Laguerre Process and Generalised Hartman-Watson Law

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LAGUERRE PROCESS AND GENERALIZED HARTMAN-WATSON LAW

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ABSTRACT. : In this paper, we study complex Wishart processes or the so-called *Laguerre processes* (X_t). We give some interest to the behaviour of the eigenvalues process, derive some useful stochastic differential equations and compute both infinitesimal generator and semi-group. We also give absolute-continuity relations between different indices. Then, we compute the density function of the generalized Hartman-Watson law as well as the law of $T_0 := \inf\{t, det(X_t) = 0\}$ when the size $m = 2$.

1. INTRODUCTION

Real Wishart processes have been introduced by M.F.Bru (cf [2]) and studied by Donati, Doumerc, Matsumoto and Yor (cf [7]) as a matrix extension of squared Bessel processes. In their survey, we encounter many expressions in which special functions of symmetric matrix argument appear, such as the Gamma function, modified Bessel functions and hypergeometric functions. However, these special functions are quite complicated: for example, in [7], the authors computed the following Laplace transform:

$$Q_x^{m+1}(\exp\left(\frac{-\nu^2}{2}\int_0^t tr(X_s^{-1})ds\right)|X_t = y) = \frac{\Gamma_m(m)}{\Gamma_m(m+1+2\nu)}det(z)^{\nu/2}\frac{{}_0F_1(m+1+2\nu,z)}{{}_0F_1(m,z)}$$

where $(X_t)_{t\geq 0}$ is a Wishart process of dimension $m+1$ starting at x, and $z = \sqrt{xy}/4t^2$. We don't know how to invert this Laplace transform. Furthermore, under $Q_x^{m-\nu}$, the tail distribution of $T_0 := \inf\{t > 0, det(X_t) = 0\}$ is given by :

$$Q_x^{m-\nu}(T_0 > t) = \frac{\Gamma_m(m)}{\Gamma_m(m+1+2\nu)}det(\frac{x}{2t})^{\nu}{}_1F_1(\nu, m+1+2\nu, -\frac{x}{2t})$$

But, we can not deduce the density of this variable from the above expression since we cannot calculate the derivative of the confluent hypergeometric function, and even integral representation doesn't lead.
Nevertheless, in the complex case, hypergeometric functions of Hermitian matrix argument (as well as orthogonal polynomials) can be expressed in terms of one-dimensional hypergeometric functions. In fact, zonal polynomials are identified with the (normalized) *Schur functions* (cf [21]) defined by

$$s_\lambda(x_1,\ldots,x_n) = \frac{det(x_i^{\lambda_j+n-j})}{det(x_i^{n-j})},$$

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where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $\leq n$ and $\lambda_1 \geq \ldots \geq \lambda_n$. Furthermore, Gross and Richards established the following result (cf [13]):

$$ {}_pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q; X) = \frac{det(x_i^{n-j}\, {}_p\mathscr{F}_q(a_1 - j + 1, \ldots, a_p - j + 1, \ldots, b_q - j + 1; x_i))}{V(X)} $$

where X is a $n \times n$ Hermitian matrix, (x_i) are its eigenvalues, ${}_p\mathscr{F}_q$ denotes the standard hypergeometric functions, ${}_pF_q$ the hypergeometric function with matrix argument and $V(X) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. Consequently, we can use all the properties of standard hypergeometric functions in order to invert the Laplace-transform of the so-called generalized Hartman-Watson law and to calculate the density of $S_0 := \frac{1}{2T_0}$. More precisely, when $m = 2$, integral representations of modified Bessel functions and contiguous relations of the hypergeometric functions (cf [4], [5], [12]) allows us to do this.
The rest of this paper consists of seven sections, which are respectively devoted to the following topics: in section 2, we introduce the Laguerre process of integer dimension. In section 3, we study the behaviour of the eigenvalues process. Then, in section 4, we define the Laguerre process of positive real index. Section 5 is devoted to the absolute-continuity relations, from which we deduce the Laplace transform of the generalized Hartman-Watson law as well as the tail distribution of T_0 in the general case. In section 6, we invert the Laplace transform, and finally, in section 7, we compute the density of S_0.

Remark. In the one-dimensional case, the Hartman-Watson law was computed by Yor (cf [29]) and for squared Bessel processes of index $-\nu$ starting at x, $S_0 \overset{\mathscr{L}}{=} \gamma_\nu / x$ where γ_ν has the following density: $\dfrac{1}{\Gamma(\nu)} u^{\nu-1} e^{-u}$.

2. Laguerre Process of integer index

Let B be a $n \times m$ complex Brownian matrix starting from B_0 , ie, $B = (B_{ij})$ where the entries B_{ij} are independent complex Brownian motions, so we can write $B = B^1 + iB^2$ where B^i, $i = 1, 2$ are two independent real Brownian matrices. We are interesting in the matrix-valued process $X_t := B_t^\star B_t$. This process verifies the following EDS:

$$ dX_t = dB_t^\star B_t + B_t^\star dB_t + 2nIdt \tag{1} $$

Definition. The process (X_t) is called the Laguerre process of size m, of dimension n and starting at $X_0 = B_0^\star B_0$, and will be denoted by $L(n, m, X_0)$.

Remarks. 1/ For $m = 1$, (X_t) is a squared Bessel process of dimension $2n$, denoted by $BESQ(2n, X_0)$.
2/ Set $X = (x_{ij})_{i,j}$. We can easily check that

$$ d(x_t)_{ii} = 2\sqrt{(x_t)_{ii}}\, d\gamma_i(t) + 2ndt \qquad 1 \leq i \leq m, $$

where (γ_i) are independent Brownian motions, thus, x_{ii} is a $BESQ(2n, (X_0)_{ii})$.
3/ From the last equation, we obtain:

$$ d(tr(X_t)) = 2\sqrt{tr(X_t)}\, d\beta_t + 2nmdt \tag{2} $$

where β is a Brownian motion. Consequently, $(tr(X_t))$ is a $BESQ(2nm, tr(X_0))$ of dimension $2nm$ starting from $tr(X_0)$.

We can deduce from equation (1) that for every $i, j, k, l \in \{1, \cdots, m\}$:

$$< dx_{ij}, dx_{kl} >_t = 2(x_{il}\delta_{kj} + x_{kj}\delta_{il})dt$$

Remark. We note that this equation is different from (I-1-5) in [2]: this difference arises from the fact that, for a complex Brownian motion γ, we have $d < \gamma, \gamma >_t = 0$ and $d < \gamma, \overline{\gamma} >_t = 2t$.

2.1. **Infinitesimal generator.** Let $H_m, \tilde{H_m^+}$ be respectively the space of $m \times m$ Hermitian matrices and the space of $m \times m$ definite positive Hermitian matrices. On the space of Hermitian matrix-argument functions, we define the matrix-valued differential operators :

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_{jk}}\right)_{j,k}, \qquad \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y_{jk}}\right)_{j,k}, \qquad \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x_{jk}} - i\frac{\partial}{\partial y_{jk}}\right)_{j,k},$$

We also define:

$$\left(\frac{\partial}{\partial z}\right)^2_{ij} := \sum_k \frac{\partial^2}{\partial z_{ik}\partial z_{kj}}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)_{ij} = \sum_k \frac{\partial^2}{\partial x_{ik}\partial y_{kj}}$$

Proposition 1. *Let f be a function satisfying:*

$$\frac{\partial f}{\partial x_{ij}} = \frac{\partial f}{\partial x_{ji}}, \quad \frac{\partial f}{\partial y_{ij}} = -\frac{\partial f}{\partial y_{ji}} \qquad \forall i, j.$$

Then, the infinitesimal generator of a Laguerre process $L(n, m, x)$ is given by:

$$\mathscr{L} = 2ntr(Re\left(\frac{\partial}{\partial z}\right)) + 2[tr(xRe\left(\frac{\partial}{\partial z}\right)^2) + tr(yIm\left(\frac{\partial}{\partial z}\right)^2)] \tag{3}$$

where $\dfrac{\partial}{\partial z}$ is the operator defined above.

Remark. Using the fact that $x^T = x$, $y^T = -y$ and $tr(AB) = tr(BA) = tr(B^T A^T)$ for any two matrices A and B, we can see that

$$\begin{aligned} tr\left(y\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) &= tr\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}y\right) \\ &= tr\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right), \end{aligned}$$

thus,

$$tr\left(yIm(\frac{\partial}{\partial z})^2\right) = 2tr\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)$$

3. Eigenvalues of Laguerre Process

In this section, we will suppose that $n \geq m$. Let us recall the following result stated in [18] and [17] (cf [2]) for the real case):

Theorem 1. *Let $\lambda_1(t), \cdots, \lambda_m(t)$ denote the eigenvalues of X_t. Suppose that at time $t = 0$, all the eigenvalues are distinct. Then, the eigenvalues process $(\lambda_1(t), \ldots, \lambda_m(t))$ satisfies the following stochastic differential system:*

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + 2\left[n + \sum_{k\neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\right] dt \qquad 1 \leq i \leq m, \qquad t < \tau,$$

where the $(\beta_i)_{1\leq i\leq m}$ are independent Brownian motions and τ is defined by $\tau := \inf\{t, \lambda_i(t) = \lambda_j(t) \text{ for some } (i,j)\}$.

Remark. With the help of the SDE verified by the eigenvalues, we can compute SDE verified by both processes $(tr(X_t))$ and $(det(X_t))$: the former is done. For the second, we find that for $t < T_0 := \inf\{t, det(X_t) = 0\}$ and for $r \in \mathbb{R}$:

$$\begin{aligned}
d(det(X_t)) &= 2det(X_t)\sqrt{tr(X_t^{-1})}d\nu_t + 2(n-m+1)det(X_t)tr(X_t^{-1})dt \\
d(\ln(det(X_t)) &= 2\sqrt{tr(X_t^{-1})}d\nu_t + 2(n-m)tr(X_t^{-1})dt, \\
d(det(X_t)^r) &= 2(det(X_t))^r\sqrt{tr(X_t^{-1})}d\nu_t + 2r(n-m+r)(det(X_t))^r tr(X_t^{-1})dt
\end{aligned}$$

so, we can see that for $n = m$, $(\ln(det(X_t))$ is a local martingale and so is $(det(X_t))^{m-n}$.

Lemma 1. *Take $X_0 \in \tilde{H_m^+}$. Then for $n \geq m$, $X_t \in \tilde{H_m^+}$.*

Proof : In fact, this result is a direct consequence of the fact that for $n = m$, $(\ln det(X_t))$ is a local martingale, and so is $((det(X_t))^{m-n})$. Hence, for $n \geq m$, these two continuous processes tend to infinity when $t \to T_0$ which is possible only if $T_0 = \infty$, because every continuous local martingale is a time-changed Brownian motion. □

Corollary 1. *If $\lambda_1(0) > \cdots > \lambda_m(0)$, then, the process defined by*

$$U(t) = \frac{1}{\prod_{i<j}(\lambda_i(t) - \lambda_j(t))}, \qquad t < \tau$$

is a local martingale.

