



HAL
open science

Simultaneous adaptation to the margin and to complexity in classification

Guillaume Lécué

► **To cite this version:**

Guillaume Lécué. Simultaneous adaptation to the margin and to complexity in classification. 2005. hal-00009241v1

HAL Id: hal-00009241

<https://hal.science/hal-00009241v1>

Preprint submitted on 29 Sep 2005 (v1), last revised 24 Oct 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Simultaneous adaptation to the margin and to complexity in classification

Guillaume Lecué
Université Paris VI

Abstract

We consider the problem of adaptation to the margin and to complexity in binary classification. We suggest a learning method with a numerically easy aggregation step. Adaptivity both to the margin and complexity in classification, usually involves empirical risk minimization or Rademacher complexities which lead to numerical difficulties. On the other hand there exist classifiers that are easy to compute and that converge with fast rates but are not adaptive. Combining these classifiers by our aggregation procedure we get numerically realizable adaptive classifiers that converge with fast rates.

AMS 1991 subject classifications: 62G05, 62G20

Key words and phrases: Classification, statistical learning, aggregation of classifiers, optimal rates, margin, complexity of classes of sets, SVM.

Short title: Aggregation of Classifiers

1 Introduction

Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. Let $D_n = ((X_i, Y_i))_{i=1, \dots, n}$ be a sample of i.i.d. random pairs of observations where $X_i \in \mathcal{X}$ and $Y_i \in \{-1, 1\}$. Denote by π the joint distribution of (X_i, Y_i) on $\mathcal{X} \times \{-1, 1\}$, and P^X the marginal distribution of X_i .

Let (X, Y) be a random pair distributed according to π and independent of the data, and let the component X of the pair be observed. The problem of statistical

learning in classification (pattern recognition) consists in predicting the corresponding value $Y \in \{-1, 1\}$.

A *prediction rule* is a measurable function $f : \mathcal{X} \mapsto \{-1, 1\}$. The *misclassification error* associated to f is

$$R(f) = \mathbb{P}(Y \neq f(X)).$$

It is well known (see, e.g., Devroye, Györfi and Lugosi (1996)) that

$$\min_f R(f) = R(f^*) = R^*,$$

where

$$f^*(x) = \text{sign}(2\eta(x) - 1)$$

and η is the *a posteriori probability* defined by

$$\eta(x) = \mathbb{P}(Y = 1|X = x),$$

for all $x \in \mathcal{X}$ (where $\text{sign}(y)$ denotes the sign of $y \in \mathbb{R}$ with the convention $\text{sign}(0) = 1$). The prediction rule f^* is called the *Bayes rule* and R^* is called the *Bayes risk*. A *classifier* is a function, $\hat{f}_n = \hat{f}_n(X, D_n)$, measurable with respect to D_n and X with values in $\{-1, 1\}$, that assigns to every sample D_n a prediction rule $\hat{f}_n(\cdot, D_n) : \mathcal{X} \mapsto \{-1, 1\}$. A key characteristic of \hat{f}_n is the value $\mathbb{E}[R(\hat{f}_n)]$ known in learning theory under the name of *generalization error*. Here

$$R(\hat{f}_n) = \mathbb{P}(Y \neq \hat{f}_n(X)|D_n).$$

The aim of statistical learning is to construct a classifier \hat{f}_n such that $\mathbb{E}[R(\hat{f}_n)]$ is as close to R^* as possible. Accuracy of a classifier \hat{f}_n is measured by the value $\mathbb{E}[R(\hat{f}_n)] - R^*$ called *excess risk* of \hat{f}_n .

Classical approach due to Vapnik and Chervonenkis (see, e.g. Devroye, Györfi and Lugosi (1996)) consists in searching for a classifier that minimizes the *empirical risk*

$$R_n(f) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{(Y_i f(X_i) \leq 0)}, \quad (1)$$

over all prediction rules f in a source class \mathcal{F} , where \mathbb{I}_A denotes the indicator of the set A . Minimizing the empirical risk (1) is computationally intractable for many

sets \mathcal{F} of classifiers, because this functional is neither convex nor continuous. Even when one wants to take a finite class \mathcal{F} , one needs that it were of large cardinality depending on the size of the sample n in order to get small excess risk. This leads to difficulties in theory and in practice. Nevertheless, we might base a tractable estimation procedure on minimization of a convex surrogate ϕ for the loss (Cortes and Vapnik (1995), Freund and Schapire (1997), Lugosi and Vayatis (2001), Friedman, Hastie and Tibshirani (2000), Bühlmann and Yu (2002)). It has been recently shown that these classification methods often give classifiers with small Bayes risk (Blanchard, Lugosi and Vayatis (2003), Scovel and Steinwart (2004,2005)). The main idea is that the sign of the minimizer of the ϕ -risk

$$A^{(\phi)}(f) = \mathbb{E}[\phi(Yf(X))],$$

where ϕ is a convex loss function and $f : \mathcal{X} \mapsto \mathbb{R}$ a measurable function, is in many cases equal to the Bayes classifier f^* . Therefore minimizing the *empirical ϕ -risk*

$$A_n^{(\phi)}(f) = \frac{1}{n} \sum_{i=1}^n \phi(Y_i f(X_i))$$

and taking $\hat{f}_n(x) = \text{sign}(\hat{F}_n(x))$ where $\hat{F}_n \in \text{Arg min}_{f \in \mathcal{F}} A_n^{(\phi)}(f)$ leads to an approximation for f^* . Here, $\text{Arg min}_{f \in \mathcal{F}} P(f)$, for a functional P , denotes the set of all $f \in \mathcal{F}$ such that $P(f) = \min_{f \in \mathcal{F}} P(f)$. Lugosi and Vayatis (2001), Blanchard, Lugosi and Vayatis (2003), Zhang (2004), Scovel and Steinwart (2004,05) and Bartlett, Jordan and McAuliffe (2003) give results on statistical properties of classifiers obtained by minimization of such a convex risk. A wide variety of classification methods in machine learning are based on this idea, in particular, on using the convex loss associated to support vector machines (Cortes and Vapnik (1995), Schölkopf and Smola (2002)),

$$\phi(x) = (1 - x)_+,$$

called the *hinge-loss*, where $z_+ = \max(0, z)$ denotes the positive part of $z \in \mathbb{R}$ or the exponential boosting loss (Freund and Schapire (1997), Lugosi and Vayatis (2001), Bühlmann and Yu (2002)),

$$\phi(x) = \exp(-x)$$

or the logit boosting loss (Friedman, Hastie and Tibshirani (2000)),

$$\phi(x) = \log_2(1 + \exp(-x)).$$

For the hinge-loss, denote by

$$A(f) = \mathbb{E}[(1 - Yf(X))_+]$$

the *hinge risk* of $f : \mathcal{X} \rightarrow \mathbb{R}$ and set

$$A^* = \inf_f A(f), \tag{2}$$

where the infimum is taken over all measurable functions f . We will call A^* the *optimal hinge risk*. One may verify that the Bayes rule f^* attains the infimum in (2) and, moreover, Zhang (2004) has shown that,

$$R(f) - R^* \leq A(f) - A^*, \tag{3}$$

for all measurable functions f with values in \mathbb{R} . Thus minimization of the *excess hinge risk*, $A(f) - A^*$, provides a reasonable alternative for minimization of excess risk.

The difficulty of classification is closely related to the behavior of the a posteriori probability η . Mammen and Tsybakov (1999), for the problem of discriminant analysis which is close to our classification problem, and Tsybakov (2004) have introduced an assumption on the closeness of η to $1/2$, called *margin assumption* (or *low noise assumption*). Under this assumption, the risk of a minimizer of the empirical risk over some fixed class \mathcal{F} converges to the minimum risk over the class with *fast rates*, namely faster than $n^{-1/2}$. In fact, with no assumption on the joint distribution π , the convergence rate of the excess risk is not faster than $n^{-1/2}$ (cf. Devroye, Györfi and Lugosi (1996)). However, under the margin assumption, it can be as fast as n^{-1} . Minimizing penalized empirical hinge risk, under this assumption, also leads to fast convergence rates (Blanchard, Bousquet and Massart (2004) Scovel and Stewart (2004, 2005)). Massart (2000), Massart and Nédélec (2003) and Massart (2004) also obtain results that can lead to fast rates in classification using penalized empirical risk in a special case of low noise assumption. Audibert and Tsybakov (2005) show that fast rates can be achieved for plug-in classifiers.

In this paper we consider the problem of adaptive classification. Mammen and Tsybakov (1999) have shown that fast rates depend on both the *margin parameter* κ and complexity ρ of the class of candidate sets for $\{x \in \mathcal{X} : \eta(x) \geq 1/2\}$. Their

results were non-adaptive supposing that κ and ρ were known. Tsybakov (2004) suggested an adaptive classifier that attains fast optimal rates, up to a logarithmic factor, without knowing κ and ρ . Tsybakov and Van de Geer (2005) suggest a penalized empirical risk minimization classifier that adaptively attains, up to a logarithmic factor, the same fast optimal rates of convergence. Tarigan and Van de Geer (2004) extend this result to l_1 -penalized empirical hinge risk minimization. Koltchinskii (2005) uses Rademacher averages to get similar result without the logarithmic factor. Related works are those of Koltchinskii (2001), Koltchinskii and Panchenko (2002), Lugosi and Wegkamp (2004).

Note that the existing papers on fast rates either suggest classifiers that can be easily implementable but are non-adaptive, or adaptive schemes that are hard to apply in practice. The aim of the present paper is to suggest an easily implementable classifier that is adaptive simultaneously to complexity and to the margin parameters and that achieves the fast rates.

The paper is organized as follows. In Section 2 we prove an oracle inequality which corresponds to the adaptation step of the procedure that we suggest. In Section 3 we apply the oracle inequality to two types of classifiers one of which is constructed by minimization on sieves (as in Tsybakov (2004)), and gives an adaptive classifier which attains fast optimal rates without logarithmic factor, and the other one is based on the support vector machines (SVM), following Scovel and Steinwart (2004, 2005). The later is realized as a computationally feasible procedure and it adaptively attains fast rates of convergence. In particular, we suggest a method of adaptive choice of the parameter of L_1 -SVM classifiers with gaussian RBF kernels. Proofs are given in Section 4.

2 Oracle inequalities

In this section we give an oracle inequality showing that a specifically defined convex combination of classifiers mimics the best classifier in a given finite set.

Suppose that we have $M \geq 2$ different classifiers $\hat{f}_1, \dots, \hat{f}_M$ taking values in $\{-1, 1\}$. The problem of model selection type aggregation, as studied in Nemirovski (2000), Yang (1999), Catoni (1997), Tsybakov (2003), consists in construction of a new clas-

sifier \tilde{f}_n (called *aggregate*) which is approximatively at least as good, with respect to the excess risk, as the best among $\hat{f}_1, \dots, \hat{f}_M$. In most of these papers the aggregation is based on splitting of the sample in two independent subsamples D_m^1 and D_l^2 of sizes m and l respectively, where $m \gg l$ and $m + l = n$. The first subsample D_m^1 is used to construct the classifiers $\hat{f}_1, \dots, \hat{f}_M$ and the second subsample D_l^2 is used to aggregate them, i.e., to construct a new classifier that mimics in a certain sense the behavior of the best among the classifiers \hat{f}_i .

In this section we will not consider the sample splitting and concentrate only on the construction of aggregates (following Nemirovski (2000), Juditsky and Nemirovski (2000), Tsybakov (2003), Birgé (2004), Bunea, Tsybakov and Wegkamp (2004)). Thus, the first subsample is fixed and instead of classifiers $\hat{f}_1, \dots, \hat{f}_M$, we have fixed prediction rules f_1, \dots, f_M . Rather than working with a part of the initial sample we will suppose, for notational simplicity, that the whole sample D_n of size n is used for the aggregation step instead of a subsample D_l^2 .

