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► **To cite this version:**

Francois Couchot. Localization of injective modules over valuation rings. Proceedings of the American Mathematical Society, 2006, 134, pp.1013–1017. hal-00009475v2

HAL Id: hal-00009475

<https://hal.science/hal-00009475v2>

Submitted on 29 Mar 2006

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LOCALIZATION OF INJECTIVE MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

ABSTRACT. It is proved that E_J is injective if E is an injective module over a valuation ring R , for each prime ideal $J \neq Z$. Moreover, if E or Z is flat, then E_Z is injective too. It follows that localizations of injective modules over h-local Prüfer domains are injective too.

If S is a multiplicative subset of a noetherian ring R , it is well known that $S^{-1}E$ is injective for each injective R -module E . The following example shows that this result is not generally true if R is not noetherian.

Example 1. Let K be a field and I an infinite set. We put $R = K^I$, $J = K^{(I)}$ and $S = \{1 - r \mid r \in J\}$. Then $R/J \cong S^{-1}R$, R is an injective module, but R/J is not injective by [5, Theorem].

However, we shall see that, for some classes of non-noetherian rings, localizations of injective modules are injective too. For instance:

Proposition 2. *Let R be a hereditary ring. For each multiplicative subset S of R and for every injective R -module E , $S^{-1}E$ is injective.*

There exist non-noetherian hereditary rings.

Proof. Let F be the kernel of the natural map: $E \rightarrow S^{-1}E$. Then E/F is injective and S -torsion-free. Let $s \in S$. We have $(0 : s) = Re$, where e is an idempotent of R . It is easy to check that $s + e$ is a non-zero-divisor. So, if $x \in E$, there exists $y \in E$ such that $x = (s + e)y$. Clearly $eE \subseteq F$. Hence $x + F = s(y + F)$. Therefore the multiplication by s in E/F is bijective, whence $E/F \cong S^{-1}E$. \square

In Proposition 2 and Example 1, R is a coherent ring. By [3, Proposition 1.2] $S^{-1}E$ is fp-injective if E is a fp-injective module over a coherent ring R , but the coherence hypothesis can't be omitted: see [3, Example p.344].

The aim of this paper is to study localizations of injective modules and fp-injective modules over a valuation ring R . Let Z be the subset of its zero-divisors. Then Z is a prime ideal. We will show the following theorem:

Theorem 3. *Let R be a valuation ring, denote by Z the set of zero divisors of R and let E be an injective (respectively fp-injective) module. Then:*

- (1) *For each prime ideal $J \neq Z$, E_J is injective (respectively fp-injective).*
- (2) *E_Z is injective (respectively fp-injective) if and only if E or Z is flat.*

1991 *Mathematics Subject Classification.* Primary 13F30, 13C11.

Key words and phrases. valuation ring, injective module, h-local domain, Prüfer domain.

In this paper all rings are associative and commutative with unity and all modules are unital. We say that an R -module E is **divisible** if, for every $r \in R$ and $x \in E$, $(0 : r) \subseteq (0 : x)$ implies that $x \in rE$, and that E is **fp-injective** (or **absolutely pure**) if $\text{Ext}_R^1(F, E) = 0$, for every finitely presented R -module F . A ring R is called **self fp-injective** if it is fp-injective as R -module. An exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is **pure** if it remains exact when tensoring it with any R -module. In this case we say that F is a **pure submodule** of E . Recall that a module E is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be **uniserial** if its submodules are linearly ordered by inclusion and a ring R is a **valuation ring** if it is uniserial as R -module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [7, Theorem 1]. Consequently a module E over a valuation ring R is fp-injective if and only if it is divisible.

An R -module F is **pure-injective** if for every pure exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

of R -modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. Then a module is injective if and only if it is pure-injective and fp-injective. A ring R is said to be an **IF-ring** if every injective module is flat. By [1, Theorem 2] R is an IF-ring if and only if R is coherent and self fp-injective.

In the sequel R is a valuation ring whose maximal ideal is P and Z is its subset of zerodivisors. Some preliminary results are needed to show Theorem 3.

Proposition 4. *Let R be a valuation ring, let E be an injective module and $r \in P$. Then E/rE is injective over R/rR .*

Proof. Let J be an ideal of R such that $Rr \subset J$ and $g : J/Rr \rightarrow E/rE$ be a nonzero homomorphism. For each $x \in E$ we denote by \bar{x} the image of x in E/rE . Let $a \in J \setminus Rr$ such that $\bar{y} = g(\bar{a}) \neq 0$. Then $(Rr : a) \subseteq (rE : y)$. Let $t \in R$ such that $r = at$. Thus $ty = rz$ for some $z \in E$. It follows that $t(y - az) = 0$. So, since $at = r \neq 0$, we have $(0 : a) \subset Rt \subseteq (0 : y - az)$. The injectivity of E implies that there exists $x \in E$ such that $y = a(x + z)$. We put $x_a = x + z$. If $b \in J \setminus Ra$ then $a(x_a - x_b) \in rE$. Hence $x_b \in x_a + (rE :_E a)$. Since E is pure-injective, by [6, Theorem 4] there exists $x \in \bigcap_{a \in J} x_a + (rE :_E a)$. It follows that $g(\bar{a}) = a\bar{x}$ for each $a \in J$. \square