Proof : We can follow in the same way of [2] but we will proceed differently. Showing that

$$U(x_1, \cdots, x_m) = \frac{1}{\prod_{i<j}(x_i - x_j)} \qquad \{x_1 > \cdots > x_m\}$$

is harmonic for the eigenvalues infinitesimal generator is equivalent to :

$$\sum_{i=1}^{m} x_i \frac{-(\partial_i^2 h)h^2 + 2(\partial_i h)^2 h}{h^4} \quad - \sum_{i=1}^{m}\left[n + \sum_{k\neq i}\frac{x_i + x_k}{x_i - x_k}\right]\frac{\partial_i h}{h^2} = 0, \tag{4}$$

where $h = 1/U$, that is :

$$\sum_{i=1}^{m} x_i \frac{-(\partial_i^2 h)h + 2(\partial_i h)^2}{h^2} \quad - \sum_{i=1}^{m} \left[n + \sum_{k \neq i} \frac{x_i + x_k}{x_i - x_k} \right] \frac{\partial_i h}{h} = 0,$$

In fact, we know that (cf [2])

$$V(x_1, \ldots, x_m) = \sum_{i<j} \log(x_i - x_j)$$

is a harmonic function, i.e, $\mathscr{A}V = 0$ where $\mathscr{A}$ denotes the eigenvalues generator of a Wishart process, hence, it follows that :

$$2\sum_{i=1}^{m} x_i \frac{(\partial_i^2 h)h - (\partial_i h)^2}{h^2} \quad + \sum_{i=1}^{m} \left[n + \sum_{k \neq i} \frac{x_i + x_k}{x_i - x_k} \right] \frac{\partial_i h}{h} = 0,$$

Thus, it remains to show that:

$$\sum_{i=1}^{m} x_i \partial_i^2 h = 0$$

which is easy to see (cf [9] p. 216). ■

Remark. An other way of thinking is to use König and O'Connell result: indeed, they showed (cf [18]), for $n \geq m$, the eigenvalues process is the h-transform of m independent $BESQ(2(n-m+1))$. Thus, if G and $\hat{G}$ denote respectively the infinitesimal generators of these two processes, then, $G(h) = 0$ and, forall C^2 function f,

$$\hat{G}(f) = \frac{1}{h} G(hf)$$

So, $\hat{G}(U) = \frac{1}{h} G(\mathbf{1}) = 0$. □

Corollary 2. *If at time $t = 0$, the eigenvalues of X are distinct, then, they will never collide, i. e, $\tau = \infty$ a. s.*

Proof : This result follows from the fact that the continuous process U tends to infinity when $t \to \tau$ which is possible only if $\tau = \infty$ a. s. (We use the same argument as before). □

3.1. **Some skew-products.** Let $(a\beta_t + \mu t)_{t \geq 0}$ a Brownian motion with drift. We set $(K_t = e^{a\beta_t + \mu t})_{t \geq 0}$, then :

$$dK_t = aK_t d\beta_t + (\mu + \frac{a^2}{2}) K_t dt$$

Define:

$$A_t = \int_0^t tr(X_s^{-1}) ds$$

Using this time-change, we have:

$$\begin{aligned} d(det(X_{A_t})) &= 2det(X_{A_t}) dW_t + 2(n-m+1) det(X_{A_t}) dt \\ d(\ln(det(X_{A_t})) &= 2dW_t + 2(n-m)dt \end{aligned}$$

where W is a standard Brownian motion. Hence, we deduce that :

$$\begin{aligned}(det(X_{A_t}))_{t\geq 0} &\overset{\mathscr{L}}{=} \left(e^{2(\beta_t+(n-m)t)}\right)_{t\geq 0} \overset{\mathscr{L}}{=} \left(R^{(n-m)}_{\int_0^t e^{2(\beta_s+(n-m)s)}ds}\right)_{t\geq 0}\\ (\ln(det(X_{A_t})))_{t\geq 0} &\overset{\mathscr{L}}{=} (2(\beta_t+(n-m)t))_{t\geq 0}\end{aligned}$$

where $(R_t)_{t\geq 0}$ is a squared Bessel process of index $(n-m)$. We can also get a look at the Brownian motions of ellipsoids already studied by Norris, Rogers and Williams (cf [23]). They considered the processes defined by :

$$F_t = G_t^T G_t \qquad \text{and} \qquad L_t = G_t G_t^T$$

where $(G_t)_{t\geq 0}$ is the right-invariant Brownian motion on $Gl(n,\mathbb{R})$ (right-invariant means that, for every $t,u\geq 0$, the law of $G_{t+u}G_t^{-1}$ does not depend on t and this right increment is independent of $\sigma(G_s,\, s\leq u)$). Let $(\gamma_i(t))_{1\leq i\leq m}$ denote the eigenvalues of F_t (or L_t), then, they never collide and we have :

$$\frac{1}{2}d(\ln\gamma_i(t)) = d\kappa_i(t) + \frac{1}{2}\sum_{k\neq i}\frac{\gamma_i(t)+\gamma_k(t)}{\gamma_i(t)-\gamma_k(t)}dt \qquad \forall t\geq 0, \qquad 1\leq i\leq m$$

where (κ_i) are independent Brownian motions, which implies that :

$$\frac{1}{2}d\ln(det(F_t)) = \sqrt{m}N_t$$

where N is a standard Brownian motion and since

$$\sum_{i=1}^m\sum_{k\neq i}\frac{\gamma_i+\gamma_k}{\gamma_i-\gamma_k} = 0$$

Thus, we get :

$$d(det(F_t)) = 2\sqrt{m}\,det(F_t)dN_t + 2m\,det(F_t)dt$$

which yields by Lamperti representation ([19]):

$$\begin{aligned}(det(F_t))_{t\geq 0} &\overset{\mathscr{L}}{=} \left(e^{2\sqrt{m}N_t}\right)_{t\geq 0}\\ &\overset{\mathscr{L}}{=} \left(\rho^{(0)}_{\int_0^t e^{2Ns}ds}\right)^{\sqrt{m}}_{t\geq 0}\end{aligned}$$

where ρ is a squared Bessel process of index 0.

3.2. **Additivity Property.**

Proposition 2. *If (X_t) and (Y_t) are two independent Laguerre processes $L(n,m,X_0)$ and $L(p,m,Y_0)$ respectively, then the process (X_t+Y_t) is a Laguerre process $L(n+p,m,X_0+Y_0)$.*

Proof: Let us write $X_t = B_t^\star B_t$ and $Y_t = R_t^\star R_t$, where (B_t) and (R_t) are, respectively, $n\times m$ and $p\times m$ independent Brownian motions. Then $N_t = \begin{pmatrix}B_t\\ R_t\end{pmatrix}$ is a $(n+p)\times m$ complex Brownian matrix and we have

$$X_t + Y_t = B_t^\star B_t + R_t^\star R_t = N_t^\star N_t. \qquad \square$$

Now, we will introduce the Laguerre processes with noninteger index δ.

4. Laguerre Processes with noninteger index

Let (X_t) be a Laguerre process $L(n, m, X_0)$ with $n \geq m$. If $X_0 \in \tilde{H_m^+}$, and if $\sqrt{X_t}$ represents the symmetric square root of X_t, it is easy to show that the matrix O defined by $O_t := \sqrt{X_t}^{-1} B_t^\star$, where $X_t = B_t^\star B_t$, verifies $O^\star O = OO^\star = I_m$. Thus,

$$d\gamma_t = O_t dB_t = \sqrt{X_t}^{-1} B_t^\star dB_t$$

is a $m \times m$ complex Brownian matrix. Replacing this expression in our equation (1), we see that X_t is governed by the following SDE:

$$dX_t = \sqrt{X_t} d\gamma_t + d\gamma_t^\star \sqrt{X_t} + 2nI_m dt$$

Theorem 2. *If (B_t) is a $m \times m$ complex Brownian matrix, then for every $X_0 \in \tilde{H_m^+}$ and $\forall \delta \geq m$, the SDE*

$$dX_t = \sqrt{X_t} dB_t + dB_t^\star \sqrt{X_t} + 2\delta I_m dt \tag{5}$$

has a unique strong solution in $\tilde{H_m^+}$. Furthermore, if the eigenvalues are distinct at time $t = 0$, then they satisfy the stochastic differential system:

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + 2\left[\delta + \sum_{k \neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\right] dt \qquad 1 \leq i \leq m,$$

where the $(\beta_i)_{1 \leq i \leq m}$ are independent Brownian motions.

Proof : The proof of the second part of the theorem is the same as before with δ instead of n. So, we have to prove the first part. First, note that $(det(X_t)), (\ln det(X_t))$ and $(det(X_t)^r)$ verify the same EDS with δ instead of n. Hence, arguing as before, we can see that $T_0 = \infty$ a.s. On the other hand, the map $a \mapsto a^{1/2}$ is analytic in $\tilde{H_m^+}$ (cf [27], p 134), so, the SDE has a unique strong solution defined on $t < T_0 = \infty$ a. s. □

Definition. Such a process is called *Laguerre process* of dimension δ, size m and initial state X_0. It will be denoted by $L(\delta, m, X_0)$.

Remarks. 1/ Any process (X_t) solution of (5) is a diffusion whose infinitesimal generator is given by:

$$\mathscr{L} = 2\delta tr(Re(\frac{\partial}{\partial z})) + 2[tr(xRe(\frac{\partial}{\partial z})^2) + tr(yIm(\frac{\partial}{\partial z})^2)]$$

2/ A simple computation shows that

$$(dX_{ij})(dX_{kl}) = 2(X_{il}\delta_{kj} + X_{kj}\delta_{il})dt,$$

for every $i, j, k, l \in \{1, \ldots, m\}$.

Now, we will focus on the existence and the unicity of equation (5) when $\delta > m - 1$ and $X_0 \in H_m^+$ (cf [2] for the real case).

4.1. **The Process X^+.** If X is a Hermitian matrix, let X^+ be the Hermitian matrix $\max(X,0)$. If we denote by (λ_i) the eigenvalues of X, then $(\lambda_i^+ = \max(\lambda_i, 0))$ are those of X^+.

Theorem 3. *For all $\delta \in \mathbb{R}_+$ and $X_0 = x \in H_m$, the stochastic differential equation*

$$dX_t = \sqrt{X_t^+}\, dB_t + dB_t^\star \sqrt{X_t^+} + 2\delta I_m dt$$

has a solution in H_m.

Proof : The mapping $a \mapsto \sqrt{a^+}$ is continuous on H_m, hence, by th. 2. 3, p159 in [14], X exists up to its explosion time. Furthermore, from

$$||\sqrt{X^+}||^2 + ||\delta I||^2 \leq \delta^2 + ||X||^2 \leq C(1 + ||X||^2),$$

we can deduce that this explosion time is infinite a. s. (cf [14] Th. 2. 4, page 163). ■

Proposition 3. *If $\lambda_1(0) > \ldots > \lambda_m(0) \geq 0$, then, $\forall\, t < S := \inf\{t, \lambda_i = \lambda_j$ for some (i, j)$\}$, the eigenvalues of X^+ verify the following differential system:*

$$d\lambda_i(t) = 2\sqrt{\lambda_i^+}(t) d\nu_i(t) + 2\left(\delta + \sum_{k \neq i} \frac{\lambda_i^+(t) + \lambda_k^+(t)}{\lambda_i(t) - \lambda_k(t)}\right) dt, \qquad 1 \leq i \leq m,$$

Proof : This differential can be shown in the same way as in Th. 1, using the above relation:

$$(dX_{ij})(dX_{kl}) = 2(X_{il}^+ \delta_{kj} + X_{kj}^+ \delta_{il}) dt.$$

Proposition 4. *If $\lambda_1(0) > \ldots > \lambda_m(0) \geq 0$, then, $\forall\, \delta > m-1$, $\forall\, t > 0$, $\lambda_m(t) \geq 0$.*

Proof : First, we note that $S = \infty$ a. s. Indeed, a similar calculus as in the first part of the proof of Corollary 1 shows that the process V defined by :

$$V(\lambda_1(t), \ldots, \lambda_m(t)) = \frac{1}{\prod_{i<j}(\lambda_i(t) - \lambda_j(t))}$$

is a local martingale. For the proof, we follow in the same way as in [2].

