

Non homogeneous Heat Equation: Identification and Regularization for the Inhomogeneous Term

by

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Keywords: Identification, Regularization, error estimations.

AMS Classifications: 35K05, 45D05, 65J20

Abstract: *We study the nonhomogeneous heat equation under the form: $u_t - u_{xx} = \varphi(t)f(x)$, where the unknown is the pair of functions (u, f) . Under various assumptions about the the function φ and the final value in $t = 1$ i.e. $g(x)$, we propose different regularizations on this ill-posed problem based on the Fourier transform associated with a Lebesgue measure. For $\varphi \not\equiv 0$ the solution is unique.*

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I. Introduction

Consider the problem: Find a pair of functions (u, f) satisfying the following equation and boundary and initial values:

$$\begin{aligned} -\frac{\partial u}{\partial t} + \Delta u &= \varphi(t)f(x), \quad (t, x) \in (0, 1) \times (0, 1) \\ u(1, t) &= 0, \quad u_x(0, t) = u_x(1, t) = 0 \\ u(x, 0) &= 0, \quad u(x, 1) = g(x) \end{aligned} \tag{1}$$

Where φ and g are two given functions.

The previous problem is equivalent to find a function f satisfying an integral equation of the 1st. kind of Volterra type:

$$g(x) = - \int_0^1 \int_0^1 N(x, 1; \xi, \tau) \varphi(\tau) f(\xi) d\xi d\tau \quad (2)$$

where $N(x, t; \xi, \tau)$ [1] is defined by

$$N(x, t; \xi, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \left(\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) + \exp\left(-\frac{(x+\xi)^2}{4(t-\tau)}\right) \right)$$

As is well-known the problem (2) (or (1)) is an ill-posed problem and its numerical solution have been discussed by various authors ([2], [3], [4], [5]). The purpose of this paper is to produce regularized solutions of this problem treated in its form (1) with an error estimates under various hypotheses on the function $\varphi(t)$ and $g(x)$.

For $\varphi \not\equiv 0$ there is uniqueness of the pair (u, f) solution of (1) (paragraph 2). In paragraph 3 we give two sorts of regularization. In fact it will be shown that if the discrepancy between $\varphi(t)$ (respectively $g(x)$) and its exact solution $\varphi_0(t)$ (respectively $g_0(x)$) is of the order ε for the $\|\cdot\|_{L^2(0,1)}$, then the discrepancy between the regularized solution $f_\varepsilon(x)$ and the exact solution $f_0(x)$ is, depending on the degree of smoothness of the exact solution $f_0(x)$, of the order $\left(\ln \frac{1}{\varepsilon}\right)^{-1}$ or $\varepsilon^{1/8}$, $0 < \varepsilon < 1$. The techniques used here is the Fourier transform associated with the variational form of (1) and a Lebesgue measure generated by the function $\varphi_0(t)$. So the proposed regularization can be applied for an integral Volterra equation of the 1st. kind of the form (0.2) where the kernel $N(x, t; \xi, \tau)$ is a solution of the heat equation.

II. Uniqueness

Let g, φ be known functions in $L^2(0, 1)$. We consider the problem of identifying a pair (u, f) satisfying

$$\left. \begin{aligned} -\frac{d}{dt} \langle u, \psi \rangle - \langle u_x, \psi_x \rangle &= \varphi \langle f, \psi \rangle \quad \forall \psi \in H^1(0, 1), \\ u(1, t) &= 0; \quad u(x, 0) = 0; \quad u(x, 1) = g(x) \end{aligned} \right\} \quad (3)$$

where $u = u(x, t)$, $f = f(x)$, $(x, t) \in [0, 1] \times [0, 1]$, $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0, 1)$.

