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Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process

Fabien Panloup*

September 30, 2005

Abstract

We investigate some recursive procedures based on an exact or “approximate” Euler scheme with decreasing step in view to computation of invariant measures of solutions to S.D.E. driven by a Lévy process. Our results are valid for a large class of S.D.E. that can be governed by Lévy processes with few moments or can have a weakly mean-reverting drift, and permit to find again the a.s. C.L.T for stable processes.

Keywords: stochastic differential equation ; Lévy process ; invariant distribution ; Euler scheme ; almost sure central limit theorem.

1 Introduction

The purpose of this paper is the computation of the invariant measure ν of a stochastic differential equation driven by a Lévy process. Practically, the aim of this work is to construct a sequence of simulatable empirical measures $(\bar{\nu}_n)_{n \geq 1}$ such that $\bar{\nu}_n(\omega, f) \rightarrow \nu(f)$ *a.s.* for a range of functions f containing bounded continuous functions, by exploiting the ergodic properties of the S.D.E. under Lyapunov assumptions.

This work is motivated by two aspects. On the one hand, we want to extend some methods investigated for Brownian diffusions (see below) to S.D.E. with jumps having a Markovian structure. Then, we know that solutions to homogeneous S.D.E driven by Lévy processes are Markov and Feller processes under classical conditions (see Theorem 1) and that, this property is not preserved if the driving process is a more general semimartingale (see [10]). On the other hand, the second interest is to propose a way to simulate invariant measures of dynamical systems which are usually used in modelling. In fact, there are many situations where a Brownian diffusion is not satisfactory because, for instance, the noise is discontinuous or is too intensive. Let us get interested in an example coming from modelling of combinations of fragmentation and coalescence phenomenas. For instance, consider a polymer formation at a critical temperature. Then, one observes that molecules break up and recombine simultaneously. This phenomena is modelled by a random process called E.F.C (Exchangeable Fragmentation-Coalescence) process. In [3], Berestycki shows that, under appropriate conditions, the dust generated by this process is a solution to a S.D.E. with mean-reversion and purely discontinuous Lévy noise. We will go back over this example in section 8.

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For other situations of modelling by S.D.E. driven by Lévy processes, we refer to Barndorff-Nielsen ([2]) for examples in financial modelling (where one can notably see the practical interest of the invariant measure of Lévy driven Ornstein-Uhlenbeck processes), Protter-Talay ([16]) (for financial examples and a telephone noise model) and Deng ([7]) who models the spot prices of electricity by a Brownian diffusion with mean-reverting perturbed by a compound Poisson noise.

As mentioned before, this problem has already been investigated in the case of Brownian diffusions. In [20], Talay approximates $\nu(f)$ by $1/N \sum_{n=1}^N f(\bar{X}_n^h)$ where $(\bar{X}_n^h)_{n \geq 1}$ is an Euler scheme with constant step h . The drawbacks of this algorithm are that it is not recursive and needs some ellipticity assumptions to be efficient. Then, our approach is inspired by a second work due to Lamberton and Pagès (see [11] and [12]) where the sequence of empirical measures is constructed from an Euler scheme with decreasing step. Denoting by $(\bar{X}_k)_{k \geq 1}$ this Euler scheme and $(\eta_k)_{k \geq 1}$ a sequence of weights such that $H_n = \sum_{k=1}^n \eta_k \xrightarrow{n \rightarrow +\infty} +\infty$, they are able to show under Lyapunov assumptions on the coefficients and easy conditions on steps and weights we don't recall, that,

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k f(\bar{X}_{k-1}) \xrightarrow{n \rightarrow +\infty} \nu(f) \quad p.s.,$$

where ν is the invariant measure of the diffusion and f is a continuous function with polynomial growth (see [11] and [12] for more details).

One of the difficulties encountered in generalizing this kind of result to S.D.E. driven by Lévy processes is the simulation of the Euler scheme because the increments of the jumps component of a Lévy process are only simulatable in particular cases. However, when "the Lévy measure is simulatable" (in a sense that we will define after), we can approximate the jumps component by a simulatable compound Poisson process obtained by truncation of the small jumps (It is also possible to stop this process at its first jump time, see Section 2.2). Note that when the truncation is small, the simulation of the increments of the compound Poisson process can be time-consuming (because of the intensity) but thanks to the decreasing step, it is possible to adjust steps and truncation such that the simulation is reasonable.

Our main result is to obtain the convergence of the sequence of empirical measures under Lyapunov assumptions in the two above situations.

2 Setting and background

1) The stochastic differential equation. Considering a positive number c and a Lévy measure π on \mathbb{R}^l , we denote by $(X_t)_{t \geq 0}$, a càdlàg process solution to the S.D.E.

$$dX_t = b^c(X_{t-})dt + \sigma(X_{t-})dW_t + \kappa(X_{t-})dZ_t^c$$

where $b^c : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a continuous vector field, σ and κ are continuous on \mathbb{R}^d with values in $\mathbb{M}_{d,l}$ (set of $d \times l$ matrices), $(W_t)_{t \geq 0}$ is a l -dimensional Brownian motion and $(Z_t^c)_{t \geq 0}$ is a purely discontinuous Lévy process independent of $(W_t)_{t \geq 0}$ with characteristic function:

$$\mathbb{E}\{e^{i \langle u, Z_t^c \rangle}\} = \exp \left[t \left(\int e^{i \langle u, y \rangle} - 1 - i \langle u, y \rangle 1_{\{|y| \leq c\}} \pi(dy) \right) \right] \quad \forall t \geq 0.$$

Defining $n_t(\omega, dy) = \sum_{0 < s \leq t} 1_{\{\Delta Z_s^c \in dy\}}$, we recall that $Z_t^c = Y_t^c + N_t^c$ where $(Y_t^c)_{t \geq 0}$ and $(N_t^c)_{t \geq 0}$ are two independent Lévy processes such that

$$Y_t^c = \int_{\{|y| \leq c\}} y(n_t(\cdot, dy) - t\pi(dy)) \quad \text{and} \quad N_t^c = \sum_{0 < s \leq t} \Delta Z_s^c 1_{\{|L_s| > c\}}.$$

$(N_t^c)_{t \geq 0}$ is a compound Poisson process with intensity $\lambda = \pi(|y| > c)$ and law of jumps $\mu(dy) = 1_{\{|y| > c\}} \pi(dy) / \pi(|y| > c)$.

Remark 1. In most papers dealing with these processes, the S.D.E. is written $dX_t = f(X_{t-})dL_t$ where $(L_t)_{t \geq 0}$ is a Lévy process. According to the Lévy-Khintchine decomposition, our formulation is a little more general but the main interest for us is to separate each part of the Lévy process because they act differently on the dynamical system. First, we isolate the drift part from the others because it usually produces the mean-reverting effect. Then, the two other parts are considered as noises and we distinguish them because they have not the same behavior.

With the above formulation, the S.D.E. does not admit a unique representation because of the truncation function (except if π is symmetrical). However, in many situations, we can rewrite the S.D.E. with a formulation which does not depend on c . So is the case if $(N_t^c)_{t \geq 0}$ is integrable (then we can compensate for the big jumps, *i.e.* set $c = +\infty$) or if $(Y_t^c)_{t \geq 0}$ has locally bounded variations (then we are not compelled to compensate the small jumps, *i.e.* it is possible to set $c = 0$).

To this end, we introduce two assumptions (\mathbf{H}_p^1) (resp. (\mathbf{H}_q^2)) on the moments of the Lévy measure which we are going to use for rewriting the S.D.E. Let $p \in \mathbb{R}^+$ and $q \in [0, 1]$:

$$(\mathbf{H}_p^1) \quad : \quad \int_{|y| > c} \pi(dy) |y|^{2p} < +\infty, \quad (\mathbf{H}_q^2) \quad : \quad \int_{|y| \leq c} \pi(dy) |y|^{2q} < +\infty.$$

These assumptions are independent of the truncation function. They provide the following indications on the moment of $(Z_t^c)_{t \geq 0}$ and on the local behavior of the small jumps. We have

$$(\mathbf{H}_p^1) \iff \mathbb{E}|Z_t^c|^{2p} < +\infty \quad \forall t \geq 0 \text{ and, } (\mathbf{H}_q^2) \iff \mathbb{E}\left\{ \sum_{0 < s \leq t} |\Delta Z_s^c|^{2q} 1_{\Delta Z_s^c \leq c} \right\} < +\infty \quad \forall t \geq 0.$$

In particular, if $p \geq 1/2$, Z_t^c is integrable and if $q \leq 1/2$, $(Z_t^c)_{t \geq 0}$ has locally integrable variations. Accordingly, we afterwards adopt the following representation for the S.D.E.:

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dW_t + \kappa(X_{t-})dZ_t \quad (1)$$

where the definition of b and Z depends on p and q : consider $p \geq 0$ and $q \in [0, 1]$ such as (\mathbf{H}_p^1) and (\mathbf{H}_q^2) are satisfied. Then, if

Case 1 : $p > 1/2$,

$$b(x) = b^c(x) + \int_{\{|y| > c\}} y \pi(dy) \kappa(x) = b^\infty(x), \quad Z_t = Z_t^\infty = Y_t + N_t \quad \text{with,}$$

$$Y_t = Y_t^c \quad \text{and} \quad N_t = N_t^c - t \int_{\{|y| > c\}} y \pi(dy).$$

Case 2 : $0 \leq p, q \leq 1/2$,

$$b(x) = b^c(x) + \int_{\{|y| \leq c\}} y \pi(dy) \kappa(x) = b^0(x), \quad Z_t = Z_t^0 = Y_t + N_t \quad \text{with,}$$

$$Y_t = Y_t^c + t \int_{\{|y| \leq c\}} y \pi(dy) \quad \text{and} \quad N_t = N_t^c.$$

Case 3 : $p \leq 1/2 < q$,

$$b(x) = b^c(x) \quad Z_t = Z_t^c = Y_t + N_t \quad \text{with,} \quad Y_t = Y_t^c \quad \text{and} \quad N_t = N_t^c.$$

Remark 2. In cases 1 and 2, the representation is really independent of c whereas in case 3, we adopt the convention which seems to be the most natural. For technical arguments, in the critical case $p = 1/2$, we choose the notation associated with case 3. In case 1, we find compound Poisson processes with moment strictly superior to 1 or 1-dimensional stable processes with index $\alpha \in (1, 2)$, case 2 contains compound Poisson processes with moment smaller than 1 and 1-dimensional stable processes with index $\alpha \in (0, 1)$. Finally, in case 3, we have for example, the 1-dimensional stable processes with index $\alpha = 1$.

We recall a result of existence and uniqueness (see e.g. [15]).

Theorem 1. *Assume that b , σ and κ are locally lipschitz functions with sublinear growth. Then, weak unicity holds for the S.D.E. (1). On the other hand, if $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space satisfying the usual conditions and X_0 is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in \mathbb{R}^d , then for any (\mathcal{F}_t) -Brownian motion $(W_t)_{t \geq 0}$, for any (\mathcal{F}_t) -measurable $(Z_t)_{t \geq 0}$ defined as previously, the S.D.E. (1) admits a unique càdlàg solution $(X_t)_{t \geq 0}$ with initial condition X_0 . Moreover, $(X_t)_{t \geq 0}$ is a Feller Markov process.*

Then, the infinitesimal generator of $(X_t)_{t \geq 0}$ is defined for all $f \in C_K^2$ by

$$Af(x) = (\nabla f | b^c)(x) + \frac{1}{2} \text{Tr}(\sigma^* D^2 f \sigma)(x) + \int \pi(dy) (f(x + \kappa(x)y) - f(x) - (\nabla f | \kappa(x)y) 1_{\{|y| \leq c\}})) . \quad (2)$$

2) Exact and approximate Euler schemes From now, we tackle the principal issue of this paper: computation of invariant measures for S.D.E. driven by a Lévy process. We want to construct a sequence of empirical measures based on the Euler scheme with decreasing step. So, our first intention is to build this Euler scheme but, compared to the Brownian diffusions, a new problem appears which is the simulation of jumps. Thus, we know that we can exactly simulate increments of the jumps part of a Lévy process only in some particular cases such as stable or compound Poisson processes. In the opposite case, we have to approximate increments of Z by simulatable random variables. To this end, we introduce a sequence of positive numbers $(u_n)_{n \geq 1}$ such as $u_n < c$ and $u_n \xrightarrow{n \rightarrow +\infty} 0$, and $(Y^{c,n})_{n \geq 1}$, a sequence of càdlàg processes defined by

$$Y_t^{c,n} = \sum_{0 < s \leq t} \Delta Y_s^c 1_{\{|\Delta Y_s^c| \in \{u_n < |y| \leq c\}\}} - t \int_{u_n < |y| \leq c} y \pi(dy) \quad \forall t \geq 0$$

$Y^{c,n}$ is a compensated compound Poisson process with parameters $\lambda_n^c = \pi(u_n < |y| \leq c)$ and $\mu_n^c = 1_{u_n < |y| \leq c} \frac{\pi(dx)}{\pi(u_n < |y| \leq c)}$. We have that $Y^{c,n} \xrightarrow{n \rightarrow +\infty} Y^c$ locally uniformly in L^2 . Then, we call (Y^n) and (Z^n) , the sequence of càdlàg processes defined by

$$Z^n = Y^n + N \quad \text{where} \quad \forall t \geq 0 \quad Y_t^n \stackrel{\mathcal{L}}{=} \begin{cases} Y_t^{c,n} & \text{in case 1 and 3} \\ Y^{c,n} + \int_{u_n < |y| \leq c} y \pi(dy) t & \text{in case 2.} \end{cases}$$

Remark 3. We notice that Z^n is the sum of a drift term and a compound Poisson process with parameters $\lambda_n = \pi(|y| > u_n)$ and $\mu_n = 1_{|y| > u_n} \frac{\pi(dx)}{\pi(|y| > u_n)}$. Then, increments of Z^n are simulatable if λ_n and the coefficient associated with the drift term can be calculated, and if μ_n is simulatable for all $n \in \mathbb{N}^*$. Classical methods of calculus or approximation of integrals and the rejection method allow a large range of Lévy processes to satisfy these conditions. If there exists $(u_n)_{n \geq 1}$ such that Z^n is simulatable for all $n \in \mathbb{N}^*$, we say that *the Lévy measure is simulatable*.

We are now able to define the *exact Euler scheme* (scheme (A)) which is used when the increments of Z are exactly simulatable and the *approximate Euler schemes* (schemes (B) and (C)) which is used when the Lévy measure is simulatable. Let $x \in \mathbb{R}^d$ and consider the below Euler schemes:

$$\bar{X}_0 = x, \quad \bar{X}_{n+1} = \bar{X}_n + \gamma_{n+1}b(\bar{X}_n) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} + \kappa(\bar{X}_n)\bar{Z}_{n+1} \quad (\text{A})$$

$$\bar{X}_0^B = x, \quad \bar{X}_{n+1}^B = \bar{X}_n^B + \gamma_{n+1}b(\bar{X}_n^B) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^B)U_{n+1} + \kappa(\bar{X}_n^B)\bar{Z}_{n+1}^B \quad (\text{B})$$

$$\bar{X}_0^C = x, \quad \bar{X}_{n+1}^C = \bar{X}_n^C + \gamma_{n+1}b(\bar{X}_n^C) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^C)U_{n+1} + \kappa(\bar{X}_n^C)\bar{Z}_{n+1}^C \quad (\text{C})$$

where in each of the schemes, $(\gamma_n)_{n \in \mathbb{N}}$ is a decreasing sequence of positive numbers satisfying

$$\lim_{n \rightarrow +\infty} \gamma_n = 0 \quad \sum_{n \geq 1} \gamma_n = +\infty,$$

$(U_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d square integrable random variables with values in \mathbb{R}^l , such that $\mathbb{E}U_1 = 0$ and $\Sigma_{U_1} = I_d$, and at last, $(\bar{Z}_n)_{n \in \mathbb{N}}$, $(\bar{Z}_n^B)_{n \in \mathbb{N}}$, and $(\bar{Z}_n^C)_{n \in \mathbb{N}}$ are sequences independent of $(U_n)_{n \in \mathbb{N}}$ of independent random variables with values in \mathbb{R}^l such that

$$\bar{Z}_n \stackrel{(\mathbb{R}^l)}{=} Z_{\gamma_n}, \quad \bar{Z}_n^B \stackrel{(\mathbb{R}^l)}{=} Z_{\gamma_n}^n \quad \text{and} \quad \bar{Z}_n^C \stackrel{(\mathbb{R}^l)}{=} Z_{\gamma_n \wedge T^n}^n \quad \forall n \in \mathbb{N}^*,$$

with $T^n = \inf\{s > 0, |\Delta Z_s^n| > 0\}$. According to these schemes, we define sequences of weighted empirical measures by

$$\bar{\nu}_n = \frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\bar{X}_{k-1}} \quad \bar{\nu}_n^B = \frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\bar{X}_{k-1}^B} \quad \text{and} \quad \bar{\nu}_n^C = \frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\bar{X}_{k-1}^C}. \quad (3)$$

where $(\eta_k)_{k \in \mathbb{N}}$ is a sequence of positive numbers satisfying $\sum_{n \geq 1} \eta_n = +\infty$ and $H_n = \sum_{k=1}^n \eta_k$.

Remark 4. In scheme (B), we replace increments of Z by increments of Z^n and in scheme (C), we stop Z^n at its first jump time because we notice that asymptotically, under conditions between $(u_n)_{n \geq 0}$ and $(\gamma_n)_{n \in \mathbb{N}^*}$, only the first jump of Z^n is important for the convergence of the empirical measures. Note that it is possible to improve the approximation of Z_t replacing the truncated jumps by a Brownian term (see [1] in the unidimensional case). We will get interested in this approximation in a future paper where we study the rate of convergence of the above empirical measures because this approximation generally improves the rate of scheme (B).

