Pivotal and Pivotal-discriminative Consequence Relations

Jonathan Ben-Naim

Pivotal and Pivotal-discriminative Consequence Relations

Jonathan Ben-Naim
LIF, CNRS
CMI 39, rue Joliot-Curie
F-13453 Marseille Cedex 13, France
jbennaim@lif.univ-mrs.fr

September 29, 2005

Abstract

A pivotal consequence relation is defined to hold between a set of formulas Γ and formula α iff every non-negligible model for Γ is a model for α. Unlike preferential consequence relations, the set of all non-negligible (or preferred) valuations is fixed and thus does not depend on the premisses under consideration. The first purpose of the present paper is to investigate pivotal consequence relations. We provide characterizations of several families in the classical framework, but also in certain three/four-valued frameworks, well-known as the paraconsistent logics J_3 and $\mathcal{FOUR}$. We show also that there is no "normal" characterization of the family of all pivotal consequence relations, in the infinite classical framework. And we show a link with X-logics. Our second purpose is to investigate a qualified version of pivotal consequence, which we call pivotal-discriminative consequence. This is defined to hold between a set of formulas Γ and formula α iff $\Gamma \mid\sim \alpha$ but $\Gamma \not\mid\sim \neg\alpha$, where $\mid\sim$ is the plain relation. We provide characterizations of several families of such relations for the classical, three, and four-valued frameworks.

1 Introduction

In [17, 18], David Makinson introduced *pivotal consequence relations* (pivotal CRs for short) and gave some of their basic properties. They are defined to hold between a set of formulas Γ and formula α iff every non-negligible model for Γ is a model for α. Unlike *preferential* CRs, the set of non-negligible (or preferred) valuations is fixed and thus does not depend on the premisses under consideration. In the present paper, the author continues Makinson's investigation, but on a more general level that covers certain three/four-valued frameworks, well-known as the paraconsistent logics J_3 and $\mathcal{FOUR}$. The motivation is that pivotal CRs represent natural ways of reasoning which are useful to handle incomplete information in the classical framework and both incomplete and inconsistent information in the J_3 and $\mathcal{FOUR}$ frameworks. We will illustrate this, together with the different advantages and drawbacks of the different frameworks, with examples in Sections 2.11.1 and 2.11.2, once the formal definitions will be made.

The first goal of the paper is to provide characterizations of several families of pivotal CRs for the classical, three and four-valued frameworks. Most of the time, these characterizations have a

purely syntactic aspect (i.e. they involve only the language and some proof systems).

Then, we will answer negatively a representation problem that was left open by Makinson, namely there is no "normal" characterization of the family of all pivotal CRs, in the infinite classical framework. Roughly speaking, we have a normal characterization iff we have found some conditions that discriminate any CR $|\sim$ only with a limited number of sets of formulas and their $|\sim$-closures. And, we will show that a certain family of pivotal CRs corresponds precisely to a certain family of X*-logics* which were introduced by Forget, Risch, and Siegel [12].

Another goal of the paper is to investigate a qualified version of pivotal CRs, which we call *pivotal-discriminative* CRs. They are defined to hold between a set of formulas Γ and formula α iff $\Gamma \ |\sim \alpha$ but $\Gamma \not|\sim \neg\alpha$, where $|\sim$ is the plain relation. They capture the idea that usually we do not want to keep contradictory conclusions. We will provide characterizations (often syntactic ones) of several families of such relations for the classical, three, and four-valued frameworks.

The rest of the paper is organized as follows. In Section 2, we give the formal definitions, an overview of the characterizations, and examples in the classical and $\mathcal{FOUR}$ frameworks. In Sections 3, 4, and 5, we give characterizations of several families of pivotal CRs. In Section 6, we show that pivotal CRs are linked to X-logics. In Section 7, we show that there is no "normal" characterization of pivotal CRs, in the infinite classical framework. In Sections 8 and 9, we give characterizations of several families of pivotal-discriminative CRs. And, Section 10 is a conclusion, summarizing and explaining what has been achieved.

2 Preliminaries

2.1 A general framework

We fix once and for all a set $\mathcal{L}$, a set $\mathcal{V}$, and a relation $\models$ on $\mathcal{V} \times \mathcal{L}$. Intuitively, $\mathcal{L}$ is a set of formulas, $\mathcal{V}$ is a set of valuations, and $\models$ is a satisfaction relation (i.e. $\forall\, x \in \mathcal{V}$, $\forall\, \alpha \in \mathcal{L}$, $x \models \alpha$ means that the formula α is satisfied in the valuation x, i.e. x is a model for α). We fix also once and for all a function $neg : \mathcal{L} \to \mathcal{L}$, a function $or : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, and a function $and : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. Intuitively, $\forall\, \alpha, \beta \in \mathcal{L}$, $neg(\alpha)$ is the negation of α, $or(\alpha, \beta)$ is the disjunction of α and β, and $and(\alpha, \beta)$ is the conjunction of α and β. We use these functions instead of the usual connectives: $\neg$, $\vee$, and $\wedge$, because we do not want nor need to assume any structure on $\mathcal{L}$. We emphasize that for the time being, *no* assumption is made about $\mathcal{L}$, $\mathcal{V}$, $\models$, neg, or, and. The idea is to obtain the most general possible results (i.e. results under the least possible assumptions on $\mathcal{L}$, neg, ...) and then to use them for instance in the particular case where $\mathcal{L}$ is a classical language, neg is the classical negation, etc. The framework is the one presented in [7]. A similar framework has been adopted in two well-known papers, [18, 15], without anticipating any of the substantive work in the present paper.

Notation 1 For all $\Gamma \subseteq \mathcal{L}$, we denote by M_Γ the set such that $M_\Gamma = \{x \in \mathcal{V} : \forall \alpha \in \Gamma, x \models \alpha\}$. Let $X \subseteq \mathcal{V}$. We denote by $T(X)$ the set such that $T(X) = \{\alpha \in \mathcal{L} : X \subseteq M_\alpha\}$. We denote by $T_d(X)$ the set such that $T_d(X) = \{\alpha \in \mathcal{L} : X \subseteq M_\alpha$ and $X \not\subseteq M_{neg(\alpha)}\}$. We denote by $\mathbf{D}$ the set such that $\mathbf{D} = \{X \subseteq \mathcal{V} : \exists \Gamma \subseteq \mathcal{L}, M_\Gamma = X\}$. We denote by $\mathbf{C}$ the set such that $\mathbf{C} = \{X \subseteq \mathcal{V} : \forall \alpha \in \mathcal{L}$, $X \not\subseteq M_\alpha$ or $X \not\subseteq M_{neg(\alpha)}\}$.

Intuitively, M_Γ is the set of all models for Γ, $T(X)$ is the set of all formulas satisfied in X, $T_d(X)$ is the set of all formulas "discriminatively" satisfied in X, $\mathbf{D}$ is the set of every set of valuations definable by a set of formulas, and $\mathbf{C}$ is the set of all "consistent" sets of valuations. Note that $\forall\, \Gamma, \Delta \subseteq \mathcal{L}$, $M_\Gamma \cap M_\Delta = M_{\Gamma \cup \Delta}$.

2.2 Consequence relations and extended consequence relations

Definition 2 We denote by $\mathcal{P}$ the power set operator. We say that $|\sim$ is a *consequence relation* (CR for short) iff $|\sim$ is a relation on $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$. For all $\Gamma \subseteq \mathcal{L}$, we denote by $\widetilde{\Gamma}$ the set such that $\widetilde{\Gamma} = \{\alpha \in \mathcal{L} : \Gamma \mid\sim \alpha\}$. We denote by $\vdash$ the CR such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\forall\, \alpha \in \mathcal{L}$, $\Gamma \vdash \alpha$ iff $M_\Gamma \subseteq M_\alpha$. For all $\Gamma \subseteq \mathcal{L}$, we denote by $\overline{\Gamma}$ the set such that $\overline{\Gamma} = \{\alpha \in \mathcal{L} : \Gamma \vdash \alpha\}$. We say that $\Gamma \subseteq \mathcal{L}$ is consistent iff $\forall\, \alpha \in \mathcal{L}$, $\Gamma \nvdash \alpha$ or $\Gamma \nvdash neg(\alpha)$. We denote by $\vdash_d$ the CR such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\forall\, \alpha \in \mathcal{L}$, $\Gamma \vdash_d \alpha$ iff $\Gamma \vdash \alpha$ and $\Gamma \nvdash neg(\alpha)$.

Intuitively, $\vdash$ is the basic CR and $\vdash_d$ is the "discriminative" CR. Note that Γ is consistent iff $M_\Gamma \in \mathbf{C}$. Note again that $\forall\, \Gamma, \Delta \subseteq \mathcal{L}$, $\overline{\Gamma} = T(M_\Gamma)$, $M_\Gamma = M_{\overline{\Gamma}}$, and $\Gamma \subseteq \overline{\Delta}$ iff $\overline{\Gamma} \subseteq \overline{\Delta}$ iff $M_\Delta \subseteq M_\Gamma$.

Definition 3 We say that $||\sim$ is an *extended consequence relation* (ECR for short) iff $||\sim$ is a relation on $\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$.

2.3 Pivotal consequence relations

Pivotal CRs were introduced by David Makinson, [17, 18]. They represent natural ways of drawing conclusions which are useful to handle incomplete and/or inconsistent information. Examples will be given in Sections 2.11.1 and 2.11.2.

Definition 4 We say that P is a *pivot* iff $P \subseteq \mathcal{V}$. We say that a pivot P is *definable* iff $P \in \mathbf{D}$. We say that a pivot P is *codefinable* iff $\mathcal{V} \setminus P \in \mathbf{D}$. Let $\mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \to \mathbf{Z}$ be a function. We say that μ is *consistency preserving* (CP for short) iff $\forall\, X \in \mathbf{Y} \cap \mathbf{D} \cap \mathbf{C}$, $\mu(X) \in \mathbf{C}$. We say that a pivot P is CP iff the function $\mu : \mathcal{P}(\mathcal{V}) \to \mathcal{P}(\mathcal{V})$ such that $\forall\, X$, $\mu(X) = X \cap P$, is CP.

Intuitively, a pivot P is a way to choose a fixed set of valuations which are considered to be the non-negligible ones. Reciprocally, every such way is represented by a pivot.

Definition 5 Let $|\sim$ be a CR. We say that $|\sim$ is a pivotal CR iff there exists a pivot P such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\widetilde{\Gamma} = T(M_\Gamma \cap P)$. We say that $|\sim$ is definable, codefinable, etc. iff $|\sim$ can be defined by a definable, codefinable, etc. pivot.

Intuitively $|\sim$ is a pivotal CR iff there exists a pivot such that $\Gamma \mid\sim \alpha$ iff the non-negligible models for Γ are models for α. Note that with preferential CRs, a valuation x belonging to two sets of valuations X and Y can be non-negligible (or preferred) in X and negligible in Y. In other words, this depends on the set of valuations that is considered, i.e. on the "context". This is not true with pivotal CRs: a valuation is non-negligible iff it is in the fixed pivot, independently of the "context".

We now give alternative definitions and intuitions for definable pivotal CRs. We say that Σ is an assumption choice iff $\Sigma \subseteq \mathcal{L}$. Intuitively, Σ is a way to choose a fixed set of formulas that is considered to hold. Reciprocally, every such way is represented by an assumption choice. Now, a CR $|\sim$ is a definable pivotal CR iff there exists an assumption choice Σ such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\forall\, \alpha \in \mathcal{L}$, $\Gamma \mid\sim \alpha$ iff $\Gamma \cup \Sigma \vdash \alpha$.

We do the same job with codefinable pivotal CRs. We say that Σ is a negative-assumption choice iff $\Sigma \subseteq \mathcal{L}$. Intuitively, Σ is a way to choose a fixed set of formulas that is considered *not* to hold. Reciprocally, every such way is represented by a negative-assumption choice. Now, under the assumption that $\forall\, \alpha, \beta \in \mathcal{L}$, $M_{or(\alpha,\beta)} = M_\alpha \cup M_\beta$, a CR $|\sim$ is a codefinable pivotal CR iff there exists a negative-assumption choice Σ such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\forall\, \alpha \in \mathcal{L}$, $\Gamma \mid\sim \alpha$ iff $\forall\, \beta \in \Sigma$, $\Gamma \vdash or(\alpha, \beta)$.

2.4 Pivotal-discriminative consequence relations

We now introduce pivotal-discriminative CRs. They capture the idea that usually we do not want to keep contradictory conclusions.

Definition 6 Let $|\sim$ be a CR. We say that $|\sim$ is a pivotal-discriminative CR iff there exists a pivot P such that $\forall\, \Gamma \subseteq \mathcal{L}$, $\widetilde{\Gamma} = T_d(M_\Gamma \cap P)$. We say that $|\sim$ is definable, codefinable, CP, etc. iff $|\sim$ can be defined by a definable, codefinable, CP, etc. pivot.