We first have

Lemma 1. *If $u \in C^1([0, 1]; L^2(0, 1)) \cap C([0, 1]; H^1(0, 1))$, $f \in L^2(0, 1)$ satisfy (3) then we have*

$$e^{\lambda^2} \int_0^1 g(x) \cos \lambda x dx = - \int_0^1 e^{\lambda^2 t} \varphi(t) dt \int_0^1 f(x) \cos \lambda x dx \quad \forall \lambda \in \mathbf{C}. \quad (4)$$

Proof

In (3)₁, by choosing $\psi(x) = \cos \lambda x$, we get

$$- \frac{d}{dt} \int_0^1 u(x, t) \cos \lambda x dx + \lambda \int_0^1 u_x(x, t) \sin \lambda x dx = \varphi(t) \int_0^1 f(x) \cos \lambda x dx. \quad (5)$$

In view of the condition $u(1, t) = 0$, we have

$$\begin{aligned} \int_0^1 u_x(x, t) \sin \lambda x dx &= u(x, t) \sin \lambda x \Big|_{x=0}^{x=1} - \lambda \int_0^1 u(x, t) \cos \lambda x dx \\ &= -\lambda \int_0^1 u(x, t) \cos \lambda x dx. \end{aligned}$$

Hence, (5) follows that

$$- \frac{d}{dt} \int_0^1 u(x, t) \cos \lambda x dx - \lambda^2 \int_0^1 u(x, t) \cos \lambda x dx = \varphi(t) \int_0^1 f(x) \cos \lambda x dx.$$

Integrating this equality from $t = 0$ to $t = 1$ and using the conditions $u(x, 0) = 0$; $u(x, 1) = g(x)$, we get (4). This completes the proof of Lemma 1.

Now, we consider the uniqueness of the solution of (3). We have

Theorem 1. *Let $u_i \in C^1([0, 1]; L^2(0, 1)) \cap C([0, 1]; H^1(0, 1))$, $f_i \in L^2(0, 1)$ ($i = 1, 2$) satisfy (3). If $\varphi \not\equiv 0$ then $(u_1, f_1) = (u_2, f_2)$.*

Proof

Put $v = u_1 - u_2$, $f = f_1 - f_2$ then v satisfy (3)₁ subject to conditions $v(1, t) = 0$; $v(x, 0) = v(x, 1) = 0$. Hence, from (4) one has

$$\int_0^1 e^{\lambda^2 t} \varphi(t) dt \int_0^1 f(x) \cos \lambda x dx = 0 \quad (6)$$

Put

$$\Phi(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} \int_0^1 \varphi(t) t^n dt, \quad F(\lambda) = \int_0^1 f(x) \cos \lambda x dx.$$

We claim that $\Phi \not\equiv 0$. In fact, if $\Phi \equiv 0$ then $\int_0^1 \varphi(t) t^n dt = 0$ for every $n = 0, 1, 2, \dots$. Using Weierstrass theorem, we have $\varphi \equiv 0$, a contradiction. Hence, $\Phi \not\equiv 0$. It follows that there is a $\lambda_0 \in \mathbf{C}$ and an $r > 0$ such that $|\Phi(\lambda)| > 0$ for every $|\lambda - \lambda_0| < r$. From (6) and the latter result, one has

$$F(\lambda) = \int_0^1 f(x) \cos \lambda x dx = 0 \quad \forall \lambda, |\lambda - \lambda_0| < r. \quad (7)$$

Since $F(\lambda)$ is an entire functions, we get in view of (7) that $F(\lambda) = 0$ for all $\lambda \in \mathbf{C}$. Putting

$$\tilde{f} = \begin{cases} \frac{1}{2}f(x) & x \in (0, 1) \\ \frac{1}{2}f(-x) & x \in (-1, 0) \\ 0 & x \notin (-1, 1) \end{cases},$$

we get that $F(\lambda)$ is the Fourier transform of \tilde{f}

$$F(\lambda) = \int_{-\infty}^{\infty} \tilde{f}(x) e^{-i\lambda x} dx. \quad (8)$$

From (7), (8), we get $\tilde{f} = 0$ a.e. on \mathbf{R} . It follows that $f = 0$ a.e. on $(0, 1)$. This completes the proof of Theorem 1.

III. Regularization

We give two regularization results

Theorem 2. *Let φ_0, g_0 be in $L^2(0, 1)$ and let (u_0, f_0) be the exact solution of (3) with φ, g replaced by φ_0, g_0 . Letting $C_0, \varepsilon > 0$, we assume that φ, g satisfy*

$$\|\varphi - \varphi_0\| < \varepsilon, \quad \|g - g_0\| < \varepsilon$$

and

$$\varphi(x) > C_0, \quad \varphi_0(x) > C_0 \text{ a.e. on } (0, 1),$$

where $\| \cdot \|$ is the norm of $L^2(0, 1)$. Putting

$$f_\varepsilon(x) = -\frac{1}{2\pi} \int_{-\lambda(\varepsilon)}^{\lambda(\varepsilon)} e^{\lambda^2} \int_0^1 g(s) \cos \lambda s ds \left(\int_0^1 e^{\lambda^2 t} \varphi(t) dt \right)^{-1} e^{i\lambda x} d\lambda$$