Our procedure is using exponential weights. The idea of exponential weights is well known, see, e.g., Augustin, Buckland and Burnham (1997), Yang (2000), Catoni (2001), Hartigan (2002) and Barron and Leung (2004). Consider the following aggregate which is a convex combination with exponential weights of M classifiers,

$$\tilde{f}_n = \sum_{j=1}^M w_j^{(n)} f_j, \quad (4)$$

where

$$w_j^{(n)} = \frac{\exp(\sum_{i=1}^n Y_i f_j(X_i))}{\sum_{k=1}^M \exp(\sum_{i=1}^n Y_i f_k(X_i))}, \quad \forall j = 1, \dots, M. \quad (5)$$

Since f_1, \dots, f_M take their values in $\{-1, 1\}$, we have,

$$w_j^{(n)} = \frac{\exp(-nA_n(f_j))}{\sum_{k=1}^M \exp(-nA_n(f_k))}, \quad (6)$$

for all $j \in \{1, \dots, M\}$, where

$$A_n(f) = \frac{1}{n} \sum_{i=1}^n (1 - Y_i f(X_i))_+ \quad (7)$$

is the empirical analog of the hinge risk. Since $A_n(f_j) = 2R_n(f_j)$ for all $j = 1, \dots, M$, these weights can be written in terms of the empirical risks of f_j 's,

$$w_j^{(n)} = \frac{\exp(-2nR_n(f_j))}{\sum_{k=1}^M \exp(-2nR_n(f_k))}, \quad \forall j = 1, \dots, M.$$

We will use the following assumption (cf. Mammen and Tsybakov (1999), Tsybakov (2004)) that will allow us to get fast learning rates for the classifiers that we aggregate.

(MA1) Margin (or low noise) assumption. *We say that the probability distribution π on the space $\mathcal{X} \times \{-1, 1\}$ satisfies the margin assumption (MA1)(κ) with margin parameter $1 \leq \kappa < +\infty$ if there exists $c > 0$ such that,*

$$\mathbb{E} \{|f(X) - f^*(X)|\} \leq c(R(f) - R^*)^{1/\kappa}, \quad (8)$$

for all measurable functions f with values in $\{-1, 1\}$.

We first give the following proposition which is valid not necessarily for the particular choice of weights given in (5).

Proposition 1. *Let assumption (MA1)(κ) hold with some $1 \leq \kappa < +\infty$. Assume that there exist two positive numbers a, b such that $M \geq an^b$. Let w_1, \dots, w_M be M statistics measurable w.r.t. the sample D_n , such that $w_j \geq 0$, for all $j = 1, \dots, M$, and $\sum_{j=1}^M w_j = 1$, ($\pi^{\otimes n}$ - a.s.). Define $\tilde{f}_n = \sum_{j=1}^M w_j f_j$, where f_1, \dots, f_M are prediction rules. There exists a constant $C > 0$ depending only on a, b and on the constant c appearing in the margin assumption (8) such that*

$$(1 - (\log M)^{-1/4})\mathbb{E} \left[A(\tilde{f}_n) - A^* \right] \leq \mathbb{E}[A_n(\tilde{f}_n) - A_n(f^*)] + Cn^{-\frac{\kappa}{2\kappa-1}}(\log M)^{7/4},$$

where f^* is the Bayes rule.

As a consequence, we obtain the following oracle inequality.

Theorem 1. *Let assumption (MA1)(κ) hold with some $1 \leq \kappa < +\infty$. Assume that we have at least a polynomial number of classifiers to aggregate, namely there exist two positive numbers a, b such that $M \geq an^b$. Then the convex aggregate \tilde{f}_n defined by (4) and (6) satisfies*

$$\mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq \left(1 + \frac{2}{\log^{1/4}(M)} \right) \left\{ 2 \min_{j=1, \dots, M} (R(f_j) - R^*) + Cn^{-\frac{\kappa}{2\kappa-1}} \log^{7/4}(M) \right\}, \quad (9)$$

for all integers $n \geq 1$, where $C > 0$ depends only on a, b and on the constant c appearing in the margin assumption (8).

Remark 1. The factor 2 multiplying $\min_{j=1,\dots,M} (R(f_j) - R^*)$ in (9) is due to the relation between the hinge excess risk and the usual excess risk (cf. inequality (3) of Zhang (2004)). The hinge-loss is more adapted for our convex aggregate, since we have the same statement without this factor, namely:

$$\mathbb{E} \left[A(\tilde{f}_n) - A^* \right] \leq \left(1 + \frac{2}{\log^{1/4}(M)} \right) \left\{ \min_{j=1,\dots,M} (A(f_j) - A^*) + Cn^{-\frac{\kappa}{2\kappa-1}} \log^{7/4}(M) \right\}.$$

Moreover, linearity of the hinge-loss on $[-1, 1]$ leads to

$$\min_{j=1,\dots,M} (A(f_j) - A^*) = \min_{f \in \text{Conv}} (A(f) - A^*),$$

where Conv is the convex hull of the set $\{f_j : j = 1, \dots, M\}$. Therefore, the excess hinge risk of \tilde{f}_n is approximately the same as the one of the best convex combination of f_j 's.

Remark 2. The key point in the proof of Theorem 1 is that, for the aggregate defined by (4) and (6)

$$A_n(\tilde{f}_n) \leq A_n(f_{j_0}) + \frac{\log(M)}{n}, \quad \forall j_0 \in \{1, \dots, M\}. \quad (10)$$

Specifically we apply this inequality for $j_0 \in \text{Arg} \min_{j=1,\dots,M} A_n(f_j)$. To show (10), only convexity of the loss is used. Thus, for other statistical problems using a convex loss this inequality will be satisfied if \tilde{f}_n is constructed via (4) and (6), where A_n should be then the corresponding empirical risk. The remainder term $\log(M)/n$ is usually smaller than the convergence rate of the ERM (Empirical Risk Minimizer), therefore, the aggregate (4) with the corresponding exponential weights approximately minimizes the empirical risk and is essentially similar to the ERM procedure. Obviously, inequality (10) is satisfied when \tilde{f}_n is the ERM aggregate defined by

$$\tilde{f}_n^R \in \text{Arg} \min_{f \in \{f_1, \dots, f_M\}} R_n(f).$$

It is a convex combination of f_j 's with weights $w_j = \mathbb{1}_{(j=\hat{j})}$ where $\hat{j} \in \text{Arg} \min_{j=1,\dots,M} A_n(f_j)$. Therefore, using Proposition 1, we obtain

$$\mathbb{E} \left[R(\tilde{f}_n^R) - R^* \right] \leq \left(1 + \frac{2}{\log^{1/4}(M)} \right) \left\{ 2 \min_{j=1,\dots,M} (R(f_j) - R^*) + Cn^{-\frac{\kappa}{2\kappa-1}} \log^{7/4}(M) \right\},$$

for all integers $n \geq 1$, where $C > 0$ depends only on a, b and c . Note that our convex aggregation method is easily implementable, which is not the case of the ERM in general. Accordingly, the aggregation method (4) is an easily computable procedure that has similar statistical properties as the ERM.

Remark 3. For a convex loss function ϕ , consider the empirical ϕ -risk $A_n^{(\phi)}(f)$. Our proof implies that the aggregate

$$\tilde{f}_n^{(\phi)}(x) = \sum_{j=1}^M w_j^\phi f_j(x) \text{ with } w_j^\phi = \frac{\exp\left(-nA_n^{(\phi)}(f_j)\right)}{\sum_{k=1}^M \exp\left(-nA_n^{(\phi)}(f_k)\right)}, \quad \forall j = 1, \dots, M,$$

satisfies the inequality (10) with $A_n^{(\phi)}$ in place of A_n . However, in general, Proposition 1 does not apply to $\tilde{f}_n^{(\phi)}$ if ϕ is not the hinge loss, because linearity of the loss $(1-x)_+$ on $[-1, 1]$ is a key point of the proof of Proposition 1.

We consider next a recursive analog of the aggregate (4). It is close to the one suggested by Yang (2000) for the density aggregation under Kullback loss and by Catoni (2004) and Bunea and Nobel (2005) for regression model with squared loss. It can be also viewed as a particular instance of the mirror descent algorithm suggested in Juditsky, Nazin, Tsybakov and Vayatis (2005). The recursive version of the aggregate (4) that we propose is the following:

$$\bar{f}_n = \frac{1}{n} \sum_{k=1}^n \tilde{f}_k = \sum_{j=1}^M \bar{w}_j f_j \tag{11}$$

where

$$\bar{w}_j = \frac{1}{n} \sum_{k=1}^n \frac{\exp(\sum_{i=1}^k Y_i f_j(X_i))}{\sum_{l=1}^M \exp(\sum_{i=1}^k Y_i f_l(X_i))}$$

which can be written as an average of the weights (5),

$$\bar{w}_j = \frac{1}{n} \sum_{k=1}^n w_j^{(k)} = \frac{1}{n} \sum_{k=1}^n \frac{\exp(-kA_k(f_j))}{\sum_{l=1}^M \exp(-kA_k(f_l))}, \tag{12}$$

for all $j = 1, \dots, M$, where $A_k(f) = (1/k) \sum_{i=1}^k (1 - Y_i f(X_i))_+$ is the empirical hinge risk of f and $w_j^{(k)}$ is the weight defined in (5), for the first k observations. This aggregate is especially useful for the on-line framework. The following theorem says that it has the same theoretical properties as the aggregate (4).

Theorem 2. *Let assumption (MA1)(κ) hold with some $1 \leq \kappa < +\infty$. Assume that there exist two positive numbers a, b such that $M \geq an^b$. Then the convex aggregate \bar{f}_n defined by (11) satisfies*

$$\mathbb{E} [R(\bar{f}_n) - R^*] \leq \left(1 + \frac{2}{\log^{1/4}(M)}\right) \left\{2 \min_{j=1, \dots, M} (R(f_j) - R^*) + C\gamma(n, \kappa) \log^{7/4}(M)\right\},$$

for all integers $n \geq 1$, where $C > 0$ depends only on a, b and c , and $\gamma(n, \kappa)$ is equal to $((2\kappa - 1)/(\kappa - 1))n^{-\frac{\kappa}{2\kappa-1}}$ if $\kappa > 1$ and to $(\log n)/n$ if $\kappa = 1$.

Remark 4. *For all $k \in \{1, \dots, n-1\}$, less observations are used to construct \tilde{f}_k than for the construction of \tilde{f}_n , thus, intuitively, we expect that \tilde{f}_n will learn better than \tilde{f}_k . In view of (11), \bar{f}_n is an average of aggregates whose performances are, a priori, worse than those of \tilde{f}_n , therefore its expected learning properties would be presumably worse than those of \tilde{f}_n . An advantage of the aggregate \bar{f}_n is in its recursive construction, but the risk behavior of \tilde{f}_n seems to be better than that of \bar{f}_n . In fact, it is easy to see that Theorem 2 is satisfied for any aggregate $\bar{f}_n = \sum_{k=1}^n w_k \tilde{f}_k$ where $w_k \geq 0$ and $\sum_{k=1}^n w_k = 1$ with $\gamma(n, \kappa) = \sum_{k=1}^n w_k k^{-\kappa/(2\kappa-1)}$, and the remainder term is minimized for $w_j = 1$ when $j = n$ and 0 elsewhere, that is for $\bar{f}_n = \tilde{f}_n$.*

Remark 5. *In this section, we have only dealt with the aggregation step. But the construction of classifiers has to take place prior to this step. This needs a split of the sample as discussed at the beginning of this section. The main drawback of this method is that only a part of the sample is used for the initial estimation. However, by using different splits of the sample and taking the average of the aggregates associated with each of them, we get a more balanced classifier which does not depend on a particular split. Since the hinge loss is linear on $[-1, 1]$, we have the same result as Theorem 1 and 2 for an average of aggregates of the form (4) and (11), respectively, for averaging over different splits of the sample.*

3 Adaptation to the margin and to complexity

In Scovel and Steinwart (2004, 2005) and Tsybakov (2004) two concepts of complexity are used. In this section we show that combining classifiers used by Tsybakov (2004) or $L1$ -SVM classifiers of Scovel and Steinwart (2004, 2005) with our aggregation

method leads to classifiers that are adaptive both to the margin parameter and to the complexity, in the two cases. Results are established for the first method of aggregation defined in (4) but they are also valid for the recursive aggregate defined in (11).

