Rankin-Cohen brackets on quasimodular forms

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RANKIN-COHEN BRACKETS ON QUASIMODULAR FORMS

FRANÇOIS MARTIN AND EMMANUEL ROYER

A . We give the algebra of quasimodular forms a collection of Rankin-Cohen operators. These operators extend those defined by Cohen on modular forms and, as for modular forms, the first of them provide a Lie structure on quasimodular forms. They also satisfy a "Leibniz rule" for the usual derivation. Rankin-Cohen operators are useful for proving arithmetic identities.

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The purpose of this paper is to present a generalisation for quasimodular forms of the Rankin-Cohen brackets for modular forms: for each $n \geq 0$, k, ℓ, s, t positive integers, we define bilinear differential operators $[\ ,\ ]_n$ sending $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t}$, and prove that these operators are unique (in a sense defined below). We have denoted $\widetilde{M}_k^{\leq s}$ the vector space of quasimodular forms of weight k and depth less or equal than s. The set of quasimodular forms of weight k and depth s is denoted $\widetilde{M}_k^s$ (see section 1.1 for the definitions).

Then we prove that the Rankin-Cohen brackets satisfy for the usual derivation ($D = \mathrm{d}/\mathrm{d}z$) the "Leibniz rule": for $f \in \widetilde{M}_k^s$ and $g \in \widetilde{M}_\ell^t$, and for each $n \geq 0$, $[f,g]_n' = [f',g]_n + [f,g']_n$. Here the Rankin-Cohen brackets are the canonical ones for f and g (those corresponding to their exact depth).

As usual, the complex upper half-plane is denoted by $\mathscr{H}$. For $k \geq 0$, $f\colon \mathscr{H} \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ the function $(f|_k\gamma)$ is defined by $(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z)$.

We introduce the following notations: let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathscr{H}$, we define

$$\mathrm{X}(\gamma, z) := \frac{c}{cz+d}$$

and

$$\mathrm{X}(\gamma)\colon z \mapsto \mathrm{X}(\gamma, z).$$

Finally, for $n \geq 0$, f, g two holomorphic functions on $\mathscr{H}$ and $\mathbf{a} = (a_r)_{1 \leq r \leq n}$ we define

$$\Phi_{\mathbf{a}}(f,g) = \sum_{r=0}^{n} a_r f^{(r)} g^{(n-r)}.$$

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The first section is a presentation of definitions and previous results concerning quasimodular forms and Rankin-Cohen brackets.

In the second section, we prove the following theorem:

Theorem 1. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0, \dots, \lfloor k/2 \rfloor\}$, $t \in \{0, \dots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{\geq 0}$. For $\mathbf{a} = (a_r)_{1 \leq r \leq n}$ defined by*

$$a_r = (-1)^r \binom{k-s+n-1}{n-r} \binom{\ell-t+n-1}{r}$$

for all r, one has

$$\Phi_{\mathbf{a}} \left(\widetilde{M}_k^{\leq s}, \widetilde{M}_\ell^{\leq t} \right) \subset \widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Moreover, if Ψ is a bilinear differential operator sending $\widetilde{M}_k^{\leq s}(\Gamma) \times \widetilde{M}_\ell^{\leq t}(\Gamma)$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t}(\Gamma)$ for all $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ a finite index subgroup, then (up to constant) $\Psi = \Phi_{\mathbf{a}}$.

Writing

$$\Phi_{n;k,s;\ell,t}(f,g) = \sum_{r=0}^{n} (-1)^r \binom{k-s+n-1}{n-r} \binom{\ell-t+n-1}{r} f^{(r)} g^{(n-r)}$$

we hence obtain a bilinear differential operator from $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t}$. For $f \in \widetilde{M}_k^s$ and $g \in \widetilde{M}_\ell^t$, we write $[f,g]_n$ instead of $\Phi_{n;k,s;\ell,t}(f,g)$.