3) About continuous time ergodicity results. Our aim is to show that the above weighted empirical measures converge to the invariant measure but before stating the corresponding results and in order to explain what kind of approach we are going to use, we do a fast background on the various methods which permit to obtain a corresponding continuous time result, *i.e.*: if ν is the invariant measure of a semi-group $(P_t)_{t \geq 0}$,

$$\nu_t(\omega, f) = \frac{1}{t} \int_0^t f(X_s^x) ds \xrightarrow{t \rightarrow +\infty} \nu(f) \quad a.s. \quad \forall x \in \mathbb{R}^d, \quad (4)$$

for every f in a range of functions containing $\mathcal{C}^b(\mathbb{R}^d)$. Under a *Has'minskii assumption*, it is well-known that if a Feller Markov process satisfies an irreducibility assumption ($\exists t_0 > 0$ such as $P_{t_0}(x, dy) = p_{t_0}(x, y)dy$ and $dy - a.s., \forall x \in \mathbb{R}^d, p_{t_0}(x, y) > 0$), existence and uniqueness of the invariant measure is ensured and (4) holds for every locally bounded function $f \in L^1(\nu)$ (see [13]). Likewise, for a Feller Markov process satisfying the same

Has'minskiï assumption and the following asymptotic flatness assumption: There exist $p > 0$, $x_0 \in \mathbb{R}^d$ and $\alpha_0 > 0$ such that for any $x \in \mathbb{R}^d$

$$\sup_{t \in \mathbb{R}^+} \mathbb{E}\{|X_t^{x_0}|^p\} < +\infty \quad \text{and} \quad \mathbb{E}\{|X_t^x - X_t^{x_0}|^p\} \leq e^{-\alpha_0 t} |x - x_0|^p$$

the same result holds for any continuous function f such that $|f(x)| \leq C(|x|^p + 1)$. Yet, these results contain two drawbacks for us. On the one hand, the proof is strongly dependent of the homogeneousness of the Markov process and therefore, is not transposable to an Euler scheme with decreasing step which is not an homogeneous Markov chain. On the other hand, irreducibility (resp. asymptotic flatness) is generally obtained under constraining conditions of regularity (resp. mean-reverting). That's why we choose to take inspiration to another approach which is based on martingale methods. In [13], this kind of result is stated for Brownian diffusions (Theorem 2) but we are able to extend it to S.D.E. driven by a Lévy process (see [14]).

3 Main results

1) Notations. Throughout this paper, we use the notation $|\cdot|$ for the Euclidean norm on \mathbb{R}^d . For any $d \times l$ matrix M , we define $\|M\|$ by: $\|M\| = \sup_{\{|x| \leq 1\}} |Mx|/|x|$. For a symmetric $d \times d$ matrix M , we set $\lambda_M = \max(0, \lambda_1, \dots, \lambda_d)$ where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of M . One notices that for each $1 \times d$ matrix x :

$$Mx^{\otimes 2} = x^* M x \leq \lambda_M |x|^2. \quad (5)$$

We denote by $\mathcal{C}_b(\mathbb{R}^d)$ (resp. $\mathcal{C}_0(\mathbb{R}^d)$) the set of bounded continuous functions on \mathbb{R}^d with values in \mathbb{R} (resp. continuous functions that go to 0 at infinity) and $\mathcal{C}_K^2(\mathbb{R}^d)$, the set of \mathcal{C}^2 functions on \mathbb{R}^d with values in \mathbb{R} and compact support. Finally, we say that f is a p -Holder function on E with values in F (where E and F are normed linear spaces) if $[f]_p = \sup_{x, y \in E} \|f(x) - f(y)\|_F / \|x - y\|_E^p < +\infty$.

2) Assumptions. Let $V : \mathbb{R}^d \rightarrow \mathbb{R}^{+*}$ be a \mathcal{C}^2 -Lyapunov function ($\lim_{|x| \rightarrow +\infty} V(x) = +\infty$) such that $|\nabla V| \leq C\sqrt{V}$ and D^2V is bounded. We set $\underline{v} := \min V$ and for $p > 0$, we define

$$\lambda_p := \frac{1}{2} \sup_{x \in \mathbb{R}^d} \lambda_{(D^2V + (p-1) \frac{\nabla V \otimes \nabla V}{V})}(x) \quad \text{and} \quad h_p := \begin{cases} [\frac{V^p}{p}]_{2p} & \text{if } p \in (0, 1/2] \\ [V^{p-1} \nabla V]_{2p-1} & \text{if } p \in (1/2, 1]. \end{cases}$$

λ_p and h_p are finite under the assumptions on V . Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$ be parameters respectively relative to the mean-reverting intensity, the moment of the Lévy process and the behavior of the small jumps. We introduce assumptions $(\mathbf{S}_{\mathbf{a}, \mathbf{p}, \mathbf{q}})$ (growth assumption) and $(\mathbf{R}_{\mathbf{a}, \mathbf{p}, \mathbf{q}})$ (mean-reverting assumption):

$$\mathbf{Assumption} (\mathbf{S}_{\mathbf{a}, \mathbf{p}, \mathbf{q}}) : \quad |b|^2 \leq CV^a \quad \begin{cases} Tr(\sigma\sigma^*) + \|\kappa\|^{2(p \vee q)} \leq CV^{a+p-1} & \text{if } p < 1 \\ Tr(\sigma\sigma^*) + \|\kappa\|^2 \leq CV^a & \text{if } p \geq 1. \end{cases}$$

Assumption $(\mathbf{R}_{\mathbf{a}, \mathbf{p}, \mathbf{q}})$: There exist $\beta \in \mathbb{R}$, $\alpha > 0$ such that

$$(\nabla V|b) + h_p \int |y|^{2p} \pi(dy) \|\kappa\|^{2p} V^{1-p} 1_{q \leq p} \leq \beta - \alpha V^a \quad \text{if } p < 1$$

$$(\nabla V|b) + c_p \lambda_p \left(Tr(\sigma\sigma^*) + \int |y|^2 \pi(dy) \|\kappa\|^2 + d_p \int |y|^{2p} \pi(dy) \frac{\|\kappa\|^{2p}}{V^{p-1}} \right) \leq \beta - \alpha V^a \quad \text{if } p \geq 1$$

where $c_p = 2^{2(p-1)\wedge 1}$ and $d_p = 0$ (resp. $[\sqrt{V}]_1^{2(p-1)}$) if $p = 1$ (resp. $p > 1$).

Remark 5. An important point of our work has been to obtain independence between the assumptions and the truncation function because we remind that c is not a structural parameter of the process. This is one of the reasons for which we introduced three cases. In fact, in cases 1 and 2, we notice that the assumptions are independent of c by construction. We have the same property in case 3 ($p \leq 1/2 < q$) because under $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$, $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$ is satisfied for any $c > 0$ if and only if it is satisfied for one c . It is due to the fact that if $p \leq 1/2$ and $q > 1/2$, condition $\|\kappa\|^{2q} = O(V^{p+a-1})$ is too constraining for κ to be a sufficiently strong mean-reverting to ensure $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$. (see end of the proof of proposition 2 in case $p < 1$ for more details)

3) Main results. Our main results are theorems 2 and 3. In theorem 2, we obtain a result under uncomplicated conditions on steps and weights. In theorem 3, we show that under more constraining conditions on steps and weights, we can relax the assumptions on the coefficients of the S.D.E. The second theorem can be very useful when the Lévy process has few moments.

Theorem 2. *Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$. Assume that $(\mathbf{H}_{\mathbf{p}}^1)$, $(\mathbf{H}_{\mathbf{q}}^2)$, $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$ and $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$ are satisfied, that $\mathbb{E}\{|U_1|^{2(p\vee 1)}\} < +\infty$ and that the sequence $(\eta_n/\gamma_n)_{n \geq 1}$ is nonincreasing. Then,*

(1) *If $p/2 + a - 1 > 0$, the sequence $(\bar{\nu}_n)_{n \geq 1}$ is almost surely tight. Moreover, if $\kappa(x) \stackrel{|x| \rightarrow +\infty}{=} o(|x|)$ and $\text{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \leq CV^{\frac{p}{2}+a-1}$, then every weak limit of this sequence is an invariant probability for the S.D.E. (1). In particular, if $(X_t)_{t \geq 0}$ admits a unique invariant probability ν , for every continuous function f such that $f = O(V^{\frac{p}{2}+a-1})$, $\lim_{n \rightarrow \infty} \bar{\nu}_n(f) = \nu(f)$.*

(2) *The same result holds for $(\bar{\nu}_n^B)_{n \geq 1}$ and under the additional condition:*

$$\pi(|y| > u_n)\gamma_n \xrightarrow{n \rightarrow +\infty} 0, \quad (6)$$

the same result holds for $(\bar{\nu}_n^C)_{n \geq 1}$.

Remark 6. Let us get interested in implications of the distribution of the Lévy measure on the above theorem. Clearly, the moment of the Lévy process is the principal structural constraint because it imposes p but, when p is fixed, we can notice that behavior of small jumps has an important influence on the conditions which ensure that every weak limit of the sequence of empirical measures is invariant. This influence is natural because, for instance, the error induced by the discretization is easier to control for a compound Poisson process than for a process which has not locally bounded variations .

Remark 7. Theorem 2 associated with scheme (B) need not satisfying condition (6). However, considering simulation, we also have to add a supplementary condition. Indeed, we see that $\pi(|y| > u_n)\gamma_n$ is the parameter of the underlying Poisson law to Z_n . Then, when we simulate the procedure, we must impose a condition such as

$$\sup_{n \in \mathbb{N}} \pi(|y| > u_n)\gamma_n < M < +\infty.$$

Below, we explain some examples where the above theorem is available. In the first one, conditions on coefficients are adapted to the case $q = 1$ (the worse case). In the second one, we choose to take $b^c = \sigma = 0$ in order to recall that a S.D.E. driven by a non-centered purely discontinuous Lévy process can generate a sufficiently strong mean-reversion for satisfying some ergodic properties. In the two situations, we take $V(x) = 1 + |x|^2$.

Example 1. Assume that assumption (\mathbf{H}_p^1) (with $p > 0$) is satisfied and that $b(x) = -\rho(x)x/|x| + \mathcal{T}(x)$ such that $C_1|x|^\ell \leq \rho(x) \leq C_2|x|^\ell$ ($0 < C_1 \leq C_2$) and $(\mathcal{T}(x)|x) = 0$, for $|x|$ sufficiently large. Assume that

$$|\mathcal{T}|^2 \leq CV^{\frac{\ell+1}{2}} \quad \text{and} \quad \begin{cases} \text{Tr}(\sigma\sigma^*) + \|\kappa\|^2 \leq CV^{\frac{p+\ell-1}{2}} & \text{if } p < 2 \\ \text{Tr}(\sigma\sigma^*) + \|\kappa\|^2 = o(V^{\frac{\ell+1}{2}}) & \text{if } p \geq 2. \end{cases}$$

Then, if $(1-p) \vee -1 < \ell \leq 1$ and $(\eta_n/\gamma_n)_{n \in \mathbb{N}^*}$ is nonincreasing, theorem 2 can be applied.

Example 2. Assume that assumption (\mathbf{H}_p^1) is satisfied with $p > 1/2$ and consider the following S.D.E.: $dX_t = \kappa(X_{t-})dZ_t^c = -\kappa(X_{t-}) \int_{\{|y|>c\}} y\pi(dy)dt + \kappa(X_{t-})dZ_t$. In order to simplify, we suppose that we are in the 1-dimensional case. We define $h = (\int_{\{|y|>c\}} y\pi(dy))$. If $h \neq 0$ and $\kappa(x) = -\text{sgn}(h)\rho(x)x/|x|$ with $C_1|x|^\ell \leq \rho(x) \leq C_2|x|^\ell$ for $|x|$ sufficiently large. Then, assumptions of theorem 2 are fulfilled if (\mathbf{H}_p^1) and (\mathbf{H}_q^2) are satisfied with $p \in (1/2, 1)$, $q < 1/2$ and $\ell \in (\frac{1-p}{1-2q}, 1)$ or $p \geq 1$, $q \leq 1/2$ and $\ell \in (-1 \vee (1-p), 1)$ or $p > 1$, $q > 1/2$ and $\ell \in (-1 \vee (1-p), 1 \wedge \frac{p-1}{2q-1})$.

The interest of Theorem 2 is to be easy to use. For example, in scheme (A), we only have to take a nonincreasing sequence $(\gamma_n)_{n \in \mathbb{N}}$ converging to 0, with infinite sum and $\eta_n = \gamma_n$. The next theorem (Theorem 3) is more constraining for steps and weights but becomes necessary if the coefficients do not satisfy the assumptions associated with Theorem 2 or if we want to obtain convergence for "less integrable" functions.

Theorem 3. Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$. Assume that (\mathbf{H}_p^1) , (\mathbf{H}_q^2) , $(\mathbf{R}_{a,p,q})$ and $(\mathbf{S}_{a,p,q})$ are satisfied, that $\mathbb{E}\{|U_1|^{2(p \vee 1)}\} < +\infty$ and that the sequence $(\eta_n/\gamma_n)_{n \geq 1}$ is nonincreasing. Then,

(1) Consider $s \in (1, 2]$ satisfying

$$\begin{cases} s > \frac{2p}{2p+(a-1)(\frac{2p-1}{p})} & \text{if } \frac{1}{2} < p \leq 1 \\ s > \frac{2p}{2p+a-1} & \text{if } p \geq 1. \end{cases} \quad (7)$$

If $p/s + a - 1 > 0$, there exist explicit sequences $(\gamma_n)_{n \geq 1}$ and $(\eta_n)_{n \geq 1}$ such that the sequence $(\bar{\nu}_n)_{n \geq 1}$ is almost surely tight. Moreover, if $\kappa(x) \stackrel{|x| \rightarrow +\infty}{=} o(|x|)$ and $\text{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \leq CV^{\frac{p}{s}+a-1}$, then every weak limit of this sequence is an invariant probability for the S.D.E. (1). In particular, if $(X_t)_{t \geq 0}$ admits a unique invariant probability ν , for every continuous function f such that $f = O(V^{\frac{p}{s}+a-1})$, $\lim_{n \rightarrow \infty} \bar{\nu}_n(f) = \nu(f)$.

(2) The same result holds for $(\bar{\nu}_n^B)_{n \geq 1}$ and under the additional condition:

$$\pi(|y| > u_n)\gamma_n \xrightarrow{n \rightarrow +\infty} 0, \quad (8)$$

the same result holds for $(\bar{\nu}_n^C)_{n \geq 1}$.

Remark 8. The family of explicit sequences $(\eta_n)_{n \geq 1}$ and $(\gamma_n)_{n \geq 1}$ satisfying sufficient conditions for the above theorem are given in the following section (see Proposition 1 for general conditions, remark 9 for conditions adapted to polynomial steps and weights and example 3 in the particular case where the mean-reverting has linear growth).

In the following example, we consider the same kind of S.D.E. as in Example 1 in the particular case $a = 1$. We give conditions on σ and κ and explicit polynomial weights and steps such that the above theorem is valid.

Example 3. Assume that (\mathbf{H}_p^1) and (\mathbf{H}_q^2) are satisfied with $p > 0$ and $q \in [0, 1]$, and consider $\gamma_n = Cn^{-r_1}$, $\eta_n = Cn^{-r_2}$ ($r_1 \leq r_2$), $s \in (1, 2]$ with

$$0 < r_1 < 2\left(1 - \frac{1}{s}\right) \quad \text{and} \quad r_2 < 1 \quad \text{or} \quad 0 < r_1 \leq 2\left(1 - \frac{1}{s}\right) \quad \text{and} \quad r_2 = 1.$$

Then, if b is defined as in example 1 with $\ell = 1$ and,

$$|\mathcal{T}|^2 \leq CV, \quad Tr(\sigma\sigma^*) + \|\kappa\|^{2q} \leq CV^{\frac{2}{s}} \quad \text{and} \quad Tr(\sigma\sigma^*) + \|\kappa\|^2 = o(V),$$

theorem 3 is valid (under less constraining assumptions on σ and κ than in example 1).

4) Organization of the proof of theorems 2 and 3. First, we prove theorems 2 and 3 for scheme (A). In section 4, we show that $(\bar{\nu}_n)_{n \in \mathbb{N}^*}$ is *a.s.* tight and in section 5, we show that every weak limit of $(\bar{\nu}_n)_{n \in \mathbb{N}^*}$ is invariant for the S.D.E. (1). Then, we observe that theorems 2 and 3 are obtained by combination of Propositions 1 and 3. At last, in section 6, we bring up the main differences in the proof of the main theorems for schemes (B) and (C).

4 A.s. tightness of $(\bar{\nu}_n(w, dx))_{n \in \mathbb{N}^*}$

The main result of this section is Proposition 1. Before stating it, we introduce function $f_{a,p}$ defined for all $s \in (1, 2]$ by

$$f_{a,p}(s) = \begin{cases} s & \text{if } s \geq 2p \\ \frac{p+a-1}{\frac{p}{s} + \frac{a-1}{2(p \wedge 1)}} \wedge s & \text{if } s < 2p \end{cases} \quad (9)$$

Suppose $p + a - 1 > 0$. $s \mapsto f_{a,p}(s)$ is a nondecreasing function which satisfies $f_{1,p}(s) = s$ for all $p > 0$ and $f_{a,p}(2) = 2$. We notice that $f_{a,p}(s) > 1$ if and only if s satisfies assumption (7).

Proposition 1. *Let $a \in (0, 1]$, $p > 0$ and $q \in (0, 1]$. Assume that (\mathbf{H}_p^1) , (\mathbf{H}_q^2) , $(\mathbf{R}_{a,p,q})$ and $(\mathbf{S}_{a,p,q})$ are satisfied. If $\mathbb{E}\{|U_1|^{2(p \vee 1)}\} < +\infty$ and $(\eta_n/\gamma_n)_{n \geq 1}$ is nonincreasing, then,*

$$(1) \quad \sup_{n \geq 1} \bar{\nu}_n(V^{\frac{p}{2} + a - 1}) < +\infty \quad a.s.$$

Consequently, if $\frac{p}{2} + a - 1 > 0$, the sequence $(\bar{\nu}_n)_{n \in \mathbb{N}}$ is *a.s.* tight.

