



HAL
open science

Localization of injective modules over valuation rings

Francois Couchot

► **To cite this version:**

| Francois Couchot. Localization of injective modules over valuation rings. 2005. hal-00009475v1

HAL Id: hal-00009475

<https://hal.science/hal-00009475v1>

Preprint submitted on 4 Oct 2005 (v1), last revised 29 Mar 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LOCALIZATION OF INJECTIVE MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

ABSTRACT. It is proved that E_J is injective if E is an injective module over a valuation ring R , for each prime ideal $J \neq Z$. Moreover, if E or Z is flat, then E_Z is injective too. It follows that localizations of injective modules over h-local Prüfer domains are injective too.

If S is a multiplicative subset of a noetherian ring R , it is well known that $S^{-1}E$ is injective for each injective R -module E . The following example shows that this result is not generally true if R is not noetherian.

Example 1. Let K be a field and I an infinite set. We put $R = K^I$, $J = K^{(I)}$ and $S = \{1 - r \mid r \in J\}$. Then $R/J \cong S^{-1}R$, R is an injective module, but R/J is not injective by [5, Theorem].

However, we shall see that, for some classes of non-noetherian rings, localizations of injective modules are injective too. For instance:

Proposition 2. *Let R be a hereditary ring. For each multiplicative subset S of R and for every injective R -module E , $S^{-1}E$ is injective.*

There exist non-noetherian hereditary rings.

Proof. Let F be the kernel of the natural map: $E \rightarrow S^{-1}E$. Then E/F is injective and S -torsion-free. Let $s \in S$. We have $(0 : s) = Re$, where e is an idempotent of R . It is easy to check that $s + e$ is a non-zero-divisor. So, if $x \in E$, there exists $y \in E$ such that $x = (s + e)y$. Clearly $eE \subseteq F$. Hence $x + F = s(y + F)$. Therefore the multiplication by s in E/F is bijective, whence $E/F \cong S^{-1}E$. \square

In Proposition 2 and Example 1, R is a coherent ring. By [3, Proposition 1.2] $S^{-1}E$ is fp-injective if E is a fp-injective module over a coherent ring R , but the coherence hypothesis can't be omitted: see [3, Example p.344].

The aim of this paper is to study localizations of injective modules and fp-injective modules over a valuation ring R . Let Z be the subset of its zero-divisors. Then Z is a prime ideal. We will show the following theorem:

Theorem 3. *Let R be a valuation ring, denote by Z the set of zero divisors of R and let E be an injective (respectively fp-injective) module. Then:*

- (1) *For each prime ideal $J \neq Z$, E_J is injective (respectively fp-injective).*
- (2) *E_Z is injective (respectively fp-injective) if and only if E or Z is flat.*

1991 *Mathematics Subject Classification.* Primary 13F30, 13C11.

Key words and phrases. valuation ring, injective module, h-local domain, Prüfer domain.

In this paper all rings are associative and commutative with unity and all modules are unital. We say that an R -module E is **divisible** if, for every $r \in R$ and $x \in E$, $(0 : r) \subseteq (0 : x)$ implies that $x \in rE$, and that E is **fp-injective** (or **absolutely pure**) if $\text{Ext}_R^1(F, E) = 0$, for every finitely presented R -module F . A ring R is called **self fp-injective** if it is fp-injective as R -module. An exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is **pure** if it remains exact when tensoring it with any R -module. In this case we say that F is a **pure** submodule of E . Recall that a module E is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be **uniserial** if its submodules are linearly ordered by inclusion and a ring R is a **valuation ring** if it is uniserial as R -module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [7, Theorem 1]. Consequently a module E over a valuation ring R is fp-injective if and only if it is divisible.

An R -module F is **pure-injective** if for every pure exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

of R -modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. Then a module is injective if and only if it is pure-injective and fp-injective. A ring R is said to be an **IF-ring** if every injective module is flat. By [1, Theorem 2] R is an IF-ring if and only if R is coherent and self fp-injective.