where $\lambda(\varepsilon) = \sqrt[7]{\pi} \varepsilon^{\frac{2}{7}(\gamma-1)}$, $0 < \gamma < 1$, then there exists a positive function $\eta(\varepsilon)$ independent of $C_0, \|g_0\|$ with $\lim_{\varepsilon \downarrow 0} \eta(\varepsilon) = 0$ and such that

$$\|2f_\varepsilon - f_0\| \leq 2C_1 \varepsilon^\gamma + 2\eta(\varepsilon).$$

where $C_1 = \frac{4}{C_0^2}(1 + C_0 + \|g_0\|)$ is a positive constant defined in terms of C_0 and $\|g_0\|$.

If we assume, in addition, that $f_0 \in H^1(0, 1)$ then the function $\eta(\varepsilon)$ can be estimated and can be taken equal to

$$\eta(\varepsilon) = \frac{(1 + \sqrt{2})}{\sqrt{\pi \lambda(\varepsilon)}} \|f_0\|_{H^1(0,1)}$$

Proof

From Lemma 1, one has

$$e^{\lambda^2} \int_0^1 g_0(x) \cos \lambda x dx = - \int_0^1 e^{\lambda^2 t} \varphi_0(t) dt \int_0^1 f_0(x) \cos \lambda x dx.$$

It follows that

$$\int_0^1 f_0(x) \cos \lambda x dx = e^{\lambda^2} \int_0^1 g_0(x) \cos \lambda x dx \left(\int_0^1 e^{\lambda^2 s} \varphi_0(s) ds \right)^{-1}.$$

Put

$$\tilde{f}_0 = \begin{cases} \frac{1}{2} f_0(x) & x \in (0, 1) \\ \frac{1}{2} f_0(-x) & x \in (-1, 0) \\ 0 & x \notin (-1, 1) \end{cases},$$

We have

$$\mathcal{F}(\tilde{f}_0)(\lambda) = \int_{-\infty}^{\infty} \tilde{f}_0(x) e^{-i\lambda x} dx = e^{\lambda^2} \int_0^1 g_0(x) \cos \lambda x dx \left(\int_0^1 e^{\lambda^2 s} \varphi_0(s) ds \right)^{-1}.$$

where $\mathcal{F}(f)$ is the Fourier transform of f :

$$\mathcal{F}(f)(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx.$$

From Plancherel theorem, we have

$$\|f_\varepsilon - \tilde{f}_0\|_{L^2(\mathbf{R})} = \frac{1}{\sqrt{2\pi}} \|\mathcal{F}(\tilde{f}_0) - \mathcal{F}(f_\varepsilon)\|_{L^2(\mathbf{R})} \quad (9)$$

On the other hand, one has,

$$\begin{aligned} \mathcal{F}(\tilde{f}_0)(\lambda) - \mathcal{F}(f_\varepsilon)(\lambda) &= \int_{-\infty}^{\infty} \tilde{f}_0(x) e^{-i\lambda x} dx - e^{\lambda^2} \int_0^1 g(x) \cos \lambda x dx \left(\int_0^1 e^{\lambda^2 s} \varphi(s) ds \right)^{-1} \\ &= e^{\lambda^2} \int_0^1 g_0(x) \cos \lambda x dx \left(\int_0^1 e^{\lambda^2 s} \varphi_0(s) ds \right)^{-1} - \\ &\quad e^{\lambda^2} \int_0^1 g(x) \cos \lambda x dx \left(\int_0^1 e^{\lambda^2 s} \varphi(s) ds \right)^{-1}. \\ &= e^{\lambda^2} \left(\int_0^1 e^{\lambda^2 s} \varphi_0(s) ds \right)^{-1} \int_0^1 (g_0(x) - g(x)) \cos \lambda x dx \\ &\quad + e^{\lambda^2} \int_0^1 g(x) \cos \lambda x dx \int_0^1 e^{\lambda^2 s} (\varphi(s) - \varphi_0(s)) ds \times \\ &\quad \times \left(\int_0^1 e^{\lambda^2 s} \varphi_0(s) ds \right)^{-1} \left(\int_0^1 e^{\lambda^2 s} \varphi(s) ds \right)^{-1}. \end{aligned}$$