(2) Let $s \in (1, 2)$ satisfying assumption (7) and consider $(\eta_n)_{n \geq 1}$ and $(\gamma_n)_{n \geq 1}$ such that

$$\left(\frac{1}{\gamma_n} \left(\frac{\eta_n}{H_n \sqrt{\gamma_n}}\right)^{f_{a,p}(s)}\right) \text{ is nonincreasing and } \sum_{n \geq 1} \left(\frac{\eta_n}{H_n \sqrt{\gamma_n}}\right)^{f_{a,p}(s)} < +\infty. \quad (10)$$

Then, $\sup_{n \geq 1} \bar{\nu}_n(V^{\frac{p}{s} + a - 1}) < +\infty$ *a.s.* and the sequence $(\bar{\nu}_n)_{n \in \mathbb{N}}$ is *a.s.* tight as soon as $p/s + a - 1 > 0$.

Remark 9. If $\gamma_n = Cn^{-r_1}$ and $\eta_n = Cn^{-r_2}$ with $r_1 \leq r_2$, then assumption (10) is fulfilled in the following cases :

$$0 < r_1 < 2\left(1 - \frac{1}{f_{a,p}(s)}\right) \quad \text{and} \quad r_2 < 1 \quad \text{or} \quad 0 < r_1 \leq 2\left(1 - \frac{1}{f_{a,p}(s)}\right) \quad \text{and} \quad r_2 = 1. \quad (11)$$

The rest of this section is devoted to the proof of this theorem. We establish a recursive control of the Euler scheme (Proposition 2) notably thanks to lemma 1. Then, we develop technical arguments close to [12] and prove Proposition 1 in subsection 4.4.

4.1 Control of the moments of the jump components

In order to control the noise induced by the jumps part, we establish a lemma where we study the behavior of the moments of Z near 0. This control is fundamental for proposition 2 and has a direct implication on the quality of assumption $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$.

Lemma 1. (i) Let $p > 0$ such that $\int_{|y|>c} \pi(dy)|y|^{2p} < +\infty$. Then,

$$\mathbb{E}\{|N_t^c|^{2p}\} = t \int_{|y|>c} |y|^{2p} \pi(dy) + \delta_c(t)t^2 \quad \text{if } p > 0$$

$$\mathbb{E}\{|N_t^c - t \int_{|y|>c} y \pi(dy)|^{2p}\} \leq t \int_{|y|>c} |y|^{2p} \pi(dy) + C_{T,p,c} t^{2(p \vee 1)} \quad \text{if } p > 1/2,$$

where δ_c is locally bounded.

(ii) Let $q \in [0, 1]$ such that $\int_{|y|\leq c} |y|^{2q} \pi(dy) < +\infty$.

$$\mathbb{E}\{|Y_t^c + t \int_{|y|\leq c} y \pi(dy)|^{2q}\} \leq t \int_{|y|\leq c} |y|^{2q} \pi(dy) \quad \text{if } q \leq 1/2$$

$$\mathbb{E}\{|Y_t^c|^{2q}\} \leq C_q t \int_{|y|\leq c} |y|^{2q} \pi(dy) + C_{T,q,c} t^2 \quad \text{if } 1/2 < q \leq 1.$$

(iii) If $\int_{|y|>c} \pi(dy)|y|^{2p} < \infty$ with $p \geq 1$, then if $p = 1$,

$$\mathbb{E}\{|Z_t|^2\} = t \int |y|^2 \pi(dy),$$

and if $p \geq 1$, for any $\epsilon > 0$, there exist $C_\epsilon > 0$ such that for any $t \leq T$,

$$\mathbb{E}\{|Z_t|^{2p}\} \leq t \left(\int |y|^{2p} \pi(dy) + \epsilon \right) + C_\epsilon t^{p \wedge 2}$$

Remark 10. The behavior of moments of the jump component is relatively different of the behavior of a Brownian motion. In particular, an important point of the behavior of jumps is that we can not find $\alpha > 1$ and $p > 0$ such that $\mathbb{E}\{|Z_t|^{2p}\} = O(t^\alpha)$ (that is a valid equality for the Brownian motion). In fact, if the above equality was valid, the Kolmogorov criterion would show that Z is continuous. That is why in $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$ with $p > 1$, we have a (parasital) component of order $2p$ coming from the jumps part.

Proof. (i) $(N_t^c)_{t \geq 0}$ is a compound Poisson process with parameters $\lambda = \pi(|y| > c)$ and $\mu(dy) = \frac{\pi(dy)}{\pi(|y|>c)}$. Then, we set $N_t^c = \sum_{n \geq 1} R_n 1_{T_n \leq t}$, where $(R_n)_{n \geq 1}$ is a sequence of i.i.d r.v.

with law μ and $(T_n)_{n \in \mathbb{N}}$ is the sequence of jump times of a Poisson process with intensity λ independent of $(R_n)_{n \geq 1}$. We have

$$\mathbb{E}\{|N_t^c|^{2p}\} = \sum_{n \geq 1} \mathbb{E}\left\{ \left| \sum_{i=1}^n R_i \right|^{2p} \right\} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = \lambda t \mathbb{E}\{|R_1|^{2p}\} F_\lambda(t)$$

$$\text{where } F(t) = e^{-\lambda t} \sum_{n \geq 0} \frac{\mathbb{E}\left\{ \left| \sum_{i=1}^{n+1} R_i \right|^{2p} \right\}}{\mathbb{E}\{|R_1|^{2p}\}} \frac{(\lambda t)^n}{(n+1)!}.$$

Using the following inequality

$$\forall (a_i)_{i=1}^n \subset \mathbb{R}^n, \quad \left| \sum_{i=1}^n a_i \right|^\alpha \leq n^{(2p) \vee 1 - 1} \sum_{i=1}^n |a_i|^{2p},$$

we obtain

$$\frac{\mathbb{E}\{|\sum_{i=1}^{n+1} R_i|^{2p}\}}{(n+1)!\mathbb{E}\{|R_1|^{2p}\}} \leq \frac{(n+1)^{(2p-1)_+}}{n!}.$$

We deduce that F is an analytic function on \mathbb{R} satisfying $F(0) = 1$ and therefore,

$$F(t) = 1 + t\delta_c(t) \text{ with } |\delta_c(t)| \leq C(T, p, c, \lambda) \quad \forall t \in [0, T].$$

The first equality follows by noticing that $\mathbb{E}\{|R_1|^{2p}\} = \frac{1}{\lambda} \int_{\{|y|>c\}} |y|^{2p} \pi(dy)$. The second inequality of (i) is obtained by using the following elementary inequality

$$\forall u, v \in \mathbb{R}^+, \forall \alpha \geq 1 \quad (u+v)^\alpha \leq u^\alpha + \alpha 2^{\alpha-1} (u^\alpha - 1)v + v^\alpha. \quad (12)$$

(ii) If $\int_{|y|\leq c} \pi(dy) |y|^{2q} < +\infty$ with $q \leq 1/2$, then Y^c has locally bounded variations. By the elementary inequality $|u+v|^\alpha \leq |u|^\alpha + |v|^\alpha$ for $\alpha \in (0, 1]$ and the compensation formula, we obtain

$$\mathbb{E}\{|Z_t^c + t \int y \pi(dy) |y|^{2q}\} = \mathbb{E}\left\{ \left| \sum_{0 < s \leq t} \Delta Y_s^c \right|^{2q} \right\} \leq \mathbb{E}\left\{ \sum_{0 < s \leq t} |\Delta Y_s^c|^{2q} \right\} = t \int_{|y|\leq c} |y|^{2q} \pi(dy).$$

Let now $q \in (1/2, 1]$. As Y^c is a martingale, we derive from the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\{|Y_t^c|^{2q}\} \leq C_q \mathbb{E}\left\{ \left(\sum_{0 < s \leq t} |\Delta Y_s^c|^2 \right)^{\frac{q}{2}} \right\}.$$

The second inequality follows by using $|u+v|^\alpha \leq |u|^\alpha + |v|^\alpha$ for $\alpha \in (0, 1]$ and the compensation formula.

(iii) M , defined by $M_t = |Z_t|^2 - t \int \pi(dy) |y|^2 \pi(dy)$ for any t , is a martingale, then, in particular,

$$\mathbb{E}\{|Z_t|^2\} = t \int |y|^2 \pi(dy). \quad (13)$$

If $p > 1$, consider $\phi_p : x \mapsto |x|^{2p}$. By Ito formula, we obtain

$$\mathbb{E}\{|Y_{t \wedge \tau_n}^c|^{2p}\} = \mathbb{E}\left\{ \sum_{0 < s \leq t \wedge \tau_n} \left(|Y_s^c|^{2p} - |Y_{s-}^c|^{2p} - (\nabla \phi_p(Y_{s-}^c) | \Delta Y_s^c) \right) \right\}$$

where $(\tau_n)_{n \geq 1}$ is a sequence of stopping times such that $\tau_n \xrightarrow{n \rightarrow +\infty} +\infty$. We have

$$\begin{aligned} \frac{\partial^2 \phi_p}{\partial x^j \partial x^i}(x) &= \alpha |x|^{2(p-1)} 1_{\{i=j\}} + 2(p-1) x_j x_i |x|^{2(p-2)}, \text{ then} \\ \Rightarrow \|D^2 \phi_p(x)\|_\infty &= \max_{i,j \in \{1,q\}} \left(\left| \frac{\partial^2 \phi_p}{\partial x^j \partial x^i}(x) \right| \right) \leq C_p |x|^{2(p-1)}. \end{aligned}$$

Consequently, by the compensation formula, the inequality: $|u+v|^\alpha \leq 2^{\alpha \vee 1 - 1} (|u|^\alpha + |v|^\alpha)$ and the fact that ΔY_s^c and Y_{s-}^c are independent, we obtain

$$\begin{aligned} \mathbb{E}\{|Y_{t \wedge \tau_n}^c|^{2p}\} &\leq C \mathbb{E}\left\{ \sum_{0 < s \leq t} \sup_{\theta \in [0,1]} |Y_{s-}^c + \theta \Delta Y_s^c|^{2(p-1)} |\Delta Y_s^c|^2 \right\} \\ &\leq C \left(\int_0^t \mathbb{E}\{|Y_s^c|^{2(p-1)}\} ds \int_{|y|\leq c} \pi(dy) |y|^2 + t \int_{|y|\leq c} |y|^{2p} \pi(dy) \right) \\ &\leq C \int_{|y|\leq c} |y|^{2p} \pi(dy) t + C \int_0^t \mathbb{E}\{|Y_s^c|^{2(p-1)}\} ds. \end{aligned}$$

Thanks to Fatou lemma, we deduce

$$\mathbb{E}\{|Y_t^c|^{2p}\} \leq C_p \int_{|y| \leq c} |y|^{2p} \pi(dy) t + C \int_0^t \mathbb{E}\{|Y_s^c|^{2(p-1)}\} ds \quad \forall t \geq 0, \quad \forall p > 1. \quad (14)$$

If $p \in [1, 2]$, we derive from Jensen inequality

$$\mathbb{E}\{|Y_s^c|^{2(p-1)}\} \leq \mathbb{E}\{|Y_s^c|^2\}^{p-1} \leq s^{p-1} \left(\int_{|y| \leq c} \pi(dy) |y|^2 \right)^{p-1}.$$

Then, by induction on $k := \inf\{l \geq 0, l \geq p-1\}$, we obtain

$$\mathbb{E}\{|Y_t^c|^{2p}\} \leq C_p \int_{|y| \leq c} |y|^{2p} \pi(dy) t + C_{p,c} t^{p \wedge 2}.$$

As $\int_{|y| \leq c} |y|^{2p} \pi(dy) \xrightarrow{c \rightarrow 0} 0$, we can choose $c_\epsilon > 0$ such that $C_p \int_{|y| \leq c} |y|^{2p} \leq \epsilon$ then using (12) and independence between Y^{c_ϵ} and N^{c_ϵ} , we deduce

$$\mathbb{E}\{|Z_t|^{2p}\} \leq \mathbb{E}\{|N_t^{c_\epsilon} - t \int_{|y| > c_\epsilon} y \pi(dy)|^{2p}\} + C \left(\mathbb{E}\{|N_t^{c_\epsilon} - t \int_{|y| > c_\epsilon} y \pi(dy)|\} \mathbb{E}\{|Y_t^{c_\epsilon}|^{2p-1}\} + \mathbb{E}\{|Y_t^{c_\epsilon}|^{2p}\} \right).$$

Standardizing ϵ , the second inequality of (iii) follows from the second inequality of (i). \square

4.2 A recursive stability relation

We first state a technical lemma useful to proposition 2.

Lemma 2. *a) If $p \in [0, 1/2]$, V^p is α -Holder for any $\alpha \in [2p, 1]$ and $\forall p \in (0, 1]$, $W = p(\nabla V)V^{p-1}$ is α -Holder for any positive $\alpha \in [2p-1, 1]$.*

b) Let $x, y \in \mathbb{R}^d$ and $\xi \in [x, x+y]$. If $p \leq 1$,

$$D^2(V^p)(\xi)y^{\otimes 2} \leq p\underline{v}^{p-1} \lambda_p |y|^2.$$

If moreover, $|y| \leq \epsilon \frac{\sqrt{V}(x)}{[\sqrt{V}]_1}$ with $\epsilon \in [0, 1]$, then,

$$D^2(V^p)(\xi)y^{\otimes 2} \leq \lambda_p (1 - \epsilon)^{2(p-1)} V^{p-1}(x) |y|^2.$$

If $p > 1$,

$$D^2(V^p)(\xi)y^{\otimes 2} \leq p\lambda_p 2^{2(p-1)\vee 1-1} \left(V^{p-1}(x) + |[\sqrt{V}]_1| |y|^{2(p-1)} \right) |y|^2.$$

Proof. Consider a continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$ such that $|f|^{\frac{1}{\alpha}}$ is Lipschitz. Then, f is α -holderian. Using this argument, a) follows. Let us now consider the inequalities in b). We have

$$D^2(V^p) = pV^{p-1}D^2V + p(p-1)V^{p-2}\nabla V \cdot \nabla V = pV^{p-1} \left(D^2V + (p-1) \frac{\nabla V \otimes \nabla V}{V} \right). \quad (15)$$

As V^{p-1} is bounded if $p \leq 1$, we derive the first inequality from elation (5). For the second one, we consider: $\xi = x + \theta y$ with $\theta \in [0, 1]$ and $|y| \leq \epsilon \frac{\sqrt{V}(x)}{[\sqrt{V}]_1}$. As \sqrt{V} is a Lipschitz function,

$$\sqrt{V}(\xi) \geq \sqrt{V}(x) - [\sqrt{V}]_1 |y| \geq (1 - \epsilon) \sqrt{V}(x) \Rightarrow V^{p-1}(\xi) \leq (1 - \epsilon)^{2(p-1)} V^{p-1}(x).$$

If $p > 1$,

$$\sqrt{V}(\xi) \leq \sqrt{V}(x) + [\sqrt{V}]_1 |y| \quad \Rightarrow \quad V^{p-1}(\xi) \leq (\sqrt{V}(x) + [\sqrt{V}]_1 |y|)^{2(p-1)}.$$

The results follows from the following inequality:

$$\forall u, v \geq 0 \quad |u + v|^\alpha \leq 2^{\alpha \vee 1 - 1} (u^\alpha + v^\alpha).$$

□

Proposition 2. *Let $p > 0$. Assume (\mathbf{H}_p^1) , (\mathbf{H}_q^2) , $(\mathbf{R}_{a,p,q})$ and $(\mathbf{S}_{a,p,q})$. Then, if $\mathbb{E}\{|U_1|^{2(p \vee 1)}\} < +\infty$, there exist $n_0 \in \mathbb{N}$, $\alpha' > 0$, $\beta' \in \mathbb{R}$ such that $\forall n \geq n_0$,*

$$\mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} \leq V^p(\bar{X}_n) + p\gamma_{n+1}V^{p-1}(\bar{X}_n)\left(\beta' - \alpha'V^a(\bar{X}_n)\right). \quad (16)$$

For this proof, one needs to investigate separately cases $p < 1$ and $p \geq 1$. We detail case $p < 1$ and if $p \geq 1$, we indicate the process of the proof which is close to Lemma 3 of [12]. One also notices that in order to show (16), one only has to show that there exist $\tilde{\beta}$ and $\tilde{\alpha}$ and a function h such that

$$\mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} \leq V^p(\bar{X}_n) + p\gamma_{n+1}V^{p-1}(\bar{X}_n)\left(\tilde{\beta} - \tilde{\alpha}V^a(\bar{X}_n)\right) + h(\gamma_{n+1})V^{a+p-1}(\bar{X}_n)$$

where $h(\gamma_{n+1}) \leq \tilde{\alpha}/2$ for n sufficiently large. Setting $\alpha' = \tilde{\alpha}/2$ and $\beta' = \tilde{\beta}$, we obtain (16).

