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## Hardy-type inequalities

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### 1. Introduction

The well-known Hardy–Sobolev inequality states that for any given domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  and any  $u \in C_c^\infty(\Omega)$ ,

$$K^2 \int_{\Omega} \frac{u^2}{|x|^2} \leq \int_{\Omega} |\nabla u|^2, \quad (1)$$

where  $K = (n - 2)/2$ . Though the constant  $K^2$  is optimal, in the sense that

$$K^2 = \inf_{u \in C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2/|x|^2},$$

equality in (1) is never achieved (by any  $u \in H_0^1(\Omega)$ ). This fact has led to the improvement of the inequality in various ways: Brezis and Vázquez [BV] first showed that if  $\Omega$  is bounded then for some  $\gamma > 0$ ,

$$\gamma \left( \int_{\Omega} |u|^p \right)^{2/p} + K^2 \int_{\Omega} \frac{u^2}{|x|^2} \leq \int_{\Omega} |\nabla u|^2, \quad (2)$$

with  $1 \leq p < 2n/(n - 2)$ . Vázquez and Zuazua [VZ] were then able to replace the  $L^p$  norm on the left hand side of (2) by a  $W^{1,q}$  norm for  $q < 2$ . Various improvements (involving e.g. weighted  $L^p$  or  $W^{1,p}$  norms) were also obtained and we refer the interested reader to [Da], [ACR], [FT], [BFT] and the references therein.

One of the consequences of inequality (2) is that the operator  $L_0 := -\Delta - \mu/|x|^2$  has a positive first eigenvalue, in the sense that

$$\inf_{\|u\|_{L^2(\Omega)}=1} \int_{\Omega} \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) > 0,$$

whenever  $\mu \leq K^2$ .

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In the first part of this work, given a compact smooth boundaryless manifold  $\Sigma \subset \Omega$  of codimension  $k \neq 2$ , we look at operators of the form

$$L = -\Delta - \frac{\mu}{d(x)^2},$$

where  $d(x) = \text{dist}(x, \Sigma)$  and  $\mu \in \mathbb{R}$ , and wonder whether an inequality similar to (2) holds.

The first results in this direction are due to Marcus, Mizel and Pinchover [MMP] and Matskewich and Sobolevskii [MS0]. They showed that if  $\Omega$  is a convex domain and  $\Sigma = \partial\Omega$  then

$$\frac{1}{4} \int_{\Omega} \frac{u^2}{d(x)^2} \leq \int_{\Omega} |\nabla u|^2. \quad (3)$$

The same authors showed that (3) did not hold in a general domain  $\Omega$  and provided examples of smooth domains  $\Omega$  such that

$$\inf_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2/d^2} < \frac{1}{4}.$$

Alternatively, Brezis and Marcus showed in [BM] that the following inequality remains true on a general (smooth bounded) domain  $\Omega$ :

$$\frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} u^2,$$

where  $C$  is some positive constant.

Finally, among many other results, Barbatis, Filippas and Tertikas [BFT, FT] extended (3) to the case where  $\Sigma \subset \Omega$  is a smooth compact manifold of codimension  $k$ , satisfying some geometric condition: they showed that if  $\Delta d^{2-k} \leq 0$  in  $\mathcal{D}'(\Omega \setminus \Sigma)$  then

$$\gamma \left( \int_{\Omega} |u|^p \right)^{2/p} + H^2 \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2,$$

where  $H = (k-2)/2$  and  $1 \leq p < 2n/(n-2)$ .

Our goal here is to drop the assumption  $\Delta d^{2-k} \leq 0$ . Our results are summarized in the following two theorems:

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set and  $\Sigma \subset \Omega$  be a compact smooth manifold without boundary of codimension  $k \neq 2$ . Let  $H = (k-2)/2$ . Then there exist  $p > 2$  and  $C > 0, \gamma > 0$  independent of  $u$  such that for any  $u \in C_c^\infty(\Omega \setminus \Sigma)$ ,*

$$\gamma \left( \int_{\Omega} |u|^p \right)^{2/p} + H^2 \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} u^2, \quad (4)$$

where  $d(x) = \text{dist}(x, \Sigma)$ ,  $1 \leq p < p_k$  and  $p_k$  is given by

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)} \quad \text{for } k > 2, \quad \frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1} \quad \text{if } k = 1.$$

**Theorem 2.** *Under the assumptions of Theorem 1, there exist  $\beta > 0$  and a neighborhood  $\Omega_\beta := \{x \in \Omega : d(x, \Sigma) < \beta\}$  of  $\Sigma$  in  $\Omega$  such that for any  $u \in C_c^\infty(\Omega_\beta \setminus \Sigma)$ ,*

$$H^2 \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2. \quad (5)$$

**Remark 1.** • If  $k \geq 3$  it follows by density that (4) and (5) hold for all  $u \in C_c^\infty(\Omega)$ .  
 • The exponent  $p_k$  appearing in Theorem 1 is probably not optimal and we expect that (4) holds for all  $1 \leq p < 2n/(n-2)$ . In fact Maz'ja [Ma, Corollary 3, Section 2.1.6] proved this result when  $\Sigma = \{x \in \mathbb{R}^n : x_1 = x_2 = \dots = x_k = 0\}$ .

As a direct consequence of Theorem 1, we see that the first eigenvalue of the operator  $L = -\Delta - \mu d^{-2}$  is finite, i.e.

$$\lambda_1 := \inf_{\|u\|_{L^2(\Omega)}=1} \int_{\Omega} \left( |\nabla u|^2 - \mu \frac{u^2}{d^2} \right) > -\infty,$$

whenever  $\mu \leq H^2$ . We proved in [DD] that in such circumstances there exists an eigenfunction  $\varphi_1$  associated to  $\lambda_1$ , i.e. a solution (in a sense which we shall make precise soon) of

$$\begin{cases} -\Delta \varphi_1 - \frac{\mu}{d^2} \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Normalizing  $\varphi_1$  by  $\|\varphi_1\|_{L^2(\Omega)} = 1$  and  $\varphi_1 > 0$ , we then investigate the behavior of  $\varphi_1$  near  $\Sigma$  and show that in a neighborhood of  $\Sigma$ , there exist constants  $C_1, C_2 > 0$  such that

$$C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1 \leq C_2 d(x)^{-\alpha(\mu)}, \quad (6)$$

where  $\alpha(\mu) = H - \sqrt{H^2 - \mu}$ .

This result enables us to treat two model applications. First we consider the quantity

$$J_\lambda := \inf_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} u^2}{\int_{\Omega} u^2 / d^2}$$

and extend a result of Brezis and Marcus [BM] stating that  $J_\lambda$  is achieved if and only if  $J_\lambda < H^2$ .

Our second application is a nonexistence result for positive solutions of the equation

$$-\Delta u - \frac{\mu}{d^2} u = u^p + \lambda,$$

completing a study started in [DN]. See Section 4.2 for details.