Theorem 4. *If $\lambda_1(0) > \ldots > \lambda_m(0) \geq 0$, then, for all $\delta > m-1$,* (5) *has a unique solution in H_m^+ in the sense of probability law.*

Proof : By Proposition 4, the solution of the SDE in Th. 3. remains positive for all $t > 0$, thus, it is a solution of (5). □

Theorem 5. *Let H_m^+ be the space of positive Hermitian matrices. Then, whenever the SDE (5) has a solution in H_m^+, for fixed t, its distribution is given by its Laplace transform:*

$$\mathbb{E}_{X_0}(\exp -(tr uX_t)) = (det(I_m + 2tu))^{-\delta} \exp(-tr(X_0(I_m + 2tu)^{-1} u)), \tag{6}$$

for all u in H_m^+.

Proof : For $s \in H_m^+$, let $g(t,s) = \Delta_t^{-\delta} \exp(-V(t,s))$, where

$$\Delta_t = det(I_m + 2ut), \quad W_t = (I_m + 2ut)^{-1} u, \quad V(t,s) = tr(sW_t),$$

First, note that $W \in H_m$. After that, we need a lemma:

Lemma 2. *The function g verifies the heat equation: $\frac{\partial g}{\partial t} = \mathscr{L} g$ where $\mathscr{L}$ is the infinitesimal generator of X.*

Proof of the lemma : If we write $s = x + iy$, then, using the fact that x is symmetric, y is skew-symmetric and W is Hermitian, we can see that $tr(sW_t) = tr(xM + iyN)$ where

$$M = \frac{W + \overline{W}}{2} \qquad N = \frac{W - \overline{W}}{2}.$$

Observing that $M^T = M$ and $N^T = -N$, we can deduce that g satisfies the conditions of Proposition 1. Besides,

$$\frac{\partial g}{\partial t} = -g(2\delta\, tr(W_t) - 2tr(sW_t^2))$$

and

$$\begin{aligned} tr(y\left(\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 g}{\partial y \partial x}\right)) &= -ig\, tr(yW^2), \\ tr(x\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2}\right)) &= g\, tr(x(M^2 + N^2)) = g\, tr(xW^2). \end{aligned}$$

Finally, noting that $tr(M) = tr(W)$, we obtain the equality. □
Now, we consider the process $(Z(t, X_t))$ defined by

$$Z(t, X_t) = g(t_1 - t, X_t), \qquad \forall\, t \leq t_1$$

for fixed t_1. From the lemma, we deduce that Z is a bounded local martingale and thus is a martingale. So, the result follows from a simple application of the optional stopping theorem. ■

Remark. From the Laplace transform, we easily deduce the additivity property for fixed t.

Corollary 3. *Let (X_t) be a Laguerre process $L(\delta, m, x)$ where $x \in \tilde{H_m}^+$. For $\delta > m - 1$, the semi-group of (X_t) is given by the following density:*

$$p_t^\delta(x, y) = \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} \exp -(\frac{1}{2t} tr(x + y))\, (det\, y)^{\delta - m}\, {}_0F_1(\delta; \frac{xy}{4t^2}) \mathbf{1}_{\{y>0\}}$$

with respect to Lebesgue measure $dy = \prod_{p \leq q} dy_{pq}^1 \prod_{p<q} dy_{pq}^2$ where $y = y^1 + iy^2$ and ${}_0F_1$ is an hypergeometric function with matrix argument(cf [6], [13]*).*

Proof : In fact, this result can be easily deduced from the case where $\delta = n$ is integer, since, in this case, X_t is a non-central complex Wishart variable $W(n, 2tI_m, x)$ (cf [15]) with density given by:

$$f_t(x, y) = \frac{1}{(2t)^{mn}\Gamma_m(n)} \exp -(\frac{1}{2t} tr(x + y))\, (det\, y)^{n - m}\, {}_0F_1(n; \frac{xy}{4t^2}) \mathbf{1}_{\{y>0\}}$$

with respect to dy. Hence, taking δ instead of n and denoting by W_t this new variable (starting from x), we can see that : (we will use $|y|$ to denote $det(y)$)

$$
\begin{aligned}
E_x(e^{-truW_t}) &= \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} e^{-\frac{trx}{2t}} \int_{y>0} \exp\left(-\frac{1}{2t} tr((I+2ut)y)\right) |y|^{\delta-m} {}_0F_1(\delta; \frac{xy}{4t^2}) dy \\
&= \frac{2t^{m\delta}|x|^{-\delta}}{\Gamma_m(\delta)} e^{-\frac{trx}{2t}} \int_{z>0} \exp(-2t\, tr(x^{-\frac{1}{2}}(I+2ut)x^{-\frac{1}{2}}z)|z|^{\delta-m} {}_0F_1(\delta; z) dz \\
&= \exp(-\frac{trx}{2t})|I+2ut|^{-\delta} \exp\left(tr(\frac{x}{2t}(I+2ut)^{-1})\right) \\
&= |I+2ut|^{-\delta} \exp\left(-\frac{1}{2t} tr((I+2ut)^{-1}(I+2ut-I))\right) \\
&= |I+2ut|^{-\delta} \exp\left(-tr(x(I+2ut)^{-1}u)\right)
\end{aligned}
$$

which is equal to (6). ■

Remarks. 1/ In the last proof, we used the change of variables $z = x^{1/2}yx^{1/2}$ which gives $dz = |x|^m dy$. For the second integral, see [11], prop XV.1.3, p 319.
2/ The expression of the semi-group extends continuously to the degenerate case, namely:

$$p_t^\delta(0_m, y) = \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} \exp -(\frac{tr(y)}{2t}) \, (det\, y)^{\delta-m} \mathbf{1}_{\{y>0\}}$$

where 0_m denotes the nul matrix.

Corollary 4. *For $\delta > m-1$, the semi-group of eigenvalues process is given by:*

$$q_t(x,y) = \frac{h(y)}{h(x)} det\left(\frac{1}{2t}\left(\frac{y_j}{x_i}\right)^{\nu/2} e^{-\frac{(x_i+y_j)}{2t}} I_\nu(\frac{\sqrt{x_i y_j}}{t})\right)$$

where $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m)$ so that $x_1 > \ldots > x_m > 0$, $y_1 > \ldots > y_m > 0$, $\delta = m + \nu$ such that $\nu > -1$, I_ν denotes the modified Bessel function and $h(x) = \prod_{i<j}(x_i - x_j)$.

Proof : The expression of the semi-group can be computed using Karlin-MacGregor formula (cf [16]) since, for $\delta > m-1$, the eigenvalues process is the h-transform of m independent $BESQ(2(\delta - m + 1))$ conditioned never to collide (cf [18]). In fact, we can extend könig and O'Connell result for any $\delta > m-1$.
Another proof is given by Péché (cf [24], p. 68). Here, we will deduce the expression of $q_t(x,y)$ from $p_t(x,y)$ following Muirhead (cf [22]), namely, by projection on the unitary group : we will use Weyl integration formula, then give a determinantal representation of hypergeometric functions of two matrix arguments.
First, we state Weyl integration formula (cf [11]) in the complex case: for any Borel function f,

$$\int_{H_m} f(A)dA = C_m \int_{U(m)} \int_{\mathbb{R}^m} f(uau^*)\alpha(du)(h(a))^2 da_1 \ldots da_m,$$

where $C_m = \dfrac{\pi^{m(m-1)}}{\Gamma_m(m)}$, $U(m)$ is the unitary group, α is the normalized Haar measure on $U(m)$, $a = diag(a_i)$, $h(a) = \prod_{i<j}(a_i - a_j)$ and $A = uau^*$. Hence, the semi-group of

eigenvalues process is given by (cf [15]):

$$\begin{aligned} q_t(x,y) &= C_m(h(y)^2)\int_{U(m)} p_t(\tilde{x},u\tilde{y}u^*)\alpha(du) \\ &= \frac{C_m(h(y)^2)}{(2t)^{m\delta}\Gamma_m(\delta)}\prod_{i,j=1}^m e^{-\frac{x_i+y_j}{2t}}\left(\prod_{i=1}^m y_j\right)^{\delta-m}\int_{U(m)} {}_0F_1(\delta;\frac{\tilde{x}u\tilde{y}u*}{4t^2})\alpha(du) \\ &= \frac{\pi^{m(m-1)}(h(y)^2)}{(2t)^{m(m+\nu)}\Gamma_m(m)\Gamma_m(m+\nu)}\prod_{i,j=1}^m e^{-\frac{x_i+y_j}{2t}}\left(\prod_{i=1}^m y_j\right)^{\nu}{}_0F_1(m+\nu;\frac{\tilde{x}}{4t^2};\tilde{y}), \end{aligned}$$

where $\tilde{y}=diag(y_j)$, x is a definite positive matrix whose eigenvalues are $x_1,\ldots,x_m$, ${}_0F_1$ in the third line is an hypergeometric function with two matrix argument (cf [13]) and $\delta=m+\nu,\nu>-1$. (For the last equality, see [15]). Next, we need a lemma.

Lemma 3. *Let $B,C\in H_m$ and let (b_i), (c_i) be respectively their eigenvalues. Then,*

$${}_pF_q((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C)=\pi^{\frac{m(m-1)}{2}(p-q-1)}\Gamma_m(m)\prod_{i=1}^p\frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)}$$
$$\prod_{j=1}^q\frac{\Gamma_m(m+\phi_j)}{(\Gamma(\phi_j+1))^m}\quad\frac{det\left({}_p\mathscr{F}_q((\mu_i+1)_{1\le i\le p},(1+\phi_j)_{1\le j\le q};b_lc_f\right)_{l,f}}{h(B)h(C)}$$

$\forall\,\mu_i,\phi_j>-1, 1\le i\le p,\ 1\le j\le q.$

Proof : Recall that the hypergeometric function of two matrix arguments is given by the following series:

$${}_pF_q((a_i)_{1\le i\le p},(e_j)_{1\le j\le q};B,C)=\sum_{k=0}^\infty\sum_\tau\frac{\prod_{i=1}^p(a_i)_\tau}{\prod_{j=1}^q(e_j)_\tau}\frac{Z_\tau(B)Z_\tau(C)}{Z_\tau(I)k!},$$

where $\tau=(k_1,\ldots,k_m)$, $k_1\ge\ldots\ge k_m$ is a partition of k, $(a)_\tau=\prod_{r=1}^m(a-r+1)_{k_r}$ is the generalized Pochammer symbol and Z_τ is the zonal polynomial. It is well known that

$$Z_\tau(B)=\frac{k!d_\tau}{(m)_\tau}s_\tau(b_1,\ldots,b_m),$$

where s_τ is the Schur function and $d_\tau=s_\tau(I)$ is the representation trace or degree (cf [13] or [11]). Substituting this expression in our series, we get:

$${}_pF_q((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C)=\sum_{k=0}^\infty\sum_\tau\frac{\prod_{i=1}^p(m+\mu_i)_\tau}{\prod_{j=1}^q(m+\phi_j)_\tau}\frac{s_\tau(B)s_\tau(C)}{(m)_\tau},$$