Remark 1.
(1) This notation is consistent with the one for modular forms, $\Phi_{n;k,0;\ell,0}(f,g)$ is equal to the standard Rankin-Cohen bracket of f and g for $f \in M_k$ and $g \in M_\ell$ (see paragraph 1.2).
(2) For $n \geq 0$, a bilinear differential operator Ψ sending $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\bigcup_v \widetilde{M}_{k+\ell+2n}^{\leq v}$ is necessarily (for weight compatibility reasons) a linear combination of $(f,g) \mapsto f^{(r)} g^{(n-r)}$, $r \in \{0, \dots, n\}$. Such a differential operator sends in principle $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t+n}$ (see lemma 6). So the operator Φ introduced before has the advantage of reducing the depth of the quasimodular form obtained, and it was not obvious that such an operator exists.

In the third section, we show that the comportement of this operator under derivation is natural, as described in our second theorem:

Theorem 2. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0, \dots, \lfloor k/2 \rfloor\}$, $t \in \{0, \dots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{>0}$. For all $f \in \widetilde{M}_k^{\leq s}$ and $g \in \widetilde{M}_\ell^{\leq t}$,*

$$\Phi_{n;k,s;\ell,t}(f,g)' = \Phi_{n;k,s;\ell+2,t+1}(f,g') + \Phi_{n;k+2,s+1;\ell,t}(f',g).$$

Remark 2. Recall (see proposition 5) that if $h \in \widetilde{M}_w^d$, then $h' \in \widetilde{M}_{w+2}^{d+1}$. The following theorem may then be rewritten as

$$[f,g]_n' = [f',g]_n + [f,g']_n.$$

For modular forms, Zagier, Cohen and Manin showed [CMZ97] that the sum of Rankin-Cohen brackets defines an associative product on the algebra $M = \prod_{k \geq 0} M_k$. In a recent paper, Bieliavski, Tang and Yao [BTY05] showed that this sum is isomorphic to the standard Moyal product. Do the

Rankin-Cohen brackets for quasimodular forms introduced here have such a geometric interpretation ? The result of theorem 2 suggests it.

The existence of Rankin-Cohen brackets, and the lowering of the depth (see remark 1) provides a new tool to obtain arithmetic identities. For example, we have $[E_2, E_2']_1 \in \widetilde{M}_8^{\leq 3}[\mathrm{SL}(2,\mathbb{Z})] = \mathbb{C}E_8 \oplus \mathbb{C}E_6' \oplus \mathbb{C}E_4''$ (see [LR05, Lemma 6]). Since $[E_2, E_2']_1$ has its Fourier coefficient of order 0 equal to 0, we actually have $[E_2, E_2']_1 \in \mathbb{C}E_6' \oplus \mathbb{C}E_4''$. By theorem 2 and $[E_2', E_2']_1 = 0$, we have $[E_2, E_2'']_1 = [E_2, E_2']_1' \in \mathbb{C}E_6'' \oplus \mathbb{C}E_4'''$, hence

$$E_2 E_2''' - 3E_2' E_2'' = -\frac{1}{10} E_4'''.$$

The Fourier developments lead then to the following proposition.

Proposition 3. *Let $n \geq 1$ be an integer,*

$$\sum_{a=1}^{n-1} a^2(3n-4a)\sigma_1(a)\sigma_1(n-a) = \frac{1}{24} n^3[\sigma_3(n) - \sigma_1(n)].$$

As usual,

$$\sigma_h(n) \coloneqq \sum_{\substack{d \in \mathbb{Z}_{>0} \\ d|n}} d^h$$

and

$$E_h(z) \coloneqq 1 - \frac{2h}{B_h} \sum_{n=1}^{+\infty} \sigma_h(n) \exp(2\pi i n z)$$

is the usual Eisenstein series of weight h.

1. O

1.1. **Quasimodular forms.** In this section, we introduce usual definitions and notations and recall some useful properties of quasimodular forms. For a more detailed introduction, see [RM05, §17].