Intuitively, $|\sim$ is a pivotal-discriminative CR iff there exists a pivot such that $\Gamma \mid\sim \alpha$ iff α is satisfied in all non-negligible models for Γ, but not its negation.

2.5 Pertinence extended consequence relations (alias X-logics)

Pertinence ECRs (alias X-logics) were introduced by Forget, Risch, and Siegel, see [12].

Definition 7 We say that Σ is a *pertinence choice* iff $\Sigma \subseteq \mathcal{L}$. In addition, we say that Σ is closed iff $\Sigma = \overline{\Sigma}$.

Intuitively, a pertinence choice Σ is a way to choose a fixed set of formulas which are considered to be the pertinent ones. Reciprocally, every such way is represented by a pertinence choice.

Definition 8 Let $||\sim$ be an ECR. We say that $||\sim$ is a *pertinence* ECR (alias X-logic) iff there exists a pertinence choice Σ such that $\forall\, \Gamma, \Delta \subseteq \mathcal{L}$, $\Gamma \mid|\sim \Delta$ iff $\overline{\Gamma \cup \Delta} \cap \Sigma \subseteq \overline{\Gamma}$. We say that $||\sim$ is closed iff $||\sim$ can be defined by a closed pertinence choice.

Intuitively, $||\sim$ is a pertinence ECR iff there exists a pertinence choice such that $\Gamma \mid|\sim \Delta$ iff every pertinent basic consequence of $\Gamma \cup \Delta$ is a basic consequence of Γ (i.e. iff the addition of Δ to Γ does not yield more pertinent formulas than with Γ alone).

2.6 Normal Characterizations

Some of our characterizations are said to be *normal*, whilst some others are not. Let us make this explicit. For every set A, we denote by $|A|$ the cardinality of A. Let now C be a family of CRs. Intuitively, we have a normal characterization of C iff we have found some conditions that determine whether a CR $|\sim$ is in C or not, only with $\lambda \leq |\mathcal{L}|$ many sets of formulas: $\Gamma_1, \ldots \Gamma_\lambda$ and their $|\sim$-closures: $\widetilde{\Gamma_1}, \ldots \widetilde{\Gamma_\lambda}$. This notion was introduced by Karl Schlechta, see e.g. [24]. We have formalized this notion in Definition 9 below.

Definition 9 Let C be a set of CRs. We have a normal characterization of C iff we have found a (finite or infinite) cardinal $\lambda \leq |\mathcal{L}|$ and a relation Φ on $\mathcal{P}(\mathcal{L})^{2\lambda}$ such that for all CR $|\sim$,

$$|\sim \in C \text{ iff } \forall\, \Gamma_1, \ldots \Gamma_\lambda \subseteq \mathcal{L},\ \Phi(\Gamma_1, \ldots \Gamma_\lambda, \widetilde{\Gamma_1}, \ldots \widetilde{\Gamma_\lambda}) \text{ holds.}$$

We specify that Φ is a relation in the straightforward set-theoretic sense (i.e. any set of 2λ-tuples of subsets of $\mathcal{L}$). Now, we turn to two remarks.

First, let C be a family of CRs and $A \subseteq \mathcal{P}(\mathcal{L})$. Suppose now that for all CR $|\sim$, $|\sim \in C$ iff $\forall\, \Gamma, \Delta \in A$, $\widetilde{\Gamma \cup \widetilde{\Delta}} = \overline{\emptyset}$. Then, we can immediately get a normal characterization of C from this. Indeed, just take the relation Φ such that $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ iff (if $\Gamma_1, \Gamma_2 \in A$ and $\Gamma_3 =$

$\Gamma_1 \cup \Gamma_5$, then $\Gamma_6 = \overline{\emptyset}$). Then, clearly $|\sim \in C$ iff $\forall\, \Gamma_1, \Gamma_2, \Gamma_3$, $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \widetilde{\Gamma_1}, \widetilde{\Gamma_2}, \widetilde{\Gamma_3})$. In other words, if there is no normal characterization, then there is no simple condition such as $\forall\, \Gamma, \Delta \in A$, $\widetilde{\Gamma \cup \widetilde{\Delta}} = \overline{\emptyset}$.

Second, suppose this time that for all CR $|\sim$, $|\sim \in C$ iff $\forall\, \Gamma$, $\widetilde{\Gamma} = T(\{x \in M_\Gamma : \forall\, \Delta,$ if $x \in M_\Delta,$ then $x \in M_{\widetilde{\Delta}}\})$. Then, it seems impossible to get a normal characterization of C from this. Roughly, the reason is that the condition depends on $2^{|\mathcal{L}|}$ many subsets of $\mathcal{L}$, whilst we need to find a relation that depends on at most $|\mathcal{L}|$ many subsets of $\mathcal{L}$.

2.7 Overview of the characterizations

The main contributions of the paper are characterizations of several families of pivotal CRs and pivotal-discriminative CRs. *Sometimes*, to show a characterization, we will need to make *some* of the following assumptions about $\mathcal{L}$, $\mathcal{V}$, $\models$, neg, or, and and:

$(A0)$ $M_{\mathcal{L}} = \emptyset$;

$(A1)$ $\forall\, \alpha, \beta \in \mathcal{L}$, $M_{or(\alpha,\beta)} = M_\alpha \cup M_\beta$;

$(A2)$ $\forall\, \alpha, \beta \in \mathcal{L}$, $M_{or(\alpha,\beta)} = M_\alpha \cup M_\beta$ and $M_{and(\alpha,\beta)} = M_\alpha \cap M_\beta$ and $M_{neg \circ neg(\alpha)} = M_\alpha$ and $M_{neg \circ or(\alpha,\beta)} = M_{and(neg(\alpha),neg(\beta))}$ and $M_{neg \circ and(\alpha,\beta)} = M_{or(neg(\alpha),neg(\beta))}$, where $\circ$ is the function composition operator (i.e. $neg \circ or(\alpha, \beta) = neg(or(\alpha, \beta))$, etc.);

$(A3)$ $\mathcal{V}$ is finite;

$(A4)$ $\forall\, \Gamma \subseteq \mathcal{L}$, $\forall\, \alpha \in \mathcal{L}$, if $\alpha \notin T(M_\Gamma)$ and $neg(\alpha) \notin T(M_\Gamma)$, then $M_\Gamma \cap M_\alpha \not\subseteq M_{neg(\alpha)}$.

But, for the time being, we do not make any assumption. Note that to show our characterizations, we will never need other assumptions about $\mathcal{L}$, $\mathcal{V}$, $\models$, neg, or, and and. And note that $(A0)$, $(A1)$, $(A2)$, $(A3)$, and $(A4)$ are not independent of each other. For instance, obviously if $(A2)$ holds, then $(A1)$ holds too. We will give normal characterizations of the following families:

- the definable pivotal CRs (in Section 3);
- the definable pivotal-discriminative CRs, under $(A2)$, $(A3)$, and $(A4)$ (in Section 8);
- the CP definable pivotal-discriminative CRs, under $(A2)$ and $(A3)$ (in Section 8).

Recall that CP means: consistency preserving, defined in Definition 4. In addition, we will give non-normal characterizations of the following families:

- the pivotal CRs (in Section 4);
- the codefinable pivotal CRs, under $(A0)$ (in Section 5);
- the pivotal-discriminative CRs, under $(A2)$, $(A3)$, and $(A4)$ (in Section 9);
- the codefinable pivotal-discriminative CRs, under $(A0)$, $(A2)$, $(A3)$, and $(A4)$ (in Section 9);
- the CP pivotal-discriminative CRs, under $(A2)$ and $(A3)$ (in Section 9);
- the CP codefinable pivotal-discriminative CRs, under $(A0)$, $(A2)$, and $(A3)$ (in Section 9).

We will see in Sections 2.8, 2.9, and 2.10, that $(A0)$, $(A1)$, $(A2)$, $(A3)$, and $(A4)$ are weak enough to hold in the classical (propositional) framework, in the $\mathcal{FOUR}$ framework, and in the J_3 framework.

2.8 The classical framework

Notation 10 We fix once and for all a set $\mathcal{A}$ of propositional symbols. We denote by $\mathcal{L}_c$ the classical language generated from $\mathcal{A}$, the constant symbols: 0, 1, and the usual connectives: $\neg$, $\vee$, $\wedge$. We denote by $\mathcal{V}_c$ the classical valuations of $\mathcal{L}_c$. We denote by $\models_c$ the classical satisfaction relation on $\mathcal{V}_c \times \mathcal{L}_c$.

Roughly speaking, $(A0)$, $(A1)$, $(A2)$, $(A3)$, and $(A4)$ hold in the classical framework. More precisely, first (1) entails $(A0)$, second (1) and (2) entail $(A1)$, $(A2)$ and $(A4)$, and third (1) and (3) entail $(A3)$, where

(1) $\mathcal{L} = \mathcal{L}_c,\ \mathcal{V} = \mathcal{V}_c$, and $\models = \models_c$;

(2) $\forall\, \alpha, \beta \in \mathcal{L},\ neg(\alpha) = \neg\alpha,\ or(\alpha, \beta) = \alpha \vee \beta,\ and(\alpha, \beta) = \alpha \wedge \beta$;

(3) $\mathcal{A}$ is finite.

2.9 The $\mathcal{FOUR}$ framework

The logic $\mathcal{FOUR}$ was introduced by Belnap in [5, 6] to deal with inconsistent and incomplete information. Intensive investigations of $\mathcal{FOUR}$ were made by e.g. Ofer Arieli and Arnon Avron [1, 2, 3]. They worked with richer languages than here, containing e.g. some constants $\bot$, $\top$ and some implication connective $\supset$ first introduced by Avron in [4]. In addition, proof systems for $\vdash$ in the $\mathcal{FOUR}$ framework have been given in [1, 2, 3]. Now, from e.g. [5, 6]:

Definition 11 We say that x is a $\mathcal{FOUR}$-valuation iff x is a function from $\mathcal{L}_c$ to $\{0, 1, \bot, \top\}$ such that $x(0) = 0$, $x(1) = 1$ and truth tables 1, 2, and 3 below hold. We denote by $\mathcal{V}_4$ the set of all $\mathcal{FOUR}$-valuations. We denote by $\models_4$ the relation on $\mathcal{V}_4 \times \mathcal{L}_c$ such that $\forall\, x \in \mathcal{V}_4, \forall\, \alpha \in \mathcal{L}_c$, $x \models_4 \alpha$ iff $x(\alpha) \in \{1, \top\}$. We call $\models_4$ the $\mathcal{FOUR}$-satisfaction relation.

α	$\neg\alpha$
0	1
1	0
$\bot$	$\bot$
$\top$	$\top$

Table 1.

α \ β	0	1	$\bot$	$\top$
0	0	1	$\bot$	$\top$
1	1	1	1	1
$\bot$	$\bot$	1	$\bot$	1
$\top$	$\top$	1	1	$\top$

$\alpha \vee \beta$

Table 2.

α \ β	0	1	$\bot$	$\top$
0	0	0	0	0
1	0	1	$\bot$	$\top$
$\bot$	0	$\bot$	$\bot$	0
$\top$	0	$\top$	0	$\top$

$\alpha \wedge \beta$

Table 3.

We found it useful to recall an intuitive meaning for $\mathcal{FOUR}$, as given by e.g. John Fox [13]. We will use this meaning in an example in Section 2.11.2. Let $x \in \mathcal{V}_4$ and $\alpha \in \mathcal{L}_c$, then

- $x(\alpha) = 0$ means: we are informed that α is false, but not informed that α is true;
- $x(\alpha) = 1$ means: we are informed that α is true, but not informed that α is false;
- $x(\alpha) = \bot$ means: we are neither informed that α is true nor informed that α is false;
- $x(\alpha) = \top$ means: we are both informed that α is true and informed that α is false.

Now consider the following *simplifying assumption*: the information about any formula can always be obtained only from the information about the propositional symbols. In other words, the initial information is the one about the propositional symbols and the information about general formulas is constructed from the initial information. Then, intuitively

(1) We are informed (inf.) that $\neg\alpha$ is true iff we are inf. that α is false;

(2) We are inf. that $\neg\alpha$ is false iff we are inf. that α is true;

(3) We are inf. that $\alpha \vee \beta$ is true iff we are inf. that α is true or inf. that β is true;

(4) We are inf. that $\alpha \vee \beta$ is false iff we are inf. that α is false and inf. that β is false;

(5) We are inf. that $\alpha \wedge \beta$ is true iff we are inf. that α is true and inf. that β is true;

(6) We are inf. that $\alpha \wedge \beta$ is false iff we are inf. that α is false or inf. that β is false.

