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# NON-EXTENDABILITY OF SEMILATTICE-VALUED MEASURES ON PARTIALLY ORDERED SETS

FRIEDRICH WEHRUNG

**ABSTRACT.** For a poset  $P$  and a distributive  $\langle \vee, 0 \rangle$ -semilattice  $S$ , a  $S$ -valued poset measure on  $P$  is a map  $\mu: P \times P \rightarrow S$  such that  $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ , and  $x \leq y$  implies that  $\mu(x, y) = 0$ , for all  $x, y, z \in P$ . In relation with congruence lattice representation problems, we consider the problem whether such a measure can be extended to a poset measure  $\bar{\mu}: \bar{P} \times \bar{P} \rightarrow S$ , for a larger poset  $\bar{P}$ , such that for all  $\mathbf{a}, \mathbf{b} \in S$  and all  $x \leq y$  in  $\bar{P}$ ,  $\bar{\mu}(y, x) = \mathbf{a} \vee \mathbf{b}$  implies that there are a positive integer  $n$  and a decomposition  $x = z_0 \leq z_1 \leq \dots \leq z_n = y$  in  $\bar{P}$  such that either  $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{a}$  or  $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{b}$ , for all  $i < n$ .

In this note we prove that this is not possible as a rule, even in case the poset  $P$  we start with is a *chain* and  $S$  has size  $\aleph_1$ . The proof uses a “monotone refinement property” that holds in  $S$  provided  $S$  is either a lattice, or countable, or strongly distributive, but fails for our counterexample. This strongly contrasts with the analogue problem for *distances* on (discrete) sets, which is known to have a positive (and even *functorial*) solution.

## 1. INTRODUCTION

In the paper [5], the author proved that for any lattice  $K$ , any distributive lattice  $S$  with zero, and any  $\langle \vee, 0 \rangle$ -homomorphism  $\varphi$  from the  $\langle \vee, 0 \rangle$ -semilattice  $\text{Con}_c K$  of all finitely generated congruences of  $K$  to  $S$ , there are a lattice  $L$ , a lattice homomorphism  $f: K \rightarrow L$ , and an isomorphism  $\alpha: \text{Con}_c L \rightarrow S$  such that  $\varphi = \alpha \circ \text{Con}_c f$ . In the paper [4], J. Tůma and the author proved that for a  $\langle \vee, 0 \rangle$ -semilattice  $S$ , this statement characterizes  $S$  being a lattice. The proof of this negative result strongly uses the lattice structure of the hypothetical lattice  $L$ , see the proof of [4, Corollary 1.3].

In the present paper, we show that for a certain semilattice  $S$  of cardinality  $\aleph_1$ , the poset structure alone is sufficient to get a related counterexample. More precisely, for a  $\langle \vee, 0 \rangle$ -semilattice  $S$ , a  $S$ -valued poset measure on a poset  $P$  is a map  $\mu: P \times P \rightarrow S$  such that  $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$  (*triangular inequality*) and  $x \leq y$  implies that  $\mu(x, y) = 0$ , for all  $x, y, z \in P$ . We say that  $\mu$  is a  $V$ -measure, if for all  $x \leq y$  in  $P$  and all  $\mathbf{a}, \mathbf{b} \in S$ , if  $\mu(y, x) \leq \mathbf{a} \vee \mathbf{b}$ , then there are a positive integer  $n$  and a decomposition  $x = z_0 \leq z_1 \leq \dots \leq z_n = y$  in  $P$  such that either  $\mu(z_{i+1}, z_i) \leq \mathbf{a}$  or  $\mu(z_{i+1}, z_i) \leq \mathbf{b}$  for all  $i < n$ . This yields the following poset analogue of the abovementioned lattice-theoretical problem.

**Problem.** *Let  $S$  be a distributive  $\langle \vee, 0 \rangle$ -semilattice. Does any  $S$ -valued poset measure on a given poset extend to some  $S$ -valued poset  $V$ -measure on a larger poset?*

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A version of this problem for so-called *distances* (instead of measures) on discrete sets (instead of posets) is stated in [3]. The answer to this related question turns out to be positive (and easy). More surprisingly, this positive solution can be made *functorial*.

Nevertheless, we prove in the present paper that the problem above has a *negative* solution. Unlike what is done in [4], we do not reach here a characterization of all lattices among distributive  $\langle \vee, 0 \rangle$ -semilattices. Our counterexample, denoted by  $\mathcal{F}(\omega_1)$  (see Corollary 4.9) is obtained as an application of a certain “free construction” used by M. Ploščica and J. Tůma in [2]. The semilattice  $D$  of [4, Section 2], which is the simplest example of a  $\langle \vee, 0 \rangle$ -semilattice which is not a lattice, does not satisfy the negative property used here. This is because  $D$  is *countable*, while we prove in Proposition 4.10 that no countable distributive  $\langle \vee, 0 \rangle$ -semilattice can have the required negative property. On the other hand, in relation to [4, Problem 4], the proof of our counterexample uses very little of the Axiom of Choice (namely, only the Axiom of *countable* choices), while the proof of the negative property of the abovementioned semilattice  $D$  established in [4, Corollary 2.4] uses the existence of an embedding from  $\omega_1$  into the reals.

## 2. BASIC CONCEPTS

For posets (i.e., partially ordered sets)  $P$  and  $Q$ , a map  $f: P \rightarrow Q$  is *isotone*, if  $x \leq y$  implies that  $f(x) \leq f(y)$ , for all  $x, y \in P$ . In addition, we say that  $f$  is *join-preserving*, if for any subset  $X$  of  $P$ , whenever the join  $\bigvee X$  of  $X$  exists in  $P$ ,  $\bigvee f[X]$  exists in  $Q$ , and  $\bigvee f[X] = f(\bigvee X)$ . For a subset  $X$  of a poset  $P$ , we shall put  $\downarrow X = \{p \in P \mid \exists x \in X \text{ such that } p \leq x\}$ , and then  $\downarrow a = \downarrow \{a\}$ , for all  $a \in P$ . We say that  $X$  is a *lower subset* of  $P$ , if  $X = \downarrow X$ .

