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Francois Couchot. Localization of injective modules over valuation rings. Proceedings of the American Mathematical Society, 2006, 134, pp.1013–1017. hal-00009475v2

**HAL Id: hal-00009475**

**<https://hal.science/hal-00009475v2>**

Submitted on 29 Mar 2006

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# LOCALIZATION OF INJECTIVE MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

**ABSTRACT.** It is proved that  $E_J$  is injective if  $E$  is an injective module over a valuation ring  $R$ , for each prime ideal  $J \neq Z$ . Moreover, if  $E$  or  $Z$  is flat, then  $E_Z$  is injective too. It follows that localizations of injective modules over h-local Prüfer domains are injective too.

If  $S$  is a multiplicative subset of a noetherian ring  $R$ , it is well known that  $S^{-1}E$  is injective for each injective  $R$ -module  $E$ . The following example shows that this result is not generally true if  $R$  is not noetherian.

**Example 1.** Let  $K$  be a field and  $I$  an infinite set. We put  $R = K^I$ ,  $J = K^{(I)}$  and  $S = \{1 - r \mid r \in J\}$ . Then  $R/J \cong S^{-1}R$ ,  $R$  is an injective module, but  $R/J$  is not injective by [5, Theorem].

However, we shall see that, for some classes of non-noetherian rings, localizations of injective modules are injective too. For instance:

**Proposition 2.** *Let  $R$  be a hereditary ring. For each multiplicative subset  $S$  of  $R$  and for every injective  $R$ -module  $E$ ,  $S^{-1}E$  is injective.*

There exist non-noetherian hereditary rings.

**Proof.** Let  $F$  be the kernel of the natural map:  $E \rightarrow S^{-1}E$ . Then  $E/F$  is injective and  $S$ -torsion-free. Let  $s \in S$ . We have  $(0 : s) = Re$ , where  $e$  is an idempotent of  $R$ . It is easy to check that  $s + e$  is a non-zerodivisor. So, if  $x \in E$ , there exists  $y \in E$  such that  $x = (s + e)y$ . Clearly  $eE \subseteq F$ . Hence  $x + F = s(y + F)$ . Therefore the multiplication by  $s$  in  $E/F$  is bijective, whence  $E/F \cong S^{-1}E$ .  $\square$

In Proposition 2 and Example 1,  $R$  is a coherent ring. By [3, Proposition 1.2]  $S^{-1}E$  is fp-injective if  $E$  is a fp-injective module over a coherent ring  $R$ , but the coherence hypothesis can't be omitted: see [3, Example p.344].

The aim of this paper is to study localizations of injective modules and fp-injective modules over a valuation ring  $R$ . Let  $Z$  be the subset of its zerodivisors. Then  $Z$  is a prime ideal. We will show the following theorem:

**Theorem 3.** *Let  $R$  be a valuation ring, denote by  $Z$  the set of zero divisors of  $R$  and let  $E$  be an injective (respectively fp-injective) module. Then:*

- (1) *For each prime ideal  $J \neq Z$ ,  $E_J$  is injective (respectively fp-injective).*
- (2)  *$E_Z$  is injective (respectively fp-injective) if and only if  $E$  or  $Z$  is flat.*

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1991 *Mathematics Subject Classification.* Primary 13F30, 13C11.

*Key words and phrases.* valuation ring, injective module, h-local domain, Prüfer domain.

In this paper all rings are associative and commutative with unity and all modules are unital. We say that an  $R$ -module  $E$  is **divisible** if, for every  $r \in R$  and  $x \in E$ ,  $(0 : r) \subseteq (0 : x)$  implies that  $x \in rE$ , and that  $E$  is **fp-injective** (or **absolutely pure**) if  $\text{Ext}_R^1(F, E) = 0$ , for every finitely presented  $R$ -module  $F$ . A ring  $R$  is called **self fp-injective** if it is fp-injective as  $R$ -module. An exact sequence  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  is **pure** if it remains exact when tensoring it with any  $R$ -module. In this case we say that  $F$  is a **pure** submodule of  $E$ . Recall that a module  $E$  is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be **uniserial** if its submodules are linearly ordered by inclusion and a ring  $R$  is a **valuation ring** if it is uniserial as  $R$ -module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [7, Theorem 1]. Consequently a module  $E$  over a valuation ring  $R$  is fp-injective if and only if it is divisible.