Note that without the simplifying assumption only the left-to-right implications of (3) and (6) do not longer hold. Now, (1) and (2) are formalized in Table 1, (3) and (4) are formalized in Table 2, and (5) and (6) are formalized in Table 3. Thus, every $\mathcal{FOUR}$-valuation represents a way to be informed about the formulas under the simplifying assumption. And clearly, every such way is represented by a $\mathcal{FOUR}$-valuation.

Roughly speaking, $(A0)$, $(A1)$, $(A2)$, $(A3)$, but *not* $(A4)$ hold in the $\mathcal{FOUR}$ framework. Indeed, first (1) entails $(A0)$, second (1) and (2) entail $(A1)$, $(A2)$, but *not* $(A4)$, and third (1) and (3) entail $(A3)$, where

(1) $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_4$, and $\models = \models_4$;

(2) $\forall\, \alpha, \beta \in \mathcal{L},\ neg(\alpha) = \neg\alpha,\ or(\alpha, \beta) = \alpha \vee \beta,\ and(\alpha, \beta) = \alpha \wedge \beta$;

(3) $\mathcal{A}$ is finite.

It can be easily checked that our assumptions still hold in the richer languages of [1, 2, 3].

2.10 The J_3 framework

The logic J_3 was introduced in [10] to answer a question posed in 1948 by S. Jaśkowski, who was interested in systematizing theories capable of containing contradictions, especially if they occur in dialectical reasoning. The step from informal reasoning under contradiction and formal reasoning with databases and information was done in [8] (also specialized for real database models in [9]), where another formulation of J_3 called **LFI1** was introduced, and its first-order version, semantics and proof theory were studied in detail. Investigations of J_3 have also been made by Avron in e.g. [4] where richer languages than here are considered. In addition, proof systems for $\vdash$ in the J_3 framework can be found in e.g. [4, 10] and the chapter IX of [11]. Now, from e.g. [10]:

Definition 12 We say that x is a J_3-valuation iff x is a function from $\mathcal{L}_c$ to $\{0, 1, \top\}$ such that $x(0) = 0$, $x(1) = 1$, and the truth tables 4, 5, and 6 below hold. We denote by $\mathcal{V}_3$ the set of all J_3-valuations. We denoted by $\models_3$ the relation on $\mathcal{V}_3 \times \mathcal{L}_c$ such that $\forall\, x \in \mathcal{V}_3$, $\forall\, \alpha \in \mathcal{L}_c$, $x \models_3 \alpha$ iff $x(\alpha) \in \{1, \top\}$. We call $\models_3$ the J_3-satisfaction relation.

α	$\neg\alpha$
0	1
1	0
$\top$	$\top$

Table 4.

α \ β	0	1	$\top$
0	0	1	$\top$
1	1	1	1
$\top$	$\top$	1	$\top$

$\alpha \vee \beta$

Table 5.

α \ β	0	1	$\top$
0	0	0	0
1	0	1	$\top$
$\top$	0	$\top$	$\top$

$\alpha \wedge \beta$

Table 6.

The same intuitive meaning as for $\mathcal{FOUR}$ is valid, except that the following is added to the simplifying assumption: for every propositional symbol, we are at least informed that it is true or informed that it is false.

Roughly speaking, $(A0)$, $(A1)$, $(A2)$, $(A3)$, and $(A4)$ hold in the J_3 framework. More precisely, first (1) entails $(A0)$, second (1) and (2) entail $(A1)$, $(A2)$, and $(A4)$, and third (1) and (3) entail $(A3)$, where

(1) $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_3$, and $\models = \models_3$;

(2) $\forall\, \alpha, \beta \in \mathcal{L},\ neg(\alpha) = \neg\alpha,\ or(\alpha, \beta) = \alpha \vee \beta,\ and(\alpha, \beta) = \alpha \wedge \beta$;

(3) $\mathcal{A}$ is finite.

It can be easily checked that our assumptions still hold in the richer languages of [4].

2.11 Different frameworks, different drawbacks

Pivotal (resp. pivotal-discriminative) CRs suffer from the following drawback: some sets of formulas are rendered useless in the sense that everything (resp. nothing) can be concluded from them. A set of formulas Γ is rendered useless in two situations:

- there does not exist a model for Γ;
- there exist models for Γ, but they are not in the pivot.

Note that in the $\mathcal{FOUR}$ and J_3 frameworks, these two situations happen less often than in the classical framework. Roughly speaking, this is due to the fact that in the $\mathcal{FOUR}$ and J_3 frameworks, there are much more valuations than in the classical framework. Consequently, it is harder to be left with no model or no model in the pivot. In particular, for all classically inconsistent $\Gamma \subseteq \mathcal{L}_c$, there are generally $\mathcal{FOUR}$ and J_3 models. Thus, in the $\mathcal{FOUR}$ and J_3 frameworks, pivotal and pivotal-discriminative CRs are paraconsistent.

On the other hand, in the paraconsistent frameworks, the Disjunctive Syllogism is not satisfied, unlike in the classical framework. The Disjunctive Syllogism says that from α and $\neg\alpha \vee \beta$, we can infer β. Let us illustrate all of this with examples.

2.11.1 A particular pivotal CR in the classical framework

Suppose that we are in the classical framework (more formally, suppose $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_c$, and $\models = \models_c$) and suppose $\mathcal{A} = \{r, q, p\}$. Recall that $\mathcal{A}$ is a set of propositional symbols introduced in Section 2.8. Intuitively, r means that Nixon is a republican, q means that Nixon is a quaker, and p

means that Nixon is a pacifist. Then, $\mathcal{V}$ is the set of the 8 following classical valuations: v_0, v_1, v_2, v_3, v_4, v_5, v_6, and v_7, which are defined in the obvious way by the following table:

	r	q	p
v_0	0	0	0
v_1	0	0	1
v_2	0	1	0
v_3	0	1	1
v_4	1	0	0
v_5	1	0	1
v_6	1	1	0
v_7	1	1	1

Now, consider the class of all republicans and the class of all quakers. Consider that a republican is normal iff he is not a pacifist and that a quaker is normal iff he is a pacifist. And, consider that a valuation is negligible if (in it) Nixon is a non-normal individual of some class. This is formalized by the following pivot $P = \{x \in \mathcal{V} : \text{if } x \models r, \text{ then } x \models \neg p, \text{ and if } x \models q, \text{ then } x \models p\}$. Let $|\sim$ be the pivotal CR defined by P.

Then, $|\sim$ leads us to "jump" to plausible conclusions from incomplete information. For instance, $r \mid\sim \neg p$ and $q \mid\sim p$. Thus, pivotal CRs can be useful to handle incomplete information.

But, we loose many sets of formulas because there is no model in the pivot. For instance: $\{q, r\} \mid\sim \alpha, \forall\, \alpha \in \mathcal{L}$. In particular, we suffer from this drawback if we face new information that contradict previous "hasty" conclusions. For instance, $\{r, p\} \mid\sim \alpha, \forall\, \alpha \in \mathcal{L}$, and $\{q, \neg p\} \mid\sim \alpha$, $\forall\, \alpha \in \mathcal{L}$. We emphasize that this is the main difference compared to preferential CRs. Indeed, with preferential CRs, we can revise previous "hasty" conclusions in the face of new information that contradict them. Examples can be found in [7]. And, a lot of sets of formulas are rendered useless because there is just no model. For instance: $\{p, \neg p, q\} \mid\sim \alpha, \forall\, \alpha \in \mathcal{L}$. In other words, $|\sim$ is not paraconsistent.

2.11.2 A particular pivotal CR in the $\mathcal{FOUR}$ framework

Suppose that we are in the $\mathcal{FOUR}$ framework (more formally, suppose $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_4$, and $\models = \models_4$) and suppose again that $\mathcal{A} = \{r, q, p\}$. Consider the same classes, etc. as in the classical case, except that this time a valuation is considered to be negligible if (in it) we are informed that Nixon is an individual of some class, but not informed that Nixon is a normal individual of that class. This is formalized by the following pivot $P = \{x \in \mathcal{V} : \text{if } x \models r, \text{ then } x \models \neg p, \text{ and if } x \models q, \text{ then } x \models p\}$. Let $|\sim$ be the pivotal CR defined by P.

Then, again $|\sim$ leads us to "jump" to plausible conclusions from incomplete information. For instance, $r \mid\sim \neg p$ and $q \mid\sim p$. In addition, less sets of formulas are rendered useless because there is no model in the pivot. For instance, $\{q, r\} \mid\sim p$ and $\{q, r\} \mid\sim \neg p$ and $\{q, r\} \mid\sim q$ and $\{q, r\} \not\mid\sim \neg q$ and $\{q, r\} \mid\sim r$ and $\{q, r\} \not\mid\sim \neg r$. In particular, we avoid trivialization if we face new information that contradict previous "hasty" conclusions. For instance, $\{r, p\} \mid\sim p$ and $\{r, p\} \mid\sim \neg p$ and $\{r, p\} \mid\sim r$ and $\{r, p\} \not\mid\sim \neg r$ and $\{r, p\} \not\mid\sim q$ and $\{r, p\} \not\mid\sim \neg q$. Another example: $\{q, \neg p\} \mid\sim \neg p$ and $\{q, \neg p\} \mid\sim p$ and $\{q, \neg p\} \mid\sim q$ and $\{q, \neg p\} \not\mid\sim \neg q$ and $\{q, \neg p\} \not\mid\sim r$ and $\{q, \neg p\} \not\mid\sim \neg r$. In addition, less sets of formulas are rendered useless because there is just no model. For instance, $\{p, \neg p, q\} \mid\sim p$ and $\{p, \neg p, q\} \mid\sim \neg p$ and $\{p, \neg p, q\} \mid\sim q$ and $\{p, \neg p, q\} \not\mid\sim \neg q$ and $\{p, \neg p, q\} \not\mid\sim r$ and $\{p, \neg p, q\} \not\mid\sim \neg r$. In fact, $|\sim$ is paraconsistent. Thus, in the $\mathcal{FOUR}$ framework, pivotal CRs can be useful to handle both incomplete and inconsistent information.

However, $|\sim$ does not satisfy the Disjunctive Syllogism. Indeed, for instance, $\{\neg r, r \vee q\} \not|\sim q$.

2.12 Related Work

The theory of pivotal CRs has been widely investigated in the classical framework. Characterizations have already been given for them. For instance, the following one appears to be part of folklore for decades: definable pivotal CRs correspond precisely to supraclassical closure operations that are compact and satisfy Disjunction in the premisses. For more details see e.g. [20, 17, 18].

In addition, David Makinson argued in [17, 18] that pivotal CRs are conceptual bridges between $\vdash$ and a principal kind of nonmonotonic CRs: *preferential* CRs, investigated in e.g. [7, 14, 16, 21, 22, 23, 24]. Similarly, Makinson argued that definable pivotal CRs are conceptual bridges between $\vdash$ and another kind of nonmonotonic CRs: *default-assumptions* CRs that were introduced by Poole in [19]. We can now add that pivotal-discriminative CRs are conceptual bridges between $\vdash_d$ and *preferential-discriminative* CRs, investigated in [7]. Note that pivotal CRs correspond to Makinson's pivotal-valuation CRs, while definable pivotal CRs correspond to Makinson's pivotal-assumption CRs.

Technically, this paper has many points in common with [7] from the present author. We will use some results from [7] as well as some results from [24]. And, we would like to emphasize that many techniques we use for pivotal CRs are inspired by techniques that Karl Schlechta used for preferential CRs.

3 Definable pivotal CRs

3.1 The necessary and sufficient conditions

In this section, we provide in our general framework a normal characterization of the family of all definable pivotal CRs. Note that another characterization of this family has already been given under the assumption that we are in the classical framework: the definable pivotal CRs correspond precisely to the supraclassical closure operations that are compact and satisfy Disjunction in the premisses. For more details see e.g. [17].

Before going further, note that *for the rest of the paper* each time we write one of the following letters: α, β, γ, Γ, Δ, Θ, x, y, z, X, Y, Z, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ (possibly with exponents and/or subscripts), we assume $\alpha, \beta, \gamma \in \mathcal{L}$ and $\Gamma, \Delta, \Theta \subseteq \mathcal{L}$ and $x, y, z \in \mathcal{V}$ and $X, Y, Z \subseteq \mathcal{V}$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$. Thus, each time we write $\forall\alpha$, $\forall X$, $\exists\alpha$, $\exists X$, etc., we mean $\forall\alpha \in \mathcal{L}$, $\forall X \subseteq \mathcal{V}$, $\exists\alpha \in \mathcal{L}$, $\exists X \subseteq \mathcal{V}$, etc.