Lemma 5. *Let R be a valuation ring, let U be a module and F a flat module. Then, for each $r, s \in R$, $F \otimes_R (sU :_U r) \cong (F \otimes_R sU :_{F \otimes_R U} r)$.*

Proof. We put $E = F \otimes_R U$. Let ϕ be the composition of the multiplication by r in U with the natural map $U \rightarrow U/sU$. Then $(sU :_U r) = \ker(\phi)$. It follows that $F \otimes_R (sU :_U r)$ is isomorphic to $\ker(\mathbf{1}_F \otimes \phi)$ since F is flat. We easily check that $\mathbf{1}_F \otimes \phi$ is the composition of the multiplication by r in E with the natural map $E \rightarrow E/sE$. It follows that $F \otimes_R (sU :_U r) \cong (sE :_E r)$. \square

Proposition 6. *Let R be a valuation ring. Then every pure-injective R -module F satisfies the following property: if $(x_i)_{i \in I}$ is a family of elements of F and $(A_i)_{i \in I}$ a family of ideals of R such that the family $\mathcal{F} = (x_i + A_i F)_{i \in I}$ has the finite intersection property, then \mathcal{F} has a non-empty intersection. The converse holds if F is flat.*

Proof. Let $i \in I$ such that A_i is not finitely generated. By [2, Lemma 29] either $A_i = Pr_i$ or $A_i = \bigcap_{c \in R \setminus A_i} cR$. If, $\forall i \in I$ such that A_i is not finitely generated, we replace $x_i + A_i F$ by $x_i + r_i F$ in the first case, and by the family $(x_i + cF)_{c \in R \setminus A_i}$ in the second case, we deduce from \mathcal{F} a family \mathcal{G} which has the finite intersection property. Since F is pure-injective, it follows that there exists $x \in F$ which belongs to each element of the family \mathcal{G} by [6, Theorem 4]. We may assume that the family $(A_i)_{i \in I}$ has no smallest element. So, if A_i is not finitely generated, there exists $j \in I$ such that $A_j \subset A_i$. Let $c \in A_i \setminus PA_j$ such that $x_j + cF \in \mathcal{G}$. Then $x - x_j \in cF \subseteq A_i F$ and $x_j - x_i \in A_i F$. Hence $x - x_i \in A_i F$ for each $i \in I$.

Conversely, if F is flat then by Lemma 5 we have $(sF :_F r) = (sR : r)F$ for each $s, r \in R$. We use [6, Theorem 4] to conclude. \square

Proposition 7. *Let R be a valuation ring and let F be a flat pure-injective module. Then:*

- (1) $F \otimes_R U$ is pure-injective if U is a uniserial module.
- (2) For each prime ideal J , F_J is pure-injective.

Proof.

(1). Let $E = F \otimes_R U$. We use [6, Theorem 4] to prove that E is pure-injective. Let $(x_i)_{i \in I}$ be a family of elements of F such that the family $\mathcal{F} = (x_i + N_i)_{i \in I}$ has the finite intersection property, where $N_i = (s_i E :_E r_i)$ and $r_i, s_i \in R, \forall i \in I$.

First we assume that $U = R/A$ where A is a proper ideal of R . So $E \cong F/AF$. If $s_i \notin A$ then $N_i = (s_i F :_F r_i)/AF = (Rs_i : r_i)F/AF$. We set $A_i = (Rs_i : r_i)$ in this case. If $s_i \in A$ then $N_i = (AF :_F r_i)/AF = (A : r_i)F/AF$. We put $A_i = (A : r_i)$ in this case. For each $i \in I$, let $y_i \in F$ such that $x_i = y_i + AF$. It is obvious that the family $(y_i + A_i F)_{i \in I}$ has the finite intersection property. By Proposition 6 this family has a non-empty intersection. Then \mathcal{F} has a non-empty intersection too.

Now we assume that U is not finitely generated. It is obvious that \mathcal{F} has a non-empty intersection if $x_i + N_i = E, \forall i \in I$. Now assume there exists $i_0 \in I$ such that $x_{i_0} + N_{i_0} \neq E$. Let $I' = \{i \in I \mid N_i \subseteq N_{i_0}\}$ and $\mathcal{F}' = (x_i + N_i)_{i \in I'}$. Then \mathcal{F} and \mathcal{F}' have the same intersection. By Lemma 5 $N_{i_0} = F \otimes_R (s_{i_0} U :_U r_{i_0})$. It follows that $(s_{i_0} U :_U r_{i_0}) \subset U$ because $N_{i_0} \neq E$. Hence $\exists u \in U$ such that $x_{i_0} + N_{i_0} \subseteq F \otimes_R Ru$. Then, $\forall i \in I', x_i + N_i \subseteq F \otimes_R Ru$. We have $F \otimes_R Ru \cong F/(0 : u)F$. From the first part of the proof $F/(0 : u)F$ is pure-injective. So we may replace R with $R/(0 : u)$ and assume that $(0 : u) = 0$. Let $A_i = ((s_i U :_U r_i) : u), \forall i \in I'$. Thus $N_i = A_i F, \forall i \in I'$. By Proposition 6 \mathcal{F}' has a non-empty intersection. So \mathcal{F} has a non-empty intersection too.