A  $\langle \vee, 0 \rangle$ -semilattice  $S$  is *distributive*, if  $c \leq a \vee b$  in  $S$  implies that there are  $x \leq a$  and  $y \leq b$  in  $S$  such that  $c \leq x \vee y$ . A distributive  $\langle \vee, 0 \rangle$ -semilattice  $S$  is *strongly distributive*, if every element of  $S$  is the join of a finite set of join-irreducible elements of  $S$ ; equivalently,  $S$  is isomorphic to the semilattice of all finitely generated lower subsets of some poset.

We shall denote by  $\text{otp } P$  the order-type of a well-ordered set  $P$ . Hence  $\text{otp } P$  is an ordinal. We shall also use standard set-theoretical notation and terminology, referring the reader to [1] for further information. In particular, we shall denote by  $\omega_1$  the first uncountable ordinal. A subset  $C$  of  $\omega_1$  is *closed unbounded*, if  $C$  is unbounded in  $\omega_1$  and the join of any nonempty bounded subset of  $C$  belongs to  $C$ . It is well-known that the closed unbounded subsets form a countably complete filterbasis on  $\omega_1$ , see [1, Lemma 7.4]. Hence containing a closed unbounded is a notion of “largeness” for subsets of  $\omega_1$ .

## 3. FREE DISTRIBUTIVE EXTENSION OF A $\langle \vee, 0 \rangle$ -SEMI LATTICE

There are several non-equivalent definitions of what should be the “free distributive extension” of a given  $\langle \vee, 0 \rangle$ -semilattice. The one that we shall use is introduced in [2, Section 2]. Let us first recall the construction.

For a  $\langle \vee, 0 \rangle$ -semilattice  $S$ , we shall put  $\mathcal{C}(S) = \{\langle u, v, w \rangle \in S^3 \mid w \leq u \vee v\}$ . A *finite* subset  $x$  of  $\mathcal{C}(S)$  is *reduced*, if it satisfies the following conditions:

- (1)  $x$  contains exactly one diagonal triple, that is, a triple of the form  $\langle u, u, u \rangle$ ; we put  $u = \pi(x)$ .
- (2)  $\langle u, v, w \rangle \in x$  and  $\langle v, u, w \rangle \in x$  implies that  $u = v = w$ , for all  $u, v, w \in S$ .

- (3)  $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x} \setminus \{\langle \pi(\mathbf{x}), \pi(\mathbf{x}), \pi(\mathbf{x}) \rangle\}$  implies that  $\mathbf{u}, \mathbf{v}, \mathbf{w} \not\leq \pi(\mathbf{x})$ , for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S$ .

We denote by  $\mathcal{R}(S)$  the set of all reduced subsets of  $\mathcal{C}(S)$ , endowed with the partial ordering  $\leq$  defined by

$$\mathbf{x} \leq \mathbf{y} \iff \forall \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x} \setminus \mathbf{y}, \text{ either } \mathbf{u} \leq \pi(\mathbf{y}) \text{ or } \mathbf{w} \leq \pi(\mathbf{y}). \quad (3.1)$$

Furthermore, we shall identify  $\mathbf{x}$  with the element  $\{\langle \mathbf{x}, \mathbf{x}, \mathbf{x} \rangle\}$  of  $\mathcal{R}(S)$ , for all  $\mathbf{x} \in S$ . For set-theoretical purists, this can for example be done by replacing  $\mathcal{R}(S)$  by the disjoint union of  $S$  with the set of non-singletons in  $\mathcal{R}(S)$ . The disjointness can easily be achieved by a suitable modification of the standard definition of a triple. We shall use the symbol  $\bowtie$  to denote the canonical generators of  $\mathcal{R}(S)$ , so that

$$\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{cases} \mathbf{u}, & \text{if } \mathbf{u} = \mathbf{v} = \mathbf{w}, \\ \{\langle 0, 0, 0 \rangle, \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle\}, & \text{otherwise,} \end{cases} \quad \text{for all } \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S).$$

Observe that the canonical map  $\pi: \mathcal{R}(S) \rightarrow S$  is *isotone* and that the restriction of  $\pi$  to  $S$  is the identity. Furthermore,  $\pi(\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})) = 0$ , for any non-diagonal  $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$ . The following is an easy consequence of (3.1).

$$\mathbf{x} \leq \mathbf{y} \iff \mathbf{x} \leq \pi(\mathbf{y}), \quad \text{for all } \langle \mathbf{x}, \mathbf{y} \rangle \in S \times \mathcal{R}(S). \quad (3.2)$$

We recall the standard facts established in [2] about this construction.

**Proposition 3.1.**

- (1) For any  $\langle \vee, 0 \rangle$ -semilattice  $S$ ,  $\mathcal{R}(S)$  is a  $\langle \vee, 0 \rangle$ -semilattice, and the inclusion map from  $S$  into  $\mathcal{R}(S)$  is a  $\langle \vee, 0 \rangle$ -embedding.
- (2) For  $\langle \vee, 0 \rangle$ -semilattices  $S$  and  $T$ , every  $\langle \vee, 0 \rangle$ -homomorphism  $f: S \rightarrow T$  extends to a unique  $\langle \vee, 0 \rangle$ -homomorphism  $\mathcal{R}(f): \mathcal{R}(S) \rightarrow \mathcal{R}(T)$  such that  $\mathcal{R}(f)(\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})) = \bowtie(f(\mathbf{u}), f(\mathbf{v}), f(\mathbf{w}))$ , for all  $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$ .
- (3) The assignment  $S \mapsto \mathcal{R}(S)$ ,  $f \mapsto \mathcal{R}(f)$  is a functor.