In Proposition 1.1 of [23], Karl Schlechta gave for any function $\mu : \mathbf{Y} \to \mathbf{Y}$ necessary and sufficient conditions for the existence of a preferential structure $\mathcal{Z}$ such that $\forall\ X \in \mathbf{Y}$, $\mu(X) = \mu_{\mathcal{Z}}(X)$, where $\mu_{\mathcal{Z}}(X)$ is the set of the *most preferred* valuations of X (i.e. the valuations such that at least one copy of which is minimal among the set of all copies of valuations in X). Then, Schlechta used his general Proposition 1.1 in the classical framework to provide normal characterizations of certain families of definability preserving preferential CRs. Definability preserving means that if $X \in \mathbf{D}$, then $\mu_{\mathcal{Z}}(X) \in \mathbf{D}$ too. Then, in [7], we used Proposition 1.1 in our general framework to show that the Schlechta's characterizations still hold. In addition, we used Proposition 1.1 in our general framework again to provide normal characterizations of certain families of definability preserving preferential-discriminative CRs. For more details, see [7].

The idea is now to use similar techniques with pivotal CRs. More precisely, in Lemma 13 below, we give for any function $\mu : \mathbf{Y} \to \mathbf{Z}$, necessary and sufficient conditions for the existence of a pivot P such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = X \cap P$. Then, in the present section, we will use Lemma 13 in our general framework to provide normal characterizations of certain families of definable pivotal CRs. In addition, in Section 8, we will use Lemma 13 in our general framework again to provide normal characterizations of certain families of definable pivotal-discriminative CRs.

Lemma 13 Let $\mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \to \mathbf{Z}$ be a function. Then, there exists a pivot P such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = X \cap P$ iff

$(\mu 0)$ $\forall\, X \in \mathbf{Y}$, $\mu(X) \subseteq X$;

$(\mu 1)$ $\forall\, X, Y \in \mathbf{Y}$, $\mu(Y) \cap X \subseteq \mu(X)$.

Proof Direction: "$\rightarrow$". Let $X, Y \in \mathbf{Y}$. We show $(\mu 0)$. $\mu(X) = X \cap P \subseteq X$. We show $(\mu 1)$. $\mu(Y) \cap X = Y \cap P \cap X \subseteq P \cap X = \mu(X)$.

Direction: "$\leftarrow$". Let $P = \{x : \exists\, X \in \mathbf{Y},\, x \in \mu(X)\}$. Let $X \in \mathbf{Y}$. We show $\mu(X) \subseteq X \cap P$. If $x \in \mu(X)$, then $x \in_{(\mu 0)} X$ and $x \in P$. We show $X \cap P \subseteq \mu(X)$. Let $x \in X \cap P$. Then, $\exists\, Y \in \mathbf{Y}$, $x \in \mu(Y)$. Thus, $x \in \mu(Y) \cap X \subseteq_{(\mu 1)} \mu(X)$. ∎

Proposition 14 Let $|\sim$ be a CR. Then, $|\sim$ is a definable pivotal CR iff

$(|\sim 0)$ $\forall\, \Gamma, \Delta,$ if $\overline{\Gamma} = \overline{\Delta}$, then $\widetilde{\Gamma} = \widetilde{\Delta}$;

$(|\sim 1)$ $\forall\, \Gamma,\ \overline{\widetilde{\Gamma}} = \widetilde{\Gamma}$;

$(|\sim 2)$ $\forall\, \Gamma,\ \Gamma \subseteq \widetilde{\Gamma}$;

$(|\sim 3)$ $\forall\, \Gamma, \Delta,\ \widetilde{\Gamma} \subseteq \overline{\widetilde{\Delta} \cup \Gamma}$.

Proof Direction: "$\rightarrow$". There exists a definable pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} = T(M_\Gamma \cap P)$. We show $(|\sim 0)$. If $\overline{\Gamma} = \overline{\Delta}$, then $M_\Gamma = M_\Delta$, thus $\widetilde{\Gamma} = T(M_\Gamma \cap P) = T(M_\Delta \cap P) = \widetilde{\Delta}$. We show $(|\sim 1)$. $\overline{\widetilde{\Gamma}} = T(M_{T(M_\Gamma \cap P)}) = T(M_\Gamma \cap P) = \widetilde{\Gamma}$. We show $(|\sim 2)$. $\Gamma \subseteq T(M_\Gamma) \subseteq T(M_\Gamma \cap P) = \widetilde{\Gamma}$.
Let $\mu : \mathbf{D} \to \mathbf{D}$ be the function such that $\forall\, X \in \mathbf{D}$, $\mu(X) = X \cap P$. As $P, X \in \mathbf{D}$, $X \cap P \in \mathbf{D}$, thus μ is well-defined. By Lemma 13, we get $(\mu 0)$ and $(\mu 1)$. Moreover, $\forall\, \Gamma$, $\mu(M_\Gamma) \in \mathbf{D}$, thus $\mu(M_\Gamma) = M_{T(\mu(M_\Gamma))} = M_{T(M_\Gamma \cap P)} = M_{\widetilde{\Gamma}}$. We show $(|\sim 3)$. $\forall\, \Gamma, \Delta$, $M_{\widetilde{\Delta}} \cap M_\Gamma = \mu(M_\Delta) \cap M_\Gamma \subseteq_{(\mu 1)} \mu(M_\Gamma) = M_{\widetilde{\Gamma}}$.

Direction: "$\leftarrow$". Let $\mu : \mathbf{D} \to \mathbf{D}$ be the function such that $\forall\, \Gamma$, $\mu(M_\Gamma) = M_{\widetilde{\Gamma}}$. If $M_\Gamma = M_\Delta$, then $\overline{\Gamma} = \overline{\Delta}$, thus $M_{\widetilde{\Gamma}} =_{(|\sim 0)} M_{\widetilde{\Delta}}$. Thus, μ is well-defined. We show $(\mu 0)$. $\forall\, \Gamma$, $\mu(M_\Gamma) = M_{\widetilde{\Gamma}} \subseteq_{(|\sim 2)} M_\Gamma$. We show $(\mu 1)$. $\forall\, \Gamma, \Delta$, $\mu(M_\Delta) \cap M_\Gamma = M_{\widetilde{\Delta}} \cap M_\Gamma = M_{\widetilde{\Delta} \cup \Gamma} \subseteq_{(|\sim 3)} M_{\widetilde{\Gamma}} = \mu(M_\Gamma)$. Thus, by Lemma 13, there exists a pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} =_{(|\sim 1)} \overline{\widetilde{\Gamma}} = T(M_{\widetilde{\Gamma}}) = T(\mu(M_\Gamma)) = T(M_\Gamma \cap P)$. In addition, $P = M_\emptyset \cap P = \mu(M_\emptyset) \in \mathbf{D}$. ∎

Note that the conditions: $(|\sim 0)$, $(|\sim 1)$, $(|\sim 2)$, and $(|\sim 3)$ depend only on $|\sim$, $\vdash$ and $\mathcal{L}$. As, $\vdash$ has been defined semantically (in Section 2.2), these conditions have a semantic aspect. In parallel, as we have a proof system for $\vdash$ in the classical, $\mathcal{FOUR}$, and J_3 frameworks, the conditions have also a syntactic aspect in these frameworks.

3.2 Normal characterization

We now give in a straightforward manner a normal characterization of definable pivotal CRs. Let Φ be the relation over $\mathcal{P}(\mathcal{L})^4$ such that $\forall\, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff

- if $\overline{\Gamma_1} = \overline{\Gamma_2},$ then $\Gamma_3 = \Gamma_4$;
- $\overline{\Gamma_3} = \Gamma_3$;
- $\Gamma_1 \subseteq \Gamma_3$;
- $\Gamma_3 \subseteq \overline{\Gamma_4 \cup \Gamma_1}$.

Then, by Proposition 14, $|\sim$ is a definable pivotal CR iff $\forall\, \Gamma_1, \Gamma_2$, $\Phi(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$.

4 Pivotal CRs

In this section, we give a non-normal characterization of the family of all pivotal CRs. We cannot use Lemma 13 to characterize the pivotal CRs as we did to characterize the definable pivotal CRs in Proposition 14. Roughly speaking, this is due to the fact that, in the "$\rightarrow$" direction, from $\widetilde{\Gamma} = T(\mu(M_\Gamma))$, we can no longer conclude $\mu(M_\Gamma) = M_{\widetilde{\Gamma}}$, as $\mu(M_\Gamma)$ is no longer necessarily in $\mathbf{D}$. Thus, we cannot get the conditions on $|\sim$ from the conditions on μ (clearly, the "$\leftarrow$" direction still holds, as if $|\sim$ is a definable pivotal CR, then it is also a pivotal CR).

Karl Schlechta encountered a similar problem in [24] when dealing with preferential CRs (not necessarily definability preserving). Here is his remedy. In Proposition 5.2.5 of [24], he gave for any function $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$, necessary and sufficient conditions for the existence of a preferential structure $\mathcal{Z}$ such that $\forall\, X \in \mathbf{Y}$, $\mu(X)$ is the smallest element of $\mathbf{Y}$ that contains $\mu_{\mathcal{Z}}(X)$ (the most preferred valuations of X). Note that it is assumed that $\mathbf{Y}$ is closed under arbitrary intersections and finite unions, and $\emptyset, \mathcal{V} \in \mathbf{Y}$. Then, Schlechta used his general Proposition 5.2.5 in the classical framework to provide non-normal characterizations of certain families of preferential CRs.

Then, strongly inspired by Proposition 5.2.5, we gave in Lemma 3.3 of [7] for any function $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$, necessary and sufficient conditions for the existence of a preferential structure $\mathcal{Z}$ such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$. Note that, unlike Proposition 5.2.5, Lemma 3.3 requires no further prerequisites. Then, in [7], we used Lemma 3.3 in our general framework to provide non-normal characterizations of certain families of preferential CRs and of certain families of preferential-discriminative CRs.

The idea is now to use similar techniques with pivotal CRs. More precisely, in Lemma 16 below, we give for any function $\mu : \mathbf{Y} \rightarrow \mathbf{Z}$, necessary and sufficient conditions for the existence of a pivot P such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(X \cap P)}$. Then, in the present section and in Section 5, we will use Lemma 16 in our general framework to provide non-normal characterizations of certain families of pivotal CRs. In addition, in Section 9, we will use Lemma 16 in our general framework again to provide non-normal characterizations of certain families of pivotal-discriminative CRs.

Definition 15 Let $\mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \rightarrow \mathbf{Z}$ be a function. We denote by $\mu' : \mathbf{Y} \rightarrow \mathcal{P}(\mathcal{V})$ the function such that $\forall\, X \in \mathbf{Y}$

$$\mu'(X) = \{x \in X : \forall\, Y \in \mathbf{Y},\ \text{if } x \in Y,\ \text{then } x \in \mu(Y)\}.$$

Lemma 16 Let $\mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \to \mathbf{Z}$ be a function. Then, there exists a pivot P such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(X \cap P)}$ iff

$(\mu 2)$ $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(\mu'(X))}$.

Proof Direction: "$\to$". We show $(\mu 2)$. Let $X \in \mathbf{Y}$.
Case 1: $\exists\, x \in X \cap P$, $x \notin \mu'(X)$. Then, $\exists\, Y \in \mathbf{Y}$, $x \in Y$ and $x \notin \mu(Y) = M_{T(Y \cap P)} \supseteq Y \cap P$, which is impossible.
Case 2: $X \cap P \subseteq \mu'(X)$.
Case 2.1: $\exists\, x \in \mu'(X)$, $x \notin \mu(X)$. Then, $x \in X$ and $\forall\, Y \in \mathbf{Y}$, if $x \in Y$, then $x \in \mu(Y)$, thus $x \in \mu(X)$, which is impossible.
Case 2.2: $\mu'(X) \subseteq \mu(X)$. Then, $\mu(X) = M_{T(X \cap P)} \subseteq M_{T(\mu'(X))} \subseteq M_{T(\mu(X))} = M_{T(M_{T(X \cap P)})} = M_{T(X \cap P)} = \mu(X)$.

Direction: "$\leftarrow$". Let $P = \{x : \exists\, X \in \mathbf{Y},\ x \in \mu'(X)\}$ and $X \in \mathbf{Y}$. Clearly, $\mu'(X) \subseteq X \cap P$. We show $X \cap P \subseteq \mu'(X)$. Let $x \in X \cap P$. Then, $\exists\, Y \in \mathbf{Y}$, $x \in \mu'(Y)$. Thus, $\forall\, Z \in \mathbf{Y}$, if $x \in Z$, then $x \in \mu(Z)$. Thus, by definition of μ', $x \in \mu'(X)$. Thus, $\mu(X) =_{(\mu 2)} M_{T(\mu'(X))} = M_{T(X \cap P)}$. ∎

Proposition 17 Let $|\sim$ be a CR. Then, $|\sim$ is a pivotal CR iff

$(|\sim 4)$ $\forall\, \Gamma,\ \widetilde{\Gamma} = T(\{x \in M_\Gamma : \forall\, \Delta,\ \text{if } x \in M_\Delta,\ \text{then } x \in M_{\widetilde{\Delta}}\})$.