Definition 4. *Let Γ be a congruence subgroup, $k \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$. An holomorphic function*

$$f : \mathscr{H} \to \mathbb{C}$$

is a quasimodular form of weight k, depth s on Γ if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathscr{H}$ such that

$$(f|_k \gamma) = \sum_{i=0}^{s} f_i \, \mathrm{X}(\gamma)^i \tag{1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and such that f_s is holomorphic at the cusps and not identically vanishing. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

The set of quasimodular forms of weight k and depth s for Γ is denoted by $\widetilde{M}_k^s(\Gamma)$. It is often more convenient to use the $\mathbb{C}$-vectorial space of quasimodular forms of weight k and depth $\leq s$, which is denoted by $\widetilde{M}_k^{\leq s}(\Gamma)$.

One has to explain the signification of the holomorphicity of f_s at the cusps. One can show [RM05, Lemme 119] that if f satisfies the quasimodularity condition (1), then f_s satisfies the modularity condition

$$(cz+d)^{-(k-2s)} f_s\left(\frac{az+b}{cz+d}\right) = f_s(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We ask f_s to be a modular form of weight $k-2s$ on Γ : it means that, for all $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, the function

$$z \mapsto (\gamma z+\delta)^{-(k-2s)} f_s\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$$

has a Fourier expansion of the form

$$\sum_{n=0}^{+\infty} \widehat{f_{s,M}}(n) \exp\left(2\pi i \frac{nz}{u_M}\right)$$

where

$$u_M = \inf\{u \in \mathbb{Z}_{>0} : T^u \in M^{-1}\Gamma M\}.$$

It follows that if f is a quasimodular form of weight k and depth s, non identically vanishing, then $k \geq 0$ and $s \leq k/2$.

Remark 3.
(1) With this definition, the space $M_k(\Gamma)$ of modular forms of weight k for Γ is exactly the space $\widetilde{M}_k^{\leq 0}(\Gamma)$.
(2) A basic example of quasimodular form which is not a modular form is E_2, the so-called "Eisenstein series of weight 2", which satisfies for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ the transformation property

$$(E_2|_2\gamma) = E_2 + \frac{6}{i\pi} \mathrm{X}(\gamma),$$

proving that $E_2 \in \widetilde{M}_2^{\leq 1}(\mathrm{SL}(2,\mathbb{Z}))$.

In the sequel, we will fix a congruence subgroup Γ, and write $\widetilde{M}_k^{\leq s}$ for $\widetilde{M}_k^{\leq s}(\Gamma)$ if no confusion is possible.

The space $\widetilde{M}_* = \bigcup_{k,s} \widetilde{M}_k^{\leq s}$ is equipped with a natural filtered-graded algebra structure (the graduation according to the weight, the filtration according to the depth). The canonical multiplication $(f,g) \longmapsto fg$ defines a morphism $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t} \longrightarrow \widetilde{M}_{k+\ell}^{\leq s+t}$.

If $f \in \widetilde{M}_k^{\leq s}$, the sequence $(f_i)_{i\in\mathbb{Z}}$ is defined by the quasimodularity condition

$$(f|_k\gamma) = \sum_{n=0}^{s} f_n \mathrm{X}(\gamma)^n$$

for all $\gamma \in \Gamma$ and $z \in \mathscr{H}$, if $i \in \{0,\dots,s\}$, and $f_i = 0$ for $i \notin \{0,\dots,s\}$. One can show that $f_0 = f$ and $f_i \in \widetilde{M}_{k-2i}^{\leq s-i}$ [RM05, Lemma 119].

The quasimodular forms are the natural extension of modular forms into a stable by derivation space, because of the following proposition:

Proposition 5. *The derivation* $D = \dfrac{d}{dz}$ *maps* $\widetilde{M}_k^s$ *to* $\widetilde{M}_{k+2}^{s+1}$.