An  $R$ -module  $F$  is **pure-injective** if for every pure exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

of  $R$ -modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. Then a module is injective if and only if it is pure-injective and fp-injective. A ring  $R$  is said to be an **IF-ring** if every injective module is flat. By [1, Theorem 2]  $R$  is an IF-ring if and only if  $R$  is coherent and self fp-injective.

In the sequel  $R$  is a valuation ring whose maximal ideal is  $P$  and  $Z$  is its subset of zerodivisors. Some preliminary results are needed to show Theorem 3.

**Proposition 4.** *Let  $R$  be a valuation ring, let  $E$  be an injective module and  $r \in P$ . Then  $E/rE$  is injective over  $R/rR$ .*

**Proof.** Let  $J$  be an ideal of  $R$  such that  $Rr \subset J$  and  $g : J/Rr \rightarrow E/rE$  be a nonzero homomorphism. For each  $x \in E$  we denote by  $\bar{x}$  the image of  $x$  in  $E/rE$ . Let  $a \in J \setminus Rr$  such that  $\bar{y} = g(\bar{a}) \neq 0$ . Then  $(Rr : a) \subseteq (rE : y)$ . Let  $t \in R$  such that  $r = at$ . Thus  $ty = rz$  for some  $z \in E$ . It follows that  $t(y - az) = 0$ . So, since  $at = r \neq 0$ , we have  $(0 : a) \subset Rt \subseteq (0 : y - az)$ . The injectivity of  $E$  implies that there exists  $x \in E$  such that  $y = a(x + z)$ . We put  $x_a = x + z$ . If  $b \in J \setminus Ra$  then  $a(x_a - x_b) \in rE$ . Hence  $x_b \in x_a + (rE :_E a)$ . Since  $E$  is pure-injective, by [6, Theorem 4] there exists  $x \in \cap_{a \in J} x_a + (rE :_E a)$ . It follows that  $g(\bar{a}) = a\bar{x}$  for each  $a \in J$ .  $\square$

**Lemma 5.** *Let  $R$  be a valuation ring, let  $U$  be a module and  $F$  a flat module. Then, for each  $r, s \in R$ ,  $F \otimes_R (sU :_U r) \cong (F \otimes_R sU :_{F \otimes_R U} r)$ .*

**Proof.** We put  $E = F \otimes_R U$ . Let  $\phi$  be the composition of the multiplication by  $r$  in  $U$  with the natural map  $U \rightarrow U/sU$ . Then  $(sU :_U r) = \ker(\phi)$ . It follows that  $F \otimes_R (sU :_U r)$  is isomorphic to  $\ker(\mathbf{1}_F \otimes \phi)$  since  $F$  is flat. We easily check that  $\mathbf{1}_F \otimes \phi$  is the composition of the multiplication by  $r$  in  $E$  with the natural map  $E \rightarrow E/sE$ . It follows that  $F \otimes_R (sU :_U r) \cong (sE :_E r)$ .  $\square$

**Proposition 6.** *Let  $R$  be a valuation ring. Then every pure-injective  $R$ -module  $F$  satisfies the following property: if  $(x_i)_{i \in I}$  is a family of elements of  $F$  and  $(A_i)_{i \in I}$  a family of ideals of  $R$  such that the family  $\mathcal{F} = (x_i + A_i F)_{i \in I}$  has the finite intersection property, then  $\mathcal{F}$  has a non-empty intersection. The converse holds if  $F$  is flat.*

**Proof.** Let  $i \in I$  such that  $A_i$  is not finitely generated. By [2, Lemma 29] either  $A_i = Pr_i$  or  $A_i = \cap_{c \in R \setminus A_i} cR$ . If,  $\forall i \in I$  such that  $A_i$  is not finitely generated, we replace  $x_i + A_i F$  by  $x_i + r_i F$  in the first case, and by the family  $(x_i + cF)_{c \in R \setminus A_i}$  in the second case, we deduce from  $\mathcal{F}$  a family  $\mathcal{G}$  which has the finite intersection property. Since  $F$  is pure-injective, it follows that there exists  $x \in F$  which belongs to each element of the family  $\mathcal{G}$  by [6, Theorem 4]. We may assume that the family  $(A_i)_{i \in I}$  has no smallest element. So, if  $A_i$  is not finitely generated, there exists  $j \in I$  such that  $A_j \subset A_i$ . Let  $c \in A_i \setminus PA_j$  such that  $x_j + cF \in \mathcal{G}$ . Then  $x - x_j \in cF \subseteq A_i F$  and  $x_j - x_i \in A_i F$ . Hence  $x - x_i \in A_i F$  for each  $i \in I$ .