We use a sample splitting to construct our aggregate. The first subsample $D_m^1 = ((X_1, Y_1), \dots, (X_m, Y_m))$, where $m = n - l$ and $l = \lceil an / \log n \rceil$ for a constant $a > 0$, is implemented to construct classifiers and the second subsample $D_l^2 = ((X_{m+1}, Y_{m+1}), \dots, (X_n, Y_n))$, is implemented to aggregate them by the procedure (4).

3.1 Adaptation in the framework of Tsybakov

Here we take $\mathcal{X} = \mathbb{R}^d$. Introduce the following pseudo-distance, and its empirical analogue, between the sets $G, G' \subseteq \mathcal{X}$:

$$d_\Delta(G, G') = P^X(G \Delta G') \quad , \quad d_{\Delta, \epsilon}(G, G') = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{(X_i \in G \Delta G')},$$

where $G \Delta G'$ is the symmetric difference between sets G and G' . If \mathcal{Y} is a class of subsets of \mathcal{X} , denote by $\mathcal{H}_B(\mathcal{Y}, \delta, d_\Delta)$ the δ -entropy with bracketing of \mathcal{Y} for the pseudo-distance d_Δ (cf. van de Geer (2000) p.16). We say that \mathcal{Y} has a *complexity bound* $\rho > 0$ if there exists a constant $A > 0$ such that

$$\mathcal{H}_B(\mathcal{Y}, \delta, d_\Delta) \leq A\delta^{-\rho}, \quad \forall 0 < \delta \leq 1.$$

Various examples of classes \mathcal{Y} having this property can be found in Dudley (1974), Korostelev and Tsybakov (1993), Mammen and Tsybakov (1995, 1999).

Let $(\mathcal{G}_\rho)_{\rho_{min} \leq \rho \leq \rho_{max}}$ be a collection of classes of subsets of \mathcal{X} , where \mathcal{G}_ρ has a complexity bound ρ , for all $\rho_{min} \leq \rho \leq \rho_{max}$. This collection corresponds to an a priori knowledge on π that the set $G^* = \{x \in \mathcal{X} : \eta(x) > 1/2\}$ lies in one of these classes (typically we have $\mathcal{G}_\rho \subset \mathcal{G}_{\rho'}$ if $\rho \leq \rho'$). The aim of adaptation to the margin and complexity is to propose a classifier, \tilde{f}_n , free from κ and ρ such that, if π satisfies (MA1)(κ) and $G^* \in \mathcal{G}_\rho$, then \tilde{f}_n learns with the optimal rate $n^{-\frac{\kappa}{2\kappa + \rho - 1}}$ (optimality has been established in Tsybakov (1999)), and this property holds for all values of $\kappa \geq 1$ and $\rho_{min} \leq \rho \leq \rho_{max}$. Following Tsybakov (2004), we introduce the following assumption on the collection $(\mathcal{G}_\rho)_{\rho_{min} \leq \rho \leq \rho_{max}}$.

(A1)(Complexity Assumption). Assume that $0 < \rho_{\min} < \rho_{\max} < 1$ and \mathcal{G}_ρ 's are classes of subsets of \mathcal{X} such that $\mathcal{G}_\rho \subseteq \mathcal{G}_{\rho'}$ for $\rho_{\min} \leq \rho < \rho' \leq \rho_{\max}$ and the class \mathcal{G}_ρ has complexity bound ρ . For any integer n , we define $\rho_{n,j} = \rho_{\min} + \frac{j}{N(n)}(\rho_{\max} - \rho_{\min})$, $j = 0, \dots, N(n)$, where $N(n)$ satisfies $A'_0 n^{b'} \leq N(n) \leq A_0 n^b$, for some finite $b \geq b' > 0$ and $A_0, A'_0 > 0$. Assume that for all $n \in \mathbb{N}$,

- (i) for all $j = 0, \dots, N(n)$ there exists \mathcal{N}_n^j an ϵ -net on $\mathcal{G}_{\rho_{n,j}}$ for the pseudo-distance d_Δ or $d_{\Delta, \epsilon}$, where $\epsilon = a_j n^{-\frac{1}{1+\rho_{n,j}}}$, $a_j > 0$ and $\max_j a_j < +\infty$,
- (ii) \mathcal{N}_n^j has a complexity bound $\rho_{n,j}$, for $j = 0, \dots, N(n)$.

The first subsample D_m^1 is used to construct the ERM classifiers $\hat{f}_m^j(x) = 2\mathbb{I}_{\hat{G}_m^j}(x) - 1$, where $\hat{G}_m^j \in \text{Arg min}_{G \in \mathcal{N}_m^j} R_m(2\mathbb{I}_G - 1)$ for all $j = 0, \dots, N(m)$, and the second subsample D_m^2 is used to construct the exponential weights of the aggregation procedure,

$$w_j^{(l)} = \frac{\exp\left(-lA^{[l]}(\hat{f}_m^j)\right)}{\sum_{k=1}^{N(m)} \exp\left(-lA^{[l]}(\hat{f}_m^k)\right)}, \quad \forall j = 1, \dots, N(m),$$

where $A^{[l]}(f) = (1/l) \sum_{i=m+1}^n (1 - Y_i f(X_i))_+$ is the empirical hinge risk of $f : \mathcal{X} \mapsto \mathbb{R}$ based on the subsample D_m^2 . We consider

$$\tilde{f}_n(x) = \sum_{j=1}^{N(m)} w_j^{(l)} \hat{f}_m^j(x), \quad \forall x \in \mathcal{X}. \quad (13)$$

Note that the construction of \hat{f}_m^j 's does not depend on the margin parameter κ .

Theorem 3. Assume that there exists a collection of classes $(\mathcal{G}_\rho)_{\rho_{\min} \leq \rho \leq \rho_{\max}}$ satisfying Assumption (A1). Then, the aggregate defined in (13) satisfies

$$\sup_{\pi \in \mathcal{P}_{\kappa, \rho}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C n^{-\frac{\kappa}{2\kappa + \rho - 1}}, \quad \forall n \geq 1,$$

for all $1 \leq \kappa < +\infty$ and all $\rho \in [\rho_{\min}, \rho_{\max}]$, where $C > 0$ is a constant depending only on $a, b, b', A, A_0, A'_0, \rho_{\min}, \rho_{\max}$ and κ , and $\mathcal{P}_{\kappa, \rho}$ is the set of all probability measures π on $\mathcal{X} \times \{-1, 1\}$ such that Assumption (MA1)(κ) is satisfied and $G^* \in \mathcal{G}_\rho$.

3.2 Adaptation in the framework of Scovel and Steinwart

3.2.1 The case of a continuous kernel

Scovel and Steinwart (2005) have obtained fast learning rates for SVM classifiers depending on three parameters, the *margin parameter* $0 \leq \alpha < +\infty$, the complexity exponent $0 < p \leq 2$ and the approximation exponent $0 \leq \beta \leq 1$. The margin parameter was first introduced in Mammen and Tsybakov (1999) for the problem of discriminant analysis and in Tsybakov (2004) for the classification problem, in the following way:

(MA2) Margin (or low noise) assumption. *We say that the probability distribution π on the space $\mathcal{X} \times \{-1, 1\}$ satisfies the margin assumption (MA2)(α) with margin parameter $0 \leq \alpha < +\infty$ if there exists $c_0 > 0$ such that*

$$\mathbb{P}(|2\eta(X) - 1| \leq t) \leq c_0 t^\alpha, \quad \forall t > 0. \quad (14)$$

As shown in Tsybakov (2004), the margin assumptions (MA1) and (MA2) are closely related: inequality (8) follows from (14) with $\kappa = \frac{1+\alpha}{\alpha}$ and $c = 2c_0^{-1/\alpha} \alpha(\alpha+1)^{-1-1/\alpha}$ for $\alpha > 0$.

Let \mathcal{X} be a compact metric space. Let H be a reproducing kernel Hilbert space (RKHS) over \mathcal{X} (see, e.g., Cristianini and Shawe-Taylor (2000), Schölkopf and Smola (2002)), B_H its closed unit ball. Denote by $\mathcal{N}(B_H, \epsilon, L_2(P_n^X))$ the ϵ -covering number of B_H w.r.t. the canonical distance of $L_2(P_n^X)$, the L_2 -space w.r.t. the empirical measure, P_n^X , on X_1, \dots, X_n . Introduce the following assumptions as in Scovel and Steinwart (2005):

(A2) *There exists $a_0 > 0$ and $0 < p \leq 2$ such that for any integer n ,*

$$\sup_{D_n \in (\mathcal{X} \times \{-1, 1\})^n} \log \mathcal{N}(B_H, \epsilon, L_2(P_n^X)) \leq a_0 \epsilon^{-p}, \quad \forall \epsilon > 0,$$

Note that the supremum is taken over all the samples of size n and the bound is assuming for any n . Every RKHS satisfies (A2) with $p = 2$ (cf. Scovel and Steinwart (2005)). We define the *approximation error function* of the L1-SVM as

$$a(\lambda) \stackrel{\text{def}}{=} \inf_{f \in H} (\lambda \|f\|_H^2 + A(f)) - A^*.$$

(A3) We say that the RKHS H , approximates π with exponent $0 \leq \beta \leq 1$, if there exists a constant $C_0 > 0$ such that

$$a(\lambda) \leq C_0 \lambda^\beta, \quad \forall \lambda > 0.$$

Note that every RKHS approximates every probability measure with exponent $\beta = 0$ and the other extremal case $\beta = 1$ is equivalent to the fact that the Bayes classifier f^* belongs to the RKHS (cf. Scovel and Steinwart (2005)). Furthermore, $\beta > 1$ only for probability measures such that $\mathbb{P}(\eta(X) = 1/2) = 1$ (cf. Scovel and Steinwart (2005)). Such distributions are not interesting for the classification problem. If (A2) and (A3) hold, the parameter (p, β) can be considered as a complexity parameter characterizing π and H .