The extension  $\mathcal{R}(S)$  is defined in such a way that  $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \mathbf{u}$  and  $\mathbf{w} = \bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) \vee \bowtie(\mathbf{v}, \mathbf{u}, \mathbf{w})$ , for all  $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathcal{C}(S)$ . Hence, putting  $\mathcal{R}^0(S) = S$  and  $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$  for each  $n$ , we obtain that the increasing union  $\mathcal{D}(S) = \bigcup (\mathcal{R}^n(S) \mid n < \omega)$  is a distributive  $\langle \vee, 0 \rangle$ -semilattice, extending  $S$ . Furthermore, putting  $\mathcal{D}(f) = \bigcup (\mathcal{D}^n(f) \mid n < \omega)$  for each  $\langle \vee, 0 \rangle$ -homomorphism  $f$ , we obtain that  $\mathcal{D}$  is a functor. The proof of the following lemma is straightforward.

**Lemma 3.2.** Let  $S$  be a  $\langle \vee, 0 \rangle$ -semilattice and let  $\langle S_i \mid i \in I \rangle$  be a family of  $\langle \vee, 0 \rangle$ -subsemilattices of  $S$ . The following statements hold:

- (1)  $\mathcal{R}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{R}(S_i)$  and  $\mathcal{D}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{D}(S_i)$ .
- (2) If  $I$  is an upward directed poset and  $\langle S_i \mid i \in I \rangle$  is isotone, then  $\mathcal{R}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{R}(S_i)$  and  $\mathcal{D}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{D}(S_i)$ .

**Definition 3.3.** For a  $\langle \vee, 0 \rangle$ -semilattice  $S$  and an element  $\mathbf{x} \in \mathcal{D}(S)$ , we define the *rank* of  $\mathbf{x}$ , denoted by  $\text{rk } \mathbf{x}$ , as the least natural number  $n$  such that  $\mathbf{x} \in \mathcal{R}^n(S)$ , and the *complexity* of  $\mathbf{x}$ , denoted by  $\|\mathbf{x}\|$ , by  $\|\mathbf{x}\| = 0$  if  $\mathbf{x} \in S$ , and

$$\|\mathbf{x}\| = \sum (\|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\| + 1 \mid \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle \in \mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{D}(S) \setminus S.$$

4. THE SEMILATTICES  $\mathcal{S}(\Lambda)$  AND  $\mathcal{F}(\Lambda)$ 

For any chain  $\Lambda$ , we shall denote by  $\mathcal{S}(\Lambda)$  the  $\langle \vee, 0 \rangle$ -semilattice defined by generators  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}_i$ , for  $i \in \Lambda$ , and relations  $\mathbf{c}_i \leq \mathbf{a} \vee \mathbf{b}$  and  $\mathbf{c}_i \leq \mathbf{c}_j$ , for all  $i \leq j$  in  $\Lambda$ . Hence the elements of  $\mathcal{S}(\Lambda)$  either belong to  $\mathcal{S}(\emptyset) = \{0, \mathbf{a}, \mathbf{b}, \mathbf{a} \vee \mathbf{b}\}$  or have the form  $\mathbf{c}_i$ ,  $\mathbf{a} \vee \mathbf{c}_i$ , or  $\mathbf{b} \vee \mathbf{c}_i$ , for some  $i \in \Lambda$ . We shall identify  $\mathcal{S}(X)$  with the  $\langle \vee, 0 \rangle$ -subsemilattice of  $\mathcal{S}(\Lambda)$  generated by  $\mathcal{S}(\emptyset) \cup \{\mathbf{c}_i \mid i \in X\}$ , for any  $X \subseteq \Lambda$ .

For chains  $X$  and  $Y$ , any isotone map  $f: X \rightarrow Y$  gives rise to a unique  $\langle \vee, 0 \rangle$ -homomorphism  $\mathcal{S}(f): \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$  fixing  $\mathbf{a}$  and  $\mathbf{b}$  and sending  $\mathbf{c}_i$  to  $\mathbf{c}_{f(i)}$ , for all  $i \in X$ . Of course, the assignment  $\Lambda \mapsto \mathcal{S}(\Lambda)$ ,  $f \mapsto \mathcal{S}(f)$  is a functor.

We denote by  $\mathcal{F} = \mathcal{D} \circ \mathcal{S}$  the composition of the two functors  $\mathcal{D}$  and  $\mathcal{S}$ .

The proof of the following lemma is straightforward.

**Lemma 4.1.** *Let  $\Lambda$  be a chain and let  $\langle X_i \mid i \in I \rangle$  be a family of subsets of  $\Lambda$ . The following statements hold:*

- (1)  $\mathcal{S}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \mathcal{S}(X_i)$ .
- (2) *If  $I$  is an upward directed poset and  $\langle X_i \mid i \in I \rangle$  is isotone, then  $\mathcal{S}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \mathcal{S}(X_i)$ .*

As an easy consequence of Lemmas 3.2 and 4.1, we get the following.

**Lemma 4.2.** *Let  $\Lambda$  be a chain. Then for any  $\mathbf{x} \in \mathcal{F}(\Lambda)$ , there exists a least (with respect to the inclusion) subset  $X$  of  $\Lambda$  such that  $\mathbf{x} \in \mathcal{F}(X)$ ; this subset is finite.*

We denote by  $\text{supp}(\mathbf{x})$  the subset given by Lemma 4.2, and we call it the *support* of  $\mathbf{x}$ .