Conversely, if  $F$  is flat then by Lemma 5 we have  $(sF :_F r) = (sR : r)F$  for each  $s, r \in R$ . We use [6, Theorem 4] to conclude.  $\square$

**Proposition 7.** *Let  $R$  be a valuation ring and let  $F$  be a flat pure-injective module. Then:*

- (1)  $F \otimes_R U$  is pure-injective if  $U$  is a uniserial module.
- (2) For each prime ideal  $J$ ,  $F_J$  is pure-injective.

**Proof.**

(1). Let  $E = F \otimes_R U$ . We use [6, Theorem 4] to prove that  $E$  is pure-injective. Let  $(x_i)_{i \in I}$  be a family of elements of  $F$  such that the family  $\mathcal{F} = (x_i + N_i)_{i \in I}$  has the finite intersection property, where  $N_i = (s_i E :_E r_i)$  and  $r_i, s_i \in R$ ,  $\forall i \in I$ .

First we assume that  $U = R/A$  where  $A$  is a proper ideal of  $R$ . So  $E \cong F/AF$ . If  $s_i \notin A$  then  $N_i = (s_i F :_F r_i)/AF = (Rs_i : r_i)F/AF$ . We set  $A_i = (Rs_i : r_i)$  in this case. If  $s_i \in A$  then  $N_i = (AF :_F r_i)/AF = (A : r_i)F/AF$ . We put  $A_i = (A : r_i)$  in this case. For each  $i \in I$ , let  $y_i \in F$  such that  $x_i = y_i + AF$ . It is obvious that the family  $(y_i + A_i F)_{i \in I}$  has the finite intersection property. By Proposition 6 this family has a non-empty intersection. Then  $\mathcal{F}$  has a non-empty intersection too.

Now we assume that  $U$  is not finitely generated. It is obvious that  $\mathcal{F}$  has a non-empty intersection if  $x_i + N_i = E$ ,  $\forall i \in I$ . Now assume there exists  $i_0 \in I$  such that  $x_{i_0} + N_{i_0} \neq E$ . Let  $I' = \{i \in I \mid N_i \subseteq N_{i_0}\}$  and  $\mathcal{F}' = (x_i + N_i)_{i \in I'}$ . Then  $\mathcal{F}$  and  $\mathcal{F}'$  have the same intersection. By Lemma 5  $N_{i_0} = F \otimes_R (s_{i_0} U :_U r_{i_0})$ . It follows that  $(s_{i_0} U :_U r_{i_0}) \subset U$  because  $N_{i_0} \neq E$ . Hence  $\exists u \in U$  such that  $x_{i_0} + N_{i_0} \subseteq F \otimes_R Ru$ . Then,  $\forall i \in I'$ ,  $x_i + N_i \subseteq F \otimes_R Ru$ . We have  $F \otimes_R Ru \cong F/(0 : u)F$ . From the first part of the proof  $F/(0 : u)F$  is pure-injective. So we may replace  $R$  with  $R/(0 : u)$  and assume that  $(0 : u) = 0$ . Let  $A_i = ((s_i U :_U r_i) : u)$ ,  $\forall i \in I'$ . Thus  $N_i = A_i F$ ,  $\forall i \in I'$ . By Proposition 6  $\mathcal{F}'$  has a non-empty intersection. So  $\mathcal{F}$  has a non-empty intersection too.

(2). We apply (1) by taking  $U = R_J$ .  $\square$

### Proof of Theorem 3

Let  $J$  be a prime ideal and  $E$  a module. If  $E$  is fp-injective,  $E$  is a pure submodule of an injective module  $M$ . It follows that  $E_J$  is a pure submodule of  $M_J$ . So, if  $M_J$  is injective we conclude that  $E_J$  is fp-injective. Now we assume that  $E$  is injective.

(1). Suppose that  $J \subset Z$ . Let  $s \in Z \setminus J$ . Then there exists  $0 \neq r \in J$  such that  $sr = 0$ . Hence  $rE$  is contained in the kernel of the natural map:  $E \rightarrow E_J$ . Moreover  $R_J = (R/rR)_J$  and  $E_J = (E/rE)_J$ . By Proposition 4,  $E/rE$  is injective over  $R/rR$  and by [2, Theorem 11]  $R/rR$  is an IF-ring. So  $E/rE$  is flat over  $R/rR$ . From Proposition 7 we deduce that  $E_J$  is pure-injective and by [3, Proposition 1.2]  $E_J$  is fp-injective. So  $E_J$  is injective.