In the sequel R is a valuation ring whose maximal ideal is P and Z is its subset of zerodivisors. Some preliminary results are needed to show Theorem 3.

Proposition 4. *Let R be a valuation ring, let E be an injective module and $r \in P$. Then E/rE is injective over R/rR .*

Proof. Let J be an ideal of R such that $Rr \subset J$ and $g : J/Rr \rightarrow E/rE$ be a nonzero homomorphism. For each $x \in E$ we denote by \bar{x} the image of x in E/rE . Let $a \in J \setminus Rr$ such that $\bar{y} = g(\bar{a}) \neq 0$. Then $(Rr : a) \subseteq (rE : y)$. Let $t \in R$ such that $r = at$. Thus $ty = rz$ for some $z \in E$. It follows that $t(y - az) = 0$. So, since $at = r \neq 0$, we have $(0 : a) \subset Rt \subseteq (0 : y - az)$. The injectivity of E implies that there exists $x \in E$ such that $y = a(x + z)$. We put $x_a = x + z$. If $b \in J \setminus Ra$ then $a(x_a - x_b) \in rE$. Hence $x_b \in x_a + (rE :_E a)$. Since E is pure-injective, by [6, Theorem 4] there exists $x \in \bigcap_{a \in J} x_a + (rE :_E a)$. It follows that $g(\bar{a}) = a\bar{x}$ for each $a \in J$. \square

Lemma 5. *Let R be a valuation ring, let U be a module and F a flat module. Then, for each $r, s \in R$, $F \otimes_R (sU :_U r) \cong (F \otimes_R sU :_{F \otimes_R U} r)$.*

Proof. We put $E = F \otimes_R U$. Let ϕ be the composition of the multiplication by r in U with the natural map $U \rightarrow U/sU$. Then $(sU :_U r) = \ker(\phi)$. It follows that $F \otimes_R (sU :_U r)$ is isomorphic to $\ker(\mathbf{1}_F \otimes \phi)$ since F is flat. We easily check that $\mathbf{1}_F \otimes \phi$ is the composition of the multiplication by r in E with the natural map $E \rightarrow E/sE$. It follows that $F \otimes_R (sU :_U r) \cong (sE :_E r)$. \square

Proposition 6. *Let R be a valuation ring. Then every pure-injective R -module F satisfies the following property: if $(x_i)_{i \in I}$ is a family of elements of F and $(A_i)_{i \in I}$ a family of ideals of R such that the family $\mathcal{F} = (x_i + A_i F)_{i \in I}$ has the finite intersection property, then \mathcal{F} has a non-empty intersection. The converse holds if F is flat.*

Proof. Let $i \in I$ such that A_i is not finitely generated. By [2, Lemma 29] either $A_i = Pr_i$ or $A_i = \bigcap_{c \in R \setminus A_i} cR$. If, $\forall i \in I$ such that A_i is not finitely generated, we replace $x_i + A_i F$ by $x_i + r_i F$ in the first case, and by the family $(x_i + cF)_{c \in R \setminus A_i}$ in the second case, we deduce from \mathcal{F} a family \mathcal{G} which has the finite intersection property. Since F is pure-injective, it follows that there exists $x \in F$ which belongs to each element of the family \mathcal{G} by [6, Theorem 4]. We may assume that the family $(A_i)_{i \in I}$ has no smallest element. So, if A_i is not finitely generated, there exists $j \in I$ such that $A_j \subset A_i$. Let $c \in A_i \setminus PA_j$ such that $x_j + cF \in \mathcal{G}$. Then $x - x_j \in cF \subseteq A_i F$ and $x_j - x_i \in A_i F$. Hence $x - x_i \in A_i F$ for each $i \in I$.

Conversely, if F is flat then by Lemma 5 we have $(sF :_F r) = (sR : r)F$ for each $s, r \in R$. We use [6, Theorem 4] to conclude. \square

Proposition 7. *Let R be a valuation ring and let F be a flat pure-injective module. Then:*

- (1) $F \otimes_R U$ is pure-injective if U is a uniserial module.
- (2) For each prime ideal J , F_J is pure-injective.

Proof.