Let H be a RKHS with a continuous kernel on \mathcal{X} satisfying (A2) with a parameter $0 < p < 2$. Define the $L1$ -SVM classifier by

$$\hat{f}_n^\lambda(x) = \text{sign}(\hat{F}_n^\lambda(x)), \forall x \in \mathcal{X}, \text{ where } \hat{F}_n^\lambda \in \text{Arg min}_{f \in H} (\lambda \|f\|_H^2 + A_n(f)) \quad (15)$$

and $\lambda > 0$ is called the regularization parameter. Assume that the underlying probability measure π belongs to the set $\mathcal{Q}_{\alpha, \beta}$ of all the probability measures on $\mathcal{X} \times \{-1, 1\}$ satisfying (MA2)(α) with $\alpha \geq 0$ and (A3) with a complexity parameter (p, β) where $0 < \beta \leq 1$. It has been shown in Scovel and Steinwart (2005) that the $L1$ -SVM classifier, $\hat{f}_n^{\lambda_n^{\alpha, \beta}}$, where the regularization parameter is $\lambda_n^{\alpha, \beta} = n^{-\frac{4(\alpha+1)}{(2\alpha+p\alpha+4)(1+\beta)}}$ satisfies the following excess risk bound: for any $\epsilon > 0$, there exists $C > 0$ depending only on α, p, β and ϵ such that

$$\mathbb{E} \left[R(\hat{f}_n^{\lambda_n^{\alpha, \beta}}) - R^* \right] \leq C n^{-\frac{4\beta(\alpha+1)}{(2\alpha+p\alpha+4)(1+\beta)} + \epsilon}, \quad \forall n \geq 1. \quad (16)$$

Remark that if $\beta = 1$, that is $f^* \in H$, then the learning rate in (16) is (up to an ϵ) $n^{-2(\alpha+1)/(2\alpha+p\alpha+4)}$ which is a fast rate since $2(\alpha+1)/(2\alpha+p\alpha+4) \in [1/2, 1)$.

To construct the classifier $\hat{f}_n^{\lambda_n^{\alpha, \beta}}$ we need to know parameters α and β that are not available in practice. Thus, it is important to construct a classifier, free from these parameters, which has the same behavior as $\hat{f}_n^{\lambda_n^{\alpha, \beta}}$, if the underlying distribution π belongs to $\mathcal{Q}_{\alpha, \beta}$. Below we give such a construction.

Since the RKHS H is given, the implementation of the $L1$ -SVM classifier \hat{f}_n^λ only requires the knowledge of the regularization parameter λ . A straightforward

application of our method would be to aggregate a family of classifiers $(\hat{f}_n^{\lambda_n, \beta})_{\alpha, \beta}$ with (α, β) belonging to some sufficiently dense grid. However this is not natural because α and β are not intrinsically related to the definition of the $L1$ -SVM method. Thus to provide an easily implementable procedure, using our aggregation method, it is natural to combine $L1$ -SVM classifiers constructed for different values of λ in a finite grid. We now define such a procedure.

We consider the $L1$ -SVM classifiers \hat{f}_m^λ , defined in (15) for the subsample D_m^1 , where λ lies in the grid

$$\mathcal{G}(l) = \{\lambda_{l,k} = l^{-\phi_{l,k}} : \phi_{l,k} = 1/2 + k\Delta^{-1}, k = 0, \dots, \lfloor 3\Delta/2 \rfloor\},$$

where we set $\Delta = l^{b_0}$ with some $b_0 > 0$. The subsample D_l^2 is used to aggregate these classifiers by the procedure (4), namely

$$\tilde{f}_n = \sum_{\lambda \in \mathcal{G}(l)} w_\lambda^{(l)} \hat{f}_m^\lambda \quad (17)$$

where

$$w_\lambda^{(l)} = \frac{\exp\left(\sum_{i=m+1}^n Y_i \hat{f}_m^\lambda(X_i)\right)}{\sum_{\lambda' \in \mathcal{G}(l)} \exp\left(\sum_{i=m+1}^n Y_i \hat{f}_m^{\lambda'}(X_i)\right)} = \frac{\exp\left(-lA^{[l]}(\hat{f}_m^\lambda)\right)}{\sum_{\lambda' \in \mathcal{G}(l)} \exp\left(-lA^{[l]}(\hat{f}_m^{\lambda'})\right)},$$

and $A^{[l]}(f) = (1/l) \sum_{i=m+1}^n (1 - Y_i f(X_i))_+$.

Theorem 4. *Let H be a RKHS with a continuous kernel on a compact metric space \mathcal{X} satisfying (A2) with a parameter $0 < p < 2$. Let K be a compact subset of $(0, +\infty) \times (0, 1]$. Then, the classifier \tilde{f}_n , defined in (17), satisfies*

$$\sup_{\pi \in \mathcal{Q}_{\alpha, \beta}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C n^{-\frac{4\beta(\alpha+1)}{(2\alpha+p\alpha+4)(1+\beta)} + \epsilon}$$

for all $(\alpha, \beta) \in K$ and $\epsilon > 0$, where $\mathcal{Q}_{\alpha, \beta}$ is the set of all probability measures on $\mathcal{X} \times \{-1, 1\}$ satisfying (MA2)(α) and (A2) with a complexity parameter (p, β) and $C > 0$ is a constant depending only on ϵ, p, K, a and b_0 .

3.2.2 The case of the Gaussian RBF kernel

In this subsection we apply our aggregation procedure to $L1$ -SVM classifiers using *Gaussian RBF kernel*. Let \mathcal{X} be the closed unit ball of the space \mathbb{R}^{d_0} endowed

with the Euclidean norm $\|x\| = \left(\sum_{i=1}^{d_0} x_i^2\right)^{1/2}$, $\forall x = (x_1, \dots, x_{d_0}) \in \mathbb{R}^{d_0}$. Gaussian RBF kernel is defined as $K_\sigma(x, x') = \exp(-\sigma^2\|x - x'\|^2)$ for $x, x' \in \mathcal{X}$ where σ is a parameter and σ^{-1} is called the *width* of the gaussian kernel. The RKHS associated to K_σ is denoted by H_σ .

Scovel and Steinwart (2004) introduced the following assumption:

(GNA) Geometric noise assumption. *There exist $C_1 > 0$ and $\gamma > 0$ such that*

$$\mathbb{E} \left[|2\eta(X) - 1| \exp\left(-\frac{\tau(X)^2}{t}\right) \right] \leq C_1 t^{\frac{\gamma d_0}{2}}, \quad \forall t > 0.$$

Here τ is a function on \mathcal{X} with values in \mathbb{R} which measures the distance between a given point x and the decision boundary, namely,

$$\tau(x) = \begin{cases} d(x, G_0 \cup G_1), & \text{if } x \in G_{-1}, \\ d(x, G_0 \cup G_{-1}), & \text{if } x \in G_1, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{X}$, where $G_0 = \{x \in \mathcal{X} : \eta(x) = 1/2\}$, $G_1 = \{x \in \mathcal{X} : \eta(x) > 1/2\}$ and $G_{-1} = \{x \in \mathcal{X} : \eta(x) < 1/2\}$. Here $d(x, A)$ denotes the Euclidean distance from a point x to the set A . If π satisfies Assumption (GNA) for a $\gamma > 0$, we say that π has a *geometric noise exponent* γ .

The $L1$ -SVM classifier associated to the gaussian RBF kernel with width σ^{-1} and regularization parameter λ is defined by $\hat{f}_n^{(\sigma, \lambda)} = \text{sign}(\hat{F}_n^{(\sigma, \lambda)})$ where $\hat{F}_n^{(\sigma, \lambda)}$ is given by (15) with $H = H_\sigma$. Using the standard development related to SVM (cf. Schölkopf and Smola (2002)), we may write $\hat{F}_n^{(\sigma, \lambda)}(x) = \sum_{i=1}^n \hat{C}_i K_\sigma(X_i, x)$, $\forall x \in \mathcal{X}$, where $\hat{C}_1, \dots, \hat{C}_n$ are solutions of the following maximization problem

$$\max_{0 \leq 2\lambda C_i Y_i \leq n^{-1}} \left\{ 2 \sum_{i=1}^n C_i Y_i - \sum_{i,j=1}^n C_i C_j K_\sigma(X_i, X_j) \right\},$$

that can be obtained using a standard quadratic programming software. According to Scovel and Steinwart (2004), if the underlying probability measure π on $\mathcal{X} \times \{-1, 1\}$, satisfies the margin assumption (MA2)(α) with margin parameter $0 \leq \alpha < +\infty$ and Assumption (GNA) with a geometric noise exponent $\gamma > 0$, the classifier $\hat{f}_n^{(\sigma_n^{\alpha, \gamma}, \lambda_n^{\alpha, \gamma})}$ where regularization parameter and width are defined by

$$\lambda_n^{\alpha, \gamma} = \begin{cases} n^{-\frac{\gamma+1}{2\gamma+1}} & \text{if } \gamma \leq \frac{\alpha+2}{2\alpha}, \\ n^{-\frac{2(\gamma+1)(\alpha+1)}{2\gamma(\alpha+2)+3\alpha+4}} & \text{otherwise,} \end{cases} \quad \text{and } \sigma_n^{\alpha, \gamma} = (\lambda_n^{\alpha, \gamma})^{-\frac{1}{(\gamma+1)d_0}},$$

satisfies

$$\mathbb{E} \left[R(\hat{f}_n^{(\sigma_n^{\alpha,\gamma}, \lambda_n^{\alpha,\gamma})}) - R^* \right] \leq C \begin{cases} n^{-\frac{\gamma}{2\gamma+1}+\epsilon} & \text{if } \gamma \leq \frac{\alpha+2}{2\alpha}, \\ n^{-\frac{2\gamma(\alpha+1)}{2\gamma(\alpha+2)+3\alpha+4}+\epsilon} & \text{otherwise,} \end{cases} \quad (18)$$

for all $\epsilon > 0$, where $C > 0$ is a constant which depends only on α, γ and ϵ . Remark that fast rates are obtained only for $\gamma > (3\alpha + 4)/(2\alpha)$.

To construct the classifier $\hat{f}_n^{(\sigma_n^{\alpha,\gamma}, \lambda_n^{\alpha,\gamma})}$ we need to know parameters α and γ , which are not available in practice. Like in Subsection 3.2.1 we use our procedure to obtain a classifier which is adaptive to the margin and to the geometric noise parameters. Our aim is to provide an easily computable adaptive classifier. We propose the following method based on a grid for (σ, λ) . We consider the finite sets

$$\mathcal{M}(l) = \left\{ (\varphi_{l,p_1}, \psi_{l,p_2}) = \left(\frac{p_1}{2\Delta}, \frac{p_2}{\Delta} + \frac{1}{2} \right) : p_1 = 1, \dots, 2\lfloor \Delta \rfloor; p_2 = 1, \dots, \lfloor \Delta/2 \rfloor \right\},$$

where we let $\Delta = l^{b_0}$ for some $b_0 > 0$, and

$$\mathcal{N}(l) = \{(\sigma_{l,\varphi}, \lambda_{l,\psi}) = (l^{\varphi/d_0}, l^{-\psi}) : (\varphi, \psi) \in \mathcal{M}(l)\}.$$

We construct the family of classifiers $(\hat{f}_m^{(\sigma,\lambda)} : (\sigma, \lambda) \in \mathcal{N}(l))$ using the observations of the subsample D_m^1 and we aggregate them by the procedure (4) using D_l^2 , namely

$$\tilde{f}_n = \sum_{(\sigma,\lambda) \in \mathcal{N}(l)} w_{\sigma,\lambda}^{(l)} \hat{f}_m^{(\sigma,\lambda)} \quad (19)$$

where

$$w_{\sigma,\lambda}^{(l)} = \frac{\exp\left(\sum_{i=m+1}^n Y_i \hat{f}_m^{(\sigma,\lambda)}(X_i)\right)}{\sum_{(\sigma',\lambda') \in \mathcal{N}(l)} \exp\left(\sum_{i=m+1}^n Y_i \hat{f}_m^{(\sigma',\lambda')}(X_i)\right)} \quad (20)$$

which can be written in terms of the empirical risk as

$$w_{\sigma,\lambda}^{(l)} = \frac{\exp\left(-lA^{[l]}(\hat{f}_m^{(\sigma,\lambda)})\right)}{\sum_{(\sigma',\lambda') \in \mathcal{N}(l)} \exp\left(-lA^{[l]}(\hat{f}_m^{(\sigma',\lambda')})\right)},$$

for all $(\sigma, \lambda) \in \mathcal{N}(l)$ and $A^{[l]}(f) = (1/l) \sum_{i=m+1}^n (1 - Y_i f(X_i))_+$.