Assume that  $Z \subset J$ . We set

$$F = \{x \in E \mid J \subset (0 : x)\} \quad \text{and} \quad G = \{x \in E \mid J \subseteq (0 : x)\}.$$

Let  $x \in E$  and  $s \in R \setminus J$  such that  $sx \in F$  (respectively  $G$ ). Then  $sJ \subset (0 : x)$  (respectively  $sJ \subseteq (0 : x)$ ). Since  $s \notin J$  we have  $sJ = J$ . Consequently  $x \in F$  (respectively  $G$ ). Thus the multiplication by  $s$  in  $E/F$  (and  $E/G$ ) is bijective because  $E$  is injective. So  $E/F$  and  $E/G$  are modules over  $R_J$  and  $E_J \cong E/F$ . We have  $G \cong \text{Hom}_R(R/J, E)$ . It follows that  $E/G \cong \text{Hom}_R(J, E)$ . But  $J$  is a flat module. Thus  $E/G$  is injective. Let  $A$  be an ideal of  $R_J$  and  $f : A \rightarrow E/F$  an homomorphism. Then there exists an homomorphism  $g : R_J \rightarrow E/G$  such that  $g \circ u = p \circ f$  where  $u : A \rightarrow R_J$  and  $p : E/F \rightarrow E/G$  are the natural maps. It follows that there exists an homomorphism  $h : R_J \rightarrow E/F$  such that  $g = p \circ h$ . It is easy to check that  $p \circ (f - h \circ u) = 0$ . So there exists an homomorphism  $\ell : A \rightarrow G/F$  such that  $v \circ \ell = f - h \circ u$  where  $v : G/F \rightarrow E/F$  is the inclusion map. First assume that  $A$  is finitely generated over  $R_J$ . We have  $A = R_J a$ . If  $0 \neq \ell(a) = y + F$ , where  $y \in G$ , then  $(0 : a) \subseteq Z \subseteq J = (0 : y)$ . Since  $E$  is injective there exists  $x \in E$  such that  $y = ax$ . Hence  $f(a) = a(h(1) + (x + F))$ . Now suppose that  $A$  is not finitely generated over  $R_J$ . If  $a \in A$  then there exist  $b \in A$  and  $r \in J$  such that  $a = rb$ . We get that  $\ell(a) = r\ell(b) = 0$ . Hence  $f = h \circ u$ .

(2). Let the notations be as above. Then  $E_Z = E/F$ . If  $Z$  is flat, we do as above to show that  $E_Z$  is injective. If  $E$  is flat then  $F = 0$ , whence  $E_Z = E$ . Now, assume that  $E_Z$  is fp-injective and  $Z$  is not flat. By [2, Theorem 10]  $R_Z$  is an IF-ring. It follows that  $E_Z$  is flat. Consequently  $F$  is a pure submodule of  $E$ . Suppose there exists  $0 \neq x \in F$ . If  $0 \neq s \in Z$  then  $(0 : s) \subseteq Z \subset (0 : x)$ . So, there exists  $y \in E$  such that  $x = sy$ . By [2, Lemma 2]  $(0 : y) = s(0 : x) \subseteq Z$ . Since  $F$  is a pure submodule, we may assume that  $y \in F$ . Whence  $Z \subset (0 : y)$ . We get a contradiction. Hence  $F = 0$  and  $E$  is flat.  $\square$

Now we give a consequence of Theorem 3. Recall that a domain  $R$  is said to be **h-local** if  $R/I$  is semilocal for every nonzero ideal  $I$ , and if  $R/P$  is local for every nonzero prime ideal  $P$ , [4].

**Corollary 8.** *Let  $R$  be a h-local Prüfer domain. For each multiplicative subset  $S$  of  $R$  and for every injective  $R$ -module  $E$ ,  $S^{-1}E$  is injective.*

**Proof.** By [4, Theorem 24]  $E_P$  is injective for each maximal ideal  $P$ . Since  $R_P$  is a valuation domain, we deduce from Theorem 3 that  $E_J$  is injective for each prime ideal  $J$ . It is easy to check that  $S^{-1}R$  is a h-local Prüfer domain. So, by [4, Theorem 24]  $S^{-1}E$  is injective.  $\square$

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