Proof of proposition 2 if $p < 1$. First, we consider cases 1 and 2 and we decompose $\Delta\bar{X}_{n+1} = \bar{X}_{n+1} - \bar{X}_n$ as follows:

$$\bar{X}_{n+1,k} = \bar{X}_n + \sum_{i=1}^k \Delta\bar{X}_{n+1,i} \quad \text{with} \quad \begin{cases} \Delta\bar{X}_{n+1,1} = \gamma_{n+1}b(\bar{X}_n) \\ \Delta\bar{X}_{n+1,2} = \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} \\ \Delta\bar{X}_{n+1,3} = \kappa(\bar{X}_n)\bar{Z}_{n+1}. \end{cases} \quad (17)$$

According to the above decomposition, we show the three followings steps:

i) There exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$

$$\mathbb{E}\{V^p(\bar{X}_{n+1,1}) - V^p(\bar{X}_n)/\mathcal{F}_n\} \leq p\gamma_{n+1}\frac{(\nabla V|b)}{V^{1-p}}(\bar{X}_n) + C\gamma_{n+1}^2V^{a+p-1}(\bar{X}_n). \quad (18)$$

ii) For every $\epsilon > 0$, there exists $n_{2,\epsilon} \in \mathbb{N}$ such that for every $n \geq n_{2,\epsilon}$,

$$\mathbb{E}\{V^p(\bar{X}_{n+1,2}) - V^p(\bar{X}_{n+1,1})/\mathcal{F}_n\} \leq \epsilon\gamma_{n+1}V^{a+p-1}(\bar{X}_n) + C_\epsilon^1\gamma_{n+1}. \quad (19)$$

iii) For every $\epsilon > 0$, there exists $n_{3,\epsilon}$ such that

$$\mathbb{E}\{V^p(\bar{X}_{n+1,3}) - V^p(\bar{X}_{n+1,2})/\mathcal{F}_n\} \leq \begin{cases} \gamma_{n+1}\|\kappa(\bar{X}_n)\|^{2p}\left(ph_p \int |y|^{2p}\pi(dy) + \epsilon\right) & \text{if } q \leq p \\ \epsilon\gamma_{n+1}V^{a+p-1}(\bar{X}_n) + C_\epsilon^2\gamma_{n+1} & \text{if } q > p \end{cases} \quad (20)$$

i) We derive from Taylor formula that

$$\mathbb{E}\{V^p(\bar{X}_{n+1,1}) - V^p(\bar{X}_n)/\mathcal{F}_n\} = p\gamma_{n+1}\frac{(\nabla V|b)}{V^{1-p}}(\bar{X}_n) + \frac{1}{2}\mathbb{E}\{D^2(V^p)(\xi_{n+1}^1)(\Delta\bar{X}_{n+1,1})^{\otimes 2}/\mathcal{F}_n\},$$

where $\xi_{n+1}^1 \in [\bar{X}_n, \bar{X}_n + \gamma_{n+1}b(\bar{X}_n)]$. Set $x = \bar{X}_n$ and $y = \gamma_{n+1}b(\bar{X}_n)$. As $\gamma_n \xrightarrow{n \rightarrow +\infty} 0$ and $|b| \leq C\sqrt{V}$, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$, $|y| \leq \frac{\sqrt{V}(x)}{2[\sqrt{V}]_1}$ a.s. Then, i) follows from the second inequality of lemma 2, b) applied with $\epsilon = 1/2$.

ii) U_{n+1} is independent of \mathcal{F}_n and $\mathbb{E}\{U_{n+1}\} = 0$. Then, by Taylor formula, we obtain

$$\mathbb{E}\{V^p(\bar{X}_{n+1,2}) - V^p(\bar{X}_{n+1,1})/\mathcal{F}_n\} = \frac{1}{2}\mathbb{E}\{D^2(V^p)(\xi_{n+1}^2)(\Delta\bar{X}_{n+1,2})^{\otimes 2}/\mathcal{F}_n\} \text{ with } \xi_{n+1}^2 \in [\bar{X}_{n+1,1}; \bar{X}_{n+1,2}].$$

Set $y = \sqrt{\gamma_{n+1}}\sigma(x)U_{n+1}$. As $\|\sigma(x)\| \leq C_\sigma\sqrt{V}(x)$, then the conditions of the second inequality of lemma 2 are satisfied if $|U_{n+1}| \leq \rho_{n+1} = \frac{2}{C_\sigma[\sqrt{V}]_1\sqrt{\gamma_{n+1}}}$. Therefore,

$$\mathbb{E}\{D^2(V^p)(\xi_{n+1}^2)(\Delta\bar{X}_{n+1,2})^{\otimes 2}1_{\{|U_{n+1}| \leq \rho_{n+1}\}}/\mathcal{F}_n\} \leq CV^{p-1}(\bar{X}_n)Tr(\sigma\sigma^*)(\bar{X}_n) \leq CV^{a+2(p-1)}(\bar{X}_n).$$

according to assumption $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$. By the first inequality of lemma 2, we also have

$$\mathbb{E}\{D^2(V^p)(\xi_{n+1}^2)(\Delta\bar{X}_{n+1,2})^{\otimes 2}1_{\{|U_{n+1}| > \rho_{n+1}\}}/\mathcal{F}_n\} \leq CV^{p+a-1}(\bar{X}_n)\mathbb{E}\{|U_{n+1}|^2 1_{\{|U_{n+1}| > \rho_{n+1}\}}\}.$$

Now, consider $\epsilon > 0$. On the one hand, there exists $C_\epsilon > 0$ such that $V^{a+2(p-1)} \leq \epsilon V^{a+p-1} + C_\epsilon$ and on the other hand, there exists $n_{2,\epsilon} \in \mathbb{N}$ such that for every $n \geq n_{2,\epsilon}$, $\mathbb{E}\{|U_{n+1}|^2 1_{\{|U_{n+1}| > \rho_{n+1}\}}\} \leq \epsilon$ because $\rho_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $U_1 \in L^2$. ii) follows.

iii). Let us first consider **Case 1** where $p \in (1/2, 1]$ and $q \in (0, 1]$. From Taylor formula, there exist $\xi_{n+1}^2 \in [\bar{X}_{n+1,2}, \bar{X}_{n+1,2} + \kappa(\bar{X}_n)\bar{Y}_{n+1}]$ and $\xi_{n+1}^3 \in [\bar{X}_{n+1,2} + \kappa(\bar{X}_n)\bar{Y}_{n+1}, \bar{X}_{n+1}]$ such that

$$\begin{aligned} V^p(\bar{X}_{n+1}) - V^p(\bar{X}_{n+1,2}) &= p\left(W(\bar{X}_{n+1,2})|\kappa(\bar{X}_n)\bar{Y}_{n+1}\right) + p\left(W(\xi_{n+1}^3) - W(\bar{X}_{n+1,2})|\kappa(\bar{X}_n)\bar{Y}_{n+1}\right) \\ &+ p\left(W(\bar{X}_{n+1,2} + \kappa(\bar{X}_n)\bar{Y}_{n+1})|\kappa(\bar{X}_n)\bar{N}_{n+1}\right) + p\left(W(\xi_{n+1}^3) - W(\bar{X}_{n+1,2} + \kappa(\bar{X}_n)\bar{Y}_{n+1})|\kappa(\bar{X}_n)\bar{N}_{n+1}\right). \end{aligned}$$

with $W = V^{p-1}\nabla V$. \bar{Y}_{n+1} , \bar{N}_{n+1} and $\sigma(\mathcal{F}_n, U_{n+1})$ are independent and in this case, \bar{Y}_{n+1} and \bar{N}_{n+1} have mean 0. Therefore, we derive from the first assumption of lemma 2

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1}) - V^p(\bar{X}_{n+1,2})/\mathcal{F}_n\} &\leq p[W]_{2(p\vee q)-1}\|\kappa(\bar{X}_n)\|^2(p\vee q)\mathbb{E}\{|\bar{Y}_{n+1}|^{2(p\vee q)}\} \\ &+ ph_p\|\kappa(\bar{X}_n)\|^{2p}\mathbb{E}\{|\bar{N}_{n+1}|^{2p}\} \end{aligned}$$

with $h_p = [W]_{2p-1}$. By lemma 1 (inequalities (i).2 and (ii).2) and assumption $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$, we obtain

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1}) - V^p(\bar{X}_{n+1,2})/\mathcal{F}_n\} &\leq ph_p \int_{|y|>c} |y|^{2p}\pi(dy)\|\kappa(\bar{X}_n)\|^{2p} \\ &+ (C_{p,q} \int_{|y|\leq c} |y|^{2(p\vee q)}\pi(dy) + \delta_c(\gamma_{n+1})\gamma_{n+1}^{2p})V^{a+p-1}(\bar{X}_n). \end{aligned}$$

By construction, this inequality is valid for any $c > 0$. As $\int_{|y|\leq c} |y|^{2(p\vee q)}\pi(dy) \xrightarrow{c \rightarrow 0} 0$ under $(\mathbf{H}_{\mathbf{q}}^2)$ and $\gamma_n \xrightarrow{n \rightarrow +\infty} 0$, there exist $c_\epsilon > 0$ and $n_{3,\epsilon} \in \mathbb{N}$ such that for every $n \geq n_{3,\epsilon}$,

$$C_{p,q} \int_{|y|\leq c_\epsilon} |y|^{2(p\vee q)}\pi(dy) + \delta_{c_\epsilon}(\gamma_{n+1})\gamma_{n+1}^{2p} \leq \epsilon.$$

Then, iii) follows if $q \leq p$. If $q > p$, we have $\|\kappa\|^{2p} \stackrel{|x| \rightarrow +\infty}{=} o(V^{a+p-1})$. Therefore, for every $\epsilon > 0$, $\|\kappa\|^{2p} \leq \epsilon V^{a+p-1} + C_\epsilon$ and the result is also obvious.

We prove now iii) in **case 2** (where $p \leq 1/2$ and $q \leq 1/2$). From lemma 2, V^p is α -Holder for every $\alpha \in [2p, 1]$. Therefore, we have

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1}) - V^p(\bar{X}_{n+1,2})/\mathcal{F}_n\} &\leq [V]_{2(p \vee q)} \|\kappa(\bar{X}_n)\|^{2(p \vee q)} \mathbb{E}\{|\bar{Y}_{n+1}|^{2(p \vee q)}\} \\ &\quad + p h_p \|\kappa(\bar{X}_n)\|^{2p} \mathbb{E}\{|\bar{N}_{n+1}|^{2p}\}. \end{aligned}$$

where $h_p = [V^p]_{2p}/p$. Using inequalities (i).1 and (ii).1 of lemma 1, iii) follows by the same process as case 1.

Summing (18), (19) and (20), we obtain: for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for every $n \geq n_\epsilon$,

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} &\leq V^p(\bar{X}_n) + p\gamma_{n+1} \left(V^{p-1}(\nabla V|b)(\bar{X}_n) + 1_{q \leq p} h_p \int \pi(dy) |y|^{2p} \|\kappa(\bar{X}_n)\|^{2p} \right) \\ &\quad + \gamma_{n+1} (\epsilon V^{p+a-1}(\bar{X}_n) + C_\epsilon). \end{aligned}$$

We derive from **(R_{a,p,q})** that

$$\mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} \leq V^p(\bar{X}_n) + p\gamma_{n+1} V^{p-1}(\bar{X}_n) \left(\beta - \alpha V^a(\bar{X}_n) \right) + \gamma_{n+1} (\epsilon V^{p+a-1}(\bar{X}_n) + C_\epsilon)$$

and the remark at the beginning of the proof allows us to conclude.

The proof is a little different in **case 3** because the decomposition introduced at the beginning of the proof now depends on the truncation. That's why we set here for $0 < c' \leq c$

$$\Delta \bar{X}_{n+1,1}^{c'} = \gamma_{n+1} b^{c'}(\bar{X}_n), \quad \Delta \bar{X}_{n+1,3}^{c'} = \kappa(\bar{X}_n) \left(\bar{Z}_{n+1} + \int_{c' < |y| \leq c} y \pi(dy) \right),$$

and $\Delta \bar{X}_{n+1,2}^{c'} = \Delta \bar{X}_{n+1,2}^{c'}$. We still have : $\Delta \bar{X}_{n+1} = \sum_{i=1}^3 \Delta \bar{X}_{n+1,i}^{c'}$. Using lemmas 1 and 2, we check that for every $\epsilon > 0$, it is possible to choose c'_ϵ sufficiently small such that i), ii) and iii) are void with the associated decomposition, and finally, we obtain

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} &\leq V^p(\bar{X}_n) + p\gamma_{n+1} \left(V^{p-1}(\nabla V|b^{c'_\epsilon})(\bar{X}_n) + \gamma_{n+1} (\epsilon V^{p+a-1}(\bar{X}_n) + C_\epsilon) \right) \\ &\leq V^p(\bar{X}_n) + p\gamma_{n+1} \left(V^{p-1}(\nabla V|b^c)(\bar{X}_n) + \gamma_{n+1} (\epsilon V^{p+a-1}(\bar{X}_n) + C_\epsilon^1 + C_\epsilon^2 \|\kappa(\bar{X}_n)\|) \right). \end{aligned}$$

because $b^{c'_\epsilon} = b^c - \kappa(x) \int_{c'_\epsilon < |y| \leq c} y \pi(dy)$. Now, under **(S_{a,p,q})**, we have

$$\|\kappa(x)\| \leq C V^{\frac{p+a-1}{2q}} = |x|^{-\rightarrow +\infty} o(V^{p+a-1}(x)).$$

because $q > 1/2$. Using **(R_{a,p,q})** and the remark at the beginning of the proof, the result also follows in case 3.

Proof of proposition 2 if $p \geq 1$: Thanks to Taylor formula with order 2, we derive from independence and mean properties that

$$\mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} = V^p(\bar{X}_n) + p\gamma_{n+1} V^{p-1}(\bar{X}_n) (\nabla V|b)(\bar{X}_n) + \mathbb{E}\{D^2(V^p)(\xi_{n+1}) |\Delta \bar{X}_{n+1}|^2 / \mathcal{F}_n\}. \quad (21)$$

where $\xi_{n+1} \in [\bar{X}_n, \bar{X}_{n+1}]$. From lemma 2, we have

$$D^2(V^p)(\xi_{n+1}) \Delta \bar{X}_{n+1}^{\otimes 2} \leq p \lambda_p 2^{(p-1) \vee 1-1} \left(V^{p-1}(\bar{X}_n) |\Delta \bar{X}_{n+1}|^2 + [\sqrt{V}]_1^{2(p-1)} |\Delta \bar{X}_{n+1}|^{2p} 1_{p>1} \right).$$

Then, using lemma 1.(iii), independence properties and assumption $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$, we show that

$$\mathbb{E}\{V^{p-1}(\bar{X}_n)|\Delta\bar{X}_{n+1}|^2/\mathcal{F}_n\} \leq \gamma_{n+1}V^{p-1}(\bar{X}_n)\left(\text{Tr}(\sigma\sigma^*(\bar{X}_n))+\int|y|^2\pi(dy)\|\kappa(\bar{X}_n)\|^2\right)+C\gamma_{n+1}^2V^{a+p-1}(\bar{X}_n). \quad (22)$$

Next, by the following inequality,

$$\begin{aligned} \forall u, v \geq 0 \quad \forall \alpha > 1 \quad (u+v)^\alpha &\leq u^\alpha + \alpha 2^{\alpha-1}(u^{\alpha-1}v + v^\alpha), \text{ we deduce} \\ |\Delta\bar{X}_{n+1}|^{2p} &\leq \|\kappa(\bar{X}_n)\|^{2p} \cdot |\bar{Z}_{n+1}|^{2p} \\ &+ C_p\left(\|\kappa(\bar{X}_n)\|^{2p-1}|\bar{Z}_{n+1}|^{2p-1}\sqrt{\gamma_{n+1}}V^{\frac{\alpha}{2}}(\bar{X}_n)(1+|U_{n+1}|) + \gamma_{n+1}^p V^{ap}(\bar{X}_n)(1+|U_{n+1}|^{2p})\right). \end{aligned}$$

and we derive from lemma 1.(iii) and assumption $(\mathbf{S}_{\mathbf{a},\mathbf{p},\mathbf{q}})$

$$\mathbb{E}\{|\Delta\bar{X}_{n+1}|^{2p}/\mathcal{F}_n\} \leq \gamma_{n+1}\int|y|^{2p}\pi(dy)\|\kappa(\bar{X}_n)\|^{2p} + \gamma_{n+1}(\epsilon + C_\epsilon\gamma_{n+1}^{\frac{1}{2}\wedge(p-1)})V^{ap}(\bar{X}_n). \quad (23)$$

Summing (22) and (23), we obtain a control of $\mathbb{E}\{D^2(V^p)(\xi_{n+1})\Delta\bar{X}_{n+1}^{\otimes 2}/\mathcal{F}_n\}$ and we deduce from (21), the following inequality

$$\begin{aligned} \mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} &\leq V^p(\bar{X}_n) + C\gamma_{n+1}(\epsilon + C_\epsilon\gamma_{n+1}^{(p-\frac{1}{2})\wedge 1})V^{a+p-1}(\bar{X}_n) \\ &+ p\gamma_{n+1}V^{p-1}(\bar{X}_n)\left((\nabla V|b))(\bar{X}_n) + c_p\lambda_p V^{p-1}(\bar{X}_n)\left(\text{Tr}(\sigma\sigma^*)(\bar{X}_n) + a_2\|\kappa(\bar{X}_n)\|^2 + a_{2p}\frac{\|\kappa\|^{2p}}{V^{p-1}}(\bar{X}_n)\right)\right) \end{aligned}$$

Recognizing assumption $(\mathbf{R}_{\mathbf{a},\mathbf{p},\mathbf{q}})$, we choose ϵ sufficiently small in order to use the remark at the beginning of the proof and conclude.

4.3 Some technical consequences of Proposition 2

At this step, we want to derive L^p -control results for $(V(X_n))_{n \in \mathbb{N}}$ from the inequality (16). When $a = 1$, it is easy to show that $(V(X_n))_{n \in \mathbb{N}}$ is bounded in L^p but when $a < 1$, we are not able to prove the same result. However, Lambertson and Pagès ([12]) showed that it was possible to compensate that incompetence by a weaker result (see lemma 4 of [12]). Then, we use again these ideas with a little different approach showing that a supermartingale property can be derived from (16).