Proof Direction: "$\to$". There exists a pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} = T(M_\Gamma \cap P)$. Let $\mu : \mathbf{D} \to \mathbf{D}$ be the function such that $\forall\, X \in \mathbf{D}$, $\mu(X) = M_{T(X \cap P)}$. By Lemma 16, we get $(\mu 2)$. In addition, $\forall\, \Gamma$, $\mu(M_\Gamma) = M_{\widetilde{\Gamma}}$. We show $(|\sim 4)$. Let $\Gamma \subseteq \mathcal{L}$. Then, $\widetilde{\Gamma} = T(M_\Gamma \cap P) = T(M_{T(M_\Gamma \cap P)}) = T(\mu(M_\Gamma)) =_{(\mu 2)} T(M_{T(\mu'(M_\Gamma))}) = T(\mu'(M_\Gamma)) = T(\{x \in M_\Gamma : \forall\, Y \in \mathbf{D}, \text{ if } x \in Y, \text{ then } x \in \mu(Y)\}) = T(\{x \in M_\Gamma : \forall\, \Delta, \text{if } x \in M_\Delta, \text{then } x \in M_{\widetilde{\Delta}}\})$.

Direction: "$\leftarrow$". Let $\mu : \mathbf{D} \to \mathbf{D}$ be the function such that $\forall\, \Gamma$, $\mu(M_\Gamma) = M_{\widetilde{\Gamma}}$. If $M_\Gamma = M_\Theta$, then $\widetilde{\Gamma} =_{(|\sim 4)} \widetilde{\Theta}$, thus μ is well-defined. We show that μ satisfies $(\mu 2)$. Let $\Gamma \subseteq \mathcal{L}$. $\mu(M_\Gamma) = M_{\widetilde{\Gamma}} =_{(|\sim 4)} M_{T(\{x \in M_\Gamma : \forall\, \Delta, \text{ if } x \in M_\Delta, \text{ then } x \in M_{\widetilde{\Delta}}\})} = M_{T(\{x \in M_\Gamma : \forall\, Y \in \mathbf{D}, \text{ if } x \in Y, \text{ then } x \in \mu(Y)\})} = M_{T(\mu'(M_\Gamma))}$. Thus, by Lemma 16, there exists a pivot P such that $\widetilde{\Gamma} =_{(|\sim 4)} T(M_{\widetilde{\Gamma}}) = T(\mu(M_\Gamma)) = T(M_{T(M_\Gamma \cap P)}) = T(M_\Gamma \cap P)$. ∎

We cannot get a normal characterization of pivotal CRs from Proposition 17, because $(|\sim 4)$ depends on too many subsets of $\mathcal{L}$. Recall that more explanations have been given in Section 2.6.

5 Codefinable pivotal CRs

In this section, we give a non-normal characterization of the family of all codefinable pivotal CRs, under the assumption that $(A0)$ holds.

Lemma 18 Let $(A0)$ holds, $\mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{D} \subseteq \mathbf{Y}$ and $\mu : \mathbf{Y} \to \mathbf{Z}$ be a function. Then, there exists a codefinable pivot P such that $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(X \cap P)}$ iff $(\mu 2)$ and

$(\mu 3)$ $\mathcal{V} \setminus \mu'(\mathcal{V}) \in \mathbf{D}$.

Proof Direction: "$\to$". We show

(0) μ satisfies $(\mu 2)$;

(1) $\mu'(\mathcal{V}) = P$;

(2) μ satisfies $(\mu 3)$.

Proof of (0). Obvious by Lemma 16.
Proof of (1). Direction: "$\subseteq$". Suppose $\exists\, x \in \mu'(\mathcal{V})$, $x \notin P$. Then, $x \in \mathcal{V} \setminus P \in \mathbf{D} \subseteq \mathbf{Y}$. However, $\forall Y \in \mathbf{Y}$, if $x \in Y$, then $x \in \mu(Y)$. Thus, $x \in \mu(\mathcal{V} \setminus P) = M_{T((\mathcal{V}\setminus P)\cap P)} = M_{T(\emptyset)} = M_{\mathcal{L}} =_{(A0)} \emptyset$, which is impossible.
Direction: "$\supseteq$". Suppose $\exists\, x \in P$, $x \notin \mu'(\mathcal{V})$. As $x \in \mathcal{V}$, $\exists\, Y \in \mathbf{Y}$, $x \in Y$ and $x \notin \mu(Y) = M_{T(Y\cap P)} \supseteq Y \cap P$, which is impossible.
Proof of (2). $\mathcal{V} \setminus \mu'(\mathcal{V}) =_{(1)} \mathcal{V} \setminus P \in \mathbf{D}$, as P is codefinable.

Direction: "$\leftarrow$". Let $P = \{x : \exists\, X \in \mathbf{Y},\ x \in \mu'(X)\}$. We show

(0) $\forall\, X \in \mathbf{Y}$, $\mu'(X) = X \cap P$;

(1) $\mathcal{V} \setminus P \in \mathbf{D}$;

(2) $\forall\, X \in \mathbf{Y}$, $\mu(X) = M_{T(X\cap P)}$.

Proof of (0). Direction: "$\subseteq$". Obvious.
Direction: "$\supseteq$". Let $x \in X \cap P$. Then, $\exists\, Y \in \mathbf{Y}$, $x \in \mu'(Y)$. Thus, $\forall\, Z \in \mathbf{Y}$, if $x \in Z$, then $x \in \mu(Z)$. Thus, by definition of μ', $x \in \mu'(X)$
Proof of (1). $\mathcal{V} \setminus P = \mathcal{V} \setminus (\mathcal{V} \cap P) =_{(0)} \mathcal{V} \setminus \mu'(\mathcal{V})$, as $\mathcal{V} = M_{\emptyset} \in \mathbf{D} \subseteq \mathbf{Y}$. But, $\mathcal{V} \setminus \mu'(\mathcal{V}) \in_{(\mu 3)} \mathbf{D}$.
Proof of (2). $\mu(X) =_{(\mu 2)} M_{T(\mu'(X))} =_{(0)} M_{T(X\cap P)}$ ∎

Proposition 19 Let $(A0)$ holds and $|\sim$ be a CR. Then, $|\sim$ is a codefinable pivotal CR iff $(|\sim 4)$, and

$(|\sim 5)$ $\mathcal{V} \setminus \{x \in \mathcal{V} : \forall\, \Delta,\ \text{if } x \in M_{\Delta},\ \text{then } x \in M_{\widetilde{\Delta}}\} \in \mathbf{D}$.

Proof Direction: "$\rightarrow$". First, by Proposition 17, we get $(|\sim 4)$. Now, there is a codefinable pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} = T(M_{\Gamma} \cap P)$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall\, X \in \mathbf{D}$, $\mu(X) = M_{T(X\cap P)}$. By Lemma 18, $(\mu 3)$ holds. Moreover $\forall\, \Gamma$, $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}}$. We show $(|\sim 5)$.
$\mathcal{V} \setminus \{x : \forall\, \Delta, \text{if } x \in M_{\Delta}, \text{then } x \in M_{\widetilde{\Delta}}\} = \mathcal{V} \setminus \{x : \forall\, Y \in \mathbf{D}, \text{if } x \in Y, \text{then } x \in \mu(Y)\} = \mathcal{V} \setminus \mu'(\mathcal{V}) \in_{(\mu 3)} \mathbf{D}$.

Direction: "$\leftarrow$". Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall\, \Gamma$, $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}}$. If $M_{\Gamma} = M_{\Theta}$, then $\widetilde{\Gamma} =_{(|\sim 4)} \widetilde{\Theta}$, thus μ is well-defined. We now show $(\mu 2)$. Let $\Gamma \subseteq \mathcal{L}$. Then, $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}} =_{(|\sim 4)} M_{T(\{x\in M_{\Gamma} : \forall\, \Delta, \text{ if } x\in M_{\Delta}, \text{ then } x\in M_{\widetilde{\Delta}}\})} = M_{T(\mu'(M_{\Gamma}))}$. We show $(\mu 3)$. $\mathcal{V} \setminus \mu'(\mathcal{V}) = \mathcal{V} \setminus \{x : \forall\, \Delta,\ \text{if } x \in M_{\Delta},\ \text{then } x \in M_{\widetilde{\Delta}}\} \in_{(|\sim 5)} \mathbf{D}$. Thus, by Lemma 18, there exists a codefinable pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} =_{(|\sim 4)} T(M_{\widetilde{\Gamma}}) = T(\mu(M_{\Gamma})) = T(M_{T(M_{\Gamma}\cap P)}) = T(M_{\Gamma} \cap P)$. ∎

We cannot get a normal characterization of codefinable pivotal CRs from Proposition 19, because $(|\sim 4)$ and $(|\sim 5)$ depend on too many subsets of $\mathcal{L}$. Recall that more explanations have been given in Section 2.6.

6 A link with X-logics

In this section, we define several families of pivotal ECRs and give easy characterizations for them. Then, we will show that codefinable pivotal ECRs are precisely closed pertinence ECRs (alias closed X-logics), under the assumption that $(A1)$ holds.

Definition 20 Let $||\sim$ be an ECR. We say that $||\sim$ is a pivotal ECR iff there exists a pivot P such that $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta$. We say that $||\sim$ is definable, codefinable, etc. iff $||\sim$ can be defined by a definable, codefinable, etc. pivot.

Proposition 21 Let $||\sim$ be an ECR. Then, $||\sim$ is a codefinable pivotal ECR iff

(0) $|\sim$ is a codefinable pivotal CR;

(1) $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $\forall\, \alpha \in \Delta, \Gamma\ |\sim \alpha$;

where $|\sim$ is the CR such that $\forall\, \Gamma, \alpha, \Gamma\ |\sim \alpha$ iff $\Gamma\ ||\sim \{\alpha\}$.

Proof Direction: "$\rightarrow$". There exists a codefinable pivot P such that $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta$. We show (0). $\forall\, \Gamma, \alpha, \Gamma\ |\sim \alpha$ iff $\Gamma\ ||\sim \{\alpha\}$ iff $M_\Gamma \cap P \subseteq M_\alpha$. We show (1). $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta = \bigcap_{\alpha\in\Delta} M_\alpha$ iff $\forall\, \alpha \in \Delta$, $M_\Gamma \cap P \subseteq M_\alpha$ iff $\forall\, \alpha \in \Delta, \Gamma\ |\sim \alpha$, as shown in the proof of (0).

Direction: "$\leftarrow$". By (0), there exists a codefinable pivot P such that $\forall \Gamma, \alpha, \Gamma\ |\sim \alpha$ iff $M_\Gamma \cap P \subseteq M_\alpha$. We show that $||\sim$ is a codefinable pivotal ECR. $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff$_{(1)}$ $\forall\, \alpha \in \Delta, \Gamma\ |\sim \alpha$ iff $\forall\, \alpha \in \Delta$, $M_\Gamma \cap P \subseteq M_\alpha$ iff $M_\Gamma \cap P \subseteq \bigcap_{\alpha\in\Delta} M_\alpha = M_\Delta$. ∎

Consequently, under $(A0)$, by Propositions 19 and 21, we get a characterization of codefinable pivotal ECRs. It is easy to see that by the same technique, we can get characterizations of pivotal ECRs, definable pivotal ECRs, etc.

Notation 22 $\forall\, \Gamma, \Delta$, we denote by $or(\Gamma, \Delta)$ the set s.t. $or(\Gamma, \Delta) = \{or(\alpha, \beta) : \alpha \in \Gamma, \beta \in \Delta\}$.

Proposition 23 Let $(A1)$ holds and $\Gamma, \Delta \subseteq \mathcal{L}$. Then, $M_\Gamma \cup M_\Delta = M_{or(\Gamma,\Delta)}$.

Proof Direction: "$\subseteq$". Let $x \in M_\Gamma \cup M_\Delta$. We show $x \in M_{or(\Gamma,\Delta)}$. Let $\delta \in or(\Gamma, \Delta)$. Then, $\exists\, \alpha \in \Gamma, \exists\, \beta \in \Delta$, $\delta = or(\alpha, \beta)$. But, $x \in M_\alpha \cup M_\beta =_{(A1)} M_{or(\alpha,\beta)} = M_\delta$.
Direction: "$\supseteq$". Suppose $\exists\, x \in M_{or(\Gamma,\Delta)}$, $x \notin M_\Gamma \cup M_\Delta$. Then, $\exists\, \alpha \in \Gamma$, $x \notin M_\alpha$ and $\exists\, \beta \in \Delta$, $x \notin M_\beta$, thus, $x \notin M_\alpha \cup M_\beta =_{(A1)} M_{or(\alpha,\beta)}$, however $or(\alpha, \beta) \in or(\Gamma, \Delta)$, thus $x \notin M_{or(\Gamma,\Delta)}$ which is impossible. ∎

Proposition 24 Let $(A1)$ holds. Then, codefinable pivotal ECRs are precisely closed pertinence ECRs.