Now, we write:

$$\begin{aligned}(m+\mu_i)_\tau &= \prod_{r=1}^{m} \frac{\Gamma(\mu_i+m+k_r-r+1)}{\Gamma(\mu_i+m-r+1)} \\ &= \prod_{r=1}^{m} \frac{\Gamma(\mu_i+1+k_r+\delta_r)}{\Gamma(\mu_i+m-r+1)} \\ &= \pi^{m(m-1)/2} \frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)} \prod_{r=1}^{m} (\mu_i+1)_{k_r+\delta_r}\end{aligned}$$

where $\delta_r = m-r$. Doing the same thing for each $(m+\phi_j)_\tau$ and for $(m)_\tau$, we can see that:

$$\begin{aligned}{}_pF_q((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C) = \pi^\beta \Gamma_m(m) \prod_{i=1}^{p} \frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)} \prod_{j=1}^{q} \frac{(\Gamma(\phi_j+1))^m}{\Gamma_m(m+\phi_j)} \\ \sum_{k=0}^{\infty} \sum_{\tau} \prod_{r=1}^{m} \left(\frac{\prod_{i=1}^{p} (\mu_i+1)_{k_r+\delta_r}}{\prod_{j=1}^{q} (\phi_j+1)_{k_r+\delta_r}} \right) \quad \frac{s_\tau(B)s_\tau(C)}{\prod_{r=1}^{m}(1)_{k_r+\delta_r}}\end{aligned}$$

where $\beta = \left(\frac{m(m-1)}{2}\right)(p-q-1)$. Then, we use the following result (Hua formula, cf [10]):

Lemma 4. *Given an entire function f, i.e, $f(z) = \sum_{k=0}^{\infty} e_k z^k$, then*

$$\frac{det(f(b_i c_j))_{i,j}}{h(B)h(C)} = \sum_{k=0}^{\infty} \sum_{\tau} \left(\prod_{r=1}^{m} e_{k_r+\delta_r} \right) \frac{s_\tau(B)s_\tau(C)}{s_\tau(I_m)}.$$

Thus, we get:

$$\begin{aligned}{}_pF_q((m+\mu_i)_{1\le i\le p},&(m+\phi_j)_{1\le j\le q};B,C) = \pi^{\frac{m(m-1)}{2}(p-q-1)} \Gamma_m(m) \prod_{i=1}^{p} \frac{\Gamma(\mu_i+1)}{\Gamma_m(m+\mu_i)} \\ &\prod_{j=1}^{q} \frac{\Gamma_m(m+\phi_j)}{\Gamma(\phi_j+1)} \quad \frac{det\left(\sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(\mu_i+1)_k}{\prod_{j=1}^{q}(\phi_j+1)_k} \frac{(b_l c_p)^k}{k!}\right)_{l,p}}{h(B)h(C)} \\ &= \pi^{\frac{m(m-1)}{2}(p-q-1)} \Gamma_m(m) \prod_{i=1}^{p} \frac{\Gamma(\mu_i+1)}{\Gamma_m(m+\mu_i)} \prod_{j=1}^{q} \frac{\Gamma_m(m+\phi_j)}{\Gamma(\phi_j+1)} \\ &\frac{det\left({}_p\mathscr{F}_q((\mu_i+1)_{1\le i\le p},(1+\phi_j)_{1\le j\le q};b_l c_f)_{l,f}\right.}{h(B)h(C)} \qquad \square\end{aligned}$$

Remark. For $p = 0$ and $q \geq 1$, we have:

$$ {}_0F_q((m+\phi_j)_{1\leq j\leq q}; B, C) = \pi^{-\frac{m(m-1)}{2}(q+1)}\Gamma_m(m)\prod_{j=1}^{q}\frac{\Gamma_m(m+\phi_j)}{(\Gamma(\phi_j+1))^m} \frac{det\left({}_0\mathscr{F}_q((1+\phi_j)_{1\leq j\leq q}; b_l c_f)_{l,f}\right)}{h(B)h(C)}, $$

and similarly,

$$ {}_0F_0(B, C) = \frac{\Gamma_m(m)}{\pi^{\frac{m(m-1)}{2}}}\frac{det(e^{b_l c_f})_{l,f}}{h(B)h(C)} $$

which can be viewed as Harish-Chandra formula.

We now proceed to the end of the proof. Taking $p = 0$, $q = 1$, $B = \dfrac{\tilde{x}}{4t^2}$, $C = \tilde{y}$, we get:

$$ {}_0F_1(m+\nu; \frac{\tilde{x}}{4t^2}; \tilde{y}) = \frac{(4t^2)^{m(m-1)/2}\Gamma_m(m+\nu)\Gamma_m(m)}{\pi^{m(m-1)}(\Gamma(\nu+1))^m}\frac{det\left({}_0\mathscr{F}_1((\nu+1); \frac{x_i y_j}{4t^2})\right)}{h(x)h(y)} $$

The expression of $q_t(x, y)$ follows from a simple computation and from the fact that:

$$ \frac{{}_0\mathscr{F}_1((\nu+1); x_i y_j/4t^2))_{i,j}}{\Gamma(\nu+1)} = \left(\frac{2t}{\sqrt{x_i y_j}}\right)^{\nu} I_\nu(\frac{\sqrt{x_i y_j}}{t}) \qquad \blacksquare $$

Proposition 5. *The measure defined by $\rho(dx) = (det(x))^{\delta-m}\,dx$ on $\tilde{H_m}^+$ is invariant under the semi-group, i. e, $\rho P_t = \rho$.*

Proof: Denote by P_t the semi-group of Laguerre process $L(\delta, m, x)$ for $\delta > m-1$. Then, we have to show that

$$ \int_{x>0} P_t f(x)\rho(dx) = \int_{y>0} f(y)\rho(dy) \qquad \forall\, f \in C_0(\tilde{H_m}^+). $$

Indeed, by Corollary 3, we have:

$$ \begin{aligned} \int_{x>0} P_t f(x)\rho(dx) &= \frac{1}{(2t)^{m\delta}\Gamma_m(m)}\int_{y>0} e^{-tr(\frac{y}{2t})}|y|^{\delta-m} f(y)\left(\int_{x>0} e^{-tr(\frac{x}{2t})}|x|^{\delta-m}{}_0F_1(\delta, \frac{xy}{4t^2})dx\right)dy \\ &= \frac{(2t)^{m\delta}}{\Gamma_m(m)}\int_{y>0} e^{-tr(\frac{y}{2t})}|y|^{-m} f(y)\left(\int_{x>0} e^{-tr(2ty^{-1/2}zy^{-1/2})}|z|^{\delta-m}{}_0F_1(\delta, z)dz\right)dy \\ &= \int_{y>0} f(y)e^{-tr(\frac{y}{2t})}|y|^{\delta-m}{}_1F_1(\delta, \delta, \frac{y}{2t})dy \\ &= \int_{y>0} f(y)e^{-tr(\frac{y}{2t})}|y|^{\delta-m}e^{tr(\frac{y}{2t})}dy \\ &= \int_{y>0} f(y)\rho(dy) \end{aligned} $$

where we used the same change of variables as in the proof of Corollary 3 and Proposition XV. 1. 3, p 319 in [11] (see remark below the proof). $\square$

Remark. For Wishart processes, it is easy to see that $\mu(dx) := (det(x))^{\frac{\delta}{2}-\frac{m+1}{2}}\mathbf{1}_{\{x>0\}}dx$ is invariant under the semi-group.

4.2. **Some Orthogonal Polynomials.**

Proposition 6. *Let $L(m+\nu, m, X_0)$, $\nu > -1$, be a Laguerre process and let $Y = (\lambda_1, \ldots, \lambda_m)$ be its eigenvalues process starting at $x \in \mathbb{R}^m$. Set :*

$$Z_t := e^{2\beta t}\, Y(\frac{1-e^{-2\beta t}}{2}) := e^{2\beta t}\, Y_{A_t} \qquad \beta \in \mathbb{R}^+,$$

then, the semi-group of (Z_t) is given by:

$$\tilde{q}_t(x,y) = \beta^{m(m+\nu)} e^{-tr(\beta x)} W(y) \sum_{k\geq 0} \sum_{\tau} \frac{L^\nu_\tau(\beta x) L^\nu_\tau(\beta y)}{\mathscr{N}_\tau} e^{-2\beta t(m-|\tau|)}, \qquad y \in \mathbb{R}^m$$

where , $W(y) = (det(y))^\nu\, (h(y))^2$, τ is a partition of k, L^ν_τ is the generalized Laguerre polynomial (see [22] *for the real case,* [1] *or* [11] *for a general setting), $\mathscr{N}_\tau$ is the normalisation constant given by :*

$$\mathscr{N}_\tau \;=\; \int_{[0,\infty[^m} (L^\nu_\tau(x))^2\, (det(y))^\nu e^{-tr(y)} (h(y))^2 dy,$$

Proof : Recall that the semi group of $(Y_t)_{t\geq 0}$ is given by :

$$q_t(x,y) = \frac{\pi^{m(m-1)}(h(y)^2)(det(y))^\nu}{(2t)^{m(m+\nu)}\Gamma_m(m)\Gamma_m(m+\nu)} e^{-\frac{tr(x+y)}{2t}}\, {}_0F_1(m+\nu; \frac{x}{4t^2}; y)$$

Using the definition of Z and setting $r = e^{-2\beta t}$, we have

$$\begin{aligned}
\tilde{q}_t(x,y) &= e^{-2\beta t m} q_{A_t}(x, e^{-2\beta t} y) \\
&= C_m(y) r^m \left(\frac{\beta}{(1-r)}\right)^{m(m+\nu)} \exp\left(-\beta tr(\frac{x}{1-r} + \frac{ry}{1-r})\right) {}_0F_1(m+\nu; \frac{\beta x}{1-r}; \frac{r\beta y}{1-r}) \\
&= C_m(y) e^{-tr(\beta x)} \beta^{m(m+\nu)} r^m \\
&\left(\frac{1}{(1-r)}\right)^{m(m+\nu)} \exp\left(-\beta tr(\frac{rx}{1-r} + \frac{ry}{1-r})\right) {}_0F_1(m+\nu; \frac{\beta x}{1-r}; \frac{r\beta y}{1-r})
\end{aligned}$$

where $C_m(y) = \dfrac{\pi^{m(m-1)}(h(y)^2)(det(y))^\nu}{\Gamma_m(m)\Gamma_m(m+\nu)}$. Then, we use the generating function (cf [1], p 201):

$$\sum_{k\geq 0}\sum_{\tau} \frac{L^\nu_\tau(u) L^\nu_\tau(v)}{\mathscr{N}_\tau} r^{|\tau|} = \frac{m!\,\pi^{m(m-1)}}{\Gamma_m(m)\Gamma_m(m+\nu)} \left(\frac{1}{(1-r)}\right)^{m(m+\nu)} \exp\left(-tr(\frac{ru}{1-r} + \frac{rv}{1-r})\right)$$
$${}_0F_1(m+\nu; \frac{u}{1-r}; \frac{rv}{1-r}), \qquad |r| < 1$$