We get after arrangements

$$\begin{aligned} |\mathcal{F}(\tilde{f}_0)(\lambda) - \mathcal{F}(f_\varepsilon)(\lambda)| &\leq \frac{\lambda^2 \varepsilon}{C_0(1 - e^{-\lambda^2})} + \frac{\varepsilon \|g\| \lambda^3 \sqrt{1 - e^{-2\lambda^2}}}{C_0^2(1 - e^{-\lambda^2})^2 \sqrt{2}} \\ &\leq 2 \frac{\lambda^2 \varepsilon}{C_0} + \frac{4\varepsilon \|g\| \lambda^3}{C_0^2} \\ &\leq \frac{4|\lambda|^3 \varepsilon}{C_0^2} (C_0 + 1 + \|g_0\|) = C_1 |\lambda|^3 \varepsilon, \quad \forall 1 \leq |\lambda| \leq 10 \end{aligned}$$

for $0 < \varepsilon < 1$. In (8) we have put $C_1 = \frac{4}{C_0^2}(1 + C_0 + \|g_0\|)$.

Similarly, for $|\lambda| \leq 1$, one has

$$|\mathcal{F}(\tilde{f}_0)(\lambda) - \mathcal{F}(f_\varepsilon)(\lambda)| \leq \frac{2\varepsilon}{C_0} + \frac{4\varepsilon \|g\|}{C_0^2} \leq C_1 \varepsilon,$$

the constant C_1 having the meaning as before in the case $|\lambda| \geq 1$.
In either cases, one has

$$|\mathcal{F}(\tilde{f}_0)(\lambda) - \mathcal{F}(f_\varepsilon)(\lambda)| \leq C_1 |\lambda|^3 \varepsilon \quad \forall \lambda \in \mathbf{R} \quad (11)$$

Noting that $\mathcal{F}(f_\varepsilon)(\lambda) = 0$ for $|\lambda| > \lambda(\varepsilon)$, we get in view of (9), (10), (11) that

$$\begin{aligned} \|f_\varepsilon - \tilde{f}_0\|_{L^2(R)}^2 &= \frac{1}{2\pi} \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda + \frac{1}{2\pi} \int_{|\lambda| < \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda) - \mathcal{F}(f_\varepsilon)(\lambda)|^2 d\lambda \\ &\leq \frac{1}{2\pi} \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda + \frac{1}{2\pi} \int_{-\lambda(\varepsilon)}^{\lambda(\varepsilon)} C_1^2 \varepsilon^2 \lambda^6(\varepsilon) d\lambda \\ &\leq \frac{1}{2\pi} \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda + \frac{1}{\pi} C_1^2 |\lambda(\varepsilon)|^7 \varepsilon^2 \\ &\leq \frac{1}{2\pi} \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda + C_1^2 \varepsilon^{2\gamma}, \end{aligned} \quad (12)$$

with $\lambda(\varepsilon)$ taken such that $\lambda(\varepsilon) = \sqrt[7]{\pi} \varepsilon^{\frac{2}{7}(\gamma-1)} \rightarrow \infty$ as $\varepsilon \rightarrow 0_+$ ($0 < \gamma < 1$).

Putting

$$\eta^2(\varepsilon) = \frac{1}{2\pi} \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda$$

we get the first estimate of Theorem 2.

Now, if $f_0 \in H^1(0, 1)$, one has

$$\begin{aligned} \mathcal{F}(\tilde{f}_0)(\lambda) &= \int_0^1 f_0(x) \cos \lambda x dx \\ &= f_0(1) \frac{\sin \lambda}{\lambda} - \frac{1}{\lambda} \int_0^1 f'_0(x) \sin \lambda x dx \quad \forall \lambda \neq 0. \end{aligned}$$

So, for $\lambda \in \mathbf{R}$, we have

$$|\mathcal{F}(\tilde{f}_0)(\lambda)| \leq \frac{\|f'_0\|}{|\lambda|} + \frac{|f_0(1)|}{|\lambda|}.$$