Lemma 3. *Let $\Phi, \Psi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that $\Phi(\bar{X}_n)$ and $\Psi(\bar{X}_n) \in L^1(\mathbb{P}) \forall n \in \mathbb{N}$ and assume that there exist $n_0 \in \mathbb{N}$, $\beta > 0$ and $\alpha > 0$ such that $\forall n \geq n_0$,*

$$\mathbb{E}\{\Phi(\bar{X}_{n+1})/\mathcal{F}_n\} \leq \Phi(\bar{X}_n) + \gamma_{n+1}(\beta - \alpha\Psi(\bar{X}_n)).$$

Consider a nonincreasing sequence $(\theta_n)_{n \in \mathbb{N}}$ of nonnegative numbers with $\sum_{n \geq 1} \theta_n \gamma_n < \infty$. Then, $(S_n)_{n \geq n_0}$ defined by $S_n = \theta_n \Phi(\bar{X}_n) + \alpha \sum_{k=1}^n \theta_k \gamma_k \Psi(\bar{X}_{k-1}) + \beta \sum_{k > n} \theta_k \gamma_k$ is a nonnegative supermartingale and therefore,

$$\sum_{n \geq 1} \theta_n \gamma_n \mathbb{E}\{\Psi(\bar{X}_{n-1})\} < +\infty \quad \text{and} \quad \mathbb{E}\{\Phi(\bar{X}_n)\} \xrightarrow{n \rightarrow +\infty} O\left(\frac{1}{\theta_n}\right).$$

Proof. Denoting by $(M_n)_{n \geq 1}$ the martingale defined by: $M_n = \Phi(\bar{X}_n) - \mathbb{E}\{\Phi(\bar{X}_n)/\mathcal{F}_{n-1}\}$, we have for $n \geq n_0$

$$\begin{aligned} \theta_{n+1}\Phi(\bar{X}_{n+1}) &\leq \theta_{n+1}(\Phi(\bar{X}_{n+1}) - \mathbb{E}\{\Phi(\bar{X}_{n+1})/\mathcal{F}_n\}) + \theta_{n+1}\mathbb{E}\{\Phi(\bar{X}_{n+1})/\mathcal{F}_n\} \\ &\leq \theta_{n+1}M_{n+1} + \theta_{n+1}(\Phi(\bar{X}_n) + \gamma_{n+1}(\beta - \alpha\Psi(\bar{X}_n))). \end{aligned}$$

As $(\theta_n)_{n \in \mathbb{N}}$ is nonincreasing, we can write

$$\begin{aligned} \theta_{n+1}\Phi(\bar{X}_{n+1}) + \alpha \sum_{k=1}^{n+1} \theta_k \gamma_k \Psi(\bar{X}_{k-1}) + \beta \sum_{k>n+1} \theta_k \gamma_k &\leq \theta_{n+1}M_{n+1} + \theta_n\Phi(\bar{X}_n) + \alpha \sum_{k=1}^n \theta_k \gamma_k \Psi(\bar{X}_{k-1}) + \beta \sum_{k>n} \theta_k \gamma_k \\ \Rightarrow \mathbb{E}\{S_{n+1}/\mathcal{F}_n\} &\leq S_n \quad \forall n \geq n_0 \end{aligned}$$

On the other hand, $S_{n_0} \in L^1$, therefore $(S_n)_{n \geq n_0}$ is a nonnegative supermartingale and the lemma follows. \square

Corollary 1. *Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$. Assume (\mathbf{H}_p^1) , (\mathbf{H}_q^2) , $(\mathbf{R}_{a,p,q})$, $(\mathbf{S}_{a,p,q})$ and $\mathbb{E}\{|U_1|^{2(p \vee 1)}\} < +\infty$. Then,*

(i) *For every nonincreasing sequence $(\theta_n)_{n \in \mathbb{N}}$ of nonnegative numbers such that $\sum_{n \geq 1} \theta_n \gamma_n < \infty$,*

$$\sum_{n \geq 1} \theta_n \gamma_n \mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\} < \infty \quad \text{and} \quad \mathbb{E}\{V^p(\bar{X}_n)\} \stackrel{n \rightarrow +\infty}{=} O\left(\frac{1}{\theta_n}\right).$$

(ii) *If $(\frac{\eta_n}{\gamma_n})_{n \in \mathbb{N}}$ is nonincreasing,*

$$\sum_{n \geq 1} \left(\frac{\eta_n}{H_n \gamma_n}\right)^2 \mathbb{E}\{|V^{\frac{p}{2}}(\bar{X}_n) - V^{\frac{p}{2}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^2\} < +\infty. \quad (24)$$

Moreover, if condition (7) and (10) are satisfied for $s \in (1, 2)$,

$$\sum_{n \geq 1} \left(\frac{\eta_n}{H_n \gamma_n}\right)^{f_{a,p}(s)} \mathbb{E}\{|V^{\frac{p}{s}}(\bar{X}_n) - V^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^{f_{a,p}(s)}\} < +\infty. \quad (25)$$

Remark 11. The additional assumptions on steps, weights and s appear in this corollary because we prove it using Chow theorem. $f_{a,p}(s)$ plays the role of the Chow exponent. Then, we need condition (7) because $f_{a,p}(s)$ must be strictly rather than 1.

Proof. (i) It suffices to check the assumptions of lemma 3 with $\Phi = V^p$ and $\Psi = V^{a+p-1}$. On the one hand, we obtain by induction that $\Phi(\bar{X}_n)$ and $\Psi(\bar{X}_n) \in L^1(\mathbb{P})$ because b , σ and κ are sublinear, $\mathbb{E}\{|U_{n+1}|^{2p}\} < +\infty$ and $\mathbb{E}\{|\bar{Z}_{n+1}|^{2p}\} < +\infty$. On the other hand, we have by proposition 2 that

$$\mathbb{E}\{V^p(\bar{X}_{n+1})/\mathcal{F}_n\} \leq V^p(\bar{X}_n) + \gamma_{n+1}V^{p-1}(\bar{X}_n) \left(\beta' - \alpha'V^a(\bar{X}_n)\right) \quad \forall n \geq n_0 \geq 1.$$

Now, we can suppose $\beta' \geq 0$. As $a > 0$ and $\lim_{|x| \rightarrow \infty} V(x) = +\infty$, $\exists C \in \mathbb{R}$ such that

$$V^{p-1} \leq \frac{\alpha'}{2\beta'} V^{p+a-1} + C. \text{ We deduce}$$

$$\mathbb{E}\{V^p(\bar{X}_n)/\mathcal{F}_{n-1}\} \leq V^p(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1})) + \gamma_n \left(\tilde{\beta}\alpha' - \frac{\alpha'}{2} V^{p+a-1}(\bar{X}_{n-1})\right) \quad (26)$$

and conclude the first part by lemma 3.

(ii). Let us begin the second part by two remarks. On the one hand, we notice that (24) is a particular case of (25) because $f_{a,p}(2) = 2$ and (10) is systematically satisfied in this case. In fact, $(\frac{1}{\gamma_n}(\frac{\eta_n}{H_n \sqrt{\gamma_n}})^2)$ is nonincreasing because $(\eta_n/\gamma_n)_{n \in \mathbb{N}}$ so is and,

$$\sum_{n \geq 1} \left(\frac{\eta_n}{H_n \sqrt{\gamma_n}}\right)^2 \leq \frac{\eta_1}{\gamma_1} \sum_{n \geq 1} \frac{\eta_n}{H_n^2} \leq \int_1^\infty \frac{dt}{t^2} < \infty.$$

Then, we only prove (25). On the other hand, we observe that it suffices to show that

$$\mathbb{E}\{|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^{f_{a,p}(s)}\} \leq C \gamma_n^{\frac{f_{a,p}(s)}{2}} \mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\}. \quad (27)$$

Indeed, if (27) is satisfied,

$$\sum_{n \geq 1} \left(\frac{\eta_n}{H_n \gamma_n} \right)^{f_{a,p}(s)} \mathbb{E}\{|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^{f_{a,p}(s)}\} \leq \sum_{n \geq 1} \left(\frac{\eta_n}{H_n \sqrt{\gamma_n}} \right)^{f_{a,p}(s)} \mathbb{E}\{V^{p+a-1}(\bar{X}_{n-1})\}. \quad (28)$$

Now, the right sum is finite because assumption (10) corresponds to assumptions associated with (i). Then, we only prove (27).

Assume first that $\mathbf{p}/s \leq \mathbf{1}/2$. In this case, $f_{a,p}(s) = s$ and $V_s^{\frac{p}{s}}$ is $2p/s$ -holderian (see lemma 2). If $q/s \leq 1/2$, then $V_s^{\frac{p}{s}}$ is also $2p/s$ -holderian. We have

$$\begin{aligned} \mathbb{E}\{|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^s / \mathcal{F}_{n-1}\} &\leq C \gamma_n^{\frac{s}{2}} \mathbb{E}\left\{ |(V_s^{\frac{p}{s}-1} \nabla V(\xi_n))| \sqrt{\gamma_n} \sigma(\bar{X}_{n-1} U_n) \right|^s / \mathcal{F}_{n-1} \Big\} \\ &\quad + C \left(\mathbb{E}\{|\kappa(\bar{X}_{n-1}) \bar{N}_n|^{2p} / \mathcal{F}_{n-1}\} + \mathbb{E}\{|\kappa(\bar{X}_{n-1}) \bar{Y}_n|^{2q} / \mathcal{F}_{n-1}\} \right) \end{aligned}$$

where $\xi_n \in [\bar{X}_{n,1}; \bar{X}_{n,2}]$. According to lemma 1, we have under (\mathbf{H}_p^1) and (\mathbf{H}_q^2)

$$\mathbb{E}\{|\bar{N}_n|^{2p}\} + \mathbb{E}\{|\bar{Y}_n|^{2q}\} = O(\gamma_n).$$

Then, thanks to assumption $(\mathbf{S}_{a,p,q})$ and using the fact that $\nabla(V_s^{\frac{p}{s}})$ is bounded, we infer

$$\mathbb{E}\{|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^s / \mathcal{F}_{n-1}\} \leq C \gamma_n^{\frac{s}{2}} V^{a+p-1}(\bar{X}_{n-1}) \quad (29)$$

and recognize (27). If $q/s > 1/2$, we have

$$\begin{aligned} \mathbb{E}\{|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))|^s / \mathcal{F}_{n-1}\} &\leq C \mathbb{E}\{|\kappa(\bar{X}_{n-1}) \bar{N}_n|^{2p} / \mathcal{F}_{n-1}\} \\ &\quad + C \gamma_n^{\frac{s}{2}} \mathbb{E}\left\{ |(V_s^{\frac{p}{s}-1} \nabla V(\xi_n))| \sqrt{\gamma_n} \sigma(\bar{X}_{n-1} U_n + \kappa(\bar{X}_{n-1}) \bar{Y}_n) \right|^s / \mathcal{F}_{n-1} \Big\} \end{aligned}$$

where $\xi_n \in [\bar{X}_{n,1}; \bar{X}_{n,1} + \sqrt{\gamma_n} \sigma(\bar{X}_{n-1} U_n + \kappa(\bar{X}_{n-1}) \bar{Y}_n)]$. Now, as $\mathbb{E}\{|\bar{Y}_n|^2\} = O(\gamma_n)$, we have by Jensen inequality

$$\mathbb{E}\{|\kappa(\bar{X}_{n-1}) \bar{Y}_n|^s / \mathcal{F}_n\} \leq C \gamma_n^{\frac{s}{2}} \|\kappa(\bar{X}_{n-1})\|^s \leq C \gamma_n^{\frac{s}{2}} V^{a+p-1}(\bar{X}_{n-1})$$

because $s \leq 2q$. (27) follows.

Assume now that $\mathbf{p}/s > \mathbf{1}/2$. Applying the following inequality,

$$\forall u, v \geq 0, \quad \forall \alpha \geq 1, \quad |u^\alpha - v^\alpha| \leq C_\alpha (|u - v| u^{\alpha-1} + |u - v|^\alpha) \quad (30)$$

with $u = \sqrt{V(\bar{X}_n)}$, $v = \sqrt{V(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))}$ and $r = (2p)/s$, we obtain

$$\begin{aligned} |V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))| &\leq C |\sqrt{V(\bar{X}_n)} - \sqrt{V(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))}| V_s^{\frac{p}{s}-\frac{1}{2}}(\bar{X}_{n-1}) \\ &\quad + C |\sqrt{V(\bar{X}_n)} - \sqrt{V(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))}|^{\frac{2p}{s}}(\bar{X}_{n-1}) \end{aligned}$$

We deduce from $(\mathbf{S}_{a,p,q})$ that

$$|\bar{X}_n - (\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))| \leq \begin{cases} C V^{\frac{a+p-1}{2p}}(\bar{X}_{n-1}) (\sqrt{\gamma_n} |U_n| + |\bar{Z}_n|) & \text{if } p < 1 \\ C V^{\frac{a}{2}}(\bar{X}_{n-1}) (\sqrt{\gamma_n} |U_n| + |\bar{Z}_n|) & \text{if } p \geq 1 \end{cases}$$

As \sqrt{V} is Lipschitz, one finds

$$|V_s^{\frac{p}{s}}(\bar{X}_n) - V_s^{\frac{p}{s}}(\bar{X}_{n-1} + \gamma_n b(\bar{X}_{n-1}))| \leq CV^r(\bar{X}_{n-1}) \left(\sqrt{\gamma_n} |U_n| + |\bar{Z}_n| + \gamma_n^{\frac{p}{s}} |U_n|^{\frac{2p}{s}} + |\bar{Z}_n|^{\frac{2p}{s}} \right)$$

$$\text{where } r = \begin{cases} \left(\frac{p}{s} + \frac{a-1}{2p} \right) \vee \left(\frac{a+p-1}{s} \right) & \text{if } p < 1 \\ \left(\frac{p}{s} + \frac{a-1}{2} \right) \vee \frac{ap}{s} & \text{if } p \geq 1. \end{cases}$$

On the one hand, we derive from lemma 1 that $\mathbb{E}\{|\bar{Z}_n|^\alpha\} = O(\gamma_n^{\frac{\alpha}{2}})$. Therefore, as $2p/s \geq 1/2$, we have

$$\mathbb{E}\left\{ \left(\sqrt{\gamma_n} |U_n| + |\bar{Z}_n| + \gamma_n^{\frac{p}{s}} |U_n|^{\frac{2p}{s}} + |\bar{Z}_n|^{\frac{2p}{s}} \right)^{f_{a,p}(s)} \right\} = O(\gamma_n^{\frac{f_{a,p}(s)}{2}}).$$

On the other hand, we notice that $r f_{a,p}(s) \leq a + p - 1$ ($f_{a,p}$ has been constructed in this way). Therefore, inequality (27) follows. \square

4.4 Proof of Proposition 1

Let $s \in (1, 2]$. By a convexity argument (see lemma 3 of [12]), it can be shown that if proposition 2 is satisfied, then the same result is satisfied for every $\bar{p} \in (0, p]$. In particular, there exists $n_0 \in \mathbb{N}$, $\hat{\alpha}$, $\hat{\beta}$, such that $\forall k \geq n_0$,

$$\mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k)/\mathcal{F}_{k-1}\} \leq V_s^{\frac{p}{s}}(\bar{X}_{k-1}) + \gamma_k V_s^{\frac{p}{s}-1}(\bar{X}_{k-1}) \left(\hat{\beta} - \hat{\alpha} V^a(\bar{X}_{k-1}) \right).$$

As in proof of corollary 1.(i), we derive from the above inequality that there exists $\tilde{\beta} \in \mathbb{R}$ such that $\forall n \geq n_0$,

$$V_s^{\frac{p}{s}+a-1}(\bar{X}_{k-1}) \leq \frac{V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k)/\mathcal{F}_{k-1}\}}{\hat{\alpha} \gamma_k} + \tilde{\beta}.$$

Then, in order to prove Proposition 1, we only have to show that

$$\sup_{n \geq n_0+1} \left(\frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} (V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k)/\mathcal{F}_{k-1}\}) \right) < \infty \quad a.s.$$

We decompose the above sum as follows:

$$\frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} (V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k)/\mathcal{F}_{k-1}\}) = \frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - V_s^{\frac{p}{s}}(\bar{X}_k) \right) \quad (31)$$

$$+ \frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_k) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k)/\mathcal{F}_{k-1}\} \right). \quad (32)$$

By Abel transform,

$$\begin{aligned} \frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - V_s^{\frac{p}{s}}(\bar{X}_k) \right) &= \frac{1}{H_n} \left(\frac{\eta_{n_0}}{\gamma_{n_0}} V_s^{\frac{p}{s}}(\bar{X}_{n_0}) - \frac{\eta_n}{\gamma_n} V_s^{\frac{p}{s}}(\bar{X}_n) \right) \\ &+ \frac{1}{H_n} \left(\sum_{k=n_0+1}^n \left(\frac{\eta_k}{\gamma_k} - \frac{\eta_{k-1}}{\gamma_{k-1}} \right) V_s^{\frac{p}{s}}(\bar{X}_{k-1}) \right) \leq \frac{1}{H_n} \frac{\eta_{n_0}}{\gamma_{n_0}} V_s^{\frac{p}{s}}(\bar{X}_{n_0}). \end{aligned}$$

because $(\frac{\eta_n}{\gamma_n})_{n \in \mathbb{N}}$ is nonincreasing. Then, as $H_n \xrightarrow{n \rightarrow +\infty} +\infty$,

$$\sup_{n \geq n_0} \left(\frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_{k-1}) - V_s^{\frac{p}{s}}(\bar{X}_k) \right) \right) < +\infty.$$

For (32), one considers the martingale $(M_n)_{n \in \mathbb{N}}$ defined by

$$M_n = \sum_{k=1}^n \frac{\eta_k}{H_k \gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_k) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k) / \mathcal{F}_{k-1}\} \right).$$

By the inequality $|u + v|^\alpha \leq 2^{\alpha-1}(|u|^\alpha + |v|^\alpha)$ and Jensen inequality,

$$\begin{aligned} \mathbb{E} \left\{ |V_s^{\frac{p}{s}}(\bar{X}_k) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k) / \mathcal{F}_{k-1}\}|^{f_{a,p}(s)} \right\} &\leq C \mathbb{E} \left\{ |V_s^{\frac{p}{s}}(\bar{X}_k) - V_s^{\frac{p}{s}}(\bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1}))|^{f_{a,p}(s)} \right\} \\ &\quad + C \mathbb{E} \left\{ |\mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1})) - V_s^{\frac{p}{s}}(\bar{X}_k) / \mathcal{F}_{k-1}\}|^{f_{a,p}(s)} \right\} \\ &\leq C \mathbb{E} \left\{ |V_s^{\frac{p}{s}}(\bar{X}_k) - V_s^{\frac{p}{s}}(\bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1}))|^{f_{a,p}(s)} \right\} \end{aligned}$$

By corollary 1, we deduce that $(M_n)_{n \in \mathbb{N}}$ is bounded in $L^{f_{a,p}(s)}$ (with $f_{a,p}(s) > 1$ from condition (7)). Therefore, thanks to Chow theorem, a.s., $M_n \xrightarrow{n \rightarrow \infty} M_\infty$ a.s. where M_∞ is finite a.s. We deduce from Kronecker lemma:

$$\frac{1}{H_n} \sum_{k=n_0+1}^n \frac{\eta_k}{\gamma_k} \left(V_s^{\frac{p}{s}}(\bar{X}_k) - \mathbb{E}\{V_s^{\frac{p}{s}}(\bar{X}_k) / \mathcal{F}_{k-1}\} \right) \xrightarrow{n \rightarrow \infty} 0.$$

5 Identification of the weak limits of $(\bar{\nu}_n(\omega, dx))_{n \geq 1}$

In this section, we show that every weak limit of $(\bar{\nu}_n(\omega, dx))_{n \geq 1}$ is an invariant distribution for $(X_t)_{t \geq 0}$. For this purpose, we deduce from the Echeverria-Weiss theorem (see [6], p. 238 and [11]) that if $A(C_K^2(\mathbb{R}^d) \subset C_0(\mathbb{R}^d))$, it suffices to show that for every weak limit ν of $(\bar{\nu}_n)_{n \geq 1}$, for every $f \in C_K^2(\mathbb{R}^d)$, $\nu(Af) = 0$. We note that $A(C_K^2(\mathbb{R}^d) \subset C_0(\mathbb{R}^d))$ if $\|\kappa(x)\| \stackrel{|x| \rightarrow +\infty}{=} o(|x|)$ (and this condition can not be improved in general). Then, the main result of this section is the following theorem.