For $r \in \mathbb{Z}_{\geq 0}$, write $f^{(r)} = D^r(f)$. The notation $f_i^{(r)}$ always means $D^r(f_i)$. The following lemma connects the transformation equation of f and $f^{(r)}$:

Lemma 6. *Let* $f \in \widetilde{M}_k^{\leq s}$, *satisfying the quasimodular condition*

$$(f|_k\gamma) = \sum_{i=0}^{s} f_i\, \mathrm{X}(\gamma)^i \qquad \text{for } \gamma \in \Gamma.$$

Then, for all $r \in \mathbb{Z}_{\geq 0}$ *and* $\gamma \in \Gamma$,

$$(f^{(r)}|_{k+2r}\gamma) = \sum_{i=0}^{s+r}\left[\sum_{j=0}^{r} j!\binom{r}{j}\binom{k+r-i+j-1}{j} f_{i-j}^{(r-j)}\right]\mathrm{X}(\gamma)^i. \tag{2}$$

Proof. The result is obtained inductively on r: it is obvious for $r = 0$, and for the induction suppose that for $r \geq 0$, formula (2) holds. For $i \in \mathbb{Z}$ let $g_i = \sum_{j=0}^{r} j!\binom{r}{j}\binom{k+r-i+j-1}{j} f_{i-j}^{(r-j)} \in \widetilde{M}_{k+2r-2i}^{\leq s+r-i}$. Then using previous proposition (which implies that $f^{(r+1)} \in \widetilde{M}_{k+2r+2}^{\leq r+s+1}$) and lemma 118 of [RM05][1] we find (recall that f_i is defined to be the identically zero function for $i \notin \{0, \dots, s\}$)

$$(f^{(r+1)}|_{k+2r+2}\gamma) = \sum_{i=0}^{s+r+1}\left(g_i' + (k+2r-i+1)g_{i-1}\right)\mathrm{X}(\gamma)^i$$

Now $g_{i-1} = \sum_{j=1}^{r+1}(j-1)!\binom{r}{j-1}\binom{k+r-i+j-1}{j-1} f_{i-j}^{(r+1-j)}$, and we obtain (denoting h_i for $g_i' + (k+2r-i+1)g_{i-1}$)

$$h_i = f_i^{(r+1)} + (k+2r-i-1)r!\binom{k+2r-i}{r} f_{i-r-1} + \sum_{j=1}^{r} f_{i-j}^{(r+1-j)} \times$$
$$\times\left(\frac{r!}{(r-j)!}\binom{k+r-i+j-1}{j} + \frac{(k+2r-i-1)r!}{(r+1-j)!}\binom{k+r-i+j-1}{j-1}\right).$$

The lemma follows by expanding the binomial coefficients. □

1.2. **Usual Rankin-Cohen brackets for modular forms.** The Rankin-Cohen brackets have been introduced by Cohen after a work of Rankin. These are bilinear differential operators, whose main property is to preserve modular forms. More precisely, for each $n \geq 0$, $(f, g) \in M_k \times M_\ell$, define the n-Rankin-Cohen bracket of f and g by

$$[f, g]_n = \sum_{r=0}^{n}(-1)^r\binom{k+n-1}{n-r}\binom{\ell+n-1}{r} f^{(r)} g^{(n-r)}.$$

Then it satisfies the following proposition

Proposition 7. (1) *The function* $[f, g]_n \in M_{k+\ell+2n}$.

(2) *Let* k, ℓ *and* n *be fixed nonnegative integers. If* Φ *is a bilinear differential operator sending* $M_k(\Gamma) \times M_\ell(\Gamma)$ *to* $M_{k+\ell+2n}(\Gamma)$ *for all* $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ *a finite index subgroup, then (up to constant)* $\Phi(f, g) = [f, g]_n$.

[1] Be careful, in this lemma the derivation differs from a $2i\pi$ factor.

For an overview of Rankin-Cohen brackets including a proof of this proposition, see for instance [Zag94], [Zag92] or [RM05].

Remark 4. The statement of the second point of the previous proposition needs explanations: the result of uniqueness is proved by using only algebraic arguments, the demonstration doesn't depend on the group Γ or on growth conditions. Of course, it is possible that for some fixed group Γ the uniqueness result doesn't hold (for instance if $M_k(\Gamma) = \{0\}$!).