*Notation 4.3.* For a chain  $\Lambda$ , isomorphic well-ordered subsets  $X$  and  $Y$  of  $\Lambda$  such that  $\text{otp } X \leq \text{otp } Y$ , and  $\mathbf{x} \in \mathcal{F}(X)$ , we set  $\mathbf{x}[Y/X] = \mathcal{F}(e_{X,Y})(\mathbf{x})$ , where  $e_{X,Y}$  denotes the unique embedding from  $X$  into  $Y$  whose range is a lower subset of  $Y$ .

Hence  $\mathbf{x}[Y/X]$  belongs to  $\mathcal{F}(Y)$ , for all  $\mathbf{x} \in \mathcal{F}(X)$ .

**Lemma 4.4.** *Let  $\Lambda$  be a chain and let  $X, Y$  be well-ordered subsets of  $\Lambda$  such that  $\text{otp } X \leq \text{otp } Y$  and  $X \cap Y$  is a lower subset of both  $X$  and  $Y$ . Then  $\mathbf{x}[Y/X] = \mathbf{x}$ , for all  $\mathbf{x} \in \mathcal{F}(X \cap Y)$ .*

*Proof.* As the set  $Z = X \cap Y$  is a lower subset of both  $X$  and  $Y$ , the homomorphism  $\mathcal{F}(e_{Z,X})$  (resp.,  $\mathcal{F}(e_{Z,Y})$ ) is the inclusion map from  $\mathcal{F}(Z)$  into  $\mathcal{F}(X)$  (resp.,  $\mathcal{F}(Y)$ ). In particular,  $\mathbf{x} = \mathcal{F}(e_{Z,X})(\mathbf{x}) = \mathcal{F}(e_{Z,Y})(\mathbf{x})$ . Therefore,

$$\mathbf{x}[Y/X] = \mathcal{F}(e_{X,Y})(\mathbf{x}) = \mathcal{F}(e_{X,Y}) \circ \mathcal{F}(e_{Z,X})(\mathbf{x}) = \mathcal{F}(e_{Z,Y})(\mathbf{x}) = \mathbf{x}. \quad \square$$

We are now reaching a crucial lemma.

**Lemma 4.5** (Interpolation Lemma). *Let  $\Lambda$  be a chain, let  $X, Y$  be finite subsets of  $\Lambda$ , and let  $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathcal{F}(X) \times \mathcal{F}(Y)$ . If  $\mathbf{x} \leq \mathbf{y}$ , then either there exists  $\mathbf{z} \in \mathcal{F}(X \cap Y)$  such that  $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$  or  $(Y \not\subseteq X \text{ and } \mathbf{c}_{\min(Y \setminus X)} \leq \mathbf{y})$ .*

*Proof.* We shall denote by  $\pi_k^l$  the canonical map from  $\mathcal{R}^l \mathcal{S}(\Lambda)$  onto  $\mathcal{R}^k \mathcal{S}(\Lambda)$ , for all natural numbers  $k \leq l$ . Put  $m = \text{rk } \mathbf{x}$  and  $n = \text{rk } \mathbf{y}$ . Observe that  $\text{supp}(\mathbf{x}) \subseteq X$  and  $\text{supp}(\mathbf{y}) \subseteq Y$ . We argue by induction on  $\|\mathbf{x}\| + \|\mathbf{y}\|$ . If either  $\text{supp}(\mathbf{x}) \subseteq Y$  or  $\text{supp}(\mathbf{y}) \subseteq X$  then either  $\mathbf{z} = \mathbf{x}$  or  $\mathbf{z} = \mathbf{y}$  belongs to  $\mathcal{F}(X \cap Y)$  and satisfies the inequalities  $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$ , so we are done. So suppose that  $\text{supp}(\mathbf{x}) \not\subseteq Y$  and

$\text{supp}(\mathbf{y}) \not\subseteq X$ . In particular,  $X \not\subseteq Y$  and  $Y \not\subseteq X$ , and both  $\text{supp}(\mathbf{x})$  and  $\text{supp}(\mathbf{y})$  are nonempty. We put  $\xi = \min(Y \setminus X)$ .

Suppose that  $m = n = 0$ , that is,  $\mathbf{x}, \mathbf{y} \in \mathcal{S}(\Lambda)$ . Pick  $i \in \text{supp}(\mathbf{x})$ . As  $\mathbf{c}_i \leq \mathbf{x} \leq \mathbf{y}$ , we obtain that either  $\mathbf{y} = \mathbf{a} \vee \mathbf{b}$  (a contradiction, as then  $\text{supp}(\mathbf{y}) = \emptyset$ ) or  $\mathbf{y} \in \{\mathbf{c}_j, \mathbf{a} \vee \mathbf{c}_j, \mathbf{b} \vee \mathbf{c}_j\}$  for some  $j \geq i$ . If  $i = j$ , then  $\text{supp}(\mathbf{x}) = \text{supp}(\mathbf{y}) = \{i\}$ , a contradiction. If  $i < j$ , then  $\xi = j$  and so  $\mathbf{c}_\xi \leq \mathbf{y}$ .

Suppose now that  $m < n$ . Then  $\mathbf{x} \leq \mathbf{y}$  means that  $\mathbf{x} \leq \pi_m^n(\mathbf{y})$  (use (3.2)). As  $\pi_m^n(\mathbf{y})$  has support contained in  $Y$  and rank at most  $m$ , it follows from the induction hypothesis that either  $\mathbf{c}_\xi \leq \pi_m^n(\mathbf{y})$  (thus, *a fortiori*,  $\mathbf{c}_\xi \leq \mathbf{y}$ ) or there exists  $\mathbf{z} \in \mathcal{F}(X \cap Y)$  such that  $\mathbf{x} \leq \mathbf{z} \leq \pi_m^n(\mathbf{y})$  (thus, *a fortiori*,  $\mathbf{z} \leq \mathbf{y}$ ).