Proposition 3. *Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$. Assume (\mathbf{H}_p^1) , (\mathbf{H}_q^2) , $(\mathbf{S}_{a,p,q})$, $\|\kappa(x)\| \stackrel{|x| \rightarrow +\infty}{=} o(|x|)$ and $(\eta_n / \gamma_n)_{n \geq 1}$ is nonincreasing. If moreover,*

$$\sup_{n \geq 1} \frac{1}{H_n} \sum_{k=1}^n \eta_k \left(\|\kappa(\bar{X}_{k-1})\|^{2q} + \text{Tr}(\sigma \sigma^*(\bar{X}_{k-1})) \right) < \infty \quad \text{and}, \quad \sum_{k \geq 1} \frac{\eta_k^2}{H_k^2 \gamma_k} \mathbb{E}\{V^{a+p-1}(\bar{X}_{k-1})\} < +\infty, \quad (33)$$

then,

$$\forall f \in C_K^2(\mathbb{R}^d), \quad \text{a.s.}, \quad \int Af d\bar{\nu}_n \xrightarrow{n \rightarrow \infty} 0. \quad (34)$$

Consequently, every weak limit of $(\bar{\nu}_n(\omega, dx))_{n \geq 1}$ is invariant for the S.D.E. (1).

Remark 12. Under assumptions of theorem 2 and 3, we see that assumption (33) is systematically satisfied thanks to Proposition 1 and corollary 1.

5.1 Proof of Proposition 3

Proof of Proposition 3 is built in two successive stages which are represented by propositions 4 and 5. Then, we observe that combination of these propositions implies Proposition 3. Before stating the first stage, we establish a lemma which will be necessary for its proof.

Lemma 4. *Let $\Phi : \mathbb{R}^d \mapsto \mathbb{R}^l$ be a continuous function with compact support, $\Psi : \mathbb{R}^d \mapsto \mathbb{R}^+$, a locally bounded function, $(h_1^\theta)_{\theta \in [0,1]}$ and $(h_2^\theta)_{\theta \in [0,1]}$ two families of functions defined on $\mathbb{R}^d \times \mathbb{R}^+$ with values in \mathbb{R}^d satisfying the two following assumptions:*

1. *There exists $\delta_0 > 0$ such that for every $\gamma \in [0, \delta_0]$, $\theta \in [0, 1]$*

$$|h_1^\theta|(x, \gamma) \xrightarrow{|x| \rightarrow +\infty} 0 \quad \text{et} \quad |h_2^\theta|(x, \gamma) \xrightarrow{|x| \rightarrow +\infty} 0.$$

- 2.

$$\sup_{x \in K, \theta \in [0,1]} |h_1^\theta(x, \gamma) - h_2^\theta(x, \gamma)| \xrightarrow{\gamma \rightarrow 0} 0 \quad \text{for every compact set } K.$$

Then, for every sequence $(x_k)_{k \in \mathbb{N}}$ de \mathbb{R}^d ,

$$\frac{1}{H_n} \sum_{k=1}^n \sup_{\theta \in [0,1]} \|\Phi(h_1^\theta(x_{k-1}, \gamma_k)) - \Phi(h_2^\theta(x_{k-1}, \gamma_k))\| \Psi(x_{k-1}) \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. Φ has compact support therefore, we derive from the first assumption that there exists $M_{\delta_0} > 0$ such that for every $|x| > M_{\delta_0}$, $\gamma \leq \delta_0$ and $\theta \in [0, 1]$,

$$\Phi(h_1^\theta(x, \gamma)) = \Phi(h_2^\theta(x, \gamma)) = 0.$$

Consider $\rho \mapsto w(\rho, \Phi) = \sup\{\eta > 0, \sup_{|x-y| \leq \eta} |\Phi(x) - \Phi(y)| \leq \rho\}$. As Φ is uniformly continuous, $w(\rho, \Phi) > 0$ for every $\rho > 0$. Thanks to the second assumption, for every $\rho > 0$, there exists $\delta_\rho \leq \delta_0$ such that for every $\gamma \leq \delta_\rho$, $\theta \in [0, 1]$,

$$\sup_{|x| \leq M_{\delta_0}} |h_1^\theta(x, \gamma) - h_2^\theta(x, \gamma)| \leq w(\rho, \Phi).$$

As $\gamma_k \xrightarrow{k \rightarrow +\infty} 0$, there exists $k_\rho \in \mathbb{N}$ such that $\gamma_k \leq \delta_\rho$ for $k \geq k_\rho$. Using that $H_n \xrightarrow{n \rightarrow +\infty} +\infty$, we deduce

$$\limsup_{n \rightarrow +\infty} \frac{1}{H_n} \sum_{k=1}^n \sup_{\theta \in [0,1]} \|\Phi(h_1^\theta(x_{k-1}, \gamma_k)) - \Phi(h_2^\theta(x_{k-1}, \gamma_k))\| \Psi(x_{k-1}) \leq \rho \sup_{|x| \leq M_{\delta_0}} \Psi(x).$$

The result follows from the local boundedness of Ψ . □

Proposition 4. *Let $q \in [0, 1]$ such that (\mathbf{H}_q^1) is satisfied. Assume that the assumptions of Proposition 3 are fulfilled. Then,*

$$a.s., \forall f \in \mathcal{C}_K^2(\mathbb{R}^d), \lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \left(\frac{\mathbb{E}\{\Delta f(\bar{X}_k) / \mathcal{F}_{k-1}\}}{\gamma_k} - Af(\bar{X}_{k-1}) \right) = 0. \quad (35)$$

Proof of proposition 4 in case 3. Let $f \in \mathcal{C}_K^2$ and decompose the infinitesimal generator in three parts:

$$\begin{aligned} A_1 f(x) &= (\nabla f|b)(x), & A_2 f(x) &= \text{Tr}(\sigma^* D^2 f \sigma)(x), & \text{and,} \\ A_3 f(x) &= \int \pi(dy) (f(x + \kappa(x)y) - f(x) - (\nabla f|\kappa(x)y) 1_{\{|y| \leq c\}}) dy. \end{aligned}$$

Taking notations (17) (associated to case 3), we then decompose the proof in three steps:

$$\begin{aligned} \text{Step 1.} \quad & \mathbb{E} \left\{ \frac{f(\bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1})) - f(\bar{X}_{k-1})}{\gamma_k} / \mathcal{F}_{k-1} \right\} = A_1 f(\bar{X}_{k-1}) + R_1(\gamma_k, \bar{X}_{k-1}) \\ & \text{with } \frac{1}{H_n} \sum_1^n \eta_k R_1(\gamma_k, \bar{X}_{k-1}) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

$$\begin{aligned} \text{Step 2.} \quad & \mathbb{E} \left\{ \frac{f(\bar{X}_{k-1,2}) - f(\bar{X}_{k-1,1})}{\gamma_k} / \mathcal{F}_{k-1} \right\} = A_2 f(\bar{X}_{k-1}) + R_2(\gamma_k, \bar{X}_{k-1}) \\ & \text{with } \frac{1}{H_n} \sum_1^n \eta_k R_2(\gamma_k, \bar{X}_{k-1}) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

$$\begin{aligned} \text{Step 3.} \quad & \frac{1}{\gamma_k} \mathbb{E} \{ f(\bar{X}_k) - f(\bar{X}_{k-1,2}) / \mathcal{F}_{k-1} \} = A_3 f(\bar{X}_{k-1}) + R_3(\gamma_k, \bar{X}_{k-1}) \\ & \text{with } \frac{1}{H_n} \sum_1^n \eta_k R_3(\gamma_k, \bar{X}_{k-1}) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Combination of the three steps implies proposition 4. We refer to proposition 4 of [12] for steps 1 and 2 and only prove the last step. As \bar{X}_{k-1} is \mathcal{F}_{k-1} -measurable and \bar{Z}_k , U_k and \mathcal{F}_{k-1} are independent, we have

$$\begin{aligned} \mathbb{E} \{ f(\bar{X}_{k-1,2} + \kappa(\bar{X}_{k-1}) \bar{Z}_k) / \mathcal{F}_{k-1} \} &= Q_{\gamma_k} f(\bar{X}_{k-1}) \quad \text{with} \\ Q_{\gamma} f(x) &= \int_{\mathbb{R}^d} \mathbb{E} \{ f(S(x, \gamma, u) + \kappa(x) Z_{\gamma}) \} \mathbb{P}_{U_1}(du), \end{aligned}$$

where $S(x, \gamma, u) = x + \gamma b(x) + \sqrt{\gamma} \sigma(x) u$. We set $V_t = S(x, \gamma, u) + \kappa(x) Z_t$. Applying Ito formula to $(f(V_t))_{t \geq 0}$, we obtain

$$f(V_t) = f(S(x, \gamma, u)) + \int_0^t \left(\nabla f(V_{s-}) | \kappa(x) dY_s \right) + \sum_{0 < s \leq t} \tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{s-}, x, \Delta Z_s),$$

$$\text{where, } \tilde{H}^f(z, x, y) = f(z + \kappa(x)y) - f(z) - (\nabla f(z) | \kappa(x)z) 1_{\{|y| \leq c\}}. \quad (36)$$

$(t, \omega) \rightarrow \int_0^t (\nabla f(V_{s-}) | \kappa(x) dY_s)$ is a martingale because ∇f is bounded. Thanks to compensation formula and using a change of variable, we obtain

$$\mathbb{E} \{ f(S(x, \gamma, u) + \kappa(x) Z_{\gamma}) \} = \mathbb{E} \{ V_{\gamma} \} = f(S(x, \gamma, u)) + \gamma \mathbb{E} \left\{ \int_0^1 dv \int \pi(dy) \tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}, x, y) \right\}. \quad (37)$$

Now, we notice that $A_3 f(x) = \int \pi(dy) \tilde{H}^f(x, x, y) = \mathbb{E} \left\{ \int_0^1 dv \int \pi(dy) \tilde{H}^f(x, x, y) \right\}$. Then,

$$\begin{aligned} \frac{1}{\gamma_k} \mathbb{E} \{ f(\bar{X}_k) - f(\bar{X}_{k-1,2}) / \mathcal{F}_{k-1} \} &= A_3 f(\bar{X}_{k-1}) + R_3(\gamma_k, \bar{X}_{k-1}) \quad \text{where,} \\ R_3(\gamma, x) &= \int_{\mathbb{R}^d} \mathbb{E} \left\{ \int_0^1 du \int_{\mathbb{R}^d} \pi(dy) \left(\tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}, x, y) - \tilde{H}^f(x, x, y) \right) \right\} \mathbb{P}_{U_1}(du). \end{aligned} \quad (38)$$

We denote by $R_{3,1}$ and $R_{3,2}$ the parts of R_3 associated to respectively, small and big jumps of \mathbf{Z} , i.e.

$$\begin{aligned} R_{3,1}(\gamma, x, c) &= \int_{\mathbb{R}^d} \mathbb{E} \left\{ \int_0^1 dv \int_{\mathbb{R}^d} \pi(dy) \left(\tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}, x, y) - \tilde{H}^f(x, x, y) \right) 1_{\{|y| \leq c\}} \right\} \mathbb{P}_{U_1}(du), \\ R_{3,2}(\gamma, x, c) &= \int_{\mathbb{R}^d} \mathbb{E} \left\{ \int_0^1 dv \int_{\mathbb{R}^d} \pi(dy) \left(\tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}, x, y) - \tilde{H}^f(x, x, y) \right) 1_{\{|y| > c\}} \right\} \mathbb{P}_{U_1}(du). \end{aligned}$$

We investigate successively $R_{3,1}$ and $R_{3,2}$. From Taylor formula, we have for every $|y| \leq c$

$$\begin{aligned} \left(\tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}, x, y) - \tilde{H}^f(x, x, y) \right) &\leq \frac{1}{2} R^x(Z, \gamma, u, v, y) |\kappa(x)y|^2, \\ \text{where } R^x(Z^c, \gamma, u, v, y) &= \sup_{\theta \in [0,1]} \|D^2 f(S(x, \gamma, u) + \kappa(x)(Z_{v\gamma} + \theta y)) - D^2 f(x + \theta \kappa(x)y)\|. \end{aligned}$$

With the notations $\Phi = D^2 f$, $\Psi(x) = \|\kappa(x)\| \cdot |y|^2$ and:

$$h_1^\theta(x, \gamma) = S(x, \gamma, u) + \kappa(x)(Z_{v\gamma} + \theta y), \quad h_2^\theta(x, \gamma) = x + \theta \kappa(x)y,$$

we show that assumptions of lemma 4 are fulfilled with fixed u, v, y and ω .

On the one hand, as $\kappa(x) \stackrel{|x| \rightarrow +\infty}{=} o(|x|)$, there exists a continuous function v such that $\kappa(x) = |x|v(x)$ and $v(x) \stackrel{|x| \rightarrow \infty}{\rightarrow} 0$. Therefore, as b and σ are sublinear,

$$\begin{cases} |S(x, \gamma, u) + \kappa(x)(Z_{v\gamma} + \theta y)| \geq |x|(1 - \gamma C_1 - (|Z_{v\gamma}| + |y|)|v(x)|) - C_2 & \text{where } C > 0 \\ |x + \theta \kappa(x)y| \geq |x|(1 - |v(x)| \cdot |y|). \end{cases} \quad (39)$$

Consider $\delta_0 > 0$ such as $1 - \delta_0 C > 0$. As Z is locally bounded because it is a càdlàg process and $v(x) \stackrel{|x| \rightarrow \infty}{\rightarrow} 0$, for every $\gamma \leq \delta_0$ and $\theta \in [0, 1]$, a.s.,

$$h_1^\theta(x, \gamma) \stackrel{|x| \rightarrow \infty}{\rightarrow} +\infty \quad \text{and} \quad h_2^\theta(x, \gamma) \stackrel{|x| \rightarrow \infty}{\rightarrow} +\infty,$$

On the other hand, consider a compact set K of \mathbb{R}^d . We have

$$\sup_{x \in K, \theta \in [0,1]} |h_1^\theta(x, \gamma) - h_2^\theta(x, \gamma)| \leq \sup_{x \in K} (|b(x)| + \|\sigma(x)\| + \|\kappa\|(x)) (\gamma + \sqrt{\gamma}|u| + |Z_{v\gamma}|) \xrightarrow{\gamma \rightarrow 0} 0 \quad \text{a.s.}$$

because b, σ, κ are locally bounded and $\lim_{t \rightarrow 0} Z_t = 0$ a.s. Therefore, by lemma 4, for any sequence $(x_k)_{k \in \mathbb{N}}$ of \mathbb{R}^d , for every $u \in \mathbb{R}^d$, $v \in [0, 1]$ and $|y| \leq c$,

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \left(\tilde{H}^f(S(x_{k-1}, \gamma_k, u) + \kappa(x_{k-1}) Z_{v\gamma_k}, x_{k-1}, y) - \tilde{H}^f(x_{k-1}, x_{k-1}, y) \right) \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

Now, as ∇f and $D^2 f$ are bounded, we derive from Taylor formula that

$$\left| H^x(Z^c, \gamma, u, y) - H^x(Z^c, 0, u, y) \right| 1_{\{|y| \leq c\}} \leq \begin{cases} 2 \|\nabla f\|_\infty \|\kappa(x)\| \cdot |y| \\ 2 \|D^2 f\|_\infty \|\kappa(x)\|^2 |y|^2. \end{cases}$$

Then, for any $q \in [1, 2]$,

$$\left| H^x(Z^c, \gamma, u, y) - H^x(Z^c, 0, u, y) \right|_{1_{\{|y| \leq c\}}} \leq C \|\kappa(x)\|^{2q} |y|^{2q}. \quad (40)$$

Thereofre, by assumption (\mathbf{H}_q^2) ($q > 1/2$ in case 3), we finally obtain by Lebesgue theorem

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k R_{3,1}(\gamma_k, x_{k-1}, c) \xrightarrow{n \rightarrow \infty} 0 \quad \text{if} \quad \sup_{n \in \mathbb{N}} \frac{1}{H_n} \sum_{k=1}^n \eta_k \|\kappa(x_{k-1})\|^{2q} < \infty.$$

Well, $\sup_{n \in \mathbb{N}} \frac{1}{H_n} \sum_{k=1}^n \eta_k \|\kappa(\bar{X}_{k-1})\|^{2q} < \infty$ a.s., therefore, $\frac{1}{H_n} \sum_{k=1}^n \eta_k R_{3,1}(\gamma_k, \bar{X}_{k-1}, c) \xrightarrow{n \rightarrow \infty} 0$ a.s. Now, let us look into $R_{3,2}$.

$$\begin{aligned} R_{3,2}(\gamma, x, c) &= \int_{\mathbb{R}^d} \mathbb{E} \left\{ \int_0^1 dv \int_{\{|y| > c\}} \pi(dy) \left(f(S(x, \gamma, u) + \kappa(x)(Z_{v\gamma} + y)) - f(x + \kappa(x)y) \right) \right\} \mathbb{P}_{U_1}(du) \\ &\quad + \int_{\mathbb{R}^d} \mathbb{E} \left\{ \int_0^1 dv \int_{\{|y| > c\}} \left(f(x) - f(S(x, \gamma, u) + \kappa(x)Z_{v\gamma}) \right) \right\} \mathbb{P}_{U_1}(du) \end{aligned}$$

One follows the same process as before. Using lemma 4, one begins by showing that for any sequence $(x_k)_{k \in \mathbb{N}}$, y such that $|y| > c$, $v \in [0, 1]$, $u \in \mathbb{R}^d$, a.s.,

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \left(f(S(x_{k-1}, \gamma_k, u) + \kappa(x_{k-1})(Z_{v\gamma_k} + y)) - f(x_{k-1} + \kappa(x_{k-1})y) \right) \xrightarrow{n \rightarrow \infty} 0 \quad (41)$$

$$\text{and} \quad \frac{1}{H_n} \sum_{k=1}^n \eta_k \left(f(x_{k-1}) - f(S(x_{k-1}, \gamma_k, u) + \kappa(x_{k-1})Z_{u\gamma_k}^c) \right) \xrightarrow{n \rightarrow \infty} 0. \quad (42)$$

By the dominated convergence theorem (which can be applied without particular conditions because $\pi(|y| > c)$ is finite and f is bounded), we deduce that for any sequence $(x_k)_{k \in \mathbb{N}}$,

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k R_{3,2}(\gamma_k, x_{k-1}, c) \xrightarrow{n \rightarrow \infty} 0 \quad a.s.$$

Step 3 is proved in case 3. Let us explain the main differences in cases 1 and 2.