(1). Let $E = F \otimes_R U$. We use [6, Theorem 4] to prove that E is pure-injective. Let $(x_i)_{i \in I}$ be a family of elements of F such that the family $\mathcal{F} = (x_i + N_i)_{i \in I}$ has the finite intersection property, where $N_i = (s_i E :_E r_i)$ and $r_i, s_i \in R, \forall i \in I$.

First we assume that $U = R/A$ where A is a proper ideal of R . So $E \cong F/AF$. If $s_i \notin A$ then $N_i = (s_i F :_F r_i)/AF = (Rs_i : r_i)F/AF$. We set $A_i = (Rs_i : r_i)$ in this case. If $s_i \in A$ then $N_i = (AF :_F r_i)/AF = (A : r_i)F/AF$. We put $A_i = (A : r_i)$ in this case. For each $i \in I$, let $y_i \in F$ such that $x_i = y_i + AF$. It is obvious that the family $(y_i + A_i F)_{i \in I}$ has the finite intersection property. By Proposition 6 this family has a non-empty intersection. Then \mathcal{F} has a non-empty intersection too.

Now we assume that U is not finitely generated. It is obvious that \mathcal{F} has a non-empty intersection if $x_i + N_i = E, \forall i \in I$. Now assume there exists $i_0 \in I$ such that $x_{i_0} + N_{i_0} \neq E$. Let $I' = \{i \in I \mid N_i \subseteq N_{i_0}\}$ and $\mathcal{F}' = (x_i + N_i)_{i \in I'}$. Then \mathcal{F} and \mathcal{F}' have the same intersection. By Lemma 5 $N_{i_0} = F \otimes_R (s_{i_0} U :_U r_{i_0})$. It follows that $(s_{i_0} U :_U r_{i_0}) \subset U$ because $N_{i_0} \neq E$. Hence $\exists u \in U$ such that $x_{i_0} + N_{i_0} \subseteq F \otimes_R Ru$. Then, $\forall i \in I', x_i + N_i \subseteq F \otimes_R Ru$. We have $F \otimes_R Ru \cong F/(0 : u)F$. From the first part of the proof $F/(0 : u)F$ is pure-injective. So we may replace R with $R/(0 : u)$ and assume that $(0 : u) = 0$. Let $A_i = ((s_i U :_U r_i) : u), \forall i \in I'$. Thus $N_i = A_i F, \forall i \in I'$. By Proposition 6 \mathcal{F}' has a non-empty intersection. So \mathcal{F} has a non-empty intersection too.

- (2). We apply (1) by taking $U = R_J$. \square

Proof of Theorem 3

Let J be a prime ideal and E a module. If E is fp-injective, E is a pure submodule of an injective module M . It follows that E_J is a pure submodule of M_J . So, if M_J is injective we conclude that E_J is fp-injective. Now we assume that E is injective.

(1). Suppose that $J \subset Z$. Let $s \in Z \setminus J$. Then there exists $0 \neq r \in J$ such that $sr = 0$. Hence rE is contained in the kernel of the natural map: $E \rightarrow E_J$. Moreover $R_J = (R/rR)_J$ and $E_J = (E/rE)_J$. By Proposition 4, E/rE is injective over R/rR and by [2, Theorem 11] R/rR is an IF-ring. So E/rE is flat over R/rR . From Proposition 7 we deduce that E_J is pure-injective and by [3, Proposition 1.2] E_J is fp-injective. So E_J is injective.

Assume that $Z \subset J$. We set

$$F = \{x \in E \mid J \subset (0 : x)\} \quad \text{and} \quad G = \{x \in E \mid J \subseteq (0 : x)\}.$$