So suppose from now on that  $m > 0$  (i.e.,  $\mathbf{x} \notin \mathcal{S}(\Lambda)$ ) and  $m \geq n$ . If  $\mathbf{x} = \bigvee_{i < k} \mathbf{x}_i$  where  $k \geq 2$  and each  $\mathbf{x}_i$  has support contained in  $X$  and complexity less than  $\|\mathbf{x}\|$ , then we apply the induction hypothesis to each inequality  $\mathbf{x}_i \leq \mathbf{y}$ , for  $i < k$ . If  $\mathbf{c}_\xi \not\leq \mathbf{y}$ , then for all  $i < k$ , there exists  $\mathbf{z}_i \in \mathcal{F}(X \cap Y)$  such that  $\mathbf{x}_i \leq \mathbf{z}_i \leq \mathbf{y}$ . Hence  $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$ , where  $\mathbf{z} = \bigvee_{i < k} \mathbf{z}_i$  belongs to  $\mathcal{F}(X \cap Y)$ . This reduces the problem to the case where  $\mathbf{x} = \bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w})$ , where  $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$  is a non-diagonal triple of elements of  $\mathcal{R}^{m-1}\mathcal{S}(X)$  of complexity less than  $\|\mathbf{x}\|$ .

If  $m > n$ , then, as  $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{x} \leq \mathbf{y}$  with  $\text{supp}(\mathbf{x}) \not\subseteq Y$ ,  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{R}^{m-1}\mathcal{S}(X)$ , and  $\mathbf{y} \in \mathcal{R}^{m-1}\mathcal{S}(Y)$ , it follows from (3.1) that either  $\mathbf{u} \leq \mathbf{y}$  or  $\mathbf{w} \leq \mathbf{y}$ . If, for example,  $\mathbf{u} \leq \mathbf{y}$ , then, by the induction hypothesis, either  $\mathbf{c}_\xi \leq \mathbf{y}$  (in which case we are done) or there exists  $\mathbf{z} \in \mathcal{F}(X \cap Y)$  such that  $\mathbf{u} \leq \mathbf{z} \leq \mathbf{y}$ . In the second case,  $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$ . The argument is similar in case  $\mathbf{w} \leq \mathbf{y}$ .

The remaining case is  $m = n > 0$ . As  $\bowtie(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{x} \leq \mathbf{y}$  with  $\text{supp}(\mathbf{x}) \not\subseteq Y$ ,  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{R}^m\mathcal{S}(X)$ , and  $\mathbf{y} \in \mathcal{R}^m\mathcal{S}(Y)$ , it follows from (3.1) that either  $\mathbf{u} \leq \pi_{m-1}^m(\mathbf{y})$  or  $\mathbf{w} \leq \pi_{m-1}^m(\mathbf{y})$ . If  $\mathbf{u} \leq \pi_{m-1}^m(\mathbf{y})$ , then, by the induction hypothesis, either  $\mathbf{c}_\xi \leq \pi_{m-1}^m(\mathbf{y})$  (thus, *a fortiori*,  $\mathbf{c}_\xi \leq \mathbf{y}$ ) or there exists  $\mathbf{z} \in \mathcal{F}(X \cap Y)$  such that  $\mathbf{u} \leq \mathbf{z} \leq \pi_{m-1}^m(\mathbf{y})$  (in which case  $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$ ). The case where  $\mathbf{w} \leq \pi_{m-1}^m(\mathbf{y})$  is similar.  $\square$

**Lemma 4.6.** *Let  $\Lambda$  be a chain and let  $X$  be a nonempty subset of  $\Lambda$  admitting a supremum, say,  $\xi$ , in  $\Lambda$ . Then  $\mathbf{c}_\xi$  is the supremum of  $\{\mathbf{c}_i \mid i \in X\}$  in  $\mathcal{F}(\Lambda)$ .*

*Proof.* Let  $\mathbf{x} \in \mathcal{F}(\Lambda)$  such that  $\mathbf{c}_i \leq \mathbf{x}$  for all  $i \in X$ , we prove that  $\mathbf{c}_\xi \leq \mathbf{x}$ . Put  $n = \text{rk } \mathbf{x}$  and  $\mathbf{y} = \pi_0^n(\mathbf{x})$ . Let  $i \in X$ . From  $\mathbf{c}_i \leq \mathbf{x}$  and  $\mathbf{c}_i \in \mathcal{S}(\Lambda)$  it follows that  $\mathbf{c}_i \leq \mathbf{y}$ . This holds for all  $i \in X$ , hence, as  $\mathbf{c}_\xi$  is clearly the supremum of  $\{\mathbf{c}_i \mid i \in X\}$  in  $\mathcal{S}(\Lambda)$ , we obtain that  $\mathbf{c}_\xi \leq \mathbf{y}$ . Therefore,  $\mathbf{c}_\xi \leq \mathbf{x}$ .  $\square$

Now we can state the main technical result of the paper. It says that  $\langle \mathbf{c}_\xi \mid \xi < \omega_1 \rangle$  is the only non-eventually constant isotone  $\omega_1$ -sequence in  $\mathcal{F}(\omega_1)$  modulo the closed unbounded filter on  $\omega_1$ .