On the other hand, since $H^1(0, 1) \hookrightarrow C[0, 1]$, there exist an $x_0 \in [0, 1]$ such that $f_0(x_0) = \int_0^1 f_0(x) dx$. We have

$$f_0(1) = f_0(x_0) + \int_{x_0}^1 f'_0(x) dx$$

Hence,

$$\begin{aligned}
|f_0(1)| &\leq \int_0^1 (|f_0(x)| + |f'_0(x)|) dx \\
&\leq \sqrt{2 \int_0^1 (|f_0(x)|^2 + |f'_0(x)|^2) dx} \\
&\leq \sqrt{2} \|f_0\|_{H^1(0,1)}.
\end{aligned}$$

Hence,

$$|\mathcal{F}(\tilde{f}_0)(\lambda)| \leq \frac{1 + \sqrt{2}}{|\lambda|} \|f_0\|_{H^1(0,1)}. \quad (13)$$

Combining (12), (13) we get

$$\begin{aligned}
\|f_\varepsilon - \tilde{f}_0\|_{L^2(\mathbf{R})}^2 &\leq \frac{(1 + \sqrt{2})^2}{\pi} \|f_0\|_{H^1(0,1)}^2 \int_{|\lambda| \geq \lambda(\varepsilon)} \frac{d\lambda}{\lambda^2} + C_1^2 \varepsilon^{2\gamma} \\
&\leq \frac{(1 + \sqrt{2})^2}{\pi} \|f_0\|_{H^1(0,1)}^2 \lambda^{-1}(\varepsilon) + C_1^2 \varepsilon^{2\gamma}
\end{aligned}$$

Since $\lambda(\varepsilon) = \sqrt[7]{\pi} \varepsilon^{\frac{2}{7}(\gamma-1)}$ and that

$$\|2f_\varepsilon - f_0\| \leq 2\|f_\varepsilon - \tilde{f}_0\|_{L^2(\mathbf{R})},$$

we can get the second estimate of Theorem 2. This completes the proof of Theorem 2.

Remark 1: Choosing $\gamma = 1/8$, we obtain $\lambda(\varepsilon) = \frac{\sqrt[7]{\pi}}{\sqrt[4]{\varepsilon}}$ and

$$\|2f_\varepsilon - f_0\| \leq 2 \left(\frac{1 + \sqrt{2}}{\sqrt[7]{\pi^4}} \|f_0\|_{H^1(0,1)} + C_1 \right) \varepsilon^{1/8}$$

The last formula gives us the best upper bound for $0 < \varepsilon < 1$ given.

Now we state and prove the last regularization result. We first put

$$\begin{aligned}
G(\lambda) &= -e^{\lambda^2} \int_0^1 g(x) \cos \lambda x dx & G_0(\lambda) &= -e^{\lambda^2} \int_0^1 g_0(x) \cos \lambda x dx \\
\Phi(\lambda) &= \int_0^1 e^{\lambda^2 t} \varphi(t) dt & \Phi_0(\lambda) &= \int_0^1 e^{\lambda^2 t} \varphi_0(t) dt \\
F(\lambda) &= \int_0^1 f(x) \cos \lambda x dx & F_0(\lambda) &= \int_0^1 f_0(x) \cos \lambda x dx
\end{aligned}$$

We have

Theorem 3 Suppose that φ_0 has the form

$$\varphi_0(t) = (1-t)^m(a + (1-t)\psi_0(t))$$

where $a \neq 0, m = 0, 1, 2, \dots, t \in (0, 1), \psi_0 \in L^2(0, 1)$. Letting $\beta \in (0, 1/2)$, we put

$$F_\varepsilon(\lambda) = \begin{cases} G(\lambda)/\Phi(\lambda) & \text{if } |\Phi(\lambda)| \geq \varepsilon^\beta, \text{ and } |\lambda| < \lambda(\varepsilon), \\ 0 & \text{if } |\Phi(\lambda)| < \varepsilon^\beta, \text{ or } |\lambda| \geq \lambda(\varepsilon) \end{cases}$$

and

$$f_\varepsilon(x) = \frac{1}{2\pi} \int_{-\lambda(\varepsilon)}^{\lambda(\varepsilon)} F_\varepsilon(\lambda) e^{i\lambda x} d\lambda.$$

Then, for each $\delta \in (0, \min\{\beta, 1 - 2\beta\})$ there exist a $C_\delta > 0, \gamma_\delta > 0$ independent of g_0, φ_0 and a function $\eta_\delta(\varepsilon)$ such that $\lim_{\varepsilon \downarrow 0} \eta_\delta(\varepsilon) = 0$ and that

$$\|2f_\varepsilon - f_0\| \leq C_\delta \varepsilon^{\gamma_\delta} + \eta_\delta(\varepsilon),$$

where $\lambda(\varepsilon) = \sqrt{\delta \ln \frac{1}{\varepsilon}}$.