The result follows from an easy computation. □

Remarks. 1/ In [1], the authors deal with Jack polynomials C^α_τ : take $\alpha = 1$ to get Schur functions and $\alpha = 2$ to get real zonal polynomials. For $\alpha = 1/2$, we get quaternion zonal polynomials.
2/ In the case $m = 1$, Z is a squared Ornstein-Ulhenbeck process (cf [25]). Its semi-group

density is given by a similar bilinear series (cf [28]). For the matrix case, we can define a process $(R_t)_{\geq 0}$ as the unique strong solution of :

$$dR_t = d\gamma_t + \beta R_t dt, \qquad R_0 = \gamma_0$$

where γ is a $n \times m$ complex matrix Brownian motion and $\beta \in \mathbb{R}^+$. Then, we consider $S_t := R_t^\star R_t$. The process $(S_t)_{t\geq 0}$ satisfies (using Itô's formula and that $(dB_t := \sqrt{S_t}^{-1} R_t^\star d\gamma_t)_{t\geq 0}$ defines a complex matrix Brownian motion) :

$$dS_t = \sqrt{S_t} dB_t + dB_t^\star \sqrt{S_t} + (2\beta S_t + 2nI_m)dt, \qquad S_0 = \gamma_0^\star \gamma_0$$

Thus, if $(X_t = B_t^\star B_t)$ is $L(n, m, x)$, then we can see from a simple computation that :

$$\left(S_t \overset{\mathscr{L}}{=} e^{2\beta t} X_{A_t}, t \geq 0\right)$$

Using Theorem 2 and Theorem 4, we can deduce that the same existence and unicity results hold for the following SDE:

$$dS_t = \sqrt{S_t} dB_t + dB_t^\star \sqrt{S_t} + (2\beta S_t + 2\delta I_m)dt, \qquad S_0 = s$$

Such a process is the complex version of the squared Ornstein-Ulhenbeck process already defined in [2], and $(Z_t)_{t\geq 0}$ is its eigenvalues process.

5. Girsanov Formula and Absolute-continuity Relations

In this section, we will discuss in the same way as in [7] to derive absolute-continuity relations between different indices.

5.1. **Positive Indices.** Take a matrix-valued Hermitian previsible process H. Let Q_x^δ be the probability law of $L(\delta, m, x)$ for $\delta > m - 1$ and $x \in \tilde{H_m^+}$. Define:

$$\begin{aligned} L_t &= \int_0^t \frac{tr(H_s dB_s + \overline{H_s}\,\overline{dB_s})}{2}, \\ \Phi_t &= \exp\left(L_t - \frac{1}{2}\int_0^t tr(H_s^2)ds\right), \end{aligned}$$

where B is a complex Brownian matrix under Q_x^δ. We can easily see that the process β defined by $\beta_t = B_t - \int_0^t H_s ds$ is a Brownian matrix under the probability

$$\mathbb{P}_x^H|_{\mathscr{F}_t} := \Phi_t \cdot Q_x^\delta|_{\mathscr{F}_t},$$

Furthermore, (X_t) is a solution of

$$dX_t = \sqrt{X_t} d\beta_t + d\beta_t^\star \sqrt{X_t} + (\sqrt{X_t} H_t + H_t \sqrt{X_t} + 2\delta I_m)dt. \tag{7}$$

Then, we consider the case when $H_t = \nu \sqrt{X_t}^{-1}$. In this case, (7) becomes

$$dX_t = \sqrt{X_t} d\beta_t + d\beta_t^\star \sqrt{X_t} + 2(\delta + \nu) I_m dt,$$

which is the SDE for a $L(\delta + \nu, m, x)$. Thus, we have:

Theorem 6. *For $\delta > m - 1$,*

$$Q_x^{\delta+\nu}|_{\mathscr{F}_t} = \exp\left(\frac{\nu}{2}\int_0^t tr(\sqrt{X_s}^{-1} dB_s + \overline{\sqrt{X_s}^{-1} dB_s}) - \frac{\nu^2}{2}\int_0^t tr({X_s}^{-1})ds\right) \cdot Q_x^\delta|_{\mathscr{F}_t}. \tag{8}$$

Proposition 7.

$$Q_x^{m+\nu}|_{\mathscr{F}_t} = \left(\frac{det(X_t)}{det(x)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t tr({X_s}^{-1})ds\right)\cdot Q_x^m|_{\mathscr{F}_t}. \tag{9}$$

Proof : We know that $\nabla_u(det(u)) = det(u)u^{-1}$, hence, $\nabla_u(\ln(det(u))) = u^{-1}$. Then, using the fact that for $\delta = m$, $(\ln(det(X_t)))$ is a local martingale, we get from Itô formula that:

$$\begin{aligned}\ln(det(X_t)) &= \ln(det(X_0)) + \int_0^t tr({X_s}^{-1}(\sqrt{X_s}dB_s + dB_s^\star\sqrt{X_s}))\\ &= \ln(det(X_0)) + \int_0^t tr(\sqrt{X_s}^{-1}dB_s + \overline{\sqrt{X_s}^{-1}dB_s}).\end{aligned}$$ □

From (9), it follows that:

Corollary 5.

$$\begin{aligned}Q_x^m(\exp\left(-\frac{\nu^2}{2}\int_0^t tr(X_s^{-1})ds\right)|X_t = y) &= \frac{det(y)}{det(x)}^{-\nu/2}\frac{p_t^{m+\nu}(x,y)}{p_t^m(x,y)}\\ &= \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)}(det(z))^{\nu/2}\frac{{}_0F_1(m+\nu,z)}{{}_0F_1(m,z)}\\ &:= \frac{\tilde{I}_\nu(z)}{\tilde{I}_0(z)},\end{aligned}$$

where $z = \dfrac{xy}{4t^2}$.

Now, we state the following asymptotic result:

Corollary 6. *Let X be a Laguerre process $L(m,m,x)$, then, as $t\to\infty$:*

$$\frac{4}{(m\log t)^2}\int_0^t tr(X_s)^{-1}ds \qquad \overset{\mathscr{L}}{\to} \qquad T_1(\beta)$$

where T_1 is the first hitting time of 1 *by a standard Brownian motion β.*

Proof : From (9), we deduce that:

$$\begin{aligned}Q_x^m(\exp\left(-\frac{2\nu^2}{(m\log t)^2}\int_0^t tr(X_s^{-1})ds\right)|X_t = ty) = \frac{\Gamma_m(m)}{\Gamma_m(m+2\nu/m\log t)}(det(xy/4t))^{\nu/m\log t}\\ \frac{{}_0F_1(m+2\nu/m\log t, xy/4t^2)}{{}_0F_1(m,xy/4t^2)}.\end{aligned}$$

Noting that $(t^m)^{-\nu/m\log t} = e^{-\nu}$, and since both hypergeometric functions converge to 1 as $t\to\infty$, we obtain:

$$Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t tr(X_s^{-1})ds|X_t = ty\right) \qquad \overset{t\to\infty}{\longrightarrow} \qquad e^{-\nu}$$

Then, since

$$\lim_{t\to\infty} t^{m^2} p_t^m(x,2y) = \lim_{t\to\infty} \frac{e^{-tr(x)/2t}}{\Gamma_m(m)} e^{-tr(y)}{}_0F_1(m, \frac{xy}{2t}) = \frac{e^{-tr(y)}}{\Gamma_m(m)}$$

we get:

$$\begin{aligned} &Q_x^m(\exp\left(-\frac{2\nu^2}{(m\log t)^2}\int_0^t tr(X_s^{-1})ds\right)) \\ &= \int_{y>0} Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t tr(X_s^{-1})ds | X_t = y\right) p_t^m(x,y)dy \\ &= \int_{y>0} Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t tr(X_s^{-1})ds | X_t = ty\right) t^{m^2} p_t^m(x,ty)dy \\ &\overset{t\to\infty}{\longrightarrow} e^{-\nu}, \end{aligned}$$

by dominated convergence Theorem. ■

5.2. **Negative Indices.** Take $0 < a \leq det(x)$. The same calculus as in section 5.1 with $H_t = -\nu\sqrt{X_t}^{-1}, 0 < \nu < 1$, shows that

$$Q_x^{m-\nu}|_{\mathscr{F}_{t\wedge T_a}} = \left(\frac{det(x)}{det(X_{t\wedge T_a})}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^{t\wedge T_a} tr({X_s}^{-1})ds\right) Q_x^m|_{\mathscr{F}_{t\wedge T_a}}$$

where $T_a := \inf\{t, det(X_t) = a\}$. Letting $a \to 0$ and using the fact that $T_0 = \infty$ a.s under Q_x^m, we get :

$$\begin{aligned} Q_x^{m-\nu}|_{\mathscr{F}_{t\wedge T_0}} &= \left(\frac{det(x)}{det(X_t)}\right)^{\nu/2} \exp\left(\frac{\nu^2}{2}\int_0^t tr({X_s}^{-1})ds\right) Q_x^m|_{\mathscr{F}_t} \\ &= \left(\frac{det(x)}{det(X_t)}\right)^{\nu} Q_x^{m+\nu}|_{\mathscr{F}_t} \end{aligned}$$

Proposition 8. *For all $t > 0$ and $0 < \nu < 1$,*

$$Q_x^{m-\nu}(T_0 > t) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} det(\frac{x}{2t})^{\nu}{}_1F_1(\nu, m+\nu, -\frac{x}{2t})$$

Proof : From the absolute-continuity relations, we deduce that:

$$Q_x^{m-\nu}(T_0 > t) = Q_x^{m+\nu}\left(\left(\frac{det(x)}{det(X_t)}\right)^{\nu}\right),$$

On the other hand, from the expression of the semi-group, we can compute:

$$\begin{aligned} Q_x^{\delta}(det(X_t)^s) &= (2t)^{ms}\frac{\Gamma_m(s+\delta)}{\Gamma_m(\delta)}{}_1F_1(-s;\delta;-\frac{x}{2t}) \\ &= (2t)^{ms}\frac{\Gamma_m(s+\delta)}{\Gamma_m(\delta)}\exp(-tr(\frac{x}{2t}))_1F_1(\delta+s;\delta;\frac{x}{2t}) \end{aligned}$$

by Kummer relation (cf Th 7. 4. 3 in [22]).Taking $s = -\nu$, we are done. ■

6. generalized Hartman-Watson law

Henceforth, we will write $\mathscr{F}$ to denote one-dimensional hypergeometric functions. We define the generalized Hartman-Watson law as the law of

$$\int_0^t tr(X_s^{-1})ds \qquad \text{under} \qquad Q_x^m(\cdot|X_t = y).$$

Its Laplace transform is given by:

$$Q_x^m(\exp\left(\frac{-\nu^2}{2}\int_0^t tr(X_s^{-1})ds\right)|X_t = y) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)}det(z)^{\nu/2}\frac{{}_0F_1(m+\nu,z)}{{}_0F_1(m,z)} \tag{10}$$

$z = xy/4t^2$. Recall that for $m = 1$, this is the well-known Hartman-Watson law and that its density was computed by Yor (cf [29]). Here, we will investigate the case $m = 2$. First, we state the following result due to Gross and Richards (cf [13]) :

$${}_0F_1(m+\nu,z) = \frac{det(z_i^{m-j}{}_0\mathscr{F}_1(m+\nu-j+1,z_i))}{V(z)},$$

where (z_i) denote the eigenvalues of z and $V(z) = \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant . Noting that $\Gamma_m(m+\nu) = \prod_{j=1}^m \Gamma(m+\nu-j+1)$, (10) is equal to:

$$(10) = \frac{det(z_i^{(m-j)/2}I_{m+\nu-j}(2\sqrt{z_i}))}{det(z_i^{(m-j)/2}I_{m-j}(2\sqrt{z_i}))}$$

Without loss of generality, we will take $t = 1$.