Denote by $\mathcal{R}_{\alpha,\gamma}$ the set of all probability measures on $\mathcal{X} \times \{-1, 1\}$ satisfying both the margin assumption (MA2)(α) with a margin parameter $\alpha > 0$ and Assumption (GNA) with a geometric noise exponent $\gamma > 0$. Define $\mathcal{U} = \{(\alpha, \gamma) \in (0, +\infty)^2 : \gamma > \frac{\alpha+2}{2\alpha}\}$ and $\mathcal{U}' = \{(\alpha, \gamma) \in (0, +\infty)^2 : \gamma \leq \frac{\alpha+2}{2\alpha}\}$.

Theorem 5. Let K be a compact subset of \mathcal{U} and K' a compact subset of \mathcal{U}' . Then the aggregate \tilde{f}_n , defined in (19), satisfies

$$\sup_{\pi \in \mathcal{R}_{\alpha, \gamma}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C \begin{cases} n^{-\frac{\gamma}{2\gamma+1} + \epsilon} & \text{if } (\alpha, \gamma) \in K', \\ n^{-\frac{2\gamma(\alpha+1)}{2\gamma(\alpha+2)+3\alpha+4} + \epsilon} & \text{if } (\alpha, \gamma) \in K, \end{cases}$$

for all $(\alpha, \gamma) \in K \cup K'$ and $\epsilon > 0$, where $C > 0$ depends only on ϵ, K, K', a and b_0 .

4 Proofs

Lemma 1. For all positive v, t and all $\kappa \geq 1$

$$t + v \geq v^{\frac{2\kappa-1}{2\kappa}} t^{\frac{1}{2\kappa}}.$$

Proof. Since \log is concave, we have $\log(ab) = (1/x)\log(a^x) + (1/y)\log(b^y) \leq \log(a^x/x + b^y/y)$ for all positive numbers a, b and x, y such that $1/x + 1/y = 1$, thus $ab \leq a^x/x + b^y/y$. Lemma 1 follows by applying this relation with $a = t^{1/(2\kappa)}, x = 2\kappa$ and $b = v^{(2\kappa-1)/(2\kappa)}$.

Proof of Proposition 1. Denote by $\gamma = (\log M)^{-1/4}$, $u = 2\gamma n^{-\frac{\kappa}{2\kappa-1}} \log^2 M$ and $W_n = (1 - \gamma)(A(\tilde{f}_n) - A^*) - (A_n(\tilde{f}_n) - A_n(f^*))$. We have:

$$\begin{aligned} \mathbb{E} [W_n] &= \mathbb{E} [W_n (\mathbb{1}_{(W_n \leq u)} + \mathbb{1}_{(W_n > u)})] \leq u + \mathbb{E} [W_n \mathbb{1}_{(W_n > u)}] \\ &= u + u \mathbb{P}(W_n > u) + \int_u^{+\infty} \mathbb{P}(W_n > t) dt \leq 2u + \int_u^{+\infty} \mathbb{P}(W_n > t) dt. \end{aligned}$$

On the other hand $(f_j)_{j=1, \dots, M}$ are prediction rules which take their values in $\{-1, 1\}$, so we have $A(f_j) = 2R(f_j)$ and $A_n(f_j) = 2R_n(f_j)$, (recall that $A^* = 2R^*$). Moreover we work in the linear part of the hinge-loss, so we have

$$\begin{aligned} \mathbb{P}(W_n > t) &= \mathbb{P} \left(\sum_{j=1}^M w_j ((A(f_j) - A^*)(1 - \gamma) - (A_n(f_j) - A_n(f^*))) > t \right) \\ &\leq \mathbb{P} \left(\max_{j=1, \dots, M} ((A(f_j) - A^*)(1 - \gamma) - (A_n(f_j) - A_n(f^*))) > t \right) \\ &\leq \sum_{j=1}^M \mathbb{P}(Z_j > \gamma(R(f_j) - R^*) + t/2), \end{aligned}$$

for all $t > u$, where $Z_j = R(f_j) - R^* - (R_n(f_j) - R_n(f^*))$ for all $j = 1, \dots, M$ (recall that $R_n(f)$ is the empirical risk defined in (1)).

Let $j \in \{1, \dots, M\}$. We can write $Z_j = (1/n) \sum_{i=1}^n (\mathbb{E}[W_{i,j}] - W_{i,j})$ where $W_{i,j} = \mathbb{1}_{(Y_i f_j(X_i) \leq 0)} - \mathbb{1}_{(Y_i f^*(X_i) \leq 0)}$. We have $|W_{i,j}| \leq 1$ and, under the margin assumption, we have $\mathbb{V}(W_{i,j}) \leq \mathbb{E}(W_{i,j}^2) = \mathbb{E}[|f_j(X) - f^*(X)|] \leq c(R(f_j) - R^*)^{1/\kappa}$ where \mathbb{V} is the symbol of the variance. By applying Bernstein's inequality we get

$$\begin{aligned} \mathbb{P}[Z_j > \epsilon] &\leq \exp\left(-\frac{n\epsilon^2}{2c(R(f_j) - R^*)^{1/\kappa} + 2\epsilon/3}\right) \\ &\leq \exp\left(-\frac{n\epsilon^2}{4c(R(f_j) - R^*)^{1/\kappa}}\right) + \exp\left(-\frac{3n\epsilon}{4}\right), \end{aligned}$$

for all $\epsilon > 0$. Denote by $u_j = u/2 + \gamma(R(f_j) - R^*)$. Then, using the usual bound of the gaussian tail we get

$$\int_u^{+\infty} \mathbb{P}(Z_j > \gamma(R(f_j) - R^*) + t/2) dt = 2 \int_{u_j}^{+\infty} \mathbb{P}(Z_j > \epsilon) d\epsilon \leq B_1 + B_2,$$

where

$$B_1 = \frac{4c(R(f_j) - R^*)^{1/\kappa}}{nu_j} \exp\left(-\frac{nu_j^2}{4c(R(f_j) - R^*)^{1/\kappa}}\right)$$

and

$$B_2 = \frac{8}{3n} \exp\left(-\frac{3nu_j}{4}\right).$$

Since $R(f_j) \geq R^*$, using Lemma 1, we obtain $u_j \geq \gamma(R(f_j) - R^*)^{\frac{1}{2\kappa}} (\log M)^{\frac{2\kappa-1}{\kappa}} n^{-1/2}$. For any $a > 0$, the mapping $x \mapsto (ax)^{-1} \exp(-ax^2)$ is decreasing on $(0, +\infty)$ thus, we have,

$$B_1 \leq \frac{4c}{\gamma\sqrt{n}} (\log M)^{-\frac{2\kappa-1}{\kappa}} \exp\left(-\frac{\gamma^2}{4c} (\log M)^{\frac{4\kappa-2}{\kappa}}\right).$$

The mapping $x \mapsto (2/a) \exp(-ax)$ is decreasing on $(0, +\infty)$, for any $a > 0$ and $u_j \geq \gamma(\log M)^2 n^{-\frac{\kappa}{2\kappa-1}}$ thus,

$$B_2 \leq \frac{8}{3n} \exp\left(-\frac{3\gamma}{4} n^{\frac{\kappa-1}{2\kappa-1}} (\log M)^2\right).$$

Since $\gamma = (\log M)^{-1/4}$, we have $\mathbb{E}(W_n) \leq 4n^{-\frac{\kappa}{2\kappa-1}} (\log M)^{7/4} + T_1 + T_2$, where

$$T_1 = \frac{4Mc}{\sqrt{n}} (\log M)^{-\frac{7\kappa-4}{4\kappa}} \exp\left(-\frac{3}{4c} (\log M)^{\frac{7\kappa-4}{2\kappa}}\right)$$

and

$$T_2 = \frac{8M}{3n} \exp \left(-(3/4)n^{\frac{\kappa-1}{2\kappa-1}} (\log M)^{7/4} \right).$$

We have $T_2 \leq f(M)(\log M)^{7/4}/n$ where $f(M) = 8M/3(\log M)^{-7/4} \exp(-3/4(\log M)^{7/4}) \leq 6$ for any integer $M \geq 1$. Moreover $\kappa/(2\kappa-1) \leq 1$ for all $1 \leq \kappa < +\infty$, so we get $T_2 \leq 6n^{-\frac{\kappa}{2\kappa-1}} (\log M)^{7/4}$ for any integers $n \geq 1$ and $M \geq 2$.

Let B be a positive number. The inequality $T_1 \leq Bn^{-\frac{\kappa}{2\kappa-1}} (\log M)^{7/4}$ is equivalent to

$$2(2\kappa-1) \left[\frac{3}{4c} (\log M)^{\frac{7\kappa-4}{2\kappa}} - \log M + \frac{7\kappa-2}{2\kappa} \log(\log M) \right] \geq \log \left((4c/B)^{2(2\kappa-1)} n \right).$$

Since we have $\frac{7\kappa-4}{2\kappa} \geq \frac{3}{2} > 1$ for all $1 \leq \kappa < +\infty$ and $M \geq an^b$ for some positive numbers a and b , there exists a constant B which depends only on a, b and c (remark that we can take B free from κ) such that, $T_1 \leq Bn^{-\frac{\kappa}{2\kappa-1}} (\log M)^{7/4}$.

Proof of Theorem 1. Observe that $(1-x)_+ = 1-x$ for $x \leq 1$. Since $Y_i \tilde{f}_n(X_i) \leq 1$ and $Y_i f_j(X_i) \leq 1$ for all $i = 1, \dots, n$ and $j = 1, \dots, M$, we have $A_n(\tilde{f}_n) = \sum_{j=1}^M w_j A_n(f_j)$ (remark that, if we do not assume that f_j 's take their values in $\{-1, 1\}$, we have $A_n(\tilde{f}_n) \leq \sum_{j=1}^M w_j A_n(f_j)$ (using the convexity of the hinge loss) which is enough for the proof). Denote by j_0 an index such that $A(f_{j_0}) = \min_{j=1, \dots, M} A(f_j)$. We have $A_n(f_j) = A_n(f_{j_0}) + \frac{1}{n} (\log(w_{j_0}) - \log(w_j))$, for all $j = 1, \dots, M$, and by multiplying the last equation by w_j and summing up over j , we get

$$A_n(\tilde{f}_n) \leq A_n(f_{j_0}) + \frac{\log M}{n}. \quad (21)$$

Since $\log(w_{j_0}) \leq 0$ and $\sum_{j=1}^M w_j \log\left(\frac{w_j}{1/M}\right) = K(w|u) \geq 0$ where $K(w|u)$ denotes the Kullback-Leiber divergence between the weights $w = (w_j)_{j=1, \dots, M}$ and uniform weights $u = (1/M)_{j=1, \dots, M}$. Let $\gamma = (\log M)^{-1/4}$. Using (21), we have

$$\begin{aligned} & \mathbb{E} \left[\left(A(\tilde{f}_n) - A^* \right) (1 - \gamma) \right] - (A(f_{j_0}) - A^*) \\ &= \mathbb{E} \left[\left(A(\tilde{f}_n) - A^* \right) (1 - \gamma) - \left(A_n(\tilde{f}_n) - A_n(f^*) \right) \right] + \mathbb{E} \left[A_n(\tilde{f}_n) - A_n(f_{j_0}) \right] \\ &\leq \mathbb{E} \left[\left(A(\tilde{f}_n) - A^* \right) (1 - \gamma) - \left(A_n(\tilde{f}_n) - A_n(f^*) \right) \right] + \frac{\log M}{n}. \end{aligned}$$

Define $W_n = \left(A(\tilde{f}_n) - A^* \right) (1 - \gamma) - \left(A_n(\tilde{f}_n) - A_n(f^*) \right)$ where f^* is the Bayes classifier. We have

$$(1 - \gamma) \left(\mathbb{E} \left[A(\tilde{f}_n) \right] - A^* \right) \leq \min_{j=1, \dots, M} (A(f_j) - A^*) + \mathbb{E} [W_n] + \frac{\log M}{n}. \quad (22)$$

According to Proposition 1, $\mathbb{E}[W_n] \leq Cn^{-\frac{\kappa}{2\kappa-1}}(\log M)^{7/4}$ where $C > 0$ is a constant depending only of a, b and c . Using (22) and the fact that $(1 - \gamma)^{-1} \leq 1 + 2\gamma$ for all $0 < \gamma < 1/2$, we get

$$\mathbb{E} \left[A(\tilde{f}_n) - A^* \right] \leq \left(1 + 2 \log^{-1/4}(M) \right) \left\{ \min_{j=1, \dots, M} (A(f_j) - A^*) + Cn^{-\frac{\kappa}{2\kappa-1}} \log^{7/4}(M) \right\}.$$

Now, using the inequality (3) of Zhang (2004) and the equality $2(R(f) - R^*) = A(f) - A^*$, which holds for all $f : \mathcal{X} \mapsto \{-1, 1\}$, we can get the assertion of Theorem 1.