Remark 2 In the case $\varphi_0 \in C^k[0, 1]$, $k \geq 1$, if we put

$$P_n \varphi_0(t) = \sum_{j=0}^{n-1} \frac{\varphi_0^{(j)}(1)}{j!} (t-1)^n,$$

(the n -th Taylor polynomial of φ_0 at $t = 1$) then the condition (14) holds if we have $P_k \varphi_0 \not\equiv 0$. So the class of functions satisfying (14) is very broad.

The proof of Theorem 3 relies on Lemma 2 and Lemma 3 followed.

Lemma 2 If

$$\varphi_0(t) = (1-t)^m(a + (1-t)\psi(t)) \tag{14}$$

with $a \neq 0$ and $\psi \in L^2(0, 1)$ then

$$\lim_{\lambda \rightarrow +\infty} \frac{\int_0^1 e^{\lambda^2 t} \varphi_0(t) dt}{e^{\lambda^2} / \lambda^{2m+2}} = m!a.$$

Proof

Put

$$J_m(\mu) = \int_0^1 (1-t)^m e^{\mu t} dt$$

We prove that

$$\lim_{\mu \rightarrow +\infty} \frac{J_m(\mu)}{\frac{e^\mu}{\mu^{m+1}}} = m! \quad m = 0, 1, 2, \dots \quad (15)$$

In fact, we shall prove the latter relation by induction. One has $J_0(\mu) = \frac{e^\mu - 1}{\mu}$. So, (15) holds for $m = 0$. Suppose (15) holds for $m = k$, we prove (15) for $m = k + 1$. In fact, one has

$$\begin{aligned} J_{k+1} &= (1-t)^{k+1} \frac{e^{\mu t}}{\mu} \Big|_0^1 + \frac{k+1}{\mu} J_k \\ &= -\frac{1}{\mu} + \frac{k+1}{\mu} J_k. \end{aligned}$$

It follows that

$$\lim_{\mu \rightarrow +\infty} \frac{J_{k+1}(\mu)}{\frac{e^\mu}{\mu^{k+2}}} = (k+1) \lim_{\mu \rightarrow +\infty} \frac{J_k(\mu)}{\frac{e^\mu}{\mu^{k+1}}} = (k+1)!.$$

This completes the proof of (15). Using (15), one has C_1 such that

$$\begin{aligned} \left| \int_0^1 e^{\lambda^2 t} \varphi_0(t) dt - a J_m(\lambda^2) \right| &\leq \int_0^1 e^{\lambda^2 t} (1-t)^{m+1} |\psi(t)| dt \\ &\leq \|\psi\|_{L^2} \sqrt{J_{2m+2}(2\lambda^2)} \\ &\leq C_1 \|\psi\|_{L^2} \sqrt{\frac{e^{2\lambda^2}}{\lambda^{4m+6}}} \text{ as } \lambda \rightarrow +\infty \end{aligned}$$

Hence

$$\lim_{\lambda \rightarrow +\infty} \frac{\int_0^1 e^{\lambda^2 t} \varphi_0(t) dt}{e^{\lambda^2} / \lambda^{2m+2}} = a \lim_{\mu \rightarrow +\infty} \frac{J_m(\lambda^2)}{\frac{e^{\lambda^2}}{\lambda^{2(m+1)}}} = m! a.$$

This completes the proof of Lemma 2.

Now we state and prove Lemma 3.

Lemma 3 *If φ_0 satisfies (14) then there exist $\gamma, \alpha_0 \in (0, 1)$ and $C_0 > 0$ such that*

$$m(B_\alpha) \leq C_0 \alpha^\gamma \quad \forall 0 < \alpha < \alpha_0.$$

Here

$$B_\alpha = \{\lambda \in \mathbf{R} : |\Phi_0(\lambda)| < \alpha\}, \quad \alpha > 0.$$

and $m(B_\alpha)$ is the Lebesgue measure of B_α .