The last purpose of this article is to extend some results in [DD]. This generalization is necessary to include the case of potentials  $a(x) = \mu \operatorname{dist}(x, \Sigma)^{-2}$ . More precisely, we shall derive estimates for solutions of the linear equation

$$\begin{cases} -\Delta u - a(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (7)$$

under the assumptions that  $a \in L^1_{\text{loc}}(\Omega)$ ,  $a$  is bounded below, i.e.

$$\text{ess inf}_{\Omega} a > -\infty,$$

and

$$\gamma \left( \int_{\Omega} |u|^r \right)^{2/r} + \int_{\Omega} a(x)u^2 \leq \int_{\Omega} |\nabla u|^2 + M \int_{\Omega} u^2, \quad (8)$$

for some  $r > 2$ ,  $\gamma > 0$ ,  $M > 0$ .

Let us now clarify what we mean by a solution of (7).

We first define the Hilbert space  $\mathcal{H}$  as the completion of  $C_c^\infty(\Omega)$  with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = (u|u)_{\mathcal{H}} := \int_{\Omega} (|\nabla u|^2 - a(x)u^2 + Mu^2),$$

where  $M$  is the same constant that appears in (8). Observe that the definition of  $\mathcal{H}$  does not change if we replace  $M$  by any larger constant.

Given  $f \in \mathcal{H}^*$ , we then say that  $u \in \mathcal{H}$  is a solution of (7) if

$$(u|v)_{\mathcal{H}} = \langle f, v \rangle_{\mathcal{H}^*, \mathcal{H}} + M(u|v)_{L^2(\Omega)} \quad \forall v \in \mathcal{H}.$$

It is convenient at this point to recall some facts that were proved in [DD]. We start by mentioning that  $\mathcal{H}$  embeds compactly in  $L^2(\Omega)$ . In particular

$$L = -\Delta - a(x)$$

has a first eigenvalue  $\lambda_1$ , which is simple.  $\lambda_1$  is not necessarily positive (Theorem 1 provides examples of potentials  $a(x) = H^2/d(x)^2$  for which in general  $\lambda_1$  can be nonpositive), but when it is, then for  $f \in \mathcal{H}^*$  problem (7) has a unique solution  $u \in \mathcal{H}$ .

We note here that uniqueness fails if one considers other classes of solutions (see an example in [DD]).

The first eigenvalue  $\lambda_1$  has an associated positive eigenfunction  $\varphi_1$  (it is not only positive a.e. but it also satisfies  $\varphi_1 \geq c \text{dist}(x, \partial\Omega)$  for some  $c > 0$ ).

Solutions in  $\mathcal{H}$  of an equation like (7) are typically unbounded (see examples in [D, DD, DN]). In [DD] we showed that if  $\lambda_1 > 0$  and  $f \geq 0$ ,  $f \not\equiv 0$  then the solution  $u \in \mathcal{H}$  of (7) is bounded below by a positive constant times  $\varphi_1$ . We also proved that if  $\lambda_1 > 0$  and  $f = 1$ , then the solution  $u$  of (7) satisfies  $u \leq C\varphi_1$  for some  $C > 0$ .

Our main result is the following:

**Theorem 3.** *Let  $0 < m < r$  and suppose that*

$$p > \frac{2r}{m(r-2)} \quad \text{and} \quad p \geq \frac{r}{r-m}.$$

*Assume that  $f \in \mathcal{H}^*$  satisfies  $\|\varphi_1^{1-m} f\|_{L^p(\Omega)} < \infty$  and that  $u \in \mathcal{H}$  is a solution of (7). Then*

$$|u(x)| \leq C(\|\varphi_1^{1-m} f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})\varphi_1(x), \quad \text{a.e. } x \in \Omega.$$

A simple corollary of this result, obtained by choosing  $m = 2$ , is the following:

**Corollary 1.** *Assume  $f \in \mathcal{H}^*$  satisfies  $f/\varphi_1 \in L^p(\Omega)$  for some  $p > r/(r-2)$  and that  $u \in \mathcal{H}$  is a solution of (7). Then*

$$|u(x)| \leq C(\|f/\varphi_1\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})\varphi_1(x), \quad a.e. x \in \Omega.$$

The paper is organized as follows: Section 2 is devoted to the proof of Theorems 1 and 2. In Section 3 we derive (6) whereas Section 4 is dedicated to the aforementioned applications. Finally, we prove Theorem 3 in Section 5.

## 2. Hardy inequalities

### 2.1. Proof of Theorem 1

The object of this subsection is to prove Theorem 1. Our arguments are based on improvements of the one-dimensional Hardy inequality inspired by [BM] and a decomposition of  $L^2$  functions in spherical harmonics taken from [VZ].

We start with a series of three lemmas, which yield a refined version of the classical Hardy inequality in  $\mathbb{R}^k$  (see (3)). The first lemma deals with radial functions:

**Lemma 1.** *Let  $k \neq 2$  and  $H = (k-2)/2$ . There exists a constant  $C > 0$  depending only on  $k$  such that*

$$\begin{aligned} & \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u^2 r^{k-1} dr \\ & \geq \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{d}{dr} (r^H u(r)) \right)^2 dr \end{aligned} \quad (9)$$

for all  $u \in C_c^\infty(0, 1/2)$ .

*Proof.* Let  $u \in C_c^\infty(0, 1/2)$  and  $v(r) = r^H u(r)$ . A standard computation yields

$$\left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} = r \left( \frac{dv}{dr} \right)^2 - H \frac{d(v^2)}{dr}. \quad (10)$$

Integrating, it follows that

$$A := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr = \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr. \quad (11)$$

Similarly, by (10) and an integration by parts,

$$\begin{aligned} B &:= \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr \\ &= \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr + 2H^2 \int_0^{1/2} \frac{u^2}{r^2} r^k dr - H \int_0^{1/2} r \left[ \frac{d(v^2)}{dr} \right] dr \\ &= \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr + (2H^2 + H) \int_0^{1/2} v^2 dr. \end{aligned} \quad (12)$$

Using integration by parts again, it follows that for given  $\varepsilon > 0$ , there exists  $C > 0$  such that

$$\begin{aligned} \int_0^{1/2} v^2 dr &= -2 \int_0^{1/2} r v \frac{dv}{dr} dr \leq C \int_0^{1/2} v^2 r dr + \varepsilon \int_0^{1/2} \left( \frac{dv}{dr} \right)^2 r dr \\ &= C \int_0^{1/2} u^2 r^{k-1} dr + \varepsilon \int_0^{1/2} \left( \frac{dv}{dr} \right)^2 r dr. \end{aligned} \quad (13)$$

Collecting (11), (12) and (13), we obtain for  $\varepsilon$  small enough

$$\begin{aligned} A - B &\geq \int_0^{1/2} r(1-r-C\varepsilon) \left( \frac{dv}{dr} \right)^2 dr - C \int_0^{1/2} u^2 r^{k-1} dr \\ &\geq \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr - C \int_0^{1/2} u^2 r^{k-1} dr. \end{aligned} \quad \square$$

The next lemma will help us deal with the nonradial part of a given function  $u : \mathbb{R}^k \rightarrow \mathbb{R}$ .

**Lemma 2.** *Let  $k \neq 2$ ,  $H = (k-2)/2$  and  $c > \bar{c} > 0$ . There exist constants  $C, \tau > 0$  depending only on  $k$  and  $\bar{c}$  such that*

$$\begin{aligned} &\int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u^2 r^{k-1} dr \\ &\geq \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right] r^k dr + \tau \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + c \frac{u^2}{r^2} \right] r^{k-1} dr \end{aligned} \quad (14)$$

for all  $u \in C_c^\infty(0, 1/2)$ .

*Proof.* It follows from (9) that if

$$D := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u^2 r^{k-1} dr \quad (15)$$

and

$$E := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right] r^k dr \quad (16)$$

then

$$\begin{aligned} D - E &\geq c \int_0^{1/2} \frac{u^2}{r^2} (1-r) r^{k-1} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr \\ &\geq \frac{c}{2} \int_0^{1/2} \frac{u^2}{r^2} r^{k-1} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr. \end{aligned} \quad (17)$$

We can also rewrite (10) as

$$r^{k-1} \left( \frac{du}{dr} \right)^2 = H^2 \frac{u^2}{r^2} r^{k-1} + r \left( \frac{dv}{dr} \right)^2 - H \frac{d(v^2)}{dr}$$

so that if  $\tau = \min \left( \frac{\bar{c}}{4H^2}, \frac{1}{4} \right)$ , then

$$\tau \int_0^{1/2} r^{k-1} \left( \frac{du}{dr} \right)^2 dr \leq \frac{\bar{c}}{4} \int_0^{1/2} \frac{u^2}{r^2} r^{k-1} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr. \quad (18)$$

It then follows from (17) and (18) that

$$D - E \geq \tau \int_0^{1/2} r^{k-1} \left( \frac{du}{dr} \right)^2 dr + \frac{c}{4} \int_0^{1/2} \frac{u^2}{r^2} r^{k-1} dr. \quad (19)$$

Hence (2) holds.  $\square$

Finally, the following lemma yields the improved Hardy inequality (in  $\mathbb{R}^k$ ) that we will be using in what follows.

**Lemma 3.** *Let  $k \neq 2$ ,  $H = (k-2)/2$  and  $\beta > 0$ . Let  $B_\beta^k$  denote the ball of  $\mathbb{R}^k$  centered at the origin and of radius  $\beta$ . There exist positive constants  $C = C(\beta, k)$ ,  $\tau = \tau(k)$  and  $\alpha = \alpha(\beta, k)$  such that*

$$\begin{aligned} & \int_{B_\beta^k} \left( |\nabla u|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C \int_{B_\beta^k} u^2 dy \\ & \geq \frac{1}{2\beta} \int_{B_\beta^k} |y| \left( |\nabla u|^2 + H^2 \frac{u^2}{|y|^2} \right) dy + \tau \int_{B_\beta^k} |\nabla(u - u_0)|^2 dy + \alpha \int_0^\beta r \left( \frac{dv_0}{dr} \right)^2 dr \end{aligned} \quad (20)$$

for all  $u \in C_c^\infty(B_\beta^k \setminus \{0\})$ , where  $u_0(r) = u_0(|y|) = \int_{\partial B_r^k} u d\sigma$  and  $v_0(r) = r^H u_0(r)$ .

*Proof.* Let  $\{f_i\}_{i=0}^\infty$  be an orthonormal basis of  $L^2(S^{k-1})$ , composed of eigenvectors of the Laplace–Beltrami operator  $\Delta|_{S^{k-1}}$ . The corresponding eigenvalues are given by  $c_i = i(k+i-2)$ ,  $i = 0, 1, 2, \dots$  (see e.g. [St]). Any  $u \in C_c^\infty(B_{1/2}^k \setminus \{0\})$  can then be written as

$$u(x) = \sum_{i=0}^\infty u_i(r) f_i(\theta)$$

where  $1/2 > r > 0$ ,  $\theta \in S^{k-1}$  and  $x = r\theta$ .

Furthermore, for  $g \in C(\mathbb{R}^+, \mathbb{R})$ ,

$$\begin{aligned} \int_{B_{1/2}^k} |\nabla u|^2 g(|y|) dy &= \int_0^{1/2} r^{k-1} g(r) dr \int_{S^{k-1}} \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right] d\theta \\ &= \sum_{i=0}^\infty \int_0^{1/2} r^{k-1} g(r) \left[ \left( \frac{du_i}{dr} \right)^2 + \frac{c_i}{r^2} u_i^2 \right] dr. \end{aligned} \quad (21)$$



For  $i = 0$ , it follows from (9) that if  $v_0(r) = r^H u_0(r)$ , then

$$\begin{aligned} & \int_0^{1/2} \left[ \left( \frac{du_0}{dr} \right)^2 - H^2 \frac{u_0^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u_0^2 r^{k-1} dr \\ & \quad \geq \int_0^{1/2} \left[ \left( \frac{dv_0}{dr} \right)^2 + H^2 \frac{v_0^2}{r^2} \right] r^k dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv_0}{dr} \right)^2 dr, \end{aligned} \quad (22)$$

while (22) implies that for  $i \geq 1$ ,

$$\begin{aligned} & \int_0^{1/2} \left[ \left( \frac{du_i}{dr} \right)^2 - (H^2 - c_i) \frac{u_i^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u_i^2 r^{k-1} dr \\ & \quad \geq \int_0^{1/2} \left[ \left( \frac{du_i}{dr} \right)^2 + (H^2 + c_i) \frac{u_i^2}{r^2} \right] r^k dr + \tau \int_0^{1/2} \left[ \left( \frac{du_i}{dr} \right)^2 + c \frac{u_i^2}{r^2} \right] r^{k-1} dr. \end{aligned} \quad (23)$$

Using (22), (23) and (24) with  $g(r) \equiv 1$  for terms involving  $r^{k-1}$  and  $g(r) = r$  for terms with  $r^k$ , we deduce (25) for  $\beta = 1/2$ . The general case is obtained by scaling.  $\square$

Next, we introduce some geometric notation that will be needed in the proof of Theorem 1. Define

$$\Omega_\beta = \{x \in \Omega \mid \text{dist}(x, \Sigma) < \beta\}.$$

We will work only with  $\beta$  small enough so that the projection  $\pi : \Omega_\beta \rightarrow \Sigma$  given by  $|\pi(x) - x| = \text{dist}(x, \Sigma)$  is well defined and smooth.

Let  $\{V_i\}_{i=1, \dots, m}$  be a family of open disjoint subsets of  $\Sigma$  such that

$$\Sigma = \bigcup_{i=1}^m \bar{V}_i, \quad |\bar{V}_i \cap \bar{V}_j| = 0 \quad \forall i \neq j.$$

We can also assume that:

(a)  $\forall i = 1, \dots, m$  there exists a smooth diffeomorphism

$$p_i : B_1^{n-k} \rightarrow U_i,$$

where  $U_i \subset \Sigma$  is open and  $\bar{V}_i \subset U_i$ ;

(b)  $p_i^{-1}(V_i)$ , which is an open set in  $\mathbb{R}^{n-k}$ , has a Lipschitz boundary; and

(c) there is a smooth choice of unit vectors  $N_1^i(\sigma), \dots, N_k^i(\sigma)$  for  $\sigma \in U_i$  which form an orthonormal frame for  $\Sigma$  on  $U_i \subset \mathbb{R}^n$ , i.e. for all  $\sigma \in U_i$ ,

$$N_j^i(\sigma) \in \mathbb{R}^n, \quad N_j^i(\sigma) \cdot N_k^i(\sigma) = \delta_{jk}, \quad N_j^i(\sigma) \cdot v = 0 \quad \forall v \in T_\sigma \Sigma.$$

Let  $W_i = p_i^{-1}(V_i)$ . For  $z \in W_i$  we will also write (abusing the notation)  $N_j^i(z) = N_j^i(p_i(z))$ . Let

$$F_i(y, z) = p_i(z) + \sum_{j=1}^k y_j N_j^i(z),$$

where  $y = (y_1, \dots, y_k) \in B_\beta^k$  and  $z \in W_i$ , so that  $F_i$  is a smooth diffeomorphism between  $B_\beta^k \times W_i$  and  $T_\beta^i$ , where

$$T_\beta^i = \pi^{-1}(V_i) \cap \Omega_\beta. \quad (24)$$

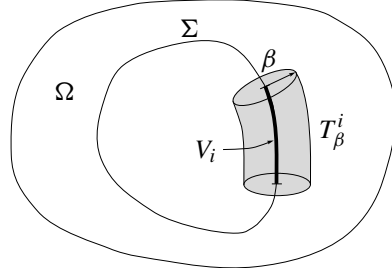


Fig. 1.

It follows from the condition  $|\bar{V}_i \cap \bar{V}_j| = 0 \forall i \neq j$  that  $|\bar{T}_\beta^i \cap \bar{T}_\beta^j| = 0 \forall i \neq j$ , and hence, for any  $f \in L^1(\Omega_\beta)$  we have

$$\int_{\Omega_\beta} f = \sum_{i=1}^m \int_{T_\beta^i} f = \sum_{i=1}^m \int_{W_i \times B_\beta^k} f \circ F_i(y, z) JF_i(y, z) dy dz, \quad (25)$$

where  $JF_i(y, z)$  stands for the Jacobian of  $F_i$  at  $(y, z)$ . We claim that

$$JF_i(y, z) = G_i(z)(1 + O(|y|)), \quad (26)$$

where  $O(|y|)$  denotes a quantity bounded by  $|y|$  (uniformly for  $z \in W_i$ ) and  $G_i(z)$  is a smooth function which is bounded away from zero. More precisely

$$G_i(z) = Jp_i(z) = \sqrt{(Dp_i(z))^* Dp_i(z)}.$$

To prove (26) it suffices to observe that  $JF_i(y, z)$  is smooth and to compute it at  $y = 0$ :

$$\begin{aligned} JF_i(0, z)^2 &= \det(DF_i(0, z))^* DF_i(0, z) \\ &= \det([D_z p_i | N_1^i, \dots, N_k^i]^* [D_z p_i | N_1^i, \dots, N_k^i]) \\ &= \det \begin{bmatrix} (D_z p_i)^* D_z p_i & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

*Proof of Theorem 1.* First, observe that it is sufficient to prove the theorem for  $u$  with support near  $\Sigma$ . Indeed, following an idea of Vázquez and Zuazua [VZ], let  $\eta \in C_c^\infty(\mathbb{R}^n)$  be such that  $\eta \equiv 1$  in  $\Omega_{\beta/2}$  and  $\text{supp}(\eta) \subset \Omega_\beta$ . Let  $u \in C_c^\infty(\Omega \setminus \Sigma)$  and write  $u = u_1 + u_2$  where  $u_1 = \eta u$ ,  $u_2 = (1 - \eta)u$ . Suppose that the conclusion of the theorem holds for  $u_1$ . Then

$$\begin{aligned} \int_{\Omega} \left( |\nabla u|^2 - H^2 \frac{u^2}{d^2} \right) &= \int_{\Omega} \left( |\nabla u_1|^2 - H^2 \frac{u_1^2}{d^2} \right) + \int_{\Omega} \left( |\nabla u_2|^2 - H^2 \frac{u_2^2}{d^2} \right) \\ &\quad + 2 \int_{\Omega} \left( \nabla u_1 \cdot \nabla u_2 - H^2 \frac{u_1 u_2}{d^2} \right). \end{aligned} \quad (27)$$

Since  $1/d$  is bounded away from  $\Sigma$  we have

$$\int_{\Omega} \left( \frac{u_2^2}{d^2} + \frac{u_1 u_2}{d^2} \right) \leq C \int_{\Omega} u^2.$$

Also note that

$$\begin{aligned} \int_{\Omega} \nabla u_1 \cdot \nabla u_2 &= \int_{\Omega} [\eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2 + u \nabla u \cdot \nabla \eta(1-2\eta)] \\ &= \int_{\Omega} [\eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2] - \frac{1}{2} \int_{\Omega_{\beta} \setminus \Omega_{\beta/2}} u^2 \nabla \cdot (\nabla \eta(1-2\eta)) \\ &\geq -C \int_{\Omega} u^2. \end{aligned} \quad (28)$$

It follows from (27), (28) that

$$\int_{\Omega} \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} \right] \geq \int_{\Omega} \left[ |\nabla u_1|^2 - H^2 \frac{u_1^2}{d^2} \right] + \int_{\Omega} |\nabla u_2|^2 - C \int_{\Omega} u^2.$$

Using (4) with  $u_1$  we conclude that

$$\int_{\Omega} \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} \right] + C \int_{\Omega} u^2 \geq \gamma \left( \int_{\Omega} |u_1|^p \right)^{2/p} + \int_{\Omega} |\nabla u_2|^2,$$

for some  $\gamma > 0$  independent of  $u$ . Hence the conclusion of the theorem for  $u$  follows easily.

Let

$$I_i = \int_{T_{\beta}^i} \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2 \right], \quad (29)$$

where  $T_{\beta}^i$  was defined in (24). In what follows we will fix  $i$  and show that there are  $p > 2$  and  $C > 0$  independent of  $u$  such that

$$\left( \int_{T_{\beta}^i} |u|^p \right)^{2/p} \leq C I_i.$$

For simplicity, and since  $i$  is fixed, we will drop the index  $i$  from all the notation that follows.

Let us introduce some additional notation:

$$\tilde{u}(y, z) = u(F(y, z)), \quad (30)$$

$$\tilde{u}_0(r, z) = \int_{\partial B_r} \tilde{u}(y, z) ds(y), \quad (31)$$

$$v_0(r, z) = r^H \tilde{u}_0(r, z). \quad (32)$$

Let us write

$$\nabla u = \nabla_N u + \nabla_T u$$

where  $\nabla_N u$  is the gradient of  $u$  in the normal direction and  $\nabla_T u$  is orthogonal to  $\nabla_N u$ . More precisely, for a point  $x = F(y, z)$ ,

$$\nabla_N u(x) = \sum_{j=1}^k \nabla u(x) \cdot N_j(z) N_j(z).$$

**Step 1.** There exists  $C > 0$  independent of  $u$  such that

$$\begin{aligned} CI \geq & \int_{W \times B_\beta^k} |\nabla_y \tilde{u}|^2 |y| dy dz + \int_{W \times B_\beta^k} |\nabla_y (\tilde{u}(y, z) - \tilde{u}_0(y, z))|^2 dy dz \\ & + \int_W \int_0^\beta \left( \frac{\partial v_0}{\partial r} \right)^2 r dr dz + \int_{W \times B_\beta^k} |(\nabla_T u) \circ F|^2 dy dz. \end{aligned} \quad (33)$$

First note that by (25), there is a constant  $C > 0$  such that

$$\begin{aligned} I \geq & \int_{W \times B_\beta^k} \left( |\nabla_N u(F(y, z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) dy dz \\ & - C \int_{W \times B_\beta^k} \left( |\nabla_N u(F(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) |y| dy dz \\ & + \int_{W \times B_\beta^k} (|\nabla_T u(F(y, z))|^2 + \tilde{u}^2) (1 - C|y|) G(z) dy dz. \end{aligned} \quad (34)$$

For fixed  $z$  we can apply Lemma 3 to the function  $\tilde{u}(\cdot, z)$ . Observe that

$$\frac{\partial \tilde{u}(y, z)}{\partial y_j} = \nabla u(F(y, z)) \cdot N_j(z)$$

and thus

$$|\nabla_y \tilde{u}(y, z)|^2 = |\nabla_N u(F(y, z))|^2.$$

Lemma 3 then yields

$$\begin{aligned} & \int_{B_\beta^k} \left( |\nabla_N u(F(y, z))|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C \int_{B_\beta^k} \tilde{u}^2 dy \\ & \geq \frac{1}{2\beta} \int_{B_\beta^k} |y| \left( |\nabla_N u(F(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) dy \\ & \quad + \tau \int_{B_\beta^k} |\nabla_y (\tilde{u} - \tilde{u}_0)|^2 dy + \alpha \int_0^\beta r \left( \frac{dv_0}{dr} \right)^2 dr. \end{aligned} \quad (35)$$

We choose (and fix once for all)  $\beta > 0$  small enough so that  $1/(2\beta) \geq C + 1$ . Then multiplying (35) by  $G(z)$ , integrating over  $W$  and combining the result with (34) we conclude that (33) holds.

**Step 2.**

$$\|\nabla v_0\|_{L^2(W \times B_\beta^2)}^2 \leq CI. \quad (36)$$

By (33) the partial derivative  $\partial v_0/\partial r$  is bounded in  $L^2(W \times B_\beta^2)$  by  $CI$ . We just have to control the derivatives  $\partial v_0/\partial z_i$ ,  $i = 1, \dots, n-k$ . But

$$\frac{\partial v_0}{\partial z_i}(r, z) = r^H \int_{\partial B_r} \frac{\partial \tilde{u}}{\partial z_i}(y, z) ds(y)$$

and

$$\frac{\partial \tilde{u}}{\partial z_i}(y, z) = \nabla u(F(y, z)) \cdot \left[ \frac{\partial p}{\partial z_i} + \sum_{j=1}^k y_j \frac{\partial N_j}{\partial z_i} \right].$$

But note that  $\partial p/\partial z_i$  is a tangent vector, hence

$$|\nabla_z \tilde{u}(y, z)| \leq C |\nabla_T u(F(y, z))| + C |y| |\nabla_N u(F(y, z))|.$$

Integrating over  $W \times B_\beta^k$  we have

$$\int_{W \times B_\beta^k} |\nabla_z \tilde{u}(y, z)|^2 dy dz \leq CI, \quad (37)$$

for some  $C$  independent of  $u$  by (33). It follows that

$$\begin{aligned} \int_{W \times B_\beta^2} |\nabla_z v_0|^2 dy dz &= \int_W \int_0^\beta r^{2H+1} \left| \int_{\partial B_r} \nabla_z \tilde{u}(y, z) ds(y) \right|^2 dr dz \\ &\leq \int_W \int_0^\beta r^{k-1} \int_{\partial B_r} |\nabla_z \tilde{u}(y, z)|^2 ds(y) dr dz \\ &\leq C \int_{W \times B_\beta^k} |\nabla_z \tilde{u}(y, z)|^2 dy dz \leq CI \end{aligned} \quad (38)$$

by (37).

**Step 3.** There is  $p > 2$  such that

$$\|\tilde{u}_0\|_{L^p(W \times B_\beta^k)}^2 \leq CI. \quad (39)$$

More precisely, for  $k \geq 3$  one can take any  $2 < p < p_k$  where  $p_k$  is given by

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)},$$

and for  $k = 1$  one can take  $2 < p \leq p_1$  where  $p_1$  is given by

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1}.$$

Using Sobolev's inequality (on  $W \times B_{\beta}^2$ ) combined with (36) we obtain

$$\int_W \int_0^{\beta} |v_0|^q r \, dr \, dz \leq CI^{q/2},$$

with  $q$  given by  $1/q = 1/2 - 1/(n - k + 2)$ . That is, in terms of  $\tilde{u}_0$  we have

$$\int_W \int_0^{\beta} |\tilde{u}_0|^q r^{qH+1} \, dr \, dz \leq CI^{q/2}. \quad (40)$$

We want an estimate for  $\int |\tilde{u}_0|^p r^{k-1} \, dr \, dz$  for some suitable  $2 < p < q$  and for this we use Hölder's inequality, distinguishing two cases:

**Case  $k \geq 3$ .** We have

$$\begin{aligned} \int_W \int_0^{\beta} |\tilde{u}_0|^p r^{k-1} \, dr \, dz &= \int_W \int_0^{\beta} |\tilde{u}_0|^p r^{\alpha} r^{k-2-\alpha} \, dr \, dz \\ &\leq C \left( \int_W \int_0^{\beta} |\tilde{u}_0|^q r^{\alpha q/p+1} \, dr, \, dz \right)^{p/q} \left( \int_0^{\beta} r^{\frac{k-2-\alpha}{1-p/q}+1} \, dr \right)^{1-p/q}. \end{aligned} \quad (41)$$

We then choose  $\alpha$  so that

$$\frac{\alpha}{p} = H = \frac{k-2}{2}.$$

In order to have the second factor on the right hand side of (41) finite we need to impose

$$\frac{k-2-\alpha}{1-p/q} > -2,$$

which is equivalent to the condition

$$\alpha < \frac{k}{1 + \frac{4}{q(k-2)}}.$$

Thus we need  $p = \alpha/H < p_k$ , where  $p_k$  is given by

$$p_k = \frac{2k}{(k-2)\left(1 + \frac{4}{q(k-2)}\right)},$$

i.e.

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)}.$$

Observe that  $p_k > 2$ . Combining then (40) and (41) finishes this case.

**Case  $k = 1$ .** In this case  $q$  is given by  $1/q = 1/2 - 1/n + 1$ , and we can choose  $p = q$ :

$$\begin{aligned} \int_W \int_0^\beta |\tilde{u}_0|^q r^{k-1} dr dz &= \int_W \int_0^\beta |\tilde{u}_0|^q dr dz \\ &\leq \int_W \int_0^\beta |\tilde{u}_0|^q r^{-q/2+1} dr dz \\ &= \int_W \int_0^\beta |\tilde{u}_0|^q r^{Hq+1} dr dz \end{aligned}$$

because  $-q/2 + 1 < 0$ .

**Step 4.**

$$\|\tilde{u} - \tilde{u}_0\|_{L^{2^*}(W \times B_\beta^k)}^2 \leq CI. \quad (42)$$

This is a consequence of Sobolev's inequality applied to the function  $\tilde{u} - \tilde{u}_0$  on the domain  $W \times B_\beta^k$ . (33) already provides a bound in  $L^2(W \times B_\beta^k)$  for  $\nabla_y(\tilde{u} - \tilde{u}_0)$ . Hence we only need to obtain a bound for the derivative of  $\tilde{u} - \tilde{u}_0$  with respect to  $z$ . In the case of the function  $\tilde{u}$  we have it already in (37). For  $\tilde{u}_0$  it is derived by a computation very similar to that at the end of Step 2. Indeed,

$$\int_{W \times B_\beta^k} |\nabla_z \tilde{u}_0|^2 dy dz = \int_W \int_0^\beta r^{k-1} \left| \int_{\partial B_r} \nabla_z \tilde{u}(y, z) ds(y) \right|^2 dr dz \leq CI,$$

which we obtain as in (38).

**Conclusion.** By (39) and (42) we see that

$$\|\tilde{u}\|_{L^p(W \times B_\beta^k)}^2 \leq CI$$

for some  $C$  independent of  $u$ . Changing variables and reintroducing the index  $i$  we have

$$\|u\|_{L^p(T_\beta^i)}^2 \leq C \int_{T_\beta^i} \left( |\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2 \right).$$

Adding these inequalities over  $i$  proves the statement of the theorem.  $\square$

## 2.2. A local version of the Hardy inequality

In this section, we show how to adapt the proof of Theorem 1 to obtain Theorem 2. We first derive variants of Lemmas 1, 2, 3.

**Lemma 4.** *Let  $k \neq 2$  and  $H = (k - 2)/2$ . There exist constants  $C, \beta_0 > 0$  such that for  $0 < \beta \leq \beta_0$ ,*

$$\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr \geq \int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr \quad (43)$$

for all  $u \in C_c^\infty(0, \beta)$ .

*Proof.* Given  $v \in C_c^\infty(0, 1/2)$ , we have

$$\int_0^{1/2} v^2 dr = -2 \int_0^{1/2} r v \frac{dv}{dr} dr \leq C \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr + \frac{1}{2} \int_0^{1/2} v^2 dr.$$

Using this and (12) we obtain

$$\int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr \leq C \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr.$$

Changing variables, it then follows that for  $u \in C_c^\infty(0, \beta)$ ,

$$\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr \leq C \beta^{k-2} \int_0^\beta r^2 \left( \frac{dv}{dr} \right)^2 dr, \quad (44)$$

while (11) becomes

$$\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr = \beta^{k-2} \int_0^\beta r \left( \frac{dv}{dr} \right)^2 dr. \quad (45)$$

If we pick  $\beta$  small, (43) follows from (44) and (45).  $\square$

A straightforward corollary of the above lemma is:

**Lemma 5.** *Let  $k \neq 2$ ,  $H = (k-2)/2$  and  $c > 0$ . There exist constants  $C, \beta_0 > 0$  such that for  $0 < \beta \leq \beta_0$ ,*

$$\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right] r^{k-1} dr \geq \int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right] r^k dr \quad (46)$$

for all  $u \in C_c^\infty(0, \beta)$ .

Combining these two lemmas, we then obtain:

**Lemma 6.** *Let  $k \neq 2$ ,  $H = (k-2)/2$  and  $\beta > 0$ . Let  $B_\beta^k$  denote the ball of  $\mathbb{R}^k$  centered at the origin and of radius  $\beta$ . There exist positive constants  $C, \beta_0$  such that for  $\beta \leq \beta_0$ ,*

$$\int_{B_\beta^k} \left( |\nabla u|^2 - H^2 \frac{u^2}{|y|^2} \right) dy \geq \frac{C}{\beta} \int_{B_\beta^k} |y| \left( |\nabla u|^2 + H^2 \frac{u^2}{|y|^2} \right) dy \quad (47)$$

for all  $u \in C_c^\infty(B_\beta^k \setminus \{0\})$ , where  $u_0(r) = u_0(|y|) = \int_{\partial B_\beta^k} u d\sigma$  and  $v_0(r) = r^H u_0(r)$ .

As in Lemma 3, for a fixed value  $\beta = \beta_0 > 0$  the proof is an application of the decomposition of a function in spherical harmonics. A simple scaling then yields the  $\beta$ -dependence of the constant appearing in (47).

*Proof of Theorem 2.* Instead of (29), we now consider

$$J_i := \int_{T_\beta^i} \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} \right]. \quad (48)$$



Using the notation of (30) we then have, by (25) and (26),

$$\begin{aligned} J_i &\geq \int_{W \times B_\beta^k} \left( |\nabla_N u(F_i(y, z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) dy dz \\ &\quad - C \int_{W \times B_\beta^k} |y| \left( |\nabla_N u(F_i(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) dy dz \geq 0, \end{aligned}$$

where we used Lemma 6 with  $\beta > 0$  small in the last inequality. Adding the above estimates over  $i$  yields the desired result.  $\square$

### 3. Remarks on the potential $a(x) = \mu \operatorname{dist}(x, \Sigma)^{-2}$

For  $0 < \mu \leq H^2$  we consider the potential

$$a(x) = \mu/d(x)^2$$

and let  $L$  denote the operator

$$Lu = -\Delta u - a(x)u.$$

Note that  $a(x)$  and  $L$  depend on  $\mu$  but we will omit this dependence from the notation.

Recall that we defined the Hilbert space  $\mathcal{H}$  as the completion of  $C_c^\infty(\Omega)$  with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\Omega} (|\nabla u|^2 - a(x)u^2 + Mu^2). \quad (49)$$

If  $\mu < H^2$  then by Theorem 1,  $\mathcal{H}$  coincides with  $H_0^1(\Omega)$ .

The main concern in this section is to obtain a precise description of the behavior near  $\Sigma$  of the first eigenfunction  $\varphi_1$  of the operator  $L$ . Indeed, we shall prove:

**Lemma 7.** *There are positive constants  $C_1, C_2$  such that*

$$C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1(x) \leq C_2 d(x)^{-\alpha(\mu)} \quad (50)$$

for  $x$  in a neighborhood of  $\Sigma$ , where  $\alpha(\mu)$  is given by

$$\alpha(\mu) = H - \sqrt{H^2 - \mu}. \quad (51)$$

Note that when  $\mu = H^2$  we have  $-\alpha(\mu) = 1 - k/2$ . Thus  $\varphi_1 \notin H_0^1(\Omega)$  in this case. Before proving the above lemma it will be necessary to show that if  $\mu = H^2$  then  $d^{1-k/2}$  (appropriately modified so that it is zero on  $\partial\Omega$ ) belongs to  $\mathcal{H}$ . We prove this and a little more next.

**Lemma 8.** *Let  $\mu = H^2$  and define*

$$v_s(x) = \eta(x)d(x)^{1-k/2}(-\log d(x))^{-s},$$

where  $\eta \in C_c^\infty(\Omega)$  is a cut-off function such that  $\eta \equiv 1$  in a neighborhood of  $\Sigma$  and  $\eta(x) = 0$  for  $d(x) \geq \operatorname{dist}(\Sigma, \partial\Omega)/2$ . Then  $v_s \in \mathcal{H}$  if and only if  $s > -1/2$ .

**Remark 2.** This lemma was stated in [VZ] in the case where  $\Sigma$  is a point.

*Proof.* Let us recall and also introduce some notation:

$$\Omega_r = \{x \in \mathbb{R}^N \mid d(x) < r\}, \quad \Sigma_r = \partial\Omega_r = \{x \in \mathbb{R}^N \mid d(x) = r\}.$$

By the Pappus theorems, the  $(N - 1)$ -dimensional area of  $\Sigma_r$  is given by

$$|\Sigma_r|_{n-1} = \omega_{k-1} r^{k-1} |\Sigma|_{n-k},$$

where  $\omega_{k-1}$  is the area of the unit sphere in  $\mathbb{R}^k$  and  $|\cdot|_j$  denotes the  $j$ -dimensional Lebesgue measure.

First we prove that  $v_s \in \mathcal{H}$  for  $s > -1/2$ . For this purpose it is enough to exhibit a sequence  $f_\varepsilon \in \mathcal{H}$  such that  $\|f_\varepsilon\|_{\mathcal{H}} \leq C$  with  $C$  independent of  $\varepsilon$  and such that  $f_\varepsilon \rightarrow v_s$  a.e. as  $\varepsilon \rightarrow 0$ ; we take

$$f_\varepsilon = \eta d^{1-k/2+\varepsilon} (-\log d)^{-s}, \quad \varepsilon > 0.$$

Clearly  $f_\varepsilon \in H_0^1(\Omega) \subset \mathcal{H}$ ,  $\int_\Omega f_\varepsilon^2 \leq C$  and  $f_\varepsilon$  is smooth away from  $\Sigma$ . Thus to estimate  $\|f_\varepsilon\|_{\mathcal{H}}$  it is sufficient to verify that for a fixed  $R > 0$  small

$$\int_{\Omega_R} |\nabla f_\varepsilon|^2 - a(x) f_\varepsilon^2 \leq C \tag{52}$$

with  $C$  independent of  $\varepsilon$ .

Near  $\Sigma$ ,  $\eta \equiv 1$  and

$$\begin{aligned} |\nabla f_\varepsilon|^2 &= d^{-k+2\varepsilon} ((1 - k/2 + \varepsilon)^2 (-\log d)^{-2s} \\ &\quad + s(2 - k + 2\varepsilon)(-\log d)^{-2s-1} + s^2(-\log d)^{-2s-2}). \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{\omega_{k-1} |\Sigma|_{n-k}} \int_{\Omega_R} |\nabla f_\varepsilon|^2 &= (1 - k/2 + \varepsilon)^2 \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s} dr \\ &\quad + s(2 - k + 2\varepsilon) \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr \\ &\quad + s^2 \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-2} dr. \end{aligned}$$

Note that the last integral on the right hand side above is bounded independently of  $\varepsilon$  for  $s > -1/2$ , that is,

$$\int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-2} dr = O(1).$$

Therefore

$$\begin{aligned} & \frac{1}{\omega_{k-1}|\Sigma|_{n-k}} \int_{\Omega_R} \left( |\nabla f_\varepsilon|^2 - H^2 \frac{f_\varepsilon^2}{d^2} \right) \\ &= \varepsilon(2-k+\varepsilon) \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s} dr \\ & \quad + s(2-k+2\varepsilon) \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr + O(1). \end{aligned} \quad (53)$$

Integrating by parts gives

$$\int_0^R r^{2\varepsilon-1} (-\log r)^{-2s} dr = \frac{1}{2\varepsilon} R^{2\varepsilon} (-\log R)^{-2s} - \frac{s}{\varepsilon} \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr$$

and substituting in (53) yields

$$\begin{aligned} & \frac{1}{\omega_{k-1}|\Sigma|_{n-k}} \int_{\Omega_R} \left( |\nabla f_\varepsilon|^2 - H^2 \frac{f_\varepsilon^2}{d^2} \right) = \frac{2-k+\varepsilon}{2} R^{2\varepsilon} (-\log R)^{-2s} \\ & \quad + \varepsilon s \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr + O(1) \\ &= \varepsilon s \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr + O(1). \end{aligned} \quad (54)$$

Integrating by parts again shows that

$$\begin{aligned} & \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-1} dr \\ &= \frac{1}{2\varepsilon} R^{2\varepsilon} (-\log R)^{-2s-1} - \frac{2s+1}{2\varepsilon} \int_0^R r^{2\varepsilon-1} (-\log r)^{-2s-2} dr = O\left(\frac{1}{\varepsilon}\right). \end{aligned}$$

After substitution in (54) we finally obtain the estimate (52). Hence  $v_s \in \mathcal{H}$  for  $s > -1/2$ .

Our argument to show that  $v_s \notin \mathcal{H}$  for  $s \leq -1/2$  relies on the intuitive idea that  $\int(-\Delta v - a(x)v + Mv)v = \|v\|_{\mathcal{H}}^2$ . To exploit this idea, let us first compute  $\Delta v_s$  near  $\Sigma$ , where  $\eta \equiv 1$ . Write

$$y(t) = t^{1-k/2} (-\log t)^{-s}.$$

Then near  $\Sigma$ , since  $|\nabla d|^2 = 1$ ,

$$\Delta v_s = y''(d) + y'(d)\Delta d.$$

We recall here the fact (see [DN]) that

$$\Delta d = \frac{k-1}{d} + g,$$

where  $g \in L^\infty$ . Hence,

$$\begin{aligned} \Delta v_s &= -H^2 d^{-k/2-1} (-\log d)^{-s} + s(s+1) d^{-k/2-1} (-\log d)^{-s-2} \\ &\quad + (1-k/2) g d^{-k/2} (-\log d)^{-s} + s g d^{-k/2} (-\log d)^{-s-1} \end{aligned}$$

so that

$$\begin{aligned} \Delta v_s + H^2 \frac{v_s}{d^2} &= s(s+1) d^{-k/2-1} (-\log d)^{-s-2} \\ &\quad + (1-k/2) g d^{-k/2} (-\log d)^{-s} + s g d^{-k/2} (-\log d)^{-s-1}. \end{aligned} \quad (55)$$

Observe that  $\Delta v_s, v_s/d^2 \in L^1(\Omega)$  and that equation (55) holds in the sense of distributions. Since we also have  $\nabla v_s \in L^1(\Omega)$ , it follows that for any  $\varphi \in C_c^\infty(\Omega)$ ,

$$\begin{aligned} (v_s | \varphi)_{\mathcal{H}} &= \int_{\Omega} \left( -\Delta v_s - H^2 \frac{v_s^2}{d^2} \right) \varphi + M \int_{\Omega} v_s \varphi \\ &\quad - \int_{\Omega_R} \left( s(s+1) d^{-k/2-1} (-\log d)^{-s-2} \right. \\ &\quad \left. + (1-k/2) g d^{-k/2} (-\log d)^{-s} + s g d^{-k/2} (-\log d)^{-s-1} \right) \varphi \\ &\quad + \int_{\Omega \setminus \Omega_R} \left( -\Delta v_s - H^2 \frac{v_s}{d^2} \right) \varphi + M \int_{\Omega} v_s \varphi. \end{aligned} \quad (56)$$

By density (56) also holds if  $\varphi$  is Lipschitz and  $\varphi = 0$  on  $\partial\Omega$ .

Let us consider first the case  $s \neq -1$ , so that  $s(s+1) \neq 0$ , and suppose that  $s < -1/2$  and  $v_s \in \mathcal{H}$ . Then there exist  $v_n \in C_c^\infty(\Omega)$  such that  $v_n \rightarrow v_s$  in  $\mathcal{H}$ . Note that since the injection  $\mathcal{H} \subset L^2(\Omega)$  is continuous, by passing to a subsequence we also have  $v_n \rightarrow v_s$  a.e. Recall from [DN, inequality (1.4) of Lemma 1.1] that for  $u \in \mathcal{H}$ , we have  $u^+ \in \mathcal{H}$  and  $\|u^+\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}$ . As a consequence  $v_n^+ \rightarrow v_s$  in  $\mathcal{H}$  and a.e. Using  $v_n^+$  in (56) we conclude that

$$\int_{\Omega_R} d^{-k/2-1} (-\log d)^{-s-2} v_n^+ \leq C$$

with  $C$  independent of  $n$ . But then Fatou's lemma implies that

$$\int_{\Omega_R} d^{-k} (-\log d)^{-2s-2} < \infty,$$

which is impossible for  $s < -1/2$ .

For the case  $s = -1$  the argument above does not work. We see that in this case, if  $\Sigma$  is flat and  $\eta \equiv 1$  in an open set then actually

$$\Delta w + H^2 \frac{w}{d^2} = 0$$

in that open set, where


$$w := v_{(-1)} = \eta d^{1-k/2} (-\log d).$$

So we argue as follows: let  $-1/2 < s < 0$ . We are going to show that  $(L + M)v_s \geq (L + M)w$  near  $\Sigma$ . If we assume that  $w \in \mathcal{H}$ , then we can apply the maximum principle and deduce that  $v_s \geq \varepsilon w$  near  $\Sigma$ , which is impossible. Indeed, by formula (55),

$$(L + M)w = -(1 - k/2)gd^{-k/2}(-\log d) + gd^{-k/2} + Md^{1-k/2}(-\log d),$$

and

$$(L + M)v_s = -s(s + 1)d^{-k/2-1}(-\log d)^{-s-2} - (1 - k/2)gd^{-k/2}(-\log d)^{-s} \\ - sgd^{-k/2}(-\log d)^{-s-1} + Md^{1-k/2}(-\log d)^{-s}.$$

Thus, for any  $\varepsilon > 0$  there is a neighborhood  $\Omega_R$  of  $\Sigma$  such that 

$$(L + M)(\varepsilon w - v_s) \leq 0 \quad \text{in } \Omega_R. \quad (57)$$

Pick  $\varepsilon > 0$  such that  $\varepsilon w - v_s \leq 0$  in  $\partial\Omega_R$ . Under the hypothesis  $w \in \mathcal{H}$  we can use a version of the maximum principle to deduce that

$$\varepsilon w - v_s \leq 0 \quad \text{in } \Omega_R.$$

Indeed, assuming  $w \in \mathcal{H}$ , we have  $(\varepsilon w - v_s)^+ \in \mathcal{H}$ . Hence the function

$$z = \begin{cases} (\varepsilon w - v_s)^+ & \text{in } \Omega_R, \\ 0 & \text{in } \Omega \setminus \Omega_R, \end{cases}$$

also belongs to  $\mathcal{H}$ . Let  $z_n \in C_c^\infty(\Omega)$  be such that  $z_n \rightarrow z$  in  $\mathcal{H}$ . Note that (57) holds in the sense of distributions and hence testing (57) with  $z_n^+$  we see that

$$(\varepsilon w - v_s | z_n^+ )_{\mathcal{H}} \leq 0.$$

Letting  $n \rightarrow \infty$  we get

$$\|z\|_{\mathcal{H}} = (\varepsilon w - v_s | z)_{\mathcal{H}} \leq 0.$$

Thus  $z \equiv 0$ , which implies that  $\varepsilon w \leq v_s$  in  $\Omega_R$ , concluding the proof of Lemma 8.  $\square$

**Remark 3.** To show that  $v_s \in \mathcal{H}$  for  $s > -1/2$  one may be tempted to use other approximating sequences, and a very natural one is

$$f_i = \min(v_s, i), \quad i = 1, 2, \dots$$

Again it would be sufficient to establish that for a fixed  $R > 0$  small

$$\int_{\Omega_R} (|\nabla f_i|^2 - a(x)f_i^2) \leq C$$

with  $C$  independent of  $i$ . For  $i$  large let  $r_i > 0$  be such that

$$r_i^{1-k/2}(-\log r_i)^{-s} = i$$

so that  $r_i \rightarrow 0$  as  $i \rightarrow \infty$ . A computation (that we omit) shows that

$$\begin{aligned} \frac{1}{w_{k-1}} \int_{\Omega_R} (|\nabla f_i|^2 - a(x) f_i^2) &= \frac{k-2}{4} (-\log r_i)^{-2s} - \frac{k-2}{2} (-\log R)^{-2s} \\ &\quad + \frac{s^2}{2s+1} ((-\log R)^{-2s-1} - (-\log r_i)^{-2s-1}). \end{aligned}$$

We see that the above quantity remains bounded as  $i \rightarrow \infty$  only for  $s \geq 0$ !

**Remark 4.** The above example shows that for  $m \in \mathbb{N}$  there exists  $v_m \in \mathcal{H}$  with  $\|v_m\|_{\mathcal{H}} = 1$  and

$$\|\min(v_m, m)\|_{\mathcal{H}} \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

Since  $v = \min(v, m) + (v - m)^+$  we also have

$$\|(v_m - m)^+\|_{\mathcal{H}} \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

$\mathcal{H}$  is thus quite different from  $H_0^1(