**Theorem 4.7.** *Let  $\sigma = \langle \mathbf{x}_\xi \mid \xi < \omega_1 \rangle$  be an isotone  $\omega_1$ -sequence of elements of  $\mathcal{F}(\omega_1)$ . Then either  $\sigma$  is eventually constant or there exists a closed unbounded subset  $C$  of  $\omega_1$  such that  $\mathbf{x}_\xi = \mathbf{c}_\xi$  for all  $\xi \in C$ .*

*Proof.* Assume that  $\sigma$  is not eventually constant. We put  $X_\xi = \text{supp}(\mathbf{x}_\xi)$  and  $n_\xi = |X_\xi|$ , for all  $\xi < \omega_1$ . So  $\mathbf{x}_\xi = \mathbf{x}'_\xi[X_\xi/n_\xi]$ , for some  $\mathbf{x}'_\xi \in \mathcal{F}(n_\xi)$ . As all sets  $X_\xi$  are finite, it follows from the  $\Delta$ -Lemma (see [1, Lemma 22.6]) that there are an uncountable subset  $I$  of  $\omega_1$  and a finite subset  $X$  of  $\omega_1$  such that  $X_\xi \cap X_\eta = X$  for all distinct  $\xi, \eta \in I$ . We may further assume without loss of generality that there are  $n < \omega$  and  $\mathbf{x} \in \mathcal{F}(n)$  such that  $n_\xi = n$  and  $\mathbf{x}'_\xi = \mathbf{x}$ , for all  $\xi \in I$ . Hence

$\mathbf{x}_\xi = \mathbf{x}[X_\xi/n]$ , for all  $\xi \in I$ . As  $\sigma$  is isotone but not eventually constant, it follows that  $X$  is a proper subset of  $X_\xi$ , for all  $\xi \in I$ . Put  $Y_\xi = X_\xi \setminus X$ . Define  $\rho(\xi)$  as the least element of  $Y_\xi$ .

For subsets  $U$  and  $V$  of  $\omega_1$ , let  $U < V$  hold, if  $u < v$  for all  $\langle u, v \rangle \in U \times V$ . By further shrinking  $I$ , we might assume that  $X < Y_\xi$ , for all  $\xi < \omega_1$ . In particular, observe that  $X = X_\xi \cap X_\eta$  is a lower subset of both  $X_\xi$  and  $X_\eta$ , for all  $\xi \neq \eta$  in  $I$ .

Let  $\xi < \eta$  in  $I$  and suppose that there exists  $\mathbf{z} \in \mathcal{F}(X)$  such that  $\mathbf{x}_\xi \leq \mathbf{z} \leq \mathbf{x}_\eta$ . Applying the embedding  $\mathcal{F}(e_{X_\xi, X_\eta})$  to the inequality  $\mathbf{x}[X_\xi/n] \leq \mathbf{z}$  and using Lemma 4.4, we obtain the inequality  $\mathbf{x}[X_\eta/n] \leq \mathbf{z}$ , so  $\mathbf{x}_\eta = \mathbf{x}[X_\eta/n] = \mathbf{z}$ , a contradiction since the left hand side has support  $X_\eta$  while the right hand side has the smaller support  $X$ . Therefore, as  $\mathbf{x}_\xi \leq \mathbf{x}_\eta$  and by Lemma 4.5, we obtain the inequality  $\mathbf{c}_{\rho(\eta)} \leq \mathbf{x}_\eta$ .

Hence, we may assume that  $\mathbf{c}_{\rho(\xi)} \leq \mathbf{x}_\xi$  for all  $\xi \in I$ . It follows that

$$\mathbf{c}_{\bar{\rho}(\xi)} \leq \mathbf{x}_\xi, \quad \text{for all } \xi < \omega_1, \quad (4.1)$$

where we put

$$\bar{\rho}(\xi) = \bigvee (\rho(\eta) \mid \eta \in I, \eta < \xi), \quad \text{for all } \xi < \omega_1.$$

As the range of  $\rho$  is unbounded, so is the range of  $\bar{\rho}$ . Hence, as  $\bar{\rho}$  is a complete join-homomorphism from  $\omega_1$  to  $\omega_1$ , the set  $C = \{\xi < \omega_1 \mid \bar{\rho}(\xi) = \xi\}$  is a closed unbounded subset of  $\omega_1$ . It follows from (4.1) that for all  $\xi \in C$ , the inequality  $\mathbf{c}_\xi \leq \mathbf{x}_\xi$  holds, and thus  $\mathbf{x}_\xi = \mathbf{c}_\xi$  by assumption.  $\square$

The following corollary expresses that  $\mathcal{F}(\omega_1)$  fails a certain “monotone refinement property”.

**Corollary 4.8.** *There are no positive integer  $n$  and no finite collection of isotone  $\omega_1$ -sequences  $\langle \mathbf{x}_{i,\xi} \mid \xi < \omega_1 \rangle$  of elements of  $\mathcal{F}(\omega_1)$ , for  $0 \leq i \leq n$ , such that*

- (1)  $\mathbf{x}_{0,\xi} = 0$  and  $\mathbf{x}_{n,\xi} = \mathbf{c}_\xi$ , for all large enough  $\xi < \omega_1$ .
- (2)  $\mathbf{x}_{i,\xi} \leq \mathbf{c}_\xi$ , for all  $i \leq n$  and all large enough  $\xi < \omega_1$ .
- (3) Either  $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{x}_{i,\xi}$  or  $\mathbf{x}_{i+1,\xi} \leq \mathbf{b} \vee \mathbf{x}_{i,\xi}$ , for all  $i < n$  and all  $\xi < \omega_1$ .