Proof

From the lemma 2 and from the analyticity of Φ_0 , the function Φ_0 has only finite zeros λ_j , $j = 1, \dots, p$. We can write

$$\Phi_0(\lambda) = \Phi_1(\lambda) \prod_{j=1}^p (\lambda - \lambda_j)^{m_j},$$

where $\Phi_1(\lambda) \neq 0$ for every $\lambda \in \mathbf{R}$. Since

$$\lim_{\mu \rightarrow +\infty} \frac{J_m(\mu)}{\frac{e^\mu}{\mu^{m+1}}} = m!,$$

we have

$$\lim_{\lambda \rightarrow \infty} \Phi_1(\lambda) = \infty.$$

It follows that there exists a $C_1 > 0$ such that

$$|\Phi_1(\lambda)| \geq C_1 \quad \forall \lambda \in \mathbf{R}.$$

Hence,

$$|\Phi_0(\lambda)| \geq C_1 \prod_{j=1}^p |\lambda - \lambda_j|^{m_j} \quad \forall \lambda \in \mathbf{R}.$$

Without loss of generality, we assume that

$$\lambda_1 < \lambda_2 < \dots < \lambda_p.$$

Put $d = \min_{1 \leq s \leq p-1} \{\lambda_{s+1} - \lambda_s\}$ and $\delta_j^2 = \frac{\alpha^{1/m_j}}{C_1^{1/m_j} d^{M/m_j}}$. For $\lambda_s + \delta_s \leq \lambda \leq \lambda_{s+1} - \delta_{s+1}$, $s = 1, \dots, p$, one has

$$\begin{aligned} |\Phi_0(\lambda)| &\geq C_1 \prod_{j=1}^p |\lambda - \lambda_j|^{m_j} \\ &\geq C_1 \delta_s^{m_s} \delta_{s+1}^{m_{s+1}} d^{M_s} = \alpha. \end{aligned}$$

where $M_s = M - m_s - m_{s+1}$, with $M = \sum_{j=1}^{p-1} m_j$. It follows that

$$\begin{aligned} B_\alpha &= \{\lambda \in \mathbf{R} : |\Phi_0(\lambda)| < \alpha\} \\ &\subset \bigcup_{s=1}^{p-1} (\lambda_s - \delta_s, \lambda_s + \delta_s) \end{aligned}$$

Hence

$$m(B_\alpha) \leq \sum_{s=1}^{p-1} 2\delta_s = 2d \sum_{j=1}^{p-1} \frac{\alpha^{1/2m_j}}{C_1^{1/2m_j} d^{M/2m_j}}.$$

Choosing $\gamma = \min_{1 \leq j \leq p} \{\frac{1}{2m_j}\}$ we complete the proof of Lemma 3.

Now, we turn to the

Proof of Theorem 3

We have

$$\|f_\varepsilon - \tilde{f}_0\|_{L^2} = \frac{1}{\sqrt{2\pi}} \|\mathcal{F}(f_\varepsilon) - \mathcal{F}(\tilde{f}_0)\|_{L^2}.$$

On the other hand,

$$\begin{aligned} \|\mathcal{F}(f_\varepsilon) - \mathcal{F}(\tilde{f}_0)\|_{L^2}^2 &= \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda \\ &\quad + \int_{-\lambda(\varepsilon)}^{\lambda(\varepsilon)} \chi_{\{|\Phi(\lambda)| < \varepsilon^\beta\}} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda \\ &\quad + \int_{-\lambda(\varepsilon)}^{\lambda(\varepsilon)} \chi_{\{|\Phi(\lambda)| \geq \varepsilon^\beta\}} |\mathcal{F}(f_\varepsilon) - \mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda \\ &\equiv I_1 + I_2 + I_3 \end{aligned}$$

We estimate I_2 , I_3 . We first have

$$\begin{aligned}
|\Phi(\lambda) - \Phi_0(\lambda)| &\leq \int_0^1 e^{\lambda^2 t} |\varphi(t) - \varphi_0(t)| dt \\
&\leq \|\varphi - \varphi_0\|_{L^2} \sqrt{\int_0^1 e^{2\lambda^2 t} dt} \\
&\leq \varepsilon \sqrt{\frac{e^{2\lambda^2} - 1}{2\lambda^2}}.
\end{aligned}$$