Proof Direction: "$\subseteq$". Let $||\sim$ be a codefinable pivotal ECR. Then, there exists a codefinable pivot P such that $\forall\, \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta$. Let $\Sigma = T(\mathcal{V} \setminus P)$. As P is codefinable, $M_\Sigma = M_{T(\mathcal{V}\setminus P)} = \mathcal{V} \setminus P$ and $\overline{\Sigma} = T(M_\Sigma) = T(\mathcal{V} \setminus P) = \Sigma$, thus Σ is a closed pertinence choice. We show that $||\sim$ is a closed pertinence ECR. $\forall \Gamma, \Delta, \Gamma\ ||\sim \Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta$ iff $M_\Gamma \subseteq M_\Delta \cup M_\Sigma$ iff $M_\Gamma \subseteq M_{\Gamma\cup\Delta} \cup M_\Sigma$ iff, by Proposition 23, $M_\Gamma \subseteq M_{or(\Gamma\cup\Delta,\Sigma)}$ iff $T(M_{or(\Gamma\cup\Delta,\Sigma)}) \subseteq T(M_\Gamma)$ iff, by Proposition 23 again, $T(M_{\Gamma\cup\Delta} \cup M_\Sigma) \subseteq T(M_\Gamma)$ iff $T(M_{\Gamma\cup\Delta}) \cap T(M_\Sigma) \subseteq T(M_\Gamma)$ iff $\overline{\Gamma \cup \Delta} \cap \overline{\Sigma} \subseteq \overline{\Gamma}$ iff $\overline{\Gamma \cup \Delta} \cap \Sigma \subseteq \overline{\Gamma}$.

Direction: "$\supseteq$". Let $||\sim$ be a closed pertinence ECR. Then, there exists a closed pertinence choice Σ such that $\forall\, \Gamma, \Delta,\ \Gamma\ ||\sim\ \Delta$ iff $\overline{\Gamma \cup \Delta} \cap \Sigma \subseteq \overline{\Gamma}$. Let $P = \mathcal{V} \setminus M_\Sigma$. Then, $\mathcal{V} \setminus P = M_\Sigma \in \mathbf{D}$, thus P is a codefinable pivot. We show that $||\sim$ is a codefinable pivotal ECR. $\forall\, \Gamma, \Delta,\ \Gamma\ ||\sim\ \Delta$ iff $\overline{\Gamma \cup \Delta} \cap \Sigma \subseteq \overline{\Gamma}$ iff $\overline{\Gamma \cup \Delta} \cap \overline{\Sigma} \subseteq \overline{\Gamma}$ iff $T(M_{\Gamma\cup\Delta}) \cap T(M_\Sigma) \subseteq T(M_\Gamma)$ iff $T(M_{\Gamma\cup\Delta} \cup M_\Sigma) \subseteq T(M_\Gamma)$ iff, by Proposition 23, $T(M_{or(\Gamma\cup\Delta,\Sigma)}) \subseteq T(M_\Gamma)$ iff $M_\Gamma \subseteq M_{or(\Gamma\cup\Delta,\Sigma)}$ iff, by Proposition 23 again, $M_\Gamma \subseteq M_{\Gamma\cup\Delta} \cup M_\Sigma$ iff $M_\Gamma \subseteq M_\Delta \cup M_\Sigma$ iff $M_\Gamma \cap (\mathcal{V} \setminus M_\Sigma) \subseteq M_\Delta$ iff $M_\Gamma \cap P \subseteq M_\Delta$. ∎

Consequently,

Proposition 25 Let $(A0)$, $(A1)$ hold and $||\sim$ be an ECR. Then $||\sim$ is a closed pertinence ECR iff

(0) $|\sim$ satisfies $(|\sim 4)$ and $(|\sim 5)$;

(1) $\forall\, \Gamma, \Delta, \Gamma\ ||\sim\ \Delta$ iff $\forall\, \alpha \in \Delta, \Gamma\ |\sim\ \alpha$;

where $|\sim$ is the CR such that $\forall\, \Gamma, \alpha, \Gamma\ |\sim\ \alpha$ iff $\Gamma\ ||\sim\ \{\alpha\}$.

Proof Obvious by Propositions 19, 21, and 24.

7 There is no normal characterization of pivotal CRs

In this section we show that it is impossible to find a normal characterization of pivotal CRs under the assumption that $(A5)$ and $(A6)$ (defined below) hold. Karl Schlechta has shown in [24] that it is impossible to get a normal characterization of preferential CRs in the infinite classical framework. We have been inspired by his techniques.

Definition 26 Let $(A5)$ and $(A6)$ be the framework assumptions such that

$(A5)$ $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_c$, and $\models = \models_c$ (i.e. we are in the classical framework);

$(A6)$ $\mathcal{A}$ is infinite (i.e. infinite number of propositional symbols).

Recall that $\mathcal{A}$ has been defined in Notation 10. We need an important lemma that Karl Schlechta gave in [24] (called there Lemma 5.2.14).

Lemma 27 From [24]. Let $(A5)$ and $(A6)$ hold and let $\mathbf{Y} \subseteq \{X \subseteq \mathcal{V} : |X| \leq |\mathcal{A}|\}$ be closed under unions of size at most $|\mathcal{A}|$ and subsets. Then, $\forall\, \Gamma$, $\exists\, X_\Gamma \in \mathbf{Y}$,

(0) $T(\bigcap_{X\in\mathbf{Y}} M_{T(M_\Gamma \setminus X)}) = T(M_\Gamma \setminus X_\Gamma)$;

(1) $\forall\, X \in \mathbf{Y}$, $T(M_\Gamma \setminus X) \subseteq T(M_\Gamma \setminus X_\Gamma)$.

Note that the subscript in X_Γ is written just to keep in mind that in the statement of the lemma, it depends on Γ. Now we come to the negative result.

Proposition 28 Let $(A5)$ and $(A6)$ hold. Then, it is impossible to find a normal characterization of pivotal CRs.

Proof Suppose on the contrary that there exist a (finite or infinite) cardinal $\lambda \leq |\mathcal{L}|$ and a relation Φ over $\mathcal{P}(\mathcal{L})^{2\lambda}$ such that for all CR $|\sim$, $|\sim$ is a pivotal CR iff $\forall \Gamma_1, \ldots \Gamma_\lambda,\ \Phi(\Gamma_1, \ldots \Gamma_\lambda, \widetilde{\Gamma_1}, \ldots \widetilde{\Gamma_\lambda})$. Then, let $\mathbf{X}$ be the set such that

$$\mathbf{X} = \{X \subseteq \mathcal{V} : |X| \leq |\mathcal{A}|\}.$$

Let $|\sim$ the CR such that $\forall\, \Gamma$,

$$\widetilde{\Gamma} = T(\bigcap_{X \in \mathbf{X}} M_{T(M_\Gamma \setminus X)}).$$

Note that $\mathbf{X} \neq \emptyset$, thus $|\sim$ is well-defined. We show

(0) $\forall\, X \subseteq \mathcal{V}$, if $|X| \leq |\mathcal{A}|$, then $T(\mathcal{V}) = T(\mathcal{V} \setminus X)$;

(1) $\exists\, \Gamma_1, \ldots \Gamma_\lambda$, $\Phi(\Gamma_1, \ldots \Gamma_\lambda, \widetilde{\Gamma_1}, \ldots \widetilde{\Gamma_\lambda})$ does not hold.

Proof of (0). Let X be such that $|X| \leq |\mathcal{A}|$. Obviously, $T(\mathcal{V}) \subseteq T(\mathcal{V} \setminus X)$. We show $T(\mathcal{V} \setminus X) \subseteq T(\mathcal{V})$ by contradiction. Suppose $\exists\, \alpha \in T(\mathcal{V} \setminus X)$, $\alpha \notin T(\mathcal{V})$. Then, $\exists\, x \in \mathcal{V}$, $x \notin M_\alpha$. Consider the set of valuations Y such that

$$Y = \{y \in \mathcal{V} : \text{for all propositional symbol } q \text{ appearing in } \alpha,\ y(q) = x(q)\}.$$

Then, $\forall\, y \in Y$, $y(\alpha) = x(\alpha)$, thus $y \notin M_\alpha$ Moreover, as the number of propositional symbols appearing in α is finite, we get $|Y| = 2^{|\mathcal{A}|}$. However $|X| \leq |\mathcal{A}|$. Thus, $|X| < |Y|$, thus $\exists\, y \in Y \setminus X \subseteq \mathcal{V} \setminus X$. Thus, $\mathcal{V} \setminus X \not\subseteq M_\alpha$, thus $\alpha \notin T(\mathcal{V} \setminus X)$, which is impossible.

Proof of (1). It suffices to show that $|\sim$ is not a pivotal CR. Suppose on the contrary that there exists a pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} = T(M_\Gamma \cap P)$. We show a contradiction. Let p be a propositional symbol.
Case 1: $\exists\, x \in P$, $x \notin M_p$. We pose $\Gamma = T(\{x\})$. Then, $M_\Gamma = \{x\}$. Thus, $\widetilde{\Gamma} = T(\{x\} \cap P) = T(\{x\})$. Thus, $p \notin \widetilde{\Gamma}$. However, $M_\Gamma \in \mathbf{X}$, thus $\bigcap_{X \in \mathbf{X}} M_{T(M_\Gamma \setminus X)} \subseteq M_{T(M_\Gamma \setminus M_\Gamma)} = M_{\mathcal{L}} = \emptyset$. Thus, $\widetilde{\Gamma} = T(\emptyset)$, thus $p \in \widetilde{\Gamma}$, which is impossible.
Case 2: $P \subseteq M_p$. Then, $\widetilde{\emptyset} = T(\bigcap_{X \in \mathbf{X}} M_{T(\mathcal{V} \setminus X)}) =_{(0)} T(\bigcap_{X \in \mathbf{X}} M_{T(\mathcal{V})}) = T(M_{T(\mathcal{V})}) = T(\mathcal{V})$. Thus, $p \notin \widetilde{\emptyset}$, as $\mathcal{V} \not\subseteq M_p$. However, $\widetilde{\emptyset} = T(\mathcal{V} \cap P) = T(P)$, thus $p \in \widetilde{\emptyset}$, which is impossible.

Now, by lemma 27, we get

(2) $\forall\, \Gamma, \exists\, X_\Gamma \in \mathbf{X}$, $\widetilde{\Gamma} = T(M_\Gamma \setminus X_\Gamma)$ and $\forall\, X \in \mathbf{X}$, $T(M_\Gamma \setminus X) \subseteq T(M_\Gamma \setminus X_\Gamma)$.

Then, Let Z be the set such that

$$Z = \bigcup_{\Gamma \in \{\Gamma_1, \ldots \Gamma_\lambda\}} X_\Gamma,$$

and let $|\sim'$ be the pivotal CR defined by the pivot $\mathcal{V} \setminus Z$. The following entails a contradiction and thus ends the proof. We show

(3) $\forall\, \Gamma \in \{\Gamma_1, \ldots \Gamma_\lambda\}$, $\widetilde{\Gamma} = T(M_\Gamma \setminus Z)$;

(4) $|\sim'$ is not a pivotal CR.

Proof of (3). Let $\Gamma \in \{\Gamma_1, \ldots \Gamma_\lambda\}$. Direction: "$\subseteq$". $X_\Gamma \subseteq Z$, thus $M_\Gamma \setminus Z \subseteq M_\Gamma \setminus X_\Gamma$, thus $\widetilde{\Gamma} =_{(2)} T(M_\Gamma \setminus X_\Gamma) \subseteq T(M_\Gamma \setminus Z)$.
Direction: "$\supseteq$". As $\lambda \leq |\mathcal{L}| = |\mathcal{A}|$ (in the infinite case, there are indeed as many propositional symbols as formulas), $|Z| \leq |\mathcal{A}|^2 = |\mathcal{A}|$, thus $Z \in \mathbf{X}$, thus $T(M_\Gamma \setminus Z) \subseteq_{(2)} T(M_\Gamma \setminus X_\Gamma) =_{(2)} \widetilde{\Gamma}$.

