotone ω_1 -sequence $\langle \mathbf{c}_\xi \mid \xi < \omega_1 \rangle$ together with the inequalities $\mathbf{c}_\xi \leq \mathbf{a} \vee \mathbf{b}$, for $\xi < \omega_1$. Suppose that $\langle \mathbf{a}_\xi \mid \xi < \omega_1 \rangle$ and $\langle \mathbf{b}_\xi \mid \xi < \omega_1 \rangle$ are isotone ω_1 -sequences in $\mathcal{F}(\omega_1)$ such that $\mathbf{c}_\xi = \mathbf{a}_\xi \vee \mathbf{b}_\xi$ while $\mathbf{a}_\xi \leq \mathbf{a}$ and $\mathbf{b}_\xi \leq \mathbf{b}$, for all $\xi < \omega_1$. Set $\mathbf{x}_{0,\xi} = 0$, $\mathbf{x}_{1,\xi} = \mathbf{a}_\xi$, and $\mathbf{x}_{2,\xi} = \mathbf{c}_\xi$, for all $\xi < \omega_1$. Then the isotone ω_1 -sequences $\langle \mathbf{x}_{i,\xi} \mid \xi < \omega_1 \rangle$, for $i \in \{0, 1, 2\}$, satisfy (1)–(3) of Corollary 4.8 (with $n = 2$), a contradiction.

Nevertheless we do not know whether the monotone refinement property of S either implies or is implied by the statement that every S -valued poset measure extends to a V -measure.

REFERENCES

- [1] T. Jech, “Set Theory”, Academic Press, Harcourt Brace Jovanovich, New York - San Francisco - London, xi + 621 p, 1978.
- [2] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices*, Contributions to General Algebra **10**, Proceedings of the Klagenfurt Conference, May 29 – June 1, 1997. Verlag Johannes Heyn, Klagenfurt 1998.
- [3] P. Růžička, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, preprint.
- [4] J. Tůma and F. Wehrung, *Unsolvable one-dimensional lifting problems for congruence lattices of lattices*, Forum Math. **14**, no. 4 (2002), 483–493.
- [5] F. Wehrung, *Forcing extensions of partial lattices*, J. Algebra **262**, no. 1 (2003), 127–193.

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