Proposition 9. *For $m = 2$, let $\lambda_1 > \lambda_2$ be the eigenvalues of $\sqrt{xy}$. Then, the density of the generalized Hartman-Watson law is given by:*

$$f(v) = \frac{\sqrt{\lambda_1\lambda_2}v}{p\pi\sqrt{2\pi v^3}}\frac{\int_0^1\int_0^\infty z\sinh(p\sqrt{1-z^2})e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y}e^{-\frac{2(y^2-\pi^2)}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dzdy}{\int_0^1\int_0^1\frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx},$$

for $v > 0$, where $p = \lambda_1 - \lambda_2$. Furthermore, if $\lambda_1 = \lambda_2 := \lambda$, then:

$$f(v) = \frac{4\lambda v e^{\frac{2\pi^2}{v}}}{\pi^2\sqrt{2\pi v^3}}\frac{\int_0^\infty g(y)e^{-\frac{2y^2}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dy}{{}_1\mathscr{F}_2(\frac{1}{2};1;2;\lambda^2)},$$

where

$$g(y) = \frac{1}{3} + \frac{\pi}{2}\frac{I_2(2\lambda\cosh y) + \mathbf{L}_2(2\lambda\cosh y)}{2\lambda\cosh y},$$

and $\mathbf{L}_2$ is the Struve function (cf [4] *or* [12]*).*

Proof : For $m = 2$, (10) becomes:

$$(10) = \frac{\lambda_1 I_{\nu+1}(\lambda_1)I_\nu(\lambda_2) - \lambda_2 I_{\nu+1}(\lambda_2)I_\nu(\lambda_1)}{\lambda_1 I_1(\lambda_1)I_0(\lambda_2) - \lambda_2 I_1(\lambda_2)I_0(\lambda_1)},$$

so, using the above integral representations (cf [4],p 46):

$$x(aI_{\nu+1}(ax)I_\nu(bx) - bI_{\nu+1}(bx)I_\nu(ax)) = (a^2 - b^2)\int_0^x uI_\nu(au)I_\nu(bu)du$$

with $x = 1, a = \lambda_1, b = \lambda_2$, and (cf [4] p 304, [12] p 734):

$$\frac{\pi}{2} I_\nu(\frac{a}{2}(\sqrt{b^2+c^2}+b)) I_\nu(\frac{a}{2}(\sqrt{b^2+c^2}-b)) = \int_0^a \frac{\cosh(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} I_{2\nu}(cx)\,dx$$

$$(a > 0, Re(\nu) > -1)$$

with $a = 1, b = (\lambda_1 - \lambda_2)u := pu$ et $c = 2\sqrt{\lambda_1\lambda_2}u$, the numerator of (10) is then equal to:

$$\frac{2}{\pi}(\lambda_1^2 - \lambda_2^2) \int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux)\,dudx.$$

Taking $\nu = 0$, the denominator is then equal to:

$$\frac{2}{\pi}(\lambda_1^2 - \lambda_2^2) \int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx.$$

Thus, (10) becomes:

$$\frac{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux)\,dudx}{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx}$$

Now, we only have to use the integral representation of $I_{2\nu}$ (cf [29]):

$$\begin{aligned} I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux) &= \frac{1}{2i\pi}\int_C e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega} e^{-2\nu\omega}\,d\omega \\ &= \frac{1}{2i\pi}\int_C e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega} \int_0^\infty \frac{2\omega e^{-v\nu^2/2}}{(2\pi v^3)^{1/2}} e^{-\frac{2\omega^2}{v}}\,dvd\omega \end{aligned}$$

where C is the contour indicated in [29], hence, the density function is given by:

$$f(v) = \frac{1}{i\pi\sqrt{2\pi v^3}} \frac{\int_0^1 \int_0^1 \int_C u\omega \frac{\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega} e^{-\frac{2\omega^2}{v}}\,dudxd\omega}{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx} \mathbf{1}_{\{v>0\}}$$

We can simplify this expression by integrating over C (cf [29]):

$$\frac{\sqrt{\lambda_1\lambda_2}v}{\pi\sqrt{2\pi v^3}} \frac{\int_0^1 \int_0^1 \int_0^\infty u^2 x \frac{\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} e^{-2\sqrt{\lambda_1\lambda_2}ux\cosh y} e^{-\frac{-2(y^2-\pi^2)}{v}} (\sinh y)\sin(\frac{4\pi y}{v})\,dudxdy}{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx}$$

Setting $z = ux$, we obtain

$$\frac{\sqrt{\lambda_1\lambda_2}v}{\pi\sqrt{2\pi v^3}} \frac{\int_0^1 \int_0^u \int_0^\infty z \frac{u\cosh(p\sqrt{u^2-z^2})}{\sqrt{u^2-z^2}} e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y} e^{-\frac{2(y^2-\pi^2)}{v}} (\sinh y)\sin(\frac{4\pi y}{v})\,dudzdy}{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx},$$

that we can integrate with respect to u to get

$$\frac{\sqrt{\lambda_1\lambda_2}v}{p\pi\sqrt{2\pi v^3}} \frac{\int_0^1 \int_0^\infty z\sinh(p\sqrt{1-z^2}) e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y} e^{-\frac{2(y^2-\pi^2)}{v}} (\sinh y)\sin(\frac{4\pi y}{v})\,dzdy}{\int_0^1 \int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)\,dudx}.$$

Now, we prove the second part. In this case, $p = 0$ and we have to calculate:

$$\frac{\lambda v e^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}} \frac{\int_0^1\int_0^1\int_0^\infty \frac{u^2 x}{\sqrt{1-x^2}} e^{-2\lambda u x\cosh y} e^{-\frac{2y^2}{v}} (\sinh y)\sin(\frac{4\pi y}{v}) du dx dy}{\int_0^1\int_0^1 \frac{u I_0(2\lambda u x)}{\sqrt{1-x^2}} du dx}$$

Setting $z = ux$, then the density is given by:

$$\frac{\lambda v e^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}} \frac{\int_0^1\int_0^\infty z\sqrt{1-z^2} e^{-2\lambda z\cosh y} e^{-\frac{2y^2}{v}} (shy) sin(\frac{4\pi y}{v}) dz dy}{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\lambda u x) du dx},$$

Integrating with respect to z, we get (cf [12] p 369):

$$\frac{\lambda v e^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}} \frac{\int_0^\infty g(y) e^{-\frac{2y^2}{v}} (shy) sin(\frac{4\pi y}{v}) dz dy}{\int_0^1\int_0^1 \frac{u I_0(2\lambda u x)}{\sqrt{1-x^2}} du dx},$$

For the denominator, we use the fact that $\frac{d}{dz}(zI_1(z)) = zI_0(z)$, which yields:

$$\int_0^1\int_0^1 \frac{u I_0(2\lambda u x)}{\sqrt{1-x^2}} du dx = \int_0^1 \frac{I_1(2\lambda x)}{2\lambda x\sqrt{1-x^2}} dx$$

Then, the following formula

$$\int_0^a x^{\alpha-1}(a^2-x^2)^{\beta-1} I_\nu(cx) dx = 2^{-\nu-1} a^{2\beta+\alpha+\nu-2} c^\nu \frac{\Gamma(\beta)\Gamma((\alpha+\nu)/2)}{\Gamma(\beta+(\alpha+\nu)/2)\Gamma(\nu+1)} {}_1\mathscr{F}_2(\frac{\alpha+\nu}{2}; \beta+\frac{\alpha+\nu}{2}; \nu+1; \frac{a^2c^2}{4}) \tag{11}$$

taken with $\alpha = 0, a = 1, \beta = 1/2, c = 2\lambda, \nu = 1$ gives:

$$\int_0^1 \frac{I_1(2\lambda x)}{2\lambda x\sqrt{1-x^2}} dx = \frac{\pi}{4} {}_1\mathscr{F}_2(\frac{1}{2}; 1; 2; \lambda^2)$$

We can treat this case in this way: we write $\lambda_1 = \lambda_2 + h$ then we replace λ_1 in (10) to get:

$$\frac{((\lambda_2+h) I_{\nu+1}(\lambda_2+h) I_\nu(\lambda_2) - \lambda_2 I_{\nu+1}(\lambda_2) I_\nu(\lambda_2+h))/h}{((\lambda_2+h) I_1(\lambda_2+h) I_0(\lambda_2) - \lambda_2 I_1(\lambda_2) I_0(\lambda_2+h))/h}.$$

Next, we let $h \to 0$. As usual, we first compute the numerator and then take $\nu = 0$. To do this, we shall evaluate :

$$\begin{aligned} A &= \lim_{h\to 0} \frac{(\lambda_2+h) I_{\nu+1}(\lambda_2+h) - \lambda_2 I_{\nu+1}(\lambda_2)}{h} \\ B &= \lim_{h\to 0} \frac{I_\nu(\lambda_2+h) - I_\nu(\lambda_2)}{h} \end{aligned}$$

which are equal respectively to $\frac{d}{dx}(xI_{\nu+1}(x))$ et $\frac{d}{dx}(I_\nu(x))$ taken for $x = \lambda$. (We'll write λ instead of λ_2).

Using the differentiation formula $\frac{d}{dx}(x^\nu I_\nu(x)) = x^\nu I_{\nu-1}(x)$(cf [20],p 110), we get:

$$\begin{aligned}\frac{d}{dx}(xI_{\nu+1}(x)) &= -\nu I_{\nu+1}(x) + xI_\nu(x)\\ \frac{d}{dx}(I_\nu(x)) &= -\frac{\nu}{x}I_\nu(x) + I_{\nu-1}(x),\end{aligned}$$

thus:

$$\begin{aligned}N &= I_\nu(\lambda)(-\nu I_{\nu+1}(\lambda) + \lambda I_\nu(\lambda)) - \lambda I_{\nu+1}(\lambda)(-\frac{\nu}{\lambda}I_\nu(\lambda) + I_{\nu-1}(\lambda))\\ &= \lambda({I_\nu}^2(\lambda) - I_{\nu+1}(\lambda)I_{\nu-1}(\lambda))\end{aligned}$$

$$A = \frac{{I_\nu}^2(\lambda) - I_{\nu+1}(\lambda)I_{\nu-1}(\lambda)}{{I_0}^2(\lambda) - I_1(\lambda)I_{-1}(\lambda)}$$

In another hand, we have the integral representation(cf [12],p 757):

$$I_\mu(z)I_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2}\cos((\mu-\nu)\theta)I_{\mu+\nu}(2z\cos\theta)d\theta, \qquad Re(\mu+\nu) > -1.$$

Consequently, the numerator is written :

$$\begin{aligned}N &= \frac{2}{\pi}\int_0^{\pi/2}(1-\cos 2\theta)I_{2\nu}(2\lambda\cos\theta)d\theta\\ &= \frac{4}{\pi}\int_0^{\pi/2}(\sin^2\theta)I_{2\nu}(2\lambda\cos\theta)d\theta\\ &= \frac{4}{\pi}\int_0^1\sqrt{1-r^2}I_{2\nu}(2\lambda r)dr,\end{aligned}$$

and so, the denominator is equal to

$$D = \frac{4}{\pi}\int_0^1\sqrt{1-r^2}I_0(2\lambda r)dr.$$

Finally, we use again the integral representation of I_ν to obtain :

$$\begin{aligned}f(u) &= \frac{\lambda u e^{2\pi^2/u}}{\pi\sqrt{2\pi u^3}}\frac{\int_0^\infty e^{-2y^2/u}\sinh(y)\sin\left(\frac{4\pi y}{u}\right)\int_0^1 r\sqrt{1-r^2}e^{-2\lambda r\cosh y}dr\,du}{\int_0^1\sqrt{1-r^2}I_0(2\lambda r)dr}\\ &= \frac{\lambda u e^{2\pi^2/u}}{\pi\sqrt{2\pi u^3}}\frac{\int_0^\infty g(y)e^{-2y^2/u}\sinh(y)\sin\left(\frac{4\pi y}{u}\right)du}{\int_0^1\sqrt{1-r^2}I_0(2\lambda r)dr},\end{aligned}$$

We can use (11) to see that the denominator is equal to:

$$\frac{\pi}{4}{}_1\mathscr{F}_2(\frac{1}{2};2;1;\lambda^2),$$

hence, we find the same expression given in the first approach.