So, if $|\Phi(\lambda)| < \varepsilon^\beta$ then

$$|\Phi_0(\lambda)| < \varepsilon \sqrt{\frac{e^{2\lambda^2} - 1}{2\lambda^2}} + \varepsilon^\beta \equiv \alpha(\varepsilon). \quad (16)$$

Now, we have

$$\mathcal{F}(\tilde{f}_0) = \int_0^1 f_0(x) \cos \lambda x dx.$$

Hence, one has

$$|\mathcal{F}(\tilde{f}_0)(\lambda)| \leq \|f_0\|_{L^2}.$$

It follows that

$$I_2 \leq \|f_0\|_{L^2} m(B_{\alpha(\varepsilon)}),$$

where $m(A)$ is the Lebesgue measure of A . Now, we estimate I_3 . If

$$|\Phi(\lambda)| \geq \varepsilon^\beta, \quad (17)$$

then by (17), one has

$$|\Phi_0(\lambda)| \geq \varepsilon^\beta - \varepsilon \sqrt{\frac{e^{2\lambda^2} - 1}{2\lambda^2}}. \quad (18)$$

In this case, we have

$$\begin{aligned}
\mathcal{F}(f_\varepsilon)(\lambda) - \mathcal{F}(\tilde{f}_0)(\lambda) &= \frac{G(\lambda)}{\Phi(\lambda)} - \frac{G_0(\lambda)}{\Phi_0(\lambda)} \\
&= \frac{G_0(\lambda)(\Phi_0(\lambda) - \Phi(\lambda)) + \Phi_0(\lambda)(G(\lambda) - G_0(\lambda))}{\Phi(\lambda)\Phi_0(\lambda)}.
\end{aligned}$$

Using (17), (18), one has

$$\begin{aligned}
|\mathcal{F}(f_\varepsilon)(\lambda) - \mathcal{F}(\tilde{f}_0)(\lambda)| &\leq \frac{\|g_0\| e^{\lambda^2} \varepsilon \sqrt{\frac{e^{2\lambda^2}-1}{2\lambda^2}} + \|\varphi_0\| \sqrt{\frac{e^{2\lambda^2}-1}{2\lambda^2}} e^{\lambda^2} \varepsilon}{\varepsilon^\beta (\varepsilon^\beta - \varepsilon \sqrt{\frac{e^{2\lambda^2}-1}{2\lambda^2}})} \\
&\leq \frac{\varepsilon^{1-2\beta} e^{\lambda^2} \sqrt{\frac{e^{2\lambda^2}-1}{2\lambda^2}} (\|g_0\| + \|\varphi_0\|)}{1 - \varepsilon^{1-\beta} \sqrt{\frac{e^{2\lambda^2}-1}{2\lambda^2}}}.
\end{aligned}$$

It follows that

$$I_3 \leq \frac{\varepsilon^{2-4\beta} e^{2\lambda(\varepsilon)^2} (e^{2\lambda(\varepsilon)^2} - 1) (\|g_0\| + \|\varphi_0\|)^2}{\lambda(\varepsilon) \left(1 - \varepsilon^{1-\beta} \sqrt{\frac{e^{2\lambda(\varepsilon)^2}-1}{2\lambda(\varepsilon)^2}}\right)^2}.$$

So, we have

$$\begin{aligned}
\|\mathcal{F}(f_\varepsilon) - \mathcal{F}(\tilde{f}_0)\|_{L^2}^2 &\leq \int_{|\lambda| \geq \lambda(\varepsilon)} |\mathcal{F}(\tilde{f}_0)(\lambda)|^2 d\lambda + \|f_0\|_{L^2}^2 m(B_{\alpha(\varepsilon)}) \\
&\quad + \frac{\varepsilon^{2-4\beta} e^{2\lambda(\varepsilon)^2} (e^{2\lambda(\varepsilon)^2} - 1) (\|g_0\| + \|\varphi_0\|)^2}{\lambda(\varepsilon) \left(1 - \varepsilon^{1-\beta} \sqrt{\frac{e^{2\lambda(\varepsilon)^2}-1}{2\lambda(\varepsilon)^2}}\right)^2}.
\end{aligned}$$

By choosing $\lambda(\varepsilon) = \sqrt{\delta \ln \frac{1}{\varepsilon}}$ and using Lemma 3, we shall complete the proof of Theorem 3.

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