Proof of Theorem 2. Since \bar{f}_n and \tilde{f}_k 's take there values in $[-1, 1]$ and $l : x \mapsto (1 - x)_+$ is linear on $[-1, 1]$, we obtain

$$A(\bar{f}_n) - A^* = \frac{1}{n} \sum_{k=1}^n \left(A(\tilde{f}_k) - A^* \right).$$

Let $k \in \{1, \dots, n\}$. Since $M \geq an^b \geq ak^b$, we have, according to Theorem 1,

$$\mathbb{E} \left[A(\tilde{f}_k) - A^* \right] \leq \left(1 + \frac{2}{\log^{1/4}(M)} \right) \left\{ \min_{j=1, \dots, M} (A(f_j) - A^*) + Ck^{-\frac{\kappa}{2\kappa-1}} \log^{7/4}(M) \right\}.$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[A(\bar{f}_n) - A^* \right] \\ & \leq \left(1 + \frac{2}{\log^{1/4}(M)} \right) \left\{ \min_{j=1, \dots, M} (A(f_j) - A^*) + C \left(\frac{1}{n} \sum_{k=1}^n k^{-\frac{\kappa}{2\kappa-1}} \right) \log^{7/4}(M) \right\}. \end{aligned}$$

Moreover, $(1/n) \sum_{k=1}^n k^{-\kappa/(2\kappa-1)} \leq \gamma(n, \kappa)$, thus, we can state the result by using the inequality (3) of Zhang (2004) and the equality $2(R(f) - R^*) = A(f) - A^*$, which holds for all $f : \mathcal{X} \mapsto \{-1, 1\}$.

Proof of Theorem 3. Let $\rho_{min} \leq \rho \leq \rho_{max}$ and $\kappa \geq 1$. Let $\rho_{m,j_0} = \min(\rho_{m,j} : \rho_{m,j} \geq \rho)$. Since $N(m) \geq A'_0 m^{b'} \geq Cl^{b'}$, where $C > 0$, using the oracle inequality, stated in Theorem 1, we have, for π satisfying (MA1)(κ),

$$\begin{aligned} & \mathbb{E} \left[R(\tilde{f}_n) - R^* | D_m^1 \right] \\ & \leq \left(1 + 2 \log^{-1/4} N(m) \right) \left\{ 2 \min_{j=1, \dots, N(m)} \left(R(\hat{f}_m^j) - R^* \right) + Cl^{-\frac{\kappa}{2\kappa-1}} \log^{7/4} N(m) \right\}, \end{aligned}$$

where C is a positive number depending only on b', a, A'_0 and c . Taking the expectation with respect to the subsample D_m^1 we have

$$\mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq \left(1 + 2 \log^{-1/4} N(m) \right) \left\{ 2\mathbb{E} \left[R(\hat{f}_m^{j_0}) - R^* \right] + Cl^{-\frac{\kappa}{2\kappa-1}} \log^{7/4} N(m) \right\}.$$

It follows from Tsybakov (2004) that, the excess risk of $\hat{f}_m^{j_0}$ satisfies

$$\sup_{\pi \in \mathcal{P}_{\kappa, \rho_{j_0}}} \mathbb{E} \left[R(\hat{f}_m^{j_0}) - R^* \right] \leq Cm^{-\frac{\kappa}{2\kappa+\rho_{j_0}-1}},$$

where C is a positive number depending only on A, c, κ, ρ_{min} and ρ_{max} (note that C does not depend on ρ_{j_0}).

Moreover we have $m \geq n(1 - a/\log 3 - 1/3)$, $N(m) \leq A_0 m^b \leq A_0 n^b$ and $l \geq an/\log n$, so that there exists a constant C depending only on $a, A_0, A'_0, b, b', \kappa, \rho_{min}$ and ρ_{max} such that

$$\sup_{\pi \in \mathcal{P}_{\kappa, \rho_{j_0}}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C \left\{ n^{-\frac{\kappa}{2\kappa+\rho_{j_0}-1}} + n^{-\frac{\kappa}{2\kappa-1}} (\log n)^{11/4} \right\}.$$

Since $\rho_{j_0} \leq \rho + N(m)^{-1} \leq \rho + (A'_0)^{-1} [n(1 - a/\log 3 - 1/3)]^{-b'}$ there exist a constant C depending only on $a, A'_0, b', \kappa, \rho_{min}$ and ρ_{max} such that for all integers n , $n^{-\frac{\kappa}{2\kappa+\rho_{j_0}-1}} \leq Cn^{-\frac{\kappa}{2\kappa+\rho-1}}$. Thus, we have

$$\sup_{\pi \in \mathcal{P}_{\kappa, \rho_{j_0}}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C \left\{ n^{-\frac{\kappa}{2\kappa+\rho-1}} + n^{-\frac{\kappa}{2\kappa-1}} (\log n)^{11/4} \right\}.$$

Theorem 2 follows directly from the last assertion seeing that $\rho \geq \rho_{min} > 0$ and $\mathcal{P}_{\kappa, \rho} \subseteq \mathcal{P}_{\kappa, \rho_{j_0}}$ since $\rho_{j_0} \geq \rho$.

Proof of Theorem 4. Define $0 < \alpha_{min} < \alpha_{max} < +\infty$ and $0 < \beta_{min} < 1$ such that $K \subset [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, 1]$. Let $(\alpha_0, \beta_0) \in K$. We consider the function on $(0, +\infty) \times (0, 1]$ with values in $(1/2, 2)$, $\phi(\alpha, \beta) = 4(\alpha + 1)/((2\alpha + p\alpha + 4)(1 + \beta))$. We take $k_0 \in \{0, \dots, \lfloor 3\Delta/2 \rfloor - 1\}$ such that

$$\phi_{l, k_0} = 1/2 + k_0\Delta^{-1} \leq \phi(\alpha_0, \beta_0) < 1/2 + (k_0 + 1)\Delta^{-1}.$$

For n greater than a constant depending only on K, p, b_0 and a there exists $\bar{\alpha}_0 \in [\alpha_{min}/2, \alpha_{max}]$ such that $\phi(\bar{\alpha}_0, \beta_0) = \phi_{l, k_0}$. Since $\alpha \mapsto \phi(\alpha, \beta_0)$ increases on \mathbb{R}^+ , we have $\bar{\alpha}_0 \leq \alpha_0$. Moreover, we have $|\phi(\alpha_1, \beta_0) - \phi(\alpha_2, \beta_0)| \geq A|\alpha_1 - \alpha_2|$, $\forall \alpha_1, \alpha_2 \in$

$[\alpha_{min}/2, \alpha_{max}]$, where $A > 0$ depends only on p and α_{max} . Thus $|\bar{\alpha}_0 - \alpha_0| \leq A^{-1}\Delta^{-1}$. We have $\bar{\alpha}_0 \leq \alpha_0$ so $\mathcal{Q}_{\alpha_0, \beta_0} \subseteq \mathcal{Q}_{\bar{\alpha}_0, \beta_0}$, hence

$$\sup_{\pi \in \mathcal{Q}_{\alpha_0, \beta_0}} \mathbb{E}[R(\tilde{f}_n) - R^*] \leq \sup_{\pi \in \mathcal{Q}_{\bar{\alpha}_0, \beta_0}} \mathbb{E}[R(\tilde{f}_n) - R^*].$$

Since $\lceil 3\Delta/2 \rceil \geq (3/2)l^{b_0}$, for π satisfying the margin assumption (MA2)($\bar{\alpha}_0$) Theorem 1 leads to

$$\begin{aligned} & \mathbb{E} \left[R(\tilde{f}_n) - R^* \mid D_m^1 \right] \\ & \leq \left(1 + 2 \log^{-1/4}(\lceil 3\Delta/2 \rceil) \right) \left\{ 2 \min_{\lambda \in \mathcal{G}(l)} \left(R(\hat{f}_m^\lambda) - R^* \right) + C_0 l^{-\frac{\bar{\alpha}_0+1}{\bar{\alpha}_0+2}} \log^{7/4}(\lceil 3\Delta/2 \rceil) \right\}, \end{aligned}$$

for all integers $n \geq 1$, where $C_0 > 0$ depends only on K, a and b_0 . Therefore, taking the expectation w.r.t. the subsample D_m^1 we get

$$\mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C_1 \left(\mathbb{E} \left[R(\hat{f}_m^{\lambda_{l, k_0}}) - R^* \right] + l^{-\frac{\bar{\alpha}_0+1}{\bar{\alpha}_0+2}} \log^{7/4}(n) \right),$$

where $\lambda_{l, k_0} = l^{-\phi_{l, k_0}}$ and $C_1 > 0$ depends only on K, a and b_0 .

Set $\Gamma : (0, +\infty) \times (0, 1] \mapsto \mathbb{R}^+$ defined by $\Gamma(\alpha, \beta) = \beta\phi(\alpha, \beta), \forall (\alpha, \beta) \in (0, +\infty) \times (0, 1]$. According to Scovel and Steinwart (2005), if $\pi \in \mathcal{Q}_{\bar{\alpha}_0, \beta_0}$ then for all $\epsilon > 0$

$$\mathbb{E} \left[R(\hat{f}_m^{\lambda_{l, k_0}}) - R^* \right] \leq C m^{-\Gamma(\bar{\alpha}_0, \beta_0) + \epsilon},$$

where $C > 0$ is a constant depending only on K, p and ϵ . Remark that C does not depend on $\bar{\alpha}_0$ and β_0 since $(\bar{\alpha}_0, \beta_0) \in [\alpha_{min}/2, \alpha_{max}] \times [\beta_{min}, 1]$ and that the constant multiplying the rate of convergence, stated in Scovel and Steinwart (2005), is uniformly bounded over (α, β) belonging to a compact subset of $(0, +\infty) \times (0, 1]$.