7. The Law of T_0

Recall that: For $0 < \nu < 1$,

$$Q_x^{m-\nu}(T_0 > t) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} det(\frac{x}{2t})^\nu {}_1F_1(\nu, m+\nu, -\frac{x}{2t})$$

Proposition 10. *Let $m = 2$ and $\lambda_1 > \lambda_2$ be the eigenvalues of x. The density of $S_0 := \dfrac{1}{2T_0}$ under $Q_x^{m-\nu}$ is given by:*

$$f(u) = \frac{(\lambda_1\lambda_2)^\nu u^{2\nu-2} e^{-(\lambda_1+\lambda_2)u}}{\Gamma(\nu+1)\Gamma(\nu)} \frac{{}_1\mathscr{F}_1(2, \nu+1, \lambda_1 u) - {}_1\mathscr{F}_1(2, \nu+1, \lambda_2 u)}{(\lambda_1 - \lambda_2)}$$

Corollary 7. *If $\lambda_1 = \lambda_2 := \lambda$, the density is written:*

$$f(u) = \frac{2\lambda^{2\nu} u^{2\nu-1} e^{-\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)} {}_1\mathscr{F}_1(\nu-1, \nu+2, -\lambda u)$$

Proof: With the help of Gross-Richards Formula, it follows that for, $m = 2$,

$$\begin{aligned} Q_x^{m-\nu}(S_0 \le u) = \frac{(\lambda_1\lambda_2)^\nu}{(\lambda_1-\lambda_2)\Gamma_2(\nu+2)} u^{2\nu} (\lambda_1 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_1 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_2 u) \\ - \lambda_2 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_2 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_1 u)), \end{aligned}$$

where $S_0 := 1/(2T_0)$. This is a C^∞ function in u, hence, we will compute its derivative to get the density. Recall that :

$$\frac{d}{dz} {}_1\mathscr{F}_1(a, b, z) = \frac{a}{b} {}_1\mathscr{F}_1(a+1, b+1, z),$$

thus, we get:

$$f(u) = \frac{d}{du} Q_x^{m-\nu}(S_0 \le u) = K(\nu, \lambda_1, \lambda_2) u^{2\nu-1}(A - B)$$

where

$$\begin{aligned} K(\nu, \lambda_1, \lambda_2) &= \frac{(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)} \\ A &= 2\nu((\lambda_1 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_1 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_2 u) \\ &- \lambda_2 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_2 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_1 u)) \\ B &= \frac{\nu}{\nu+2}((\lambda_1^2 u_1 {}_1\mathscr{F}_1(\nu+1, \nu+3, -\lambda_1 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_2 u) \\ &- \lambda_2^2 u_1 {}_1\mathscr{F}_1(\nu+1, \nu+3, -\lambda_2 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_1 u)). \end{aligned}$$

Then, we use the contiguous relation :

$$b\, {}_1\mathscr{F}_1(a, b, z) - b\, {}_1\mathscr{F}_1(a-1, b, z) = z\, {}_1\mathscr{F}_1(a, b+1, z)$$

to see that

$$\lambda_1 u_1 {}_1\mathscr{F}_1(\nu+1, \nu+3, -\lambda_1 u) = (\nu+2)({}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_1 u) - {}_1\mathscr{F}_1(\nu+1, \nu+2, -\lambda_1 u))$$

$$\lambda_2 u_1 {}_1\mathscr{F}_1(\nu+1, \nu+3, -\lambda_2 u) = (\nu+2)({}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_2 u) - {}_1\mathscr{F}_1(\nu+1, \nu+2, -\lambda_2 u))$$

implies that:

$$f(u) = K_1(\nu, \lambda_1, \lambda_2) u^{2\nu-1}(C + D - E - F)$$

avec

$$\begin{aligned} K_1(\nu,\lambda_1,\lambda_2) &= \frac{\nu(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)} \\ C &= \lambda_1 {}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u) \\ D &= \lambda_1 {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_1 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u) \\ E &= \lambda_2 {}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u) \\ F &= \lambda_2 {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_2 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u), \end{aligned}$$

Applying again the above contiguous relation yields:

$$\lambda_1 u\, {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_1 u) = (\nu+1)({}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_1 u) - {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_1 u))$$

$$\lambda_2 u\, {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_2 u) = (\nu+1)({}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_2 u) - {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_2 u))$$

$$\lambda_2 u\, {}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u) = (\nu+1)({}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u) - {}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_2 u))$$

$$\lambda_1 u\, {}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u) = (\nu+1)({}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u) - {}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_1 u))$$

Replacing in the expression of f, we obtain

$$f(u) = K_2(\nu,\lambda_1,\lambda_2)u^{2\nu-2}(G-H),$$

where

$$\begin{aligned} K_2(\nu,\lambda_1,\lambda_2) &= \frac{\nu(\nu+1)(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)} \\ G &= {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_2 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u) \\ H &= {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_1 u)\,{}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u) \end{aligned}$$

Eventually, writing

$$\begin{aligned} \Gamma_2(\nu+2) &= \Gamma(\nu+2)\Gamma(\nu+1) \\ &= \nu(\nu+1)\Gamma(\nu+1)\Gamma(\nu) \\ {}_1\mathscr{F}_1(a,a,z) &= e^{-z} \\ {}_1\mathscr{F}_1(a,b,-z) &= e^{-z}\,{}_1\mathscr{F}_1(b-a,b,z), \end{aligned}$$

we get

$$f(u) = \frac{(\lambda_1\lambda_2)^\nu u^{2\nu-2}e^{-(\lambda_1+\lambda_2)u}}{\Gamma(\nu+1)\Gamma(\nu)}\,\frac{{}_1\mathscr{F}_1(2,\nu+1,\lambda_1 u) - {}_1\mathscr{F}_1(2,\nu+1,\lambda_2 u)}{(\lambda_1-\lambda_2)}$$

Th case $\lambda_1 = \lambda_2$ is treated in the same way as before (for the Hartman-Watson law). In fact , writing $\lambda_1 = \lambda_2 + h$ and letting $h \to 0$, we see that the density is given by :

$$\begin{aligned} f(u) &= \frac{\lambda^{2\nu}u^{2\nu-2}e^{-2\lambda u}}{\Gamma(\nu+1)\Gamma(\nu)}\frac{d}{d\lambda}\,{}_1\mathscr{F}_1(2,\nu+1,\lambda u) \\ &= \frac{2\lambda^{2\nu}u^{2\nu-1}e^{-2\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)}{}_1\mathscr{F}_1(3,\nu+2,\lambda u) \\ &= \frac{2\lambda^{2\nu}u^{2\nu-1}e^{-\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)}{}_1\mathscr{F}_1(\nu-1,\nu+2,-\lambda u) \qquad \blacksquare \end{aligned}$$

Remark. Noting that:

$$Q_x^{m-\nu}(e^{-rS_0}) = r\int_0^\infty e^{-ru}Q_x^{m-\nu}(S_0 \le u)du,$$

we can derive the Laplace transform of S_0 from its distribution function. Integrals of confluent hypergeometric products appear and give rise the so-called Appell function (or Lauricella function) F_2. In fact, we have (cf [12]):

$$\int_0^\infty u^{2\nu}e^{-(\lambda_1+\lambda_2+r)u}{}_1\mathscr{F}_1(2,\nu+2,\lambda_1 u){}_1\mathscr{F}_1(2,\nu+1,\lambda_2 u)du = K\,F_2(a,b,c,d;x;y)$$

However, using the density function expression in the case $\lambda_1 = \lambda_2$, we get

$$\begin{aligned}
Q_x^{m-\nu}(e^{-rS_0}) &= \frac{2\lambda^{2\nu}}{\Gamma(\nu+2)\Gamma(\nu)}\int_0^\infty u^{2\nu-1}e^{-(r+2\lambda)u}{}_1\mathscr{F}_1(3,\nu+2,\lambda u)du \\
&\overset{(1)}{=} \frac{2\lambda^{2\nu}}{\Gamma(\nu+2)\Gamma(\nu)}\sum_{n\ge 0}\frac{(3)_n}{(\nu+2)_n}\frac{\lambda^n}{n!}\int_0^\infty u^{2\nu+n-1}e^{-(r+2\lambda)u}du \\
&= \frac{2\lambda^{2\nu}\Gamma(2\nu)}{\Gamma(\nu+2)\Gamma(\nu)(2\lambda+r)^{2\nu}}\sum_{n\ge 0}\frac{(3)_n(2\nu)_n}{(\nu+2)_n n!}\left(\frac{\lambda}{2\lambda+r}\right)^n \\
&= \frac{\Gamma(2\nu)}{2^{2\nu-1}\Gamma(\nu+2)\Gamma(\nu)}\left(\frac{2\lambda}{2\lambda+r}\right)^{2\nu}{}_2\mathscr{F}_1(2\nu,3,\nu+2;\frac{\lambda}{2\lambda+r}) \\
&\overset{(2)}{=} \frac{2\Gamma(2\nu)}{\Gamma(\nu+2)\Gamma(\nu)}{}_2\mathscr{F}_1(2\nu,\nu-1,\nu+2;-\frac{\lambda}{\lambda+r}),
\end{aligned}$$

where in (1), we used Fubini-Tonelli theorem and in (2), the Pfaff transformation. (cf [20])

We close this paper by computing the law, under $Q_x^{m+\nu}$, of

$$A_t := \inf\{u, H_u := \int_0^u tr(X_s^{-1})ds > t\}$$

taken at an exponential random time with parameter $\mu^2/2$, T_μ, independent of (X_t). Using the absolute continuity relations, we get:

$$\begin{aligned}
Q_x^{m+\nu}(A_{T_\mu} > u) &= Q_x^{m+\nu}\left(\frac{\mu^2}{2}\int_0^\infty e^{-\mu^2 t/2}\mathbf{1}_{\{A_t>u\}}du\right) \\
&= Q_x^{m+\nu}\left(\frac{\mu^2}{2}\int_0^\infty e^{-\mu^2 t/2}\mathbf{1}_{\{t>H_u\}}du\right) \\
&= Q_x^{m+\nu}(e^{-\mu^2 H_u/2}) \\
&= Q_x^m(\left(\frac{det(X_u)}{det(x)}\right)^{\nu/2}e^{-\frac{\mu^2+\nu^2}{2}H_u}),
\end{aligned}$$

hence, if we set $\sigma^2 = \mu^2 + \nu^2$,

$$Q_x^{m+\nu}(A_{T_\mu} > u) = Q_x^{m+\sigma}(\left(\frac{det(X_u)}{det(x)}\right)^{(\nu-\sigma)/2})$$
$$= (det(x))^{(\sigma-\nu)/2}(2u)^{m(\nu-\sigma)/2}\frac{\Gamma_m((m+\sigma+\nu)/2)}{\Gamma_m(m+\sigma)}{}_1F_1(\frac{\sigma-\nu}{2}, m+\sigma, -\frac{x}{2u}).$$