Let $x \in E$ and $s \in R \setminus J$ such that $sx \in F$ (respectively G). Then $sJ \subset (0 : x)$ (respectively $sJ \subseteq (0 : x)$). Since $s \notin J$ we have $sJ = J$. Consequently $x \in F$ (respectively G). Thus the multiplication by s in E/F (and E/G) is bijective because E is injective. So E/F and E/G are modules over R_J and $E_J \cong E/F$. We have $G \cong \text{Hom}_R(R/J, E)$. It follows that $E/G \cong \text{Hom}_R(J, E)$. But J is a flat module. Thus E/G is injective. Let A be an ideal of R_J and $f : A \rightarrow E/F$ an homomorphism. Then there exists an homomorphism $g : R_J \rightarrow E/G$ such that $g \circ u = p \circ f$ where $u : A \rightarrow R_J$ and $p : E/F \rightarrow E/G$ are the natural maps. It follows that there exists an homomorphism $h : R_J \rightarrow E/F$ such that $g = p \circ h$. It is easy to check that $p \circ (f - h \circ u) = 0$. So there exists an homomorphism $\ell : A \rightarrow G/F$ such that $v \circ \ell = f - h \circ u$ where $v : G/F \rightarrow E/F$ is the inclusion map. First assume that A is finitely generated over R_J . We have $A = R_J a$. If $0 \neq \ell(a) = y + F$, where $y \in G$, then $(0 : a) \subseteq Z \subseteq J = (0 : y)$. Since E is injective there exists $x \in E$ such that $y = ax$. Hence $f(a) = a(h(1) + (x + F))$. Now suppose that A is not finitely generated over R_J . If $a \in A$ then there exist $b \in A$ and $r \in J$ such that $a = rb$. We get that $\ell(a) = r\ell(b) = 0$. Hence $f = h \circ u$.

(2). Let the notations be as above. Then $E_Z = E/F$. If Z is flat, we do as above to show that E_Z is injective. If E is flat then $F = 0$, whence $E_Z = E$. Now, assume that E_Z is fp-injective and Z is not flat. By [2, Theorem 10] R_Z is an IF-ring. It follows that E_Z is flat. Consequently F is a pure submodule of E . Suppose there exists $0 \neq x \in F$. If $0 \neq s \in Z$ then $(0 : s) \subseteq Z \subset (0 : x)$. So, there exists $y \in E$ such that $x = sy$. By [2, Lemma 2] $(0 : y) = s(0 : x) \subseteq Z$. Since F is a pure submodule, we may assume that $y \in F$. Whence $Z \subset (0 : y)$. We get a contradiction. Hence $F = 0$ and E is flat. \square

Now we give a consequence of Theorem 3. Recall that a domain R is said to be **h-local** if R/I is semilocal for every nonzero ideal I , and if R/P is local for every nonzero prime ideal P , [4].

Corollary 8. *Let R be a h-local Prüfer domain. For each multiplicative subset S of R and for every injective R -module E , $S^{-1}E$ is injective.*

Proof. By [4, Theorem 24] E_P is injective for each maximal ideal P . Since R_P is a valuation domain, we deduce from Theorem 3 that E_J is injective for each prime ideal J . It is easy to check that $S^{-1}R$ is a h-local Prüfer domain. So, by [4, Theorem 24] $S^{-1}E$ is injective. \square

REFERENCES

- [1] R. Colby, *Flat injective modules*. J. Algebra **35** (1975), 239–252
- [2] F. Couchot, *Injective modules and fp-injective modules over valuations rings*. J. Algebra **267** (2003), 359–376.
- [3] F. Couchot, *Exemples d'anneaux auto-fp-injectifs*. Comm. Algebra **10(4)** (1982), 339–360.
- [4] E. Matlis, *Torsion-free modules*. University of Chicago Press (1972).
- [5] B.L. Osofsky, *Noninjective cyclic modules*. Proc. Am. Math. Soc. **19** (1968), 1383–1384.
- [6] R.B. Warfield, *Purity and algebraic compactness for modules*. Pac. J. Math. **28(3)** (1961), 689–719.
- [7] R. Warfield, *Decomposability of finitely presented modules*. Proc. Amer. Math. Soc. **25** (1970), 167–172.

LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS UMR 6139, DÉPARTEMENT DE
MATHÉMATIQUES ET MÉCANIQUE, 14032 CAEN CEDEX, FRANCE

E-mail address: `couchot@math.unicaen.fr`