The Rankin-Cohen brackets are very useful for many arithmetical applications, and appear in various mathematical domains as for instance invariant theory ([UU96] and [CMS01]) or non-commutative geometry ([CM04b] and [CM04a]).

2. R -C

In this section, we want to prove theorem 1 as stated in the introduction. We first establish a condition on **a** implying the result, in lemma 8 and then prove the existence of the Rankin-Cohen brackets (lemma 10). For s, t and n nonnegative integers, we introduce the set

$$\mathcal{E}(s,t,n) = \left\{(u,v,\alpha,\beta) \in \mathbb{Z}_{\geq 0}^4 : u \leq s,\ v \leq t,\ \alpha+\beta \leq u+v+n-s-t-1\right\}.$$

Lemma 8. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0,\dots,\lfloor k/2\rfloor\}$, $t \in \{0,\dots,\lfloor \ell/2\rfloor\}$ and $n \in \mathbb{Z}_{>0}$. For $\mathbf{a} = (a_r)_{0\leq r\leq n}$ verifying*

$$\sum_{r=0}^{n} a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$, one has

$$\Phi_{\mathbf{a}}\left(\widetilde{M}_k^{\leq s}, \widetilde{M}_\ell^{\leq t}\right) \subset \widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. Let $f \in \widetilde{M}_k^{\leq s}$ and $g \in \widetilde{M}_\ell^{\leq t}$. From lemma 6 we deduce

$$\begin{aligned}
(\Phi_{\mathbf{a}}(f,g) \underset{k+\ell+2n}{|} \gamma) &= \sum_{r=0}^{n} a_r (f^{(r)} \underset{k+2r}{|} \gamma)(g^{(n-r)} \underset{\ell+2(n-r)}{|} \gamma) \\
&= \sum_{i=0}^{s+t+n} \mathrm{X}(\gamma)^i \sum_{\substack{(i_1,i_2)\in\mathbb{Z}_{\geq 0}^2 \\ i_1+i_2=i}} \sum_{r=0}^{n} a_r \\
&\quad \times \sum_{j_1=0}^{r} j_1! \binom{r}{j_1}\binom{k+r-i_1+j_1-1}{j_1} \\
&\quad \times \sum_{j_2=0}^{n-r} j_2! \binom{n-r}{j_2}\binom{\ell+n-r-i_2+j_2-1}{j_2} \\
&\quad \times f_{i_1-j_1}^{(r-j_1)} g_{i_2-j_2}^{(n-r-j_2)}.
\end{aligned}$$

It follows that $\Phi_{\mathbf{a}}(f,g) \in \widetilde{M}^{\leq s+t}_{k+\ell+2n}$ if and only if

$$\sum_{\substack{(i_1,i_2)\in\mathbb{Z}^2_{\geq 0}\\ i_1+i_2=s+t+i}} \sum_{r=0}^{n} a_r \sum_{j_1=0}^{r}\sum_{j_2=0}^{n-r} j_1!j_2!\binom{r}{j_1}\binom{n-r}{j_2} \times \binom{k+r-i_1+j_1-1}{j_1}\binom{\ell+n-r-i_2+j_2-1}{j_2} f^{(r-j_1)}_{i_1-j_1} g^{(n-r-j_2)}_{i_2-j_2} = 0 \tag{3}$$

for all $i \in \{1,\dots,n\}$. This is easily seen to be equivalent to

$$\sum_u\sum_v \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2_{\geq 0}\\ \alpha+\beta=n+u+v-s-t-i}} \sum_r a_r (r-\alpha)!(n-r-\beta)! \times \binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{r-\alpha}\binom{\ell+n-r-v-1}{n-r-\beta} f^{(\alpha)}_u g^{(\beta)}_v = 0$$

for all $i \in \{1,\dots,n\}$, the sets of summation being determined by the binomial coefficients. Hence, $\Phi_{\mathbf{a}}\left(\widetilde{M}^{\leq s}_k, \widetilde{M}^{\leq t}_\ell\right) \in \widetilde{M}^{\leq s+t}_{k+\ell+2n}$ is implied by

$$\sum_r a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0 \tag{4}$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$. □