Proof of (4). $\forall \Gamma \in \{\Gamma_1, \ldots \Gamma_\lambda\}$, $\widetilde{\Gamma}' = T(M_\Gamma \cap (\mathcal{V} \setminus Z)) = T(M_\Gamma \setminus Z) =_{(3)} \widetilde{\Gamma}$. Thus, $\Phi(\Gamma_1, \ldots \Gamma_\lambda$, $\widetilde{\Gamma_1}', \ldots \widetilde{\Gamma_\lambda}')$ iff $\Phi(\Gamma_1, \ldots \Gamma_\lambda, \widetilde{\Gamma_1}, \ldots \widetilde{\Gamma_\lambda})$, which does not hold. Thus, $|\sim'$ is not a pivotal CR. ∎

8 Definable pivotal-discriminative CRs

8.1 The necessary and sufficient conditions

In this section, we give normal characterizations of the two following families:

- the definable pivotal-discriminative CRs, under $(A2)$, $(A3)$, and $(A4)$;
- the CP definable pivotal-discriminative CRs, under $(A2)$ and $(A3)$.

Notation 29 $\mathbb{N}$ denotes the natural numbers: $\{0, 1, 2, \ldots\}$ and $\mathbb{N}^+$ the positive natural numbers: $\{1, 2, \ldots\}$.

Definition 30 Let $|\sim$ be a CR, $\Gamma \subseteq \mathcal{L}$, and $i \in \mathbb{N}$, $i \geq 2$. Then,

$$H_1(\Gamma) \stackrel{def}{=} \{neg(\beta) : \beta \notin \widetilde{\Gamma},\ \beta \in \overline{\Gamma \cup \widetilde{\Gamma}},\ \text{and } neg(\beta) \notin \overline{\Gamma \cup \widetilde{\Gamma}}\};$$

$$H_i(\Gamma) \stackrel{def}{=} \{neg(\beta) : \begin{cases} \beta \notin \widetilde{\Gamma} \text{ and} \\ \beta \in \overline{\Gamma \cup \widetilde{\Gamma} \cup H_1(\Gamma) \cup \ldots \cup H_{i-1}(\Gamma)} \text{ and} \\ neg(\beta) \notin \overline{\Gamma \cup \widetilde{\Gamma} \cup H_1(\Gamma) \cup \ldots \cup H_{i-1}(\Gamma)} \end{cases} \};$$

$$H(\Gamma) \stackrel{def}{=} \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma).$$

Note that H should be indexed by $|\sim$, but as there will never be any ambiguity, we omit it to increase readability. We come to the representation results.

Proposition 31 Let $|\sim$ be a CR. Then,

(i) if $(A2)$, $(A3)$, and $(A4)$ hold, then $|\sim$ is a definable pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$ and $(|\sim 9)$ hold;

(ii) if $(A2)$ and $(A3)$ hold, then $|\sim$ is a CP definable pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 9)$, and $(|\sim 10)$ hold;

where

$(|\sim 6)$ $\forall \Gamma, \alpha, \beta$, if $\beta \notin \widetilde{\Gamma}$, $\beta \in \overline{\Gamma \cup \widetilde{\Gamma}}$, and $\Gamma \cup \widetilde{\Gamma} \cup \{neg(\alpha)\} \subseteq \overline{\Gamma \cup \widetilde{\Gamma} \cup \{neg(\beta)\}}$, then $\alpha \notin \widetilde{\Gamma}$;

$(|\sim 7)$ $\forall \Gamma, \alpha, \beta$, if $\alpha, \beta \notin \widetilde{\Gamma}$, $\alpha \in \overline{\Gamma \cup \widetilde{\Gamma}}$, and $\Gamma \cup \widetilde{\Gamma} \cup \{\beta\} \subseteq \overline{\Gamma \cup \widetilde{\Gamma} \cup \{neg(\alpha)\}}$, then $or(\alpha, \beta) \notin \widetilde{\Gamma}$;

($|\sim 8$) $\forall\,\Gamma, \alpha$, if $\alpha \in \widetilde{\Gamma}$, then $neg(\alpha) \notin \overline{\Gamma \cup \widetilde{\Gamma}}$;

($|\sim 9$) $\forall\,\Gamma, \Delta, \Gamma \cup \widetilde{\Gamma} \cup H(\Gamma) \subseteq \overline{\Delta \cup \widetilde{\Delta} \cup H(\Delta) \cup \Gamma}$;

($|\sim 10$) $\forall\,\Gamma$, if Γ is consistent, then $\Gamma \subseteq \widetilde{\Gamma}$, $\widetilde{\Gamma}$ is consistent, and $\overline{\widetilde{\Gamma}} = \widetilde{\Gamma}$.

Before we show Proposition 31, we need to introduce Lemmas 32 and 33 below. They are taken from [7]. Moreover, [7] has nothing to do with pivots, but with preferential structures. In fact, it seems that these lemmas can be useful to characterize any family C of CRs such that $|\sim \in C$ iff $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$, where $\mu : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{V})$ is simply a function satisfying $(\mu 0)$.

Lemma 32 From [7]. Let $(A2)$ and $(A3)$ hold, and let $|\sim$ be a CR such that ($|\sim 6$), ($|\sim 7$), and ($|\sim 8$) hold. Then, $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)})$.

Lemma 33 From [7]. Let $(A2)$ and $(A3)$ hold, $|\sim$ be a CR, and $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be a function such that $(\mu 0)$ holds and $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$. Then,

(0) ($|\sim 6$), ($|\sim 7$), ($|\sim 8$) hold;

(1) if $(A4)$ holds, then $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)}$;

(2) if μ is CP, then $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)}$.

We come to the **proof of Proposition 31**.

Proof Direction: "$\rightarrow$". For (i) and (ii), there exists a definable pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(M_\Gamma \cap P)$. We show ($|\sim 0$). If $\overline{\Gamma} = \overline{\Delta}$, then $\widetilde{\Gamma} = T_d(M_\Gamma \cap P) = T_d(M_\Delta \cap P) = \widetilde{\Delta}$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall\, X \in \mathbf{D}$, $\mu(X) = X \cap P$. As P is definable, μ is well-defined. By Lemma 13, we get $(\mu 0)$ and $(\mu 1)$. In addition, $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$. Thus, by Lemma 33 (0), ($|\sim 6$), ($|\sim 7$), and ($|\sim 8$) hold.

(i). By Lemma 33 (1), $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)}$. Thus, by $(\mu 1)$, ($|\sim 9$) holds.

(ii). In addition, P is CP, thus μ is CP, thus by Lemma 33 (2), $\forall\Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)}$. Thus, by $(\mu 1)$, ($|\sim 9$) holds. In addition, we show ($|\sim 10$). Let $\Gamma \subseteq \mathcal{L}$ be consistent. Then, $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$, thus $\mu(M_\Gamma) \in \mathbf{D} \cap \mathbf{C}$. We show $\Gamma \subseteq \widetilde{\Gamma}$. Let $\alpha \in \Gamma$. Then, $\mu(M_\Gamma) \subseteq_{(\mu 0)} M_\Gamma \subseteq M_\alpha$. Thus, $\mu(M_\Gamma) \not\subseteq M_{neg(\alpha)}$, thus $\alpha \in T_d(\mu(M_\Gamma)) = \widetilde{\Gamma}$. We show that $\widetilde{\Gamma}$ is consistent. $M_{\widetilde{\Gamma}} = M_{T_d(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))} = \mu(M_\Gamma) \in \mathbf{C}$. And, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\widetilde{\Gamma}}) = \overline{\widetilde{\Gamma}}$.

Direction: "$\leftarrow$". For (i) and (ii), ($|\sim 0$), ($|\sim 6$), ($|\sim 7$), ($|\sim 8$), and ($|\sim 9$) hold. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall\Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)}$. If $M_\Gamma = M_\Delta$, then $\widetilde{\Gamma} =_{(|\sim 0)} \widetilde{\Delta}$ and $H(\Gamma) =_{(|\sim 0)} H(\Delta)$, thus $\mu(M_\Gamma) = \mu(M_\Delta)$, thus μ is well-defined. By Lemma 32, $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$. Obviously $(\mu 0)$ holds. And, by ($|\sim 9$), $(\mu 1)$ holds. Thus, by Lemma 13, there exists a pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_\Gamma \cap P)$. In addition, $P = M_\emptyset \cap P = \mu(M_\emptyset) \in \mathbf{D}$.

(i). This has been shown in the common part above.

(ii). We show that μ is CP. Let $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$. Case 1: $H_1(\Gamma) \neq \emptyset$. Thus, $\exists\,\beta$, $\beta \notin \widetilde{\Gamma}$ and $M_{\widetilde{\Gamma}} \cap M_\Gamma \subseteq M_\beta$. As, $\Gamma \subseteq_{(|\sim 10)} \widetilde{\Gamma}$, $M_{\widetilde{\Gamma}} \cap M_\Gamma = M_{\widetilde{\Gamma}}$, thus $M_{\widetilde{\Gamma}} \subseteq M_\beta$, thus $\beta \in T(M_{\widetilde{\Gamma}}) = \overline{\widetilde{\Gamma}} =_{(|\sim 10)} \widetilde{\Gamma}$, which is impossible. Case 2: $H_1(\Gamma) = \emptyset$. Thus, $H(\Gamma) = \emptyset$. Thus, $\mu(M_\Gamma) = M_{\Gamma \cup \widetilde{\Gamma} \cup H(\Gamma)} = M_{\Gamma \cup \widetilde{\Gamma}} = M_{\widetilde{\Gamma}} \in_{(|\sim 10)} \mathbf{C}$. We show that P is CP. If $X \in \mathbf{D} \cap \mathbf{C}$, then $X \cap P = \mu(X) \in \mathbf{C}$. ∎

Note that the conditions: $(|\sim\ 0)$, $(|\sim\ 6)$, $(|\sim\ 7)$, $(|\sim\ 8)$, $(|\sim\ 9)$, and $(|\sim\ 10)$ depend only on $|\sim$, $\vdash$, $\mathcal{L}$, neg, and or. As, $\vdash$ has been defined semantically (in Section 2.2), these conditions have a semantic aspect. In parallel, as we have a proof system for $\vdash$ in the classical, $\mathcal{FOUR}$, and J_3 frameworks, the conditions have also a syntactic aspect in these frameworks.

8.2 Normal characterizations

We now give in a straightforward manner normal characterizations of definable pivotal-discriminative CRs and of CP definable pivotal-discriminative CRs. We use the following definitions: let $\Gamma_1, \Gamma_2 \subseteq \mathcal{L}$ and $i \in \mathbb{N}$, $i \geq 2$, then

$$H_1(\Gamma_1, \Gamma_2) \stackrel{def}{=} \{neg(\beta) : \beta \notin \Gamma_2, \text{ and } \beta \in \overline{\overline{\Gamma_1} \cup \overline{\Gamma_2}},\ neg(\beta) \notin \overline{\overline{\Gamma_1} \cup \overline{\Gamma_2}}\};$$

$$H_i(\Gamma_1, \Gamma_2) \stackrel{def}{=} \{neg(\beta) : \begin{cases} \beta \notin \Gamma_2 \text{ and} \\ \beta \in \overline{\overline{\Gamma_1} \cup \overline{\Gamma_2} \cup H_1(\Gamma_1, \Gamma_2) \cup \ldots \cup H_{i-1}(\Gamma_1, \Gamma_2)} \text{ and} \\ neg(\beta) \notin \overline{\overline{\Gamma_1} \cup \overline{\Gamma_2} \cup H_1(\Gamma_1, \Gamma_2) \cup \ldots \cup H_{i-1}(\Gamma_1, \Gamma_2)} \end{cases} \};$$

$$H(\Gamma_1, \Gamma_2) \stackrel{def}{=} \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma_1, \Gamma_2).$$

Let Φ be the relation over $\mathcal{P}(\mathcal{L})^4$ such that $\forall\, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff

- if $\overline{\Gamma_1} = \overline{\Gamma_2}$, then $\Gamma_3 = \Gamma_4$;
- $\forall\, \alpha, \beta$, if $\beta \notin \Gamma_3$, $\beta \in \overline{\overline{\Gamma_1} \cup \overline{\Gamma_3}}$, $\Gamma_1 \cup \Gamma_3 \cup \{neg(\alpha)\} \subseteq \overline{\overline{\Gamma_1} \cup \overline{\Gamma_3} \cup \{neg(\beta)\}}$, then $\alpha \notin \Gamma_3$;
- $\forall \alpha, \beta$, if $\alpha, \beta \notin \Gamma_3$, $\alpha \in \overline{\overline{\Gamma_1} \cup \overline{\Gamma_3}}$, $\Gamma_1 \cup \Gamma_3 \cup \{\beta\} \subseteq \overline{\overline{\Gamma_1} \cup \overline{\Gamma_3} \cup \{neg(\alpha)\}}$, then $or(\alpha, \beta) \notin \Gamma_3$;
- $\forall\, \alpha$, if $\alpha \in \Gamma_3$, then $neg(\alpha) \notin \overline{\overline{\Gamma_1} \cup \overline{\Gamma_3}}$;
- $\Gamma_1 \cup \Gamma_3 \cup H(\Gamma_1, \Gamma_3) \subseteq \overline{\overline{\Gamma_2} \cup \overline{\Gamma_4} \cup H(\Gamma_2, \Gamma_4) \cup \overline{\Gamma_1}}$.