(2). We apply (1) by taking $U = R_J$. \square

Proof of Theorem 3

Let J be a prime ideal and E a module. If E is fp-injective, E is a pure submodule of an injective module M . It follows that E_J is a pure submodule of M_J . So, if M_J is injective we conclude that E_J is fp-injective. Now we assume that E is injective.

(1). Suppose that $J \subset Z$. Let $s \in Z \setminus J$. Then there exists $0 \neq r \in J$ such that $sr = 0$. Hence rE is contained in the kernel of the natural map: $E \rightarrow E_J$. Moreover $R_J = (R/rR)_J$ and $E_J = (E/rE)_J$. By Proposition 4, E/rE is injective over R/rR and by [2, Theorem 11] R/rR is an IF-ring. So E/rE is flat over R/rR . From Proposition 7 we deduce that E_J is pure-injective and by [3, Proposition 1.2] E_J is fp-injective. So E_J is injective.

Assume that $Z \subset J$. We set

$$F = \{x \in E \mid J \subset (0 : x)\} \quad \text{and} \quad G = \{x \in E \mid J \subseteq (0 : x)\}.$$

Let $x \in E$ and $s \in R \setminus J$ such that $sx \in F$ (respectively G). Then $sJ \subset (0 : x)$ (respectively $sJ \subseteq (0 : x)$). Since $s \notin J$ we have $sJ = J$. Consequently $x \in F$ (respectively G). Thus the multiplication by s in E/F (and E/G) is bijective because E is injective. So E/F and E/G are modules over R_J and $E_J \cong E/F$. We have $G \cong \text{Hom}_R(R/J, E)$. It follows that $E/G \cong \text{Hom}_R(J, E)$. But J is a flat module. Thus E/G is injective. Let A be an ideal of R_J and $f : A \rightarrow E/F$ an homomorphism. Then there exists an homomorphism $g : R_J \rightarrow E/G$ such that $g \circ u = p \circ f$ where $u : A \rightarrow R_J$ and $p : E/F \rightarrow E/G$ are the natural maps. It follows that there exists an homomorphism $h : R_J \rightarrow E/F$ such that $g = p \circ h$. It is easy to check that $p \circ (f - h \circ u) = 0$. So there exists an homomorphism $\ell : A \rightarrow G/F$ such that $v \circ \ell = f - h \circ u$ where $v : G/F \rightarrow E/F$ is the inclusion map. First assume that A is finitely generated over R_J . We have $A = R_J a$. If $0 \neq \ell(a) = y + F$, where $y \in G$, then $(0 : a) \subseteq Z \subseteq J = (0 : y)$. Since E is injective there exists $x \in E$ such that $y = ax$. Hence $f(a) = a(h(1) + (x + F))$. Now suppose that A is not finitely generated over R_J . If $a \in A$ then there exist $b \in A$ and $r \in J$ such that $a = rb$. We get that $\ell(a) = r\ell(b) = 0$. Hence $f = h \circ u$.

(2). Let the notations be as above. Then $E_Z = E/F$. If Z is flat, we do as above to show that E_Z is injective. If E is flat then $F = 0$, whence $E_Z = E$. Now, assume that E_Z is fp-injective and Z is not flat. By [2, Theorem 10] R_Z is an IF-ring. It follows that E_Z is flat. Consequently F is a pure submodule of E . Suppose there exists $0 \neq x \in F$. If $0 \neq s \in Z$ then $(0 : s) \subseteq Z \subset (0 : x)$. So, there exists $y \in E$ such that $x = sy$. By [2, Lemma 2] $(0 : y) = s(0 : x) \subseteq Z$. Since F is a pure submodule, we may assume that $y \in F$. Whence $Z \subset (0 : y)$. We get a contradiction. Hence $F = 0$ and E is flat. \square

Now we give a consequence of Theorem 3. Recall that a domain R is said to be **h-local** if R/I is semilocal for every nonzero ideal I , and if R/P is local for every nonzero prime ideal P , [4].

Corollary 8. *Let R be a h-local Prüfer domain. For each multiplicative subset S of R and for every injective R -module E , $S^{-1}E$ is injective.*

Proof. By [4, Theorem 24] E_P is injective for each maximal ideal P . Since R_P is a valuation domain, we deduce from Theorem 3 that E_J is injective for each prime ideal J . It is easy to check that $S^{-1}R$ is a h-local Prüfer domain. So, by [4, Theorem 24] $S^{-1}E$ is injective. \square

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