Remark 5. The statement of the previous lemma is in fact an equivalence, if we ask $\Phi_{\mathbf{a}}$ to satisfy $\Phi_{\mathbf{a}}\left(\widetilde{M}^{\leq s}_k(\Gamma), \widetilde{M}^{\leq t}_\ell(\Gamma)\right) \subset \widetilde{M}^{\leq s+t}_{k+\ell+2n}(\Gamma)$ for each Γ a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$: indeed for $\{a(u,v,\alpha,\beta)\}$ a non identically zero family of complex numbers, if $\Psi\colon (f,g) \mapsto \sum_{(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)} a(u,v,\alpha,\beta) f^{(\alpha)}_u g^{(\beta)}_v$ satisfy $\Psi(\widetilde{M}^{\leq s}_k(\Gamma), \widetilde{M}^{\leq t}_\ell(\Gamma)) = 0$, then exists $M > 0$ such that the minimum of $\dim(\widetilde{M}^{\leq s}_k(\Gamma))$ and $\dim(\widetilde{M}^{\leq t}_\ell(\Gamma))$ is strictly smaller than M. However, as for modular forms, for each $A > 0$ exists Γ a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ such that $\dim \widetilde{M}^{\leq s}_k(\Gamma) > A$ and $\dim \widetilde{M}^{\leq t}_\ell(\Gamma) > A$ (recall that $k, \ell \in \mathbb{Z}_{>0}$).

We shall now give a necessary condition for $\mathbf{a}$ satisfying the condition of lemma 8.

Lemma 9. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0,\dots,\lfloor k/2\rfloor\}$, $t \in \{0,\dots,\lfloor \ell/2\rfloor\}$ and $n \in \mathbb{Z}_{>0}$. If $\mathbf{a} = (a_r)_{0\leq r\leq n}$ verifies*

$$\sum_{r=0}^{n} a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$, then there exists $\lambda \in \mathbb{C}$ such that

$$a_r = \lambda(-1)^r \binom{k+n-s-1}{n-r}\binom{\ell+n-t-1}{r}$$

for all $r \in \{0,\dots,n\}$.

Proof. Define $\mathbf{b} = (b_r)_{0\leq r\leq n}$ by

$$b_r = a_r(k+r-s-1)!(\ell+n-r-t-1)!$$

for all r. Then

$$\sum_{r=0}^{n} b_r \binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{s-u}\binom{\ell+n-r-v-1}{t-v} = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$. Choosing $u = s$, $t = v$ and $\beta = 0$ leads to $F^{(\alpha)}(1) = 0$ for all $\alpha \in \{0,\dots,n-1\}$ where F is the generating function of $\mathbf{b}$ defined by

$$F(x) = \sum_{r=0}^{n} b_r x^r.$$

This implies the existence of $\mu \in \mathbb{C}$ such that $F(x) = \mu(x-1)^n$ and thus $b_r = \mu(-1)^r\binom{n}{r}$. The result follows by defining

$$\lambda = \mu \frac{n!}{(k-s+n-1)!(\ell-t+n-1)!}.$$

□

We obtain the existence of the Rankin-Cohen operator for quasimodular forms in showing that the vector $\mathbf{a}$ we found in lemma 9 is admissible.

Lemma 10. *Let k,ℓ in $\mathbb{Z}_{>0}$, $s \in \{0,\dots,\lfloor k/2\rfloor\}$, $t \in \{0,\dots,\lfloor \ell/2\rfloor\}$ and $n \in \mathbb{Z}_{>0}$. Let $\mathbf{a} = (a_r)_{1\leq r\leq n}$ be defined by*

$$a_r = (-1)^r \binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r}.$$

Then

$$\Phi_{\mathbf{a}}\left(\widetilde{M}_k^{\leq s}, \widetilde{M}_\ell^{\leq t}\right) \subset \widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. By lemma 8 it suffices to verify