Proof of proposition 4 in cases 1 and 2. First, we examine case 2. We notice that in this case,

$$\frac{1}{\gamma_k} \mathbb{E} \{ f(\bar{X}_k) - f(\bar{X}_{k-1,2}) / \mathcal{F}_{k-1} \} = A_3 f(\bar{X}_{k-1}) + R_{3,2}(\gamma_k, \bar{X}_{k-1}, 0)$$

(41) and (42) are still valid with $c = 0$ but, as the Lévy measure is not finite (except if $q = 0$), we have to give arguments for using the dominated convergence theorem. As f is bounded and Lipschitz, it is therefore $2q$ -holderian for any $q \in [0, 1/2]$. Then,

$$\begin{aligned} |f(S(x, \gamma, u) + \kappa(x)(Z_{v\gamma} + y)) - f(S(x, \gamma, u) + \kappa(x)Z_{v\gamma})| &\leq C \|\kappa(x)\|^{2q} |y|^{2q} \\ \text{et} \quad |f(x + \kappa(x)y) - f(x)| &\leq C \|\kappa(x)\|^{2q} |y|^{2q} \end{aligned}$$

Assumption (33) allows us to conclude.

In case 1, it suffices to amount to case 3 if $q > 1/2$ and to case 2 if $q \leq 1/2$.

Proposition 5. *Let $a \in (0, 1]$, $p > 0$ and $q \in [0, 1]$ such that (\mathbf{H}_p^1) , (\mathbf{H}_q^2) and $(\mathbf{S}_{a,p,q})$ are satisfied. Assume that assumptions of Proposition 3 are fulfilled. Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{H_n} \sum_{k=1}^n \frac{\eta_k}{\gamma_k} \mathbb{E} \{ f(\bar{X}_k) - f(\bar{X}_{k-1}) / \mathcal{F}_{k-1} \} = 0 \quad a.s.$$

Proof. We do not detail the proof of this proposition which is an adaptation of proposition 3 in [12]. \square

6 Proof of main theorems for schemes (B) and (C)

The aim of this section is to bring up the main difficulties induced by the approximation of jumps in (B) and (C). We remind that for scheme (A), main theorems (theorem 2 and 3) have been proved in two successive steps. First, we got interested in tightness results (Proposition 1) and then, we proved that every weak limit was invariant for $(X_t)_{t \geq 0}$ (Proposition 3). We follow the same process for schemes (B) and (C) and will successively verify if Proposition 1 and Proposition 3 are still valid.

6.1 A.s. tightness of $\bar{\nu}_n^B(\omega, dx)$ and $\bar{\nu}_n^C(\omega, dx)$

The result of tightness associated with schemes (B) and (C) is strictly identical to Proposition 1 (In particular, assumption (6) is not necessary for tightness). Looking precisely into the proof of this theorem (for scheme (A)), we realize that the properties of jumps that we use, are: control of the moment of components of jump (lemma 1) which is fundamental for proposition 2, independence between law of Y and truncation function, and independence between $(\bar{Y}_n)_{n \in \mathbb{N}}$, $(\bar{N}_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$. We show in lemma 5 that we have the same controls as in lemma 1 for moments of components of jumps associated with schemes (B) and (C). Then, as scheme (B) satisfies the two independence properties, Proposition 1 follows in this case. In scheme (C), $(\bar{Y}_n^C)_{n \in \mathbb{N}}$ and $(\bar{N}_n^C)_{n \in \mathbb{N}}$ are not independent. It creates several technical difficulties in proof of proposition 2 in case $p < 1$ but the process of the proof is the same. Then, we only state an analogous result lemma 1.

Lemma 5. *Let T_0 be a positive number.*

(i) *Let $p > 0$ such that $\int_{|y| > c} \pi(dy) |y|^{2p} < +\infty$. Then, for any $t \leq T_0$,*

$$\mathbb{E}\{|N_{t \wedge T^n}^c|^{2p}\} \leq t \int_{|y| > c} |y|^{2p} \pi(dy) \quad \text{if } p > 0$$

$$\mathbb{E}\{|N_{t \wedge T^n}^c - (t \wedge T^n) \int_{|y| > c} y \pi(dy)|^{2p}\} \leq t \int_{|y| > c} |y|^{2p} \pi(dy) + C_{T_0, p, c} t^{2(p \wedge 1)} \quad \text{if } p > 1/2,$$

(ii) *Let T be a (\mathcal{F}_t) stopping time and $q \in [0, 1]$ such that $\int_{|y| \leq c} |y|^{2q} \pi(dy) < +\infty$. Then, for any $t \leq T_0$,*

$$\mathbb{E}\{|Y_{t \wedge T}^{c, n} + t \wedge T \int_{|y| \leq c} |y|^{2q} \pi(dy)\} \leq t \int_{|y| \leq c} |y|^{2q} \pi(dy) \quad \text{if } q \leq 1/2$$

$$\mathbb{E}\{|Y_{t \wedge T}^{c, n}|^{2q}\} \leq C_q t \int_{|y| \leq c} |y|^{2q} \pi(dy) \quad \text{if } 1/2 < q \leq 1.$$

(iii) *For every (\mathcal{F}_t) -stopping time T , $\mathbb{E}\{|Z_{t \wedge T}^n|^2\} \leq t \int |y|^2 \pi(dy)$. If $p \geq 1$, for every $\epsilon > 0$, there exists $C_\epsilon > 0$, $n_0 \in \mathbb{N}$ such that for every $t \geq T_0$ and $n \geq n_0$,*

$$\mathbb{E}\{|Z_t^n|^{2p}\} \leq t \left(\int |y|^{2p} \pi(dy) + \epsilon \right) + C_\epsilon t^{p \wedge 2} \quad \text{and,}$$

$$\mathbb{E}\{|Z_{t \wedge T^n}^n|^{2p}\} \leq t \left(\int |y|^{2p} \pi(dy) + \epsilon \right) + C_\epsilon t^{p \wedge 2}.$$

Proof. The proof is left to the reader. □

Remark 13. The fact that the control is only valid for sufficiently large n does not set any problem because inequality 16 just needs to be valid for sufficiently large n .

6.2 Identification of the limit of $(\bar{\nu}_n^B)_{n \in \mathbb{N}}$ and $(\bar{\nu}_n^C)_{n \in \mathbb{N}}$

The theorem which is obtained for $(\bar{\nu}_n^B)_{n \in \mathbb{N}}$ and $(\bar{\nu}_n^C)_{n \in \mathbb{N}}$ is strictly identical to Proposition 3 under the additional condition (6) for scheme (C). We recall that the proof of Proposition 3 is built in two steps represented by propositions 4 and 5. We notice that proposition 5 is still valid without additional difficulties. On the other hand, proof of the analogous result to proposition 4 contains some new difficulties that the three following lemmas allow us to overcome. Lemma 6 justifies the legitimacy of schemes (B) and (C). Indeed, it shows that the error produced by the approximation of the jumps part is inconsequential on the invariance of the limiting probability. Then, we state two technical lemmas. Lemma 7 ensures a uniform control in n of $(Z^n)_{n \in \mathbb{N}}$ and lemma 8 is a variant of lemma 4. We denote by $A^{k,B}$ and $A^{k,C}$ the operators on C_2^K with values in $C^b(\mathbb{R}^d, \mathbb{R})$ defined by

$$\begin{aligned} A^{k,B}f(x) &= (\nabla f|b^c)(x) + \frac{1}{2}Tr(\sigma^*D^2f\sigma)(x) + \int_{\{|y| \geq u_k\}} \tilde{H}^f(x, x, y)\pi(dy) \\ A^{k,C}f(x) &= (\nabla f|b^c)(x) + \frac{1}{2}Tr(\sigma^*D^2f\sigma)(x) + \frac{1 - e^{-\pi(D_k)\gamma_k}}{\pi(D_k)\gamma_k} \int_{\{|y| \geq u_k\}} \tilde{H}^f(x, x, y)\pi(dy). \end{aligned}$$

Lemma 6. *Assume that (\mathbf{H}_q^2) is satisfied and consider a sequence $(x_k)_{k \in \mathbb{N}}$ such that*

$$\sup_{n \geq 1} \frac{1}{H_n} \sum_{k=1}^n \eta_k \|\kappa\|^{2q}(x_{k-1}) < \infty. \quad (43)$$

Then, for every function $f \in C_2^K(\mathbb{R}^d, \mathbb{R})$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \left(Af(x_{k-1}) - A^{k,B}f(x_{k-1}) \right) &= 0 \quad \text{and if } \pi(D_n)\gamma_n \xrightarrow{n \rightarrow +\infty} 0, \\ \lim_{n \rightarrow +\infty} \frac{1}{H_n} \sum_{k=1}^n \eta_k \left(Af(x_{k-1}) - A^{k,C}f(x_{k-1}) \right) &= 0. \end{aligned}$$

Proof. We have $A^{k,B}f(x) - Af(x) = \int_{\{|y| < u_k\}} \tilde{H}^f(x, x, y)\pi(dy)$. When $q \geq 1/2$, we deduce from Taylor formula and the boundedness of ∇f and D^2f that there exists $C_q > 0$ such that

$$|\tilde{H}^f(x, x, y)|1_{\{|y| \leq u_k\}} \leq C_q \|\kappa(x)\|^{2q} |y|^{2q} 1_{\{|y| \leq u_k\}}.$$

When $q \leq 1/2$, as f is $2q$ -holderian,

$$|\tilde{H}^f(x, x, y)|1_{\{|y| \leq u_k\}} \leq [f]_{2q} \|\kappa(x)\|^{2q} |y|^{2q} 1_{\{|y| \leq u_k\}} + \sup_{x \in \text{supp}f} |\nabla f(x)| \cdot \|\kappa(x)\| \cdot |y| 1_{\{|y| \leq u_k\}}.$$

Setting $v_{k,q} = \int_{\{|y| \leq u_k\}} |y|^{2q} \pi(dy)$, we have

$$|Af(x_{k-1}) - A^{k,B}f(x_{k-1})| \leq \begin{cases} C(v_{k,q} \|\kappa(x_{k-1})\|^{2q} + v_{k,1}) & \text{if } q \leq 1/2 \\ C v_{k,q} \|\kappa(x_{k-1})\|^{2q} & \text{if } q \geq 1/2 \end{cases}$$

As $v_{k,\alpha} \xrightarrow{k \rightarrow \infty} 0$ for every $\alpha \geq q$ under assumption (\mathbf{H}_q^2) , the first result follows from (43). One deduces the second one by noticing that

$$|A^{k,B} - A^{k,C}(x)| \leq C \pi(D_k) \gamma_k (1 + \|\kappa(x)\|^{2q}).$$

□

Lemma 7. Consider $Z^n = Y^n + N$. We are able to build a sequence of càdlàg processes $(\hat{Z}^n)_{n \in \mathbb{N}}$ and \hat{Z} such that $\hat{Z}^n \stackrel{\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)}{=} Z^n$ and $\hat{Z} \stackrel{\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)}{=} Z$ with

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq s \leq T} |\hat{Z}_s^n| < +\infty \quad \forall T > 0 \quad \text{and} \quad \limsup_{n \rightarrow +\infty, \gamma \rightarrow 0} \sup_{0 \leq s \leq \gamma} |\hat{Z}_s^n| = 0 \quad \text{a.s.} \quad (44)$$

Proof. Z^n converges locally uniformly in L^2 towards Z . Then, it is clear that Z^n converges in law towards Z for the topology of uniform convergence and therefore for Skorokhod topology. Thanks to Skorokhod theorem of representation, there exists $(\hat{Z}^n)_{n \in \mathbb{N}}$ and \hat{Z} with $\hat{Z}^n \stackrel{\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)}{=} Z^n$ and $\hat{Z} \stackrel{\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)}{=} Z$ such that \hat{Z}^n tends a.s. towards \hat{Z} for Skorokhod topology. On the one hand, $f \mapsto \sup_{x \in [0, T]} |f(x)|$ is continuous for the Skorokhod topology that ensures the first assumption. On the other hand, by definition of Skorokhod topology, for every $T > 0$, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of increasing functions from $[0, T]$ to $[0, T]$ such that

$$\sup_{0 \leq s \leq T} |\lambda_n(s) - s| \xrightarrow{n \rightarrow +\infty} 0 \quad \text{et} \quad \sup_{0 \leq s \leq T} |\hat{Z}^n(\lambda_n(s)) - \hat{Z}_s| \xrightarrow{n \rightarrow +\infty} 0.$$

As \hat{Z} is right continuous and $\hat{Z}_0 = 0$, for every $\rho > 0$, there exists $\delta > 0$ such that for every $s \in [0, 2\delta]$, $|\hat{Z}_s| \leq \rho/2$. Then, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$\sup_{s \in [0, 2\delta]} |\hat{Z}^n(\lambda_n(s))| \leq \rho \quad \Rightarrow \quad \sup_{s \in [0, \inf_{n \geq n_0} \lambda_n(2\delta)]} |\hat{Z}_s^n| \leq \rho.$$

The result follows taking $n_1 \geq n_0$ such that $\lambda_n(2\delta) \geq \delta$ if $n \geq n_1$. \square

Lemma 8. Let $\Phi : \mathbb{R}^d \mapsto \mathbb{R}^l$ be a continuous function with compact support, $\Psi : \mathbb{R}^d \mapsto \mathbb{R}^+$, a locally bounded function, $(h_1^{\theta, n})_{\theta \in [0, 1], n \in \mathbb{N}^*}$ and $(h_2^{\theta, n})_{\theta \in [0, 1], n \in \mathbb{N}^*}$ be families of functions on $\mathbb{R}^d \times \mathbb{R}^+$ with values in \mathbb{R}^d satisfying the two following assumptions:

1. There exists $\delta_0 > 0$ such that for every $\gamma \in [0, \delta_0]$, $\theta \in [0, 1]$,

$$\inf_{n \geq 1} |h_1^{\theta, n}|(x, \gamma) \xrightarrow{|x| \rightarrow +\infty} +\infty \quad \text{and} \quad \inf_{n \geq 1} |h_2^{\theta, n}|(x, \gamma) \xrightarrow{|x| \rightarrow +\infty} +\infty.$$

2.

$$\sup_{x \in K, \theta \in [0, 1]} |h_1^{\theta, n}(x, \gamma) - h_2^{\theta, n}(x, \gamma)| \xrightarrow{\gamma \rightarrow 0, n \rightarrow +\infty} 0 \quad \text{for every compact set } K.$$

Then, for every sequence $(x_k)_{k \in \mathbb{N}}$ in \mathbb{R}^d ,

$$\frac{1}{H_n} \sum_{k=1}^n \sup_{\theta \in [0, 1]} \|\Phi(h_1^{\theta, n}(x_{k-1}, \gamma_k)) - \Phi(h_2^{\theta, n}(x_{k-1}, \gamma_k))\| \Psi(x_{k-1}) \xrightarrow{n \rightarrow +\infty} 0.$$

Idea of the proof of proposition 4 for schemes (B) and (C). We only want to explain why step 3 of proposition 2 is still valid for schemes (B) and (C). We use the same process but technical arguments can be different. First, consider scheme (B). We have

$$\begin{aligned} \frac{1}{\gamma_k} \mathbb{E}\{f(\bar{X}_k^B) - f(\bar{X}_{k-1,2}^B) / \mathcal{F}_{k-1}^B\} &= A_3^{k,B} f(\bar{X}_{k-1}^B) + R_3^{B,k}(\gamma_k, \bar{X}_{k-1}^B) \quad \text{où,} \\ R_3^{B,k}(\gamma, x) &= \int_{\mathbb{R}^d} \mathbb{E}\left\{ \int_0^1 du \int_{|y| > u_k} \pi(dy) \left(\tilde{H}^f(S(x, \gamma, u) + \kappa(x)Z_{v\gamma}^k, x, y) - \tilde{H}^f(x, x, y) \right) \right\} \mathbb{P}_{U_1}(du). \end{aligned} \quad (45)$$

Using lemmas 7 and 8, we are able to show that $\frac{1}{H_n} \sum_{k=1}^n \eta_k R_3^{B,k}(\gamma_k, \bar{X}_{k-1}^B) \xrightarrow{n \rightarrow \infty} 0$ a.s. and lemma 6 then ensures that step 3 is valid for scheme (B).