If we set $Y := \frac{1}{2A_{T_\mu}}$, the last expression becomes:

$$Q_x^{m+\nu}(Y < t) = (det(x))^{(\sigma-\nu)/2}t^{m(\sigma-\nu)/2}\frac{\Gamma_m((m+\sigma+\nu)/2)}{\Gamma_m(m+\sigma)}{}_1F_1(\frac{\sigma-\nu}{2}, m+\sigma, -tx),$$

and for $m = 2$,

$$Q_x^{\nu+2}(Y < t) = (det(x))^{(\sigma-\nu)/2}t^{(\sigma-\nu)}\frac{\Gamma_m((\sigma+\nu+2)/2)}{\Gamma_m(\sigma+2)}{}_1F_1(\frac{\sigma-\nu}{2}, \sigma+2, -tx).$$

Then, applying Gross-Richards formula, we obtain:

$$Q_x^{m+\nu}(Y < t) = \frac{(x_1x_2)^{(\sigma-\nu)/2)}}{x_1-x_2}\frac{((\sigma+\nu)/2)\Gamma^2((\sigma+\nu)/2)}{(\sigma+1)\Gamma^2(\sigma+1)}t^{(\sigma-\nu)/2}$$
$$[x_{11}\mathscr{F}_1(\frac{\sigma-\nu}{2}, \sigma+2, -tx_1){}_1\mathscr{F}_1(\frac{\sigma-\nu}{2}-1, \sigma+1, -tx_2) -$$
$$x_{21}\mathscr{F}_1(\frac{\sigma-\nu}{2}, \sigma+2, -tx_2){}_1\mathscr{F}_1(\frac{\sigma-\nu}{2}-1, \sigma+1, -tx_1)].$$

Where $x_1 > x_2$ denotes the eigenvalues of x. As usually, this is a C^∞ function , thus, we shall calculate its derivative to get the density. A similar calculus as before for T_0 yields:

$$f(t) = \frac{(x_1x_2)^b ab}{x_1-x_2}\frac{\Gamma^2(a)}{\Gamma(a+b+1)}t^{2b-2}[{}_1\mathscr{F}_1(b+1, a+b+1, -tx_2){}_1\mathscr{F}_1(b-1, a+b+1, -tx_1)$$
$$- {}_1\mathscr{F}_1(b+1, a+b+1, -tx_1){}_1\mathscr{F}_1(b-1, a+b+1, -tx_2)],$$

where $a = \dfrac{\sigma+\nu}{2}$ and $b = \dfrac{\sigma-\nu}{2}$.

Remark. Recall that, in the one-dimensional case, $A_{T_\mu} = \dfrac{R}{2Z}$, where $R \overset{\mathscr{L}}{=} \beta(1, a)$ and $Z \overset{\mathscr{L}}{=} \gamma_b$, both variables are independent (cf [30], p. 68).

8. Conclusion

The Gross-Richards formula has been the main ingredient in this paper, since it enables us to express more explicitly the special functions of matrix argument. The case $m = 3$ can be treated in the same way, but computation becomes very complicated. So, if we want to deal with the general case, it will be convenient to find a more explicit formula. Indeed, Schur functions can be expressed as polynomials in the elementary symmetric functions e_r or as polynomials in the completely symmetric functions h_r (cf [21], p. 19-21). More precisely, we have :

$$s_\lambda = det(e_{\lambda_i-i+j}) \qquad 1 \le i, j \le n$$
$$s_\lambda = det(h_{\lambda'_i-i+j}) \qquad 1 \le i, j \le n$$

where λ is a partition of length $\leq n$, and λ' is the conjugate of λ (cf [21], p. 2). So, using these two identities, can we establish more general results?

9. Appendix: special functions

9.1. **The hypergeometric series.** The multivariate hypergeometric function of Hermitian matrix argument is defined by:

$$ {}_pF_q((a_i)_{1\leq i\leq p},(b_j)_{1\leq j\leq q};X)=\sum_{k\geq 0}\sum_{\tau\perp k}\frac{(a_1)_\tau\cdots(a_p)_\tau}{(b_1)_\tau\cdots(b_q)_\tau}\frac{C_\tau(X)}{k!} $$

where $\tau=(k_1,\ldots,k_m)$ is a partition of weight k and length m such that $k_1\geq\ldots\geq k_m$, $(a)_\tau$ is the generalised Pochammer symbol defined by:

$$ (a)_\tau=\prod_{i=1}^{m}\frac{\Gamma(a+k_i-i+1)}{\Gamma(a-i+1)},\qquad \tau=(k_1,\ldots,k_m) $$

and $C_\tau(X)$ is the zonal polynomial of X such that :

$$ (tr(X))^k=\sum_{\tau\perp k}C_\tau(X) $$

In fact, there are several normalizations for this polynomial ([22] for the real symmetric case, [6] for the Hermitian complex case and [11] for a more general setting. This polynomial is symmetric, homogeneous, of degree k in the eigenvalues of X and is an eigenfunction of the following differential operator :

$$ \Delta_X=\sum_{i=1}^{m}x_i^2\frac{\partial^2}{\partial x_i^2}+2\sum_{i=1}^{m}\sum_{1\leq k\neq i\leq m}\frac{x_i^2}{x_i-x_k}\frac{\partial}{\partial x_i} $$

Besides, we have :

$$ C_\tau(YX)=C_\tau(\sqrt{Y}X\sqrt{Y}) $$

for any Hermitian matrix Y. In the complex case, this polynomial is identified with the Schur function s_τ.
It is well-known that, if $p=q+1$, then the hypergeometric series is convergent for $0\leq||X||<1$ ($||\cdot||$ is the norm given by the spectral radius) , if $p\leq q$, then it converges everywhere and else, it diverges ([22], [6], [11]).

9.2. **The modified Bessel function.** The modified Bessel function with index $\nu\in\mathbb{R}$ is given by the following series :

$$ I_\nu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma_m(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu},\qquad z\in\mathbb{C}. $$

It can be represented through standard hypergeometric functions ${}_0F_1$ and ${}_1F_1$:

$$ \begin{aligned} I_\nu(z)&=\frac{1}{\Gamma_m(\nu+1)}\left(\frac{z}{2}\right)^\nu{}_0F_1(\nu+1;z^2)\\ &=\frac{(z/2)^\nu}{\Gamma_m(\nu+1)}e^{-z}{}_1F_1(\nu+\frac{1}{2},2\nu+1;2z) \end{aligned} $$

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References

[1] *T. H. Baker, P. J. Forrester.* The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* **188**. 1997, 175-216.

[2] *M.F.Bru.*Wishart Process. *J. Theoretical Probability*, **4** , no. 4, 1991, 725 -751.

[3] *M.F.Bru.* Diffusions of perturbed principal component analysis. *J. Multivariate Anal.* **29**, 1989, 127-136.

[4] *Yu.A.Brychkov, O.I.Marichev, A.P.Prudnikov.* Integrals and Series, vol 2: special functions.*Gordon and Breach science publishers.*

[5] *Yu.A.Brychkov, O.I.Marichev, A.P.Prudnikov.* Integrals and Series, vol 3: more special functions.*Gordon and Breach science publishers.*

[6] *Y. Chikuze.* Partial differential equations. *Ann. Institute. Stat. Math.*, **28**, 1976, 187-199.

[7] *C.Donati-Martin, Y.Doumerc, H.Matsumoto, M.Yor.* Some properties of Wishart process and a matrix extension of the Hartman-Watson law. *Pub. R. I. M. S. Kyoto university*, **40**, No 4, 2004, 1385-1412.

[8] *H. Doss, E. Lenglart.* Sur l'existence, l'unicité et le comportement asymptotique des solutions d'équations différentielles stochastiques. *Ann. Inst. H. Poincaré*, **14**, no 2, 1978, 189-214.

[9] *Y. Doumerc.* Matrices aléatoires, processus stochastiques et groupes de réfléxions. *Ph. D. Thesis.* May 2005.

[10] *J. Faraut.* Infinite dimensional harmonic analysis and probability. To appear in *Tata Institute Lecture notes.*

[11] *J. Faraut, A. Koryani.* Analysis on Symmetric Cones.*Clarendon Press, Oxford*, 1994.

[12] *I.S.Gradshteyn, I.M.Ryzhik.* Table of integrals, series and products. Fifth Edition. *Academic Press*, 1994.

[13] *K. I. Gross, St.P.Richards.* Special functions of matrix argument, *Bull. Amer. Math. Soc.* **24**, no 2, 1991, 349-355.

[14] *N. Ikeda, S. Watanabe.* Stochastic Differential Equations, 2nd ed. *North-Holland Publishing Company.*

[15] *A. T. James.* Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Stat.* **35**, 1964, 475-501.

[16] *S. P. Karlin, G. MacGregor.* Coincidence probabilities. *Pacif. J. Math.* **9**, 1959, 1141-1164.

[17] *M. Katori, H. Tanemura.* Symmetry of matrix-valued processes and noncolliding diffusion particle systems. *J. Math. Phy.* No 45, 2004, 3058-3085.

[18] *W. König, N. O' Connell.* Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Elec. Comm. in Proba.* **6**, 2001, 107-114.

[19] *J. Lamperti.* Semi-stable Markov processes. I . *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **22**, 1972, 205-225.

[20] *N.N.Lebedev.* Special Functions And Their Applications. *Dover Publications, INC.* 1972.

[21] *L. G. Macdonald.* Symmetric Functions and Hall Polynomials, 2nd ed. *Clarendon Press, Oxford*, 1995.

[22] *R . J. Muirhead.* Aspects of Multivariate Statistical Theory. *Wiley Series in Probability*, 1982.

[23] *J. R. Norris, L. C. G. Rogers, D. Williams.* Brownian motions of ellipsoids. *Trans. Amer. Math. Soc.* **294**, 1986. 757-765.

[24] *S. Péché.* Universality of local eigenvalue statistics for random sample covariance matrices. *Ph. D. Thesis.* 2003.

[25] *J. Pitman, M. Yor.* A Decomposition of Bessel Bridges. *P. T. R. F.* **59**, 1982, 425-457.

[26] *D. Revuz, M. Yor.* Continuous Martingales And Brownian Motion, 3rd ed, Springer, 1999.

[27] *L.C.G.Rogers, D.Williams.* Diffusions, Markov Processes and Martingales. Vol 2, Itô Calculus.*Cambridge University Press*, 1987.

[28] *E. Wong.* The construction of a class of stationnary Markov. *Proceedings. The* 16^{th} *Symposium. Applied Math.* AMS. Providence. RI. 1964. 264-276.

[29] *M.Yor*. Loi de l'indice du lacet Brownien et distribution de Hartman-Watson, *P.T.R.F*, **53**, No.1, 1980, 71-95.

[30] *M. Yor*. Some Aspects of Brownian Motion. Part I : Some Special Functionals. *Birkhäuser Verlag*. 1992.