$$\sum_{\substack{(r_1,r_2)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\\ r_1+r_2=n}} \frac{(-1)^{r_1}}{r_1!r_2!}\binom{r_1}{\alpha}\binom{r_2}{\beta}\binom{k-u-1+r_1}{s-u}\binom{\ell-v-1+r_2}{t-v} = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$. Fix $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$ and define

$$F(X) = \sum_{\substack{(r_1,r_2)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\\ r_1+r_2=n}} \frac{(-1)^{r_1}}{r_1!r_2!}\binom{r_1}{\alpha}\binom{r_2}{\beta}\binom{k-u-1+r_1}{s-u}\binom{\ell-v-1+r_2}{t-v}X^n.$$

It is the coefficient of order n in the product $P_1(X)P_2(X)$ where

$$P_1(X) = \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!}\binom{r_1}{\alpha}\binom{k-u-1+r_1}{s-u}X^{r_1}$$

$$P_2(X) = \sum_{r_2=0}^{+\infty} \frac{1}{r_2!}\binom{r_2}{\beta}\binom{\ell-v-1+r_2}{t-v}X^{r_2}.$$

We have

$$P_1(X) = \frac{X^\alpha}{\alpha!}Q_1^{(\alpha)}(X) \text{ with } Q_1(X) = \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!}\binom{k-u-1+r_1}{s-u}X^{r_1}$$

and

$$Q_1(X) = \frac{X^{-k+s+1}}{(s-u)!} R_1^{(s-u)}(X) \text{ with } \begin{aligned} R_1(X) &= \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!} X^{r_1+k-u-1} \\ &= X^{k-u-1} e^{-X}. \end{aligned}$$

We therefore may write $P_1(X) = \Pi_1(X)e^{-X}$ where Π_1 is a polynomial of degree $\alpha + s - u$. Similary, $P_2(X) = \Pi_2(X)e^X$ where Π_2 is a polynomial of degree $\beta+t-v$. It follows that P_1P_2 is a polynomial of degree $\alpha+\beta+s+t-u-v$. Finally, $F(X) = 0$ (recall that, by definition, $\alpha + \beta - u - v < n - s - t$) hence the result. □

The previous lemmas prove the theorem 1, the remark 5 explaining the unicity statement.

3. R -C

In this section, we prove theorem 2. First, we remark that

$$\begin{aligned} (5)\quad \Phi_{n;k,s;\ell,t}(f,g)' = \sum_{r=0}^{n-1} (-1)^r \Big[&\binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r} \\ &- \binom{k-s+n-1}{n-r-1}\binom{\ell-t+n-1}{r+1}\Big] f^{(r+1)} g^{(n-r)} \\ &+ \binom{k-s+n-1}{n} f g^{(n+1)} + (-1)^n \binom{\ell-t+n-1}{n} f^{(n+1)} g. \end{aligned}$$

Next,

$$\begin{aligned} \Phi_{n;k,s;\ell+2,t+1}(f,g') = &\binom{k-s+n-1}{n} f g^{(n+1)} \\ &- \sum_{r=0}^{n-1} (-1)^r \binom{k-s+n-1}{n-r-1}\binom{\ell-t+n}{r+1} f^{(r+1)} g^{(n-r)} \end{aligned}$$

so that

$$\begin{aligned} (6)\quad \Phi_{n;k+2,s+1;\ell,t}(f',g) + \Phi_{n;k,s;\ell+2,t+1}(f,g') = \\ \binom{k-s+n-1}{n} f g^{(n+1)} + (-1)^n \binom{\ell-t+n-1}{n} f^{(n+1)} g \\ + \sum_{r=0}^{n-1} (-1)^r \Big[\binom{k-s+n}{n-r}\binom{\ell-t+n-1}{r} \\ - \binom{k-s+n-1}{n-r-1}\binom{\ell-t+n}{r+1}\Big] f^{(r+1)} g^{(n-r)} \end{aligned}$$

and equality from (5) and (6) follows by expanding the binomial coefficients.

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