*Proof.* We prove that for all  $i \leq n$ , there exists  $\eta_i < \omega_1$  such that  $\mathbf{x}_{i,\xi} \leq \mathbf{c}_{\eta_i}$  for all  $\xi < \omega_1$ . We argue by induction on  $i$ . For  $i = 0$  it holds by assumption, with  $\eta_0 = 0$ . Suppose that  $\mathbf{x}_{i,\xi} \leq \mathbf{c}_{\eta_i}$ , for all  $\xi < \omega_1$ . Let  $\xi > \eta_i$ . Assume, for example, that  $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{x}_{i,\xi}$ ; so  $\mathbf{x}_{i+1,\xi} \leq \mathbf{a} \vee \mathbf{c}_{\eta_i}$ . Observing that  $\mathbf{c}_\xi \not\leq \mathbf{a} \vee \mathbf{c}_{\eta_i}$ , we get that  $\mathbf{c}_\xi \not\leq \mathbf{x}_{i+1,\xi}$ . As this holds for all  $\xi > \eta_i$  and by Theorem 4.7, we obtain that  $\langle \mathbf{x}_{i+1,\xi} \mid \xi < \omega_1 \rangle$  is eventually constant, and hence below some  $\mathbf{c}_{\eta_{i+1}}$ , therefore completing the induction step.

In particular, for  $i = n$ , we obtain that the  $\omega_1$ -sequence  $\langle \mathbf{c}_\xi \mid \xi < \omega_1 \rangle$  is eventually dominated by the constant  $\mathbf{c}_{\eta_n}$ , a contradiction.  $\square$

Hence we get a negative extension property for posets.

**Corollary 4.9.** *There are a poset measure  $\mu: (\omega_1 + 1) \times (\omega_1 + 1) \rightarrow \mathcal{F}(\omega_1)$  such that  $\mu(\omega_1, 0) = \mathbf{a} \vee \mathbf{b}$  but there are no poset  $P$  containing  $\omega_1 + 1$ , no poset measure  $\bar{\mu}: P \times P \rightarrow \mathcal{F}(\omega_1)$  extending  $\mu$ , no positive integer  $n$ , and no decomposition  $0 = z_0 \leq z_1 \leq \dots \leq z_n = \omega_1$  in  $P$  such that either  $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{a}$  or  $\bar{\mu}(z_{i+1}, z_i) \leq \mathbf{b}$  for all  $i < n$ .*

*Proof.* Define  $\mu: (\omega_1 + 1) \times (\omega_1 + 1) \rightarrow \mathcal{F}(\omega_1)$  by

$$\mu(\xi, \eta) = \begin{cases} 0, & \text{if } \xi \leq \eta, \\ \mathbf{c}_\xi, & \text{if } \eta < \xi < \omega_1, \\ \mathbf{a} \vee \mathbf{b}, & \text{if } \eta < \xi = \omega_1. \end{cases} \quad \text{for all } \xi, \eta \leq \omega_1.$$

It is straightforward to verify that  $\mu$  is a poset measure on  $\omega_1 + 1$ . Suppose that  $P$ ,  $\bar{\mu}$ ,  $n$ ,  $z_0, \dots, z_n$  satisfy the given conditions. We put  $\mathbf{x}_{i,\xi} = \bar{\mu}(\xi, z_{n-i})$ , for all  $\xi < \omega_1$ . It is not hard, using the triangular inequality, to verify that the elements  $\mathbf{x}_{i,\xi}$  satisfy the assumptions (i)–(iii) of Corollary 4.8, a contradiction.  $\square$

More “amenable” semilattices do satisfy the “monotone refinement property” that, because of Corollary 4.8, fails in  $\mathcal{F}(\omega_1)$ .

**Proposition 4.10.** *Let  $S$  be a distributive  $\langle \vee, 0 \rangle$ -semilattice. If  $S$  is either a lattice, or strongly distributive, or countable, then for all  $\mathbf{a}, \mathbf{b} \in S$ , every chain  $\Lambda$ , and every isotone  $\Lambda$ -sequence  $\langle \mathbf{c}_i \mid i \in \Lambda \rangle$  of elements of  $S$  such that  $\mathbf{c}_i \leq \mathbf{a} \vee \mathbf{b}$  for all  $i \in \Lambda$ , then there are isotone  $\Lambda$ -sequences  $\langle \mathbf{a}_i \mid i \in \Lambda \rangle$  and  $\langle \mathbf{b}_i \mid i \in \Lambda \rangle$  of elements of  $S$  such that  $\mathbf{a}_i \leq \mathbf{a}$ ,  $\mathbf{b}_i \leq \mathbf{b}$ , and  $\mathbf{c}_i = \mathbf{a}_i \vee \mathbf{b}_i$ , for all  $i \in \Lambda$ .*

*Proof.* If  $S$  is a lattice the conclusion is trivial: put  $\mathbf{a}_i = \mathbf{a} \wedge \mathbf{c}_i$  and  $\mathbf{b}_i = \mathbf{b} \wedge \mathbf{c}_i$ , for all  $i \in \Lambda$ .

Now assume that  $S$  is strongly distributive. Denote by  $C_i$  the (finite) set of all maximal join-irreducible elements of  $S$  below  $\mathbf{c}_i$ , for all  $i \in \Lambda$ . Observe that  $C_i \subseteq \downarrow C_j$ , for all  $i \leq j$  in  $\Lambda$ . For every finite subset  $I$  of  $\Lambda$ , denote by  $X_I$  the set of all families  $\langle \langle A_i, B_i \rangle \mid i \in I \rangle$  such that

- (1)  $A_i \subseteq \downarrow \mathbf{a}$ ,  $B_i \subseteq \downarrow \mathbf{b}$ , and  $A_i \cup B_i = C_i$ , for all  $i \in I$ .
- (2) For all  $i \leq j$  in  $I$ ,  $A_i \subseteq \downarrow A_j$  and  $B_i \subseteq \downarrow B_j$ .