Let $\epsilon > 0$. Assume that $\pi \in \mathcal{Q}_{\alpha_0, \beta_0}$. We have $n(1 - a/\log 3 - 1/3) \leq m \leq n$, $l \geq an/\log n$ and $\Gamma(\bar{\alpha}_0, \beta_0) \leq (\bar{\alpha}_0 + 1)/(\bar{\alpha}_0 + 2) \leq 1$, therefore, there exists $C_2, C'_2 > 0$ depending only on a, b_0, K, p and ϵ such that for any n greater than a constant depending only on β_{min}, a and b_0

$$\mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C_2 \left(n^{-\Gamma(\bar{\alpha}_0, \beta_0) + \epsilon} + n^{-\frac{\bar{\alpha}_0+1}{\bar{\alpha}_0+2}} (\log n)^{11/4} \right) \leq C'_2 n^{-\Gamma(\bar{\alpha}_0, \beta_0) + \epsilon}.$$

Moreover, Γ satisfies $|\Gamma(\bar{\alpha}_0, \beta_0) - \Gamma(\alpha_0, \beta_0)| \leq B\Delta^{-1}$, where B depends only on p and α_{min} , and $\left(n^{B\Delta^{-1}} \right)_{n \in \mathbb{N}}$ is a convergent sequence (recall that $\Delta = l^{b_0}$ (cf. subsection 3.2.1)). This achieves the proof.

Proof of Theorem 5. Let $(\alpha_0, \gamma_0) \in K \cup K'$. First assume that (α_0, γ_0) belongs to $K \subset \mathcal{U}$. We consider the set

$$\mathcal{S} = \{(\varphi, \psi) \in (0, 1/2) \times (1/2, 1) : 2 - 2\psi - \varphi > 0\}.$$

Each point of \mathcal{S} is associated to a margin parameter (14) and to a geometric noise exponent by the following functions on \mathcal{S} with values in $(0, +\infty)$,

$$\bar{\alpha}(\varphi, \psi) = \frac{4\psi - 2}{2 - 2\psi - \varphi} \text{ and } \bar{\gamma}(\varphi, \psi) = \frac{\psi}{\varphi} - 1.$$

We loose for $(\varphi, \psi) \in \mathcal{S} \cap \mathcal{M}(l)$ such that $\bar{\alpha}(\varphi, \psi) \leq \alpha_0$, $\bar{\gamma}(\varphi, \psi) \leq \gamma_0$, $\bar{\alpha}(\varphi, \psi)$ is close enough to α_0 , $\bar{\gamma}(\varphi, \psi)$ is close enough to γ_0 and $\bar{\gamma}(\varphi, \psi) > (\bar{\alpha}(\varphi, \psi) + 2)/(2\bar{\alpha}(\varphi, \psi))$. Since $\gamma_0 > (\alpha_0 + 2)/(2\alpha_0)$ there exists a solution $(\varphi_0, \psi_0) \in \mathcal{S}$ of the system of equations

$$\begin{cases} \bar{\alpha}(\varphi, \psi) = \alpha_0 \\ \bar{\gamma}(\varphi, \psi) = \gamma_0 \end{cases}. \quad (23)$$

For all integers n greater than a constant depending only on K, a and b_0 , there exists $(p_{1,0}, p_{2,0}) \in \{1, \dots, 2\lfloor \Delta \rfloor\} \times \{2, \dots, \lfloor \Delta/2 \rfloor\}$ defined by

$$\varphi_{l,p_{1,0}} = \min(\varphi_{l,p} : \varphi_{l,p} \geq \varphi_0) \text{ and } \psi_{l,p_{2,0}} = \max(\psi_{l,p_2} : \psi_{l,p_2} \leq \psi_0) - \Delta^{-1}.$$

We have $2 - 2\psi_{l,p_{2,0}} - \varphi_{l,p_{1,0}} > 0$. Therefore $(\varphi_{l,p_{1,0}}, \psi_{l,p_{2,0}}) \in \mathcal{S} \cap \mathcal{M}(l)$. Define $\bar{\alpha}_0 = \bar{\alpha}(\varphi_{l,p_{1,0}}, \psi_{l,p_{2,0}})$ and $\bar{\gamma}_0 = \bar{\gamma}(\varphi_{l,p_{1,0}}, \psi_{l,p_{2,0}})$. Since (φ_0, ψ_0) satisfies (23), we have

$$\psi_{l,p_{2,0}} + \Delta^{-1} \leq \psi_0 = -\frac{\alpha_0}{2\alpha_0 + 4}\varphi_0 + \frac{1 + \alpha_0}{2 + \alpha_0} \leq -\frac{\alpha_0}{2\alpha_0 + 4}(\varphi_{l,p_{1,0}} - (2\Delta)^{-1}) + \frac{1 + \alpha_0}{2 + \alpha_0}$$

and $(\alpha_0/(2\alpha_0 + 4))(2\Delta)^{-1} \leq \Delta^{-1}$, thus

$$\psi_{l,p_{2,0}} \leq -\frac{\alpha_0}{2\alpha_0 + 4}\varphi_{l,p_{1,0}} + \frac{1 + \alpha_0}{2 + \alpha_0} \text{ so } \bar{\alpha}_0 \leq \alpha_0.$$

With a similar argument, we have $\psi_{l,p_{2,0}} \leq (\alpha_0 + 1)\varphi_{l,p_{1,0}}$, that is $\bar{\gamma}_0 \leq \gamma_0$. Now we show that $\bar{\gamma}_0 > (\bar{\alpha}_0 + 2)/(2\bar{\alpha}_0)$. Since (α_0, γ_0) belongs to a compact, (φ_0, ψ_0) and $(\varphi_{l,p_{1,0}}, \psi_{l,p_{2,0}})$ belong to a compact subset of $(0, 1/2) \times (1/2, 1)$ for n greater than a constant depending only on K, a, b_0 . Thus, there exists a constant, A , depending only on K , such that for n large enough, we have

$$|\alpha_0 - \bar{\alpha}_0| \leq A\Delta^{-1} \text{ and } |\gamma_0 - \bar{\gamma}_0| \leq A\Delta^{-1}.$$

Denote by $d_K = d(\partial\mathcal{U}, K)$, where $\partial\mathcal{U}$ is the boundary of \mathcal{U} and $d(A, B)$ denotes the Euclidean distance between sets A and B . We have $d_K > 0$ since K is a compact, $\partial\mathcal{U}$ is closed and $K \cap \partial\mathcal{U} = \emptyset$. Set $0 < \alpha_{min} < \alpha_{max} < +\infty$ and $0 < \gamma_{min} < \gamma_{max} < +\infty$ such that $K \subset [\alpha_{min}, \alpha_{max}] \times [\gamma_{min}, \gamma_{max}]$. Define $\mu = \min(\alpha_{min}/2, d_K)$ and $\mathcal{U}_\mu = \{(\alpha, \gamma) \in (0, +\infty)^2 : \alpha \geq 2\mu \text{ and } \gamma > (\alpha - \mu + 2)/(2(\alpha - \mu))\}$. We have $K \subset \mathcal{U}_\mu$ so $\gamma_0 > (\alpha_0 - \mu + 2)/(2(\alpha_0 - \mu))$. Since $(\alpha \mapsto (\alpha + 2)/(2\alpha))$ is decreasing, $\bar{\gamma}_0 > \gamma_0 - A\Delta^{-1}$ and $\alpha_0 \leq \bar{\alpha}_0 + A\Delta^{-1}$, we have $\bar{\gamma}_0 > \bar{\beta}(\bar{\alpha}_0) - A\Delta^{-1}$ where $\bar{\beta}$ is a function on $(0, 2\alpha_{max}]$ with values in $(0, +\infty)$ defined by $\bar{\beta}(\alpha) = (\alpha - (\mu - A\Delta^{-1}) + 2)/(2(\alpha - (\mu - A\Delta^{-1})))$. We have $|\bar{\beta}(\alpha_1) - \bar{\beta}(\alpha_2)| \geq (2\alpha_{max})^{-2}|\alpha_1 - \alpha_2|$ for all $\alpha_1, \alpha_2 \in (0, 2\alpha_{max}]$. Therefore

$$\bar{\beta}(\bar{\alpha}_0) - A\Delta^{-1} \geq \bar{\beta}(\bar{\alpha}_0 + 4A\alpha_{max}^2\Delta^{-1}).$$

Thus, for n greater than a constant depending only on K, a and b_0 we have $\bar{\gamma}_0 > (\bar{\alpha}_0 + 2)/(2\bar{\alpha}_0)$.

Since $\bar{\alpha}_0 \leq \alpha_0$ and $\bar{\gamma}_0 \leq \gamma_0$, we have $\mathcal{R}_{\alpha_0, \gamma_0} \subset \mathcal{R}_{\bar{\alpha}_0, \bar{\gamma}_0}$ and

$$\sup_{\pi \in \mathcal{R}_{\alpha_0, \gamma_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq \sup_{\pi \in \mathcal{R}_{\bar{\alpha}_0, \bar{\gamma}_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right].$$

If the underlying probability measure π satisfies assumption (MA2)($\bar{\alpha}_0$) we get from Theorem 1

$$\begin{aligned} & \mathbb{E} \left[R(\tilde{f}_n) - R^* | D_m^1 \right] \\ & \leq \left(1 + \frac{2}{\log^{1/4}(M(l))} \right) \left\{ 2 \min_{(\sigma, \lambda) \in \mathcal{N}(l)} \left(R(\hat{f}_m^{(\sigma, \lambda)}) - R^* \right) + C_2 l^{-\frac{\bar{\alpha}_0 + 1}{\bar{\alpha}_0 + 2}} \log^{7/4}(M(l)) \right\}, \end{aligned} \quad (24)$$

for all integers $n \geq 1$, where $C_2 > 0$ depends only on K, a and b_0 and $M(l)$ is the cardinality of $\mathcal{N}(m)$. Remark that $M(l) \geq l^{2b_0}/2$, so we can apply Theorem 1.

Let $\epsilon > 0$. Since $M(l) \leq n^{2b_0}$ and $\bar{\gamma}_0 > (\bar{\alpha}_0 + 2)/(2\bar{\alpha}_0)$, taking expectations in (24) and using the result (18) of Scovel and Steinwart (2004), for $\sigma = \sigma_{l, \varphi_{l, p_{1,0}}}$ and $\lambda = \lambda_{l, \psi_{l, p_{2,0}}}$, we obtain

$$\sup_{\pi \in \mathcal{R}_{\bar{\alpha}_0, \bar{\gamma}_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C \left(m^{-\Theta(\bar{\alpha}_0, \bar{\gamma}_0) + \epsilon} + l^{-\frac{\bar{\alpha}_0 + 1}{\bar{\alpha}_0 + 2}} \log^{7/4}(n) \right),$$

where $\Theta : \mathcal{U} \mapsto \mathbb{R}$ is defined, for all $(\alpha, \gamma) \in \mathcal{U}$, by $\Theta(\alpha, \gamma) = (2\gamma(\alpha + 1))/(2\gamma(\alpha + 2) + 3\alpha + 4)$ and $C > 0$ depends only on a, b_0, K and ϵ . Remark that, as in Scovel and

Steinwart (2005), the constant before the rate of convergence in (18) is uniformly bounded on every compact of \mathcal{U} . We have $\Theta(\bar{\alpha}_0, \bar{\gamma}_0) \leq \Theta(\alpha_0, \gamma_0) \leq \Theta(\bar{\alpha}_0, \bar{\gamma}_0) + 2A\Delta^{-1}$, $m \geq n(1 - a/\log 3 - 1/3)$ and $\left(m^{2A\Delta^{-1}}\right)_{n \in \mathbb{N}}$ is a convergent sequence, so there exists $C_1 > 0$ depending only on K, a, b_0 such that

$$m^{-\Theta(\bar{\alpha}_0, \bar{\gamma}_0)} \leq C_1 n^{-\Theta(\alpha_0, \gamma_0)}, \quad \forall n \geq 1.$$

Similar argument as at the end of the proof of Theorem 4 and the fact that $\Theta(\alpha, \gamma) < (\alpha + 1)/(\alpha + 2)$ for all $(\alpha, \gamma) \in \mathcal{U}$, leads to the result of the first part of Theorem 5.