For scheme (C), using the stopping time theorem, we show that

$$\frac{1}{\gamma_k} \mathbb{E}\{f(\bar{X}_k^C) \bar{Z}_k^C - f(\bar{X}_{k-1,2}^C) / \mathcal{F}_{k-1}^C\} = A_3^k f(\bar{X}_{k-1}^C) + R_3^{C,k}(\gamma_k, \bar{X}_{k-1}^C) \quad \text{where}$$

$$R_3^{C,k}(\gamma, x) = \int_{\mathbb{R}^d} \int_0^1 \int_{D_k} \mathbb{E}\left\{ \tilde{H}^f(S(x, \gamma, u) + \kappa(x) Z_{v\gamma}^k, x, y) - \tilde{H}^f(x, x, y) \right\} e^{-\lambda_k v} \pi(dy) dv \mathbb{P}_{U_1}(du).$$

where $\lambda_k = \pi(D_k) \gamma_k$. Then, by lemma 6, it suffices to show that $\frac{1}{H_n} \sum_{k=1}^n \eta_k R_3^{C,k}(\gamma_k, \bar{X}_{k-1}^C) \xrightarrow{n \rightarrow \infty} 0$, that is deduced from lemmas 7 and 8. \square

7 An a.s C.L.T. for non square-integrables random variables

The ‘‘classical’’ a.s. C.L.T due to Brosamler ([5]) and Schatte ([18]) is the following result. Let $(U_n)_{n \in \mathbb{N}^*}$ be a sequence of i.i.d. random variables with values in \mathbb{R}^d such that $\mathbb{E} U_1 = 0$ and $\Sigma_{U_1} = I_d$. Then,

$$\mathbb{P} - p.s \quad \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} \delta_{\frac{1}{\sqrt{k}} \sum_{i=1}^k U_i} \xrightarrow{(\mathbb{R}^d)} \mathcal{N}(0, I_d).$$

This result is naturally associated with the Central Limit Theorem which expresses the fact that every square-integrable random variable is in the domain of normal attraction of the normal law. In the case where one has no more square-integrability, Berkes, Horvath and Khoshnevisan ([4]) obtained an extension of this result associated with the non square-integrable attractive laws which are stable laws (with index $\alpha \in (0, 2)$). We are going to show that theorem 2 allows us to find again this extension.

Denote by $(Z_t^{\alpha,c})_{t \geq 0}$ a symmetrical one-dimensional α -stable process such that the characteristic function ϕ of $Z_1^{\alpha,c}$ satisfies $\phi(u) = e^{-\rho|u|^\alpha}$ where $\rho = 2c \int_0^{+\infty} y^{-\alpha} \sin y dy$. Consider a sequence of symmetrical i.i.d. r.v $(V_n)_{n \in \mathbb{N}}$ with

$$\mathbb{P}(V_1 \geq x) = \frac{c}{x^\alpha} + \delta(x)(x^{-\alpha}(\ln x)^{-\gamma}) \quad \text{where } x > 0, \gamma > 0 \text{ and } \delta(x) \xrightarrow{x \rightarrow +\infty} 0. \quad (46)$$

Under these conditions, we know that $\frac{V_1 + \dots + V_n}{n^{\frac{1}{\alpha}}} \xrightarrow{(\mathbb{R})} \bar{Z}_1^{\alpha,c}$ (see Gnedenko-Kolmogorov, [8]).

We have the following result:

Theorem 4. *Let $(\eta_k)_{k \in \mathbb{N}^*}$ be a nonincreasing sequence with infinite sum such that $(k\eta_k)_{k \in \mathbb{N}^*}$ is nonincreasing and $\mu = \mathcal{L}(Z_1^{\alpha,c})$. Then, if $\gamma > \frac{1}{\alpha}$,*

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\frac{V_1 + \dots + V_k}{k^{\frac{1}{\alpha}}}} \xrightarrow{(\mathbb{R})} \mu \quad a.s. \quad \text{and in particular,} \quad \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} \delta_{\frac{V_1 + \dots + V_k}{k^{\frac{1}{\alpha}}}} \xrightarrow{(\mathbb{R})} \mu. \quad a.s.$$

In order to prove this theorem, we first need an almost sure invariance principle due to Stout ([19]).

Proposition 6. *Let $(V_n)_{n \geq 1}$ and $(\bar{Z}_n)_{n \geq 1}$ be sequences of i.i.d. random variables such that $\bar{Z}_1 \stackrel{(\mathbb{R})}{=} Z_1^{\alpha,c}$ and V_1 is define as above. Then, if $\gamma > \frac{1}{\alpha}$, there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and sequences $(\hat{V}_n)_{n \geq 1}$ and $(\hat{Z}_n)_{n \geq 1}$ with $\hat{V}_1 = V_1$ and $\hat{Z}_1 = \bar{Z}_1$ such that*

$$\sum_{i=1}^n \hat{Z}_i - \sum_{i=1}^n \hat{V}_i \stackrel{n \rightarrow +\infty}{=} o(n^{\frac{1}{\alpha}} (\ln n)^{-\rho}) \quad \text{with } 0 < \rho < \gamma - \frac{1}{\alpha}.$$

Proof of theorem 4. First, we assume that $V_1 = \bar{Z}_1 \stackrel{\mathcal{L}}{=} Z_1^{\alpha,c}$. Consider

$$S_0 = 0 \quad \text{and} \quad S_n = \frac{\bar{Z}_1 + \dots + \bar{Z}_n}{n^{\frac{1}{\alpha}}} \quad \forall n \geq 1.$$

We notice that $S_{n+1} = S_n - \frac{1}{\alpha}\gamma_{n+1}S_n + \gamma_{n+1}^{\frac{1}{\alpha}}\bar{Z}_{n+1} + R_{n+1}$ with $\gamma_n = \frac{1}{n}$ and $R_{n+1} = O(\frac{1}{n^2}S_n)$. The idea of the proof is to compare $(\bar{Z}_n)_{n \geq 0}$ with the (*exact*) Euler scheme (A) with initial value 0 associated with the S.D.E. $(E_{\alpha,c})$ defined by $dX_t = -\frac{1}{\alpha}X_t dt + dZ_t^{\alpha,c}$. As $(Z_t^{\alpha,c})_{t \geq 0}$ is a self-similar process with index $\frac{1}{\alpha}$, its Euler scheme can be written

$$\bar{X}_0 = 0 \quad \text{and} \quad \bar{X}_{n+1} = \bar{X}_n - \frac{1}{\alpha}\gamma_{n+1}\bar{X}_n + \gamma_{n+1}^{\frac{1}{\alpha}}\bar{Z}_{n+1}.$$

Assumptions of theorem 2 are clearly fulfilled with $a = 1$ for any $p \in (0, \alpha/2)$ and $q \in (\alpha/2, 1)$. Now, we know that $(E_{\alpha,c})$ admits a unique invariant measure μ such that $\mu = \mathcal{L}(Z_1^{\alpha,c})$ (see [17], p188), therefore, by theorem 2,

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\bar{X}_{k-1}} \xrightarrow{(\mathbb{R})} \mu. \quad (47)$$

Then, consider $\Delta_n = S_n - \bar{X}_n$. We have $\Delta_0 = 0$ and $\Delta_{n+1} = (1 - \frac{1}{\alpha(n+1)})\Delta_n + R_{n+1}$. Setting $k_0 = \inf\{k \geq 0, k - \frac{1}{\alpha} > 0\}$, we deduce that $\Delta_n = \frac{\Delta_{k_0}}{c_n} + \frac{1}{c_n} \sum_{k=k_0+1}^n c_k R_k$ for every $n \geq k_0 + 1$ with $c_n = \prod_{k=k_0+1}^n (1 - \frac{1}{\alpha k})^{-1}$. We show that $\Delta_n \xrightarrow{n \rightarrow +\infty} 0$ *a.s.* We notice that

$$c_n = \exp\left(-\sum_{k=k_0+1}^n \ln\left(1 - \frac{1}{\alpha k}\right)\right) = \exp\left(\frac{1}{\alpha} \sum_{k=k_0}^n 1/k + O\left(\frac{1}{k^2}\right)\right) \underset{n \rightarrow +\infty}{\sim} C' n^{\frac{1}{\alpha}}.$$

On the one hand, if $\alpha > 1$, \bar{Z}_1 is integrable, therefore,

$$\sum_{k \geq 1} \mathbb{E}\{|R_k|\} \leq C \sum_{k \geq 1} \frac{k}{k^{\frac{1}{\alpha}+2}} \mathbb{E}\{|\bar{Z}_1|\} \leq C \sum_{k \geq 1} \frac{1}{k^{1+\frac{1}{\alpha}}} < +\infty$$

We deduce that $\sum_{k \geq 1} |R_k| < +\infty$ and as $c_n \xrightarrow{n \rightarrow +\infty} +\infty$, we derive from Kronecker lemma that

$$\frac{1}{c_n} \sum_{k=k_0+1}^n c_k R_k \xrightarrow{n \rightarrow +\infty} 0 \quad \Rightarrow \quad \Delta_n \xrightarrow{n \rightarrow +\infty} 0 \quad \text{a.s. if } \alpha > 1.$$

On the other hand, if $\alpha \leq 1$, \bar{Z}_1 has a moment of order θ for every $\theta < \alpha$. It follows:

$$\mathbb{E}\{|R_k|^\theta\} \leq C \frac{n}{n^{2\theta + \frac{\theta}{\alpha}}} \mathbb{E}\{|X_1|^\theta\} = \frac{C'}{n^{\theta(2 + \frac{1}{\alpha}) - 1}}$$

Therefore, if θ satisfies $\theta(2 + \frac{1}{\alpha}) - 1 > 1$, i.e. if $\frac{2\alpha}{2+\alpha} < \theta < \alpha$, we have $\sum_{k \geq 1} |R_k|^\theta < +\infty$ that implies that

$$\frac{1}{c_n^\theta} \sum_{k=k_0+1}^n c_k^\theta |R_k|^\theta \xrightarrow{n \rightarrow +\infty} 0$$

Now, as $\theta \leq 1$, $\left|\frac{1}{c_n} \sum_{k=k_0+1}^n c_k R_k\right|^\theta \leq \frac{1}{c_n^\theta} \sum_{k=k_0+1}^n c_k^\theta |R_k|^\theta$. It follows that $\Delta_n \xrightarrow{n \rightarrow +\infty} 0$ *a.s.* We derive from (47) that

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\frac{\bar{Z}_1 + \dots + \bar{Z}_k}{k^{\frac{1}{\alpha}}}} \xrightarrow{n \rightarrow +\infty} \mu \quad \text{a.s.} \quad (48)$$

Now, consider a sequence $(V_n)_{n \geq 0}$ of i.i.d. symmetrical random variables satisfying (46). As (48) is almost surely valid and $(\bar{Z}_n)_{n \geq 1}$ is a sequence of i.i.d. random variables, we notice that (48) is also valid for every sequence $(\hat{Z}_n)_{n \geq 1}$ of i.i.d random variables such that $\hat{Z}_1 \stackrel{\mathcal{L}}{=} \bar{Z}_1$. By proposition 6, we obtain that there exists a sequence of i.i.d. random variables $(\hat{V}_n)_{n \geq 0}$ such that $V_1 = \hat{V}_1$ and:

$$\frac{1}{H_n} \sum_{k=1}^n \eta_k \delta_{\frac{\hat{V}_1 + \dots + \hat{V}_k}{k^{\frac{1}{\alpha}}}} \xrightarrow{n \rightarrow +\infty} \mu \quad \text{a.s.} \quad (49)$$

As $(V_n)_{n \geq 1}$ and $(\hat{V}_n)_{n \geq 1}$ are sequence of i.i.d. random variables such that $V_1 = \hat{V}_1$, (49) is also right for $(V_n)_{n \geq 1}$. \square

8 Simulations

Example 1. Denote by $(Z_t)_{t \geq 0}$ a Cauchy process with parameter 1 (with Lévy measure defined by $\pi(dy) = 1/y^2 dy$) and consider the O-U process solution to $dX_t = -X_t dt + dZ_t$ corresponding to $(E_{1,1})$ defined in the previous subsection. The unique invariant measure of $(X_t)_{t \geq 0}$ is the Cauchy law (see [17], p.188) and the assumptions of theorem 2 are fulfilled with $V(x) = 1 + x^2$, $a = 1$ and every $p \in (0, 1/2)$ and $q \in (1/2, 1)$. Therefore,

$$\bar{\nu}_n(f), \bar{\nu}_n^B(f), \bar{\nu}_n^C(f) \xrightarrow{n \rightarrow +\infty} \int \frac{f(x)}{\pi(1+x^2)} dx \quad \text{a.s.}$$

for every f such that $f = O(|x|^{\frac{1}{2}-\epsilon})$ for every $\epsilon > 0$. In the following figures, one compares the theoretical density of the invariant measure with the density obtained by convolution of each of the empirical measures by a Gaussian kernel for $N = 5.10^4$. We choose $\eta_n = \gamma_n = 1/\sqrt{n}$, $u_n = \sqrt{\gamma_n}$ (in order to have $\pi(D_n)\gamma_n \rightarrow 0$) and t indicates the CPU time.

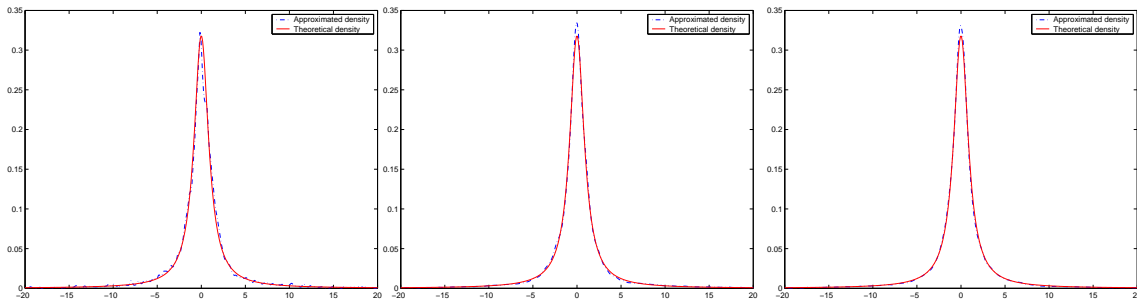


Figure 1: Sch. (A), $t = 12.5$ Figure 2: Sch. (B), $t = 16.6$ Figure 3: Sch. (C), $t = 16.4$

We observe that the best rate is obtained for the exact Euler scheme. In order to have a more precise idea of the differences between the three Euler schemes, let us observe $n \mapsto \nu_n(f)$ where $f(x) = |x|^{0.4}$ for several choices of polynomial steps. We set $\gamma_n = \eta_n = 1/n^\theta$ and $u_n = \gamma_n$ (resp. $\beta = 0.5$) for scheme (B)(resp. for schemes (C)). We are able to show that these choice of truncations give the best rate (see [14]).

We notice that among the tested steps, the best rate seems to be obtained for $\theta = 0.3$. Notably, in schemes (B) and (C), we see that if the step is slow, the stabilization is slow and if the step is too fast ($\theta = 0.1$), there are not sufficient variations to correct the error.

Example 2. Now, we study the following S.D.E:

$$dX_t = (1 - X_{t-})dt - X_{t-}dZ_t$$

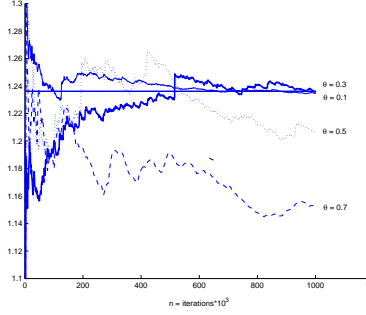


Figure 4: Scheme (A)

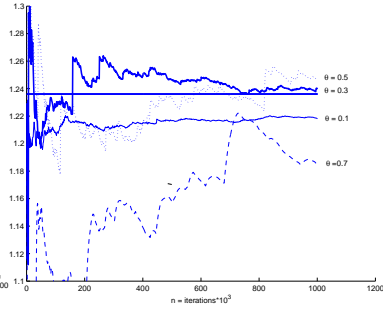


Figure 5: Scheme (B)

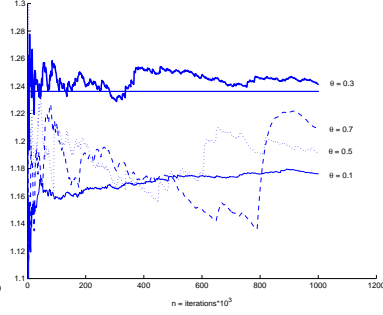


Figure 6: Scheme (C)

where $(Z_t)_{t \geq 0}$ is a drift-free subordinator with Lévy measure π defined by

$$\pi(dy) = \frac{f_{\frac{3}{2}, \frac{1}{2}}(y)}{y^2} dy,$$

where $f_{a,b}$ is the density of the $\beta(a,b)$ -distribution. This S.D.E. models the dust generated by a particular E.F.C. process (see Introduction) whose sudden dislocations do not create dust, having parameters (according to the notations of [3]):

$$c_k = 0, \quad c_e = 1 \quad \nu_{coag}(dy) = f_{\frac{3}{2}, \frac{1}{2}}(y) dy.$$

We notice that $(\mathbf{S}_{1,1,\frac{1}{2}})$ and $(\mathbf{R}_{1,1,\frac{1}{2}})$ are satisfied with $V(x) = 1 + x^2$ but the problem is that we do not have $\kappa(x) = o(x)$. However, as $\text{supp } \pi$ is restrained to $[0, 1]$ without singularities in 0 or 1, we are able to show that assumption $\kappa(x) = o(x)$ is not necessary in this case. Then, let us represent the computation of the invariant measure obtained for scheme (B) and (C). (We are not able to simulate scheme (A) in that case).

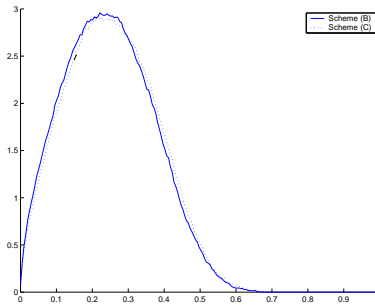


Figure 7: Approximated density, $N = 10^6$

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