We claim that  $X_I$  is nonempty, for every finite subset  $I$  of  $\Lambda$ . We argue by induction on  $|I|$ . The conclusion is obvious for  $I = \emptyset$ . For  $I = \{i\}$ , put  $A_i = \downarrow \mathbf{a} \cap C_i$  and  $B_i = \downarrow \mathbf{b} \cap C_i$ . Now suppose that  $I = \{i\} \cup J$ , where  $i < j$  for all  $j \in J$  and  $J$  is nonempty. By induction hypothesis, there exists an element  $\langle \langle A_k, B_k \rangle \mid k \in J \rangle$  in  $X_J$ . Put  $j = \min J$ ,  $A_i = \downarrow A_j \cap C_i$ , and  $B_i = \downarrow B_j \cap C_i$ . It is straightforward to verify that  $\langle \langle A_k, B_k \rangle \mid i \in I \rangle$  belongs to  $X_I$ . This completes the induction step.

It follows that the set  $\Omega_I$  of all families  $\langle \langle A_i, B_i \rangle \mid i \in \Lambda \rangle$  of elements of the Cartesian product  $\Omega = \prod (\mathfrak{P}(C_i) \times \mathfrak{P}(C_i) \mid i \in \Lambda)$  (where  $\mathfrak{P}(X)$  denotes the powerset of a set  $X$ ) whose restriction to  $I$  belongs to  $X_I$  is nonempty, for every finite subset  $I$  of  $\Lambda$ . Endow  $\Omega$  with the product topology of the discrete topologies on all (finite) sets  $\mathfrak{P}(C_i) \times \mathfrak{P}(C_i)$ . By Tychonoff’s Theorem,  $\Omega$  is compact. Hence the intersection of all  $\Omega_I$ , for  $I$  a finite subset of  $\Lambda$ , is nonempty. Let  $\langle \langle A_i, B_i \rangle \mid i \in \Lambda \rangle$  be an element of that intersection. Then the collection of all elements  $\mathbf{a}_i = \bigvee A_i$  and  $\mathbf{b}_i = \bigvee B_i$ , for  $i \in \Lambda$ , satisfies the required conditions.

Assume, finally, that  $S$  is countable. Define an equivalence relation  $\equiv$  on  $\Lambda$  by  $i \equiv j$  iff  $\mathbf{c}_i = \mathbf{c}_j$ , for all  $i, j \in \Lambda$ , and denote by  $[i]$  the  $\equiv$ -equivalence class of  $i$ , for any  $i \in \Lambda$ . Putting  $\mathbf{c}_{[i]} = \mathbf{c}_i$  makes it possible to replace  $\Lambda$  by  $\Lambda/\equiv$ . In particular, as  $S$  is countable,  $\Lambda$  becomes countable as well. Now write  $\Lambda = \bigcup (\Lambda_n \mid n < \omega)$ , where  $\langle \Lambda_n \mid n < \omega \rangle$  is an increasing sequence of finite subsets of  $\Lambda$  with  $|\Lambda_n| = n$ , for all  $n < \omega$ . Denote by  $Y_n$  the set of all families  $\langle \langle \mathbf{a}_l, \mathbf{b}_l \rangle \mid l \in \Lambda_n \rangle$  such that  $\mathbf{a}_i \leq \mathbf{a}$ ,  $\mathbf{b}_i \leq \mathbf{b}$ ,  $\mathbf{c}_i = \mathbf{a}_i \vee \mathbf{b}_i$ , and  $i \leq j$  implies that  $\mathbf{a}_i \leq \mathbf{a}_j$  and  $\mathbf{b}_i \leq \mathbf{b}_j$ , for all  $i \leq j$  in  $\Lambda_n$ . Suppose that we are given an element of  $Y_n$  as above, and denote by  $k$  the unique

element of  $Y_{n+1} \setminus Y_n$ . Suppose, for example, that  $\min \Lambda_n < k < \max \Lambda_n$ , and denote by  $i$  (resp.,  $j$ ) the largest (resp., least) element of  $\Lambda_n$  below  $k$  (resp., above  $k$ ). As  $\mathbf{c}_k \leq \mathbf{c}_j = \mathbf{a}_j \vee \mathbf{b}_j$ , there are  $\mathbf{a}' \leq \mathbf{a}_j$  and  $\mathbf{b}' \leq \mathbf{b}_j$  such that  $\mathbf{c}_k = \mathbf{a}' \vee \mathbf{b}'$ . Put  $\mathbf{a}_k = \mathbf{a}_i \vee \mathbf{a}'$  and  $\mathbf{b}_k = \mathbf{b}_i \vee \mathbf{b}'$ . Then  $\langle \langle \mathbf{a}_l, \mathbf{b}_l \rangle \mid l \in \Lambda_{n+1} \rangle$  is an element of  $Y_{n+1}$ . So every element of  $Y_n$  extends to an element of  $Y_{n+1}$ . The proof is even easier in case either  $k < \min \Lambda_n$  or  $k > \max \Lambda_n$ . Hence we have constructed inductively a family  $\langle \langle \mathbf{a}_i, \mathbf{b}_i \rangle \mid i \in \Lambda \rangle$  whose restriction to  $\Lambda_n$  belongs to  $Y_n$ , for all  $n < \omega$ . Therefore, the elements  $\mathbf{a}_i$  and  $\mathbf{b}_i$ , for  $i \in \Lambda$ , are as required.  $\square$

#### REFERENCES

- [1] T. Jech, “Set Theory”, Academic Press, Harcourt Brace Jovanovich, New York - San Francisco - London, xi + 621 p, 1978.
- [2] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices*, Contributions to General Algebra **10**, Proceedings of the Klagenfurt Conference, May 29 – June 1, 1997. Verlag Johannes Heyn, Klagenfurt 1998.
- [3] P. Růžicka, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, preprint.
- [4] J. Tůma and F. Wehrung, *Unsolvable one-dimensional lifting problems for congruence lattices of lattices*, Forum Math. **14**, no. 4 (2002), 483–493.
- [5] F. Wehrung, *Forcing extensions of partial lattices*, J. Algebra **262**, no. 1 (2003), 127–193.

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