Let now $(\alpha_0, \gamma_0) \in K'$. Let $\alpha'_{max} > 0$ be such that $\forall (\alpha, \gamma) \in K', \alpha \leq \alpha'_{max}$. Take $p_{1,0} \in \{1, \dots, 2\lfloor \Delta \rfloor\}$ such that $\varphi_{l, p_{1,0}} = \min(\varphi_{l, p} : \varphi_{l, p} \geq (2\gamma_0 + 1)^{-1} \text{ and } p \in 4\mathbb{N})$, where $4\mathbb{N}$ is the set of all integers multiple of 4. For large values of n , $p_{1,0}$ exists and $p_{1,0} \in 4\mathbb{N}$. We denote by $\bar{\gamma}_0 \in (0, +\infty)$ such that $\varphi_{l, p_{1,0}} = (2\bar{\gamma}_0 + 1)^{-1}$, we have $\bar{\gamma}_0 \leq \gamma_0$ thus $\mathcal{R}_{\alpha_0, \gamma_0} \subseteq \mathcal{R}_{\alpha_0, \bar{\gamma}_0}$ and

$$\sup_{\pi \in \mathcal{R}_{\alpha_0, \gamma_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq \sup_{\pi \in \mathcal{R}_{\alpha_0, \bar{\gamma}_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right].$$

If the underlying probability measure π satisfies the margin assumption (14) with the margin parameter α_0 then, using Theorem 1, we obtain

$$\begin{aligned} & \mathbb{E} \left[R(\tilde{f}_n) - R^* | D_m^1 \right] \\ & \leq \left(1 + \frac{2}{\log^{1/4}(M(l))} \right) \left\{ 2 \min_{(\sigma, \lambda) \in \mathcal{N}(l)} \left(R(\hat{f}_m^{(\sigma, \lambda)}) - R^* \right) + Cl^{-\frac{\alpha_0 + 1}{\alpha_0 + 2}} \log^{7/4}(M(l)) \right\}, \end{aligned} \quad (25)$$

for all integers $n \geq 1$, where $C > 0$ depends only on a and b_0 , and $M(l)$ is the cardinality of $\mathcal{N}(l)$.

Let $\epsilon > 0$ and $p_{2,0} \in \{1, \dots, \lfloor \Delta/2 \rfloor\}$ defined by $p_{2,0} = p_{1,0}/4$ (note that $p_{1,0} \in 4\mathbb{N}$). We have

$$\sigma_{l, \varphi_{l, p_{1,0}}} = \left(\lambda_{l, \psi_{l, p_{2,0}}} \right)^{-\frac{1}{d_0(\bar{\gamma}_0 + 1)}}.$$

Since $\bar{\gamma}_0 \leq (\alpha_0 + 2)/(2\alpha_0)$, using the result (18) of Scovel and Steinwart (2004) we have, for $\sigma = \sigma_{l, \varphi_{l, p_{1,0}}}$ and $\lambda = \lambda_{l, \psi_{l, p_{2,0}}}$,

$$\mathbb{E} \left[R(\hat{f}_m^{(\sigma, \lambda)}) - R^* \right] \leq Cm^{-\bar{\Gamma}(\bar{\gamma}_0) + \epsilon},$$

where $\bar{\Gamma} : (0, +\infty) \mapsto \mathbb{R}$ is the function defined by $\bar{\Gamma}(\gamma) = \gamma/(2\gamma + 1)$ for all $\gamma \in (0, +\infty)$ and $C > 0$ depends only on a, b_0, K' and ϵ . Remark that, as in the first part of the proof, we can uniformly bound the constant before the rate of convergence in (18) on every compact subset of \mathcal{U}' . Since $M(l) \leq n^{2b_0}$, taking the expectation, in (25), we find

$$\sup_{\pi \in \mathcal{R}_{\alpha_0, \bar{\gamma}_0}} \mathbb{E} \left[R(\tilde{f}_n) - R^* \right] \leq C \left(m^{-\Gamma(\bar{\gamma}_0) + \epsilon} + l^{-\frac{\alpha_0 + 1}{\alpha_0 + 2}} \log^{7/4}(n) \right),$$

where $C > 0$ depends only on a, b_0, K' and ϵ . Moreover $|\gamma_0 - \bar{\gamma}_0| \leq 2(2\alpha'_{max} + 1)^2 \Delta^{-1}$ so $|\bar{\Gamma}(\bar{\gamma}_0) - \bar{\Gamma}(\gamma_0)| \leq 2(2\alpha_{max} + 1)\Delta^{-1}$. To achieve the proof we use same argument as for the first part of the proof.

Acknowledgement

I want to thank my advisor, Alexandre Tsybakov, for spending many hours to help me for this work, giving me many ideas, advices and a lot of interesting papers.

References

- [1] Audibert, J.-Y. and Tsybakov, A.B. (2005). Fast learning rates for plug-in classifiers under the margin condition. Available at <http://www.proba.jussieu.fr/mathdoc/preprints/index.html#2005> (Preprint PMA-998).
- [2] Augustin, N.H. , Buckland, S.T. and Burnham, K.P. (1997). Model selection: An integral part of inference. *Biometrics*, Vol. 53, pp. 603-618.
- [3] Barron, A. and Leung, G. (2004). Information Theory and Mixing Least-Squares Regressions. Manuscript.
- [4] Bartlett, P.L., Freund, Y., Lee, W.S. and Schapire, R.E. (1998). Boosting the margin: a new explanation for the effectiveness of voting methods. *Ann. Statist.* **26** 1651-1686.
- [5] Bartlett, P. , Jordan, M. and McAuliffe, J. (2003). Convexity, Classification and Risk Bounds. Available at <http://stat-www.berkeley.edu/tech-reports/638.pdf>
- [6] Birgé, L. (2004). Model selection via testing: an alternative to (penalized) maximum likelihood estimators. Available at <http://www.proba.jussieu.fr/mathdoc/textes/PMA-862.pdf>. To appear in *Annales of IHP*.

- [7] Blanchard, G. , Bousquet, O. , and Massart, P. (2004). Statistical Performance of Support Vector Machines. Available at <http://mahery.math.u-psud.fr/~blanchard/publi/>.
- [8] Blanchard, G. , Lugosi, G. , and Vayatis, N. (2003). On the rate of convergence of regularized boosting classifiers. *JMLR* 4(Oct): 861-894.
- [9] Bühlmann, P. and Yu, B., Analyzing bagging. (2002). *Ann. Statist.* **30** , No. 4, 927- 961.
- [10] Bunea, F. , Tsybakov, A.B. and Wegkamp, M. (2004). Aggregation for regression learning. Preprint n.948, Laboratoire de Probabilités et Modèles aléatoires, Universités Paris 6 and Paris 7 (available at <http://www.proba.jussieu.fr/mathdoc/preprints>).
- [11] Catoni, O. (2001). Statistical learning theory and stochastic optimization. Lecture notes from Saint-Flour Probability Summer School XXXI-2001. Series: Lecture Notes in Mathematics, Vol. 1851.
- [12] Cristianini, N. and Shawe-Taylor, J. (2002). *An introduction to Support Vector Machines*. Cambridge University Press.
- [13] Devroye, L. , Györfi, L. and Lugosi, G. (1996). A Probabilistic Theory of Pattern Recognition. *Springer, New York, Berlin, Heidelberg*.
- [14] Friedman, J. , Hastie, T. and Tibshirani, R. (2000). Additive logistic regression: a statistical view of boosting. *Ann. Statist.*, **28**, 337-407.
- [15] Hartigan, J.A. (2002). Bayesian regression using Akaike priors. *Yale University, New Haven*. Preprint.
- [16] Juditsky, A., Nazin, A., Tsybakov, A.B. and Vayatis, N. (2005). Online aggregation with Mirror-Descent algorithm. Preprint n.987, Laboratoire de Probabilités et Modèles aléatoires, Universités Paris 6 and Paris 7 (available at <http://www.proba.jussieu.fr/mathdoc/preprints/index.html#2005>).
- [17] Koltchinskii, V. (2001). Rademacher penalties and structural risk minimization. *IEEE Trans. Inform. Theory* **47**, 1902-1914.
- [18] Koltchinskii, V. (2005). Local Rademacher Complexities and Oracle Inequalities in Risk Minimization. To appear in *Ann. Statist.*
- [19] Koltchinskii, V. and Panchenko, D. (2002). Empirical margin distributions and bounding the generalization error of combined classifiers. *Ann. Statist.* **30**, 1-50.
- [20] Koltchinskii, V. and Panchenko, D. (2000). Rademacher penalties and bounding the risk of function learning. High dimension probability, II (Seattle, WA, 1999), 443-457, Progr. Probab., 47, Birkhäuser Boston, MA.

- [21] Lugosi, G. and Vayatis, N. (2004). On the Bayes-risk consistency of regularized boosting methods. *Ann. Statist.* **32**, No. 1, 30-55.
- [22] Lugosi, G. and Wegkamp, M. (2004). Complexity regularization via localized random penalties. *Ann. Statist.* **32**, No. 4, 1679-1697
- [23] Mammen, E. and Tsybakov, A.B. (1999). Smooth discrimination analysis. *Ann. Statist.* **27**, 1808 - 1829.
- [24] Massart, P. (2000). Some applications of concentration inequalities to Statistics. Probability Theory. *Annales de la Faculté des Sciences de Toulouse n2, 245-303* (volume spécial dédié à Michel Talagrand).
- [25] Massart, P. (2004). Concentration inequalities and Model Selection. *Lectures notes of Saint Flour*.
- [26] Massart, P. and Nédélec, E. (2003). Risk Bound for Statistical Learning. Preprint. <http://www.math.u-psud.fr/~massart/page5.html>
- [27] Samarov, A. and Tsybakov, A.B. (2005). Aggregation of density estimators and dimension reduction. To appear in "Festschrift in honor of Kjell Doksum".
- [28] Schölkopf, B. and Smola, A. (2002). *Learning with kernels*. MIT Press, Cambridge.
- [29] Scovel, C. and Steinwart, I. (2004). Fast Rates for Support Vector Machines using Gaussian Kernels, Los Alamos National Laboratory Technical Report LA-UR 04-8796, submitted to Annals of Statistics.
- [30] Scovel, C. and Steinwart, I. (2005). Fast Rates for Support Vector Machines, Los Alamos National Laboratory Technical Report LA-UR 05-0451, submitted to COLT.
- [31] Tarigan, B. and van de Geer, S.A. (2004). Adaptivity of Support Vector Machines with l_1 Penalty. PASCAL. Technical Report Volume MI 2004-14, 2004.
- [32] Tsybakov, A.B. (2004). Optimal aggregation of classifiers in statistical learning. *Ann. Statist.* **32** No. 1, 135-166.
- [33] Tsybakov, A.B. and van de Geer S.A. (2005). Square root penalty: adaptation to the margin in classification and in edge estimation. *Ann. Statist.* **33**, 1203 - 1224.
- [34] van de Geer, S. (2000). *Applications of Empirical Process Theory*. Cambridge Univ. Press.
- [35] Yang, Y. (2000). Combining different regression procedures for adaptive regression. *Journal of multivariate Analysis* Vol. 74, pp. 135-161

- [36] Yang, Y. (2000). Mixing strategies for density estimation. *Ann. Statist.* **28**, No. 1, 75-87.
- [37] Zhang, T. (2004). Statistical behavior and consistency of classification methods based on convex risk minimization. *Ann. Statist.* **32** , No. 1, 56-85.

GUILLAUME LECUÉ
LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES
UNIVERSITÉ PARIS 6
4, PL.JUSSIEU, BP 188, 75252 PARIS, FRANCE
email: lecue@ccr.jussieu.fr