Then, under $(A2)$, $(A3)$, and $(A4)$, by Proposition 31 (i), $|\sim$ is a definable pivotal-discriminative CR iff $\forall\, \Gamma_1, \Gamma_2$, $\Phi(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$. Let Φ' be the relation over $\mathcal{P}(\mathcal{L})^4$ such that $\forall\, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, $\Phi'(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ and

$$\text{if } \Gamma_1 \text{ is consistent, then } \Gamma_1 \subseteq \Gamma_3,\ \Gamma_3 \text{ is consistent, and } \overline{\Gamma_3} = \Gamma_3.$$

Then, under $(A2)$ and $(A3)$, by Proposition 31 (ii), $|\sim$ is a CP definable pivotal-discriminative CR iff $\forall\, \Gamma_1, \Gamma_2$, $\Phi'(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$.

9 General and codefinable pivotal-discriminative CRs

In this section, we give non-normal characterizations of the four following families:

- the pivotal-discriminative CRs, under $(A2)$, $(A3)$, and $(A4)$;
- the codefinable pivotal-discriminative CRs, under $(A0)$, $(A2)$, $(A3)$, and $(A4)$;
- the CP pivotal-discriminative CRs, under $(A2)$ and $(A3)$;

- the CP codefinable pivotal-discriminative CRs, under $(A0)$, $(A2)$ and $(A3)$.

Proposition 34 Let $|\sim$ be a CR. Then,

(i) if $(A2)$, $(A3)$, and $(A4)$ hold, then $|\sim$ is a pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 11)$ hold;

(ii) if $(A0)$, $(A2)$, $(A3)$, and $(A4)$ hold, then $|\sim$ is a codefinable pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 11)$, and $(|\sim 12)$ hold;

(iii) if $(A2)$ and $(A3)$ hold, then $|\sim$ is a CP pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 10)$, and $(|\sim 11)$ hold;

(iv) if $(A0)$, $(A2)$, and $(A3)$ hold, then $|\sim$ is a CP codefinable pivotal-discriminative CR iff $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 10)$, $(|\sim 11)$, and $(|\sim 12)$ hold;

where

$(|\sim 11)$ $\forall\, \Gamma,\ M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)} = M_{T(\{x\in M_\Gamma : \forall\, \Delta, \text{ if } x\in M_\Delta, \text{ then } x\in M_{\Delta\cup\widetilde{\Delta}\cup H(\Delta)}\})}$;

$(|\sim 12)$ $\mathcal{V}\setminus\{x\in\mathcal{V} : \forall\, \Delta,\ \text{if } x\in M_\Delta,\ \text{then } x\in M_{\Delta\cup\widetilde{\Delta}\cup H(\Delta)}\}\in\mathbf{D}$.

Proof Direction: "$\rightarrow$". For (i), (ii), (iii), and (iv), there exists a pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(M_\Gamma\cap P)$. We get obviously $(|\sim 0)$. Let $\mu : \mathbf{D}\rightarrow\mathbf{D}$ be the function such that $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{T(M_\Gamma\cap P)}$. As $\forall\, X$, $X\subseteq M_\alpha$ iff $M_{T(X)}\subseteq M_\alpha$, we get $\widetilde{\Gamma} = T_d(M_\Gamma\cap P) = T_d(M_{T(M_\Gamma\cap P)}) = T_d(\mu(M_\Gamma))$. In addition, $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{T(M_\Gamma\cap P)}\subseteq M_{T(M_\Gamma)} = M_\Gamma$, thus $(\mu 0)$ holds. Thus, by Lemma 33 (0), $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$ hold. In addition, by Lemma 16, $(\mu 2)$ holds.

(i). By Lemma 33 (1), $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)}$. Thus, By $(\mu 2)$, $(|\sim 11)$ hold.

(ii). As above, $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)}$ and $(|\sim 11)$ holds. In addition to the common part, P is codefinable, thus by Lemma 18, $(\mu 3)$ holds, thus $(|\sim 12)$ holds.

(iii). In addition to the common part, P is CP. We show that μ is CP. Let $M_\Gamma\in\mathbf{D}\cap\mathbf{C}$. Then, $M_\Gamma\cap P\in\mathbf{C}$, thus $\mu(M_\Gamma) = M_{T(M_\Gamma\cap P)}\in\mathbf{C}$, as $M_\Gamma\cap P\subseteq M_\alpha$ iff $M_{T(M_\Gamma\cap P)}\subseteq M_\alpha$. Therefore, by Lemma 33 (2), $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)}$. Thus, by $(\mu 2)$, $(|\sim 11)$ holds. We show $(|\sim 10)$. Verbatim the same reasoning as for "$\rightarrow$" (ii) of Proposition 31.

(iv). As above $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)}$, and $(|\sim 11)$ and $(|\sim 10)$ hold. In addition to the common part P is codefinable, thus by Lemma 18, $(\mu 3)$ holds, thus $(|\sim 12)$ holds.

Direction: "$\leftarrow$". For (i), (ii), (iii), and (iv), we have $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 11)$. Let $\mu : \mathbf{D}\rightarrow\mathbf{D}$ be the function such that $\forall\,\Gamma$, $\mu(M_\Gamma) = M_{\Gamma\cup\widetilde{\Gamma}\cup H(\Gamma)}$. If $M_\Gamma = M_\Delta$, then $\widetilde{\Gamma} =_{(|\sim 0)}\widetilde{\Delta}$ and $H(\Gamma) =_{(|\sim 0)} H(\Delta)$, thus μ is well-defined. By Lemma 32, $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$. And, by $(|\sim 11)$, $(\mu 2)$ holds.

(i). By Lemma 16, there exists a pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_{T(M_\Gamma\cap P)}) = T_d(M_\Gamma\cap P)$.

(ii). By $(|\sim 12)$, $(\mu 3)$ holds. Thus, by Lemma 18, there exists a codefinable pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_{T(M_\Gamma\cap P)}) = T_d(M_\Gamma\cap P)$.

(iii). By Lemma 16, there exists a pivot P such that $\forall\,\Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_{T(M_\Gamma\cap P)}) = T_d(M_\Gamma\cap P)$. We show that, by $(|\sim 10)$, μ is CP. Verbatim the same reasoning as "$\leftarrow$" (ii) of Proposition 31. We show that P is CP. Let $X\in\mathbf{D}\cap\mathbf{C}$. Then, $M_{T(X\cap P)} = \mu(X)\in\mathbf{C}$. Thus, $X\cap P\in\mathbf{C}$, as $M_{T(X\cap P)}\subseteq M_\alpha$ iff $X\cap P\subseteq M_\alpha$.

(iv). By $(|\sim 12)$, $(\mu 3)$ holds. Thus, by Lemma 18, there exists a codefinable pivot P such that $\forall\, \Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_{T(M_\Gamma \cap P)}) = T_d(M_\Gamma \cap P)$. In addition, by verbatim the same reasoning as just above, μ is CP, thus P is CP. ∎

We cannot get a normal characterization for any of these four families from proposition 34, because $(|\sim 11)$ and $(|\sim 12)$ depend on too many sets Δ. Recall that more explanations have been given in Section 2.6.

10 Summary and conclusion

Pivotal and pivotal-discriminative CRs formalize natural ways of reasoning. They are useful to handle incomplete information in the classical framework, and both incomplete and inconsistent information in the J_3 and $\mathcal{FOUR}$ frameworks. The main goal of this work was to characterize several families of them in a general framework that covers all of the classical, three, and four-valued frameworks. We have also answered negatively a representation problem that was left open by Makinson, namely there is no normal characterization of the family of all pivotal CRs, in the infinite classical framework. And we have shown that codefinable pivotal ECRs are precisely closed pertinence ECRs (alias X-logics).

Technically, this paper has many points of commonality with [7]. We have used some results of [7] and it seems that some results of the present papers can be used in future work. More precisely, Lemmas 13, 16, and 18 have an interest of their own. Indeed, we used them in both the basic and the discriminative case. This suggests that they may be useful to characterize some other families of CRs based on pivots.

Similarly, Lemmas 32 and 33 which are taken from [7] were used there to characterize some families of preferential-discriminative CRs and are used here to characterize some families of pivotal-discriminative CRs. In fact, it seems that these lemmas can be useful to characterize any family C of CRs such that $|\sim \in C$ iff $\forall\, \Gamma$, $\widetilde{\Gamma} = T_d(\mu(M_\Gamma))$, where $\mu : \mathcal{P}(\mathcal{V}) \to \mathcal{P}(\mathcal{V})$ is simply a function satisfying $(\mu 0)$.

Let us finish by a constructive self-criticism: some of our conditions are are probably too complex (and ugly) to be used efficiently. Simplifying them could be the goal of a future work.

11 Acknowledgments

I would like to thank Karl Schlechta very much for his hints, advice, constructive criticism, and more. I would also like to thank David Makinson for his kind comments and suggestions and Nicolas Ollinger for the discussions about the $\mathcal{FOUR}$ framework.

References

[1] O. Arieli and A. Avron. Logical bilattices and inconsistent data. In *Proc. 9th IEEE Anual Symp. on Logic in Computer Science*, pages 468–476. IEEE Press, 1994.

[2] O. Arieli and A. Avron. Reasoning with logical bilattices. *Journal of Logic, Language and Information*, 5(1):25–63, 1996.

[3] O. Arieli and A. Avron. The value of the four values. *Artificial Intelligence*, 102:97–141, 1998.

[4] A. Avron. Natural 3-valued logics : characterization and proof theory. *The Journal of Symbolic Logic*, 56(1):276–294, 1991.

[5] N. D. Belnap. How computers should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*, pages 30–56. Oriel Press, 1977.

[6] N. D. Belnap. A useful four-valued logic. In J.M. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 7–37. Oriel Press, 1977.

[7] J. Ben-Naim. Preferential and Preferential-discriminative Consequence Relations. *Journal of Logic and Computation*, 15(3):263–294, 2005.

[8] W. A. Carnielli, J. Marcos, and S. de Amo. Formal inconsistency and evolutionary databases. *Logic and Logical Philosophy*, 8:115–152, 2000.

[9] S. de Amo, W. A. Carnielli, and J. Marcos. A logical framework for integrating inconsistent information in multiple databases. In Thomas Eiter and Berlin Klaus-Dieter Schewe, Springer-Verlag, editors, *Second International Symposium on Foundations of Information and Knowledge Systems*, pages 67–84, 2002.

[10] I.M.L. D'Ottaviano and N.C.A. da Costa. Sur un problème de Jaśkowski, Sciences. In *Comptes Rendus de l'Académie des Sciences de Paris*, volume 270, pages 1349–1353, 1970.

[11] R.L. Epstein. *The Semantic foundation of logic. Vol. I: propositional logics*. Kluwer Academic Publisher, 1990.

[12] L. Forget, V. Risch, and P. Siegel. Preferential logics are X-logics. *Journal of Logic and Computation*, 11:71–83, 2001.

[13] J. Fox. Motivation and Demotivation of a Four-Valued Logic. *Notre Dame Journal of Formal Logic*, 31(1):76–80, 1989.

[14] S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models, and cumulative logics. *Artificial Intelligence*, 44(1-2):167–207, 1990.

[15] D. Lehmann. Nonmonotonic Logics and Semantics. *Journal of Logic and Computation*, 11(2):229–256, 2001.

[16] D. Lehmann and M. Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55(1):1–60, 1992.

[17] D. C. Makinson. Bridges between classical and nonmonotonic logic. *Logic Journal of the IGPL*, pages 69–96, 2003.

[18] D. C. Makinson. *Bridges from Classical to Nonmonotonic Logic*. London: Kings College Publications, 2005.

[19] D. Poole. A logical framework for default reasoning. *Artificial Intelligence*, 36:27–47, 1988.

[20] H. Rott. *Change, Choice and Inference: A Study of Belief Revision and Nonmonotonic Reasoning*. Clarenton Press, Oxford UK (Oxford Logic Guides 42), 2001.

[21] K. Schlechta. Some results on classical preferential models. *Journal of Logic and Computation*, 2(6):675–686, 1992.

[22] K. Schlechta. Some completeness results for stoppered and ranked classical preferential models. *Journal of Logic and Computation*, 6(4):599–622, 1996.

[23] K. Schlechta. New techniques and completeness results for preferential structures. *The Journal of Symbolic Logic*, 65(2):719–746, 2000.

[24] K. Schlechta. *Coherent systems*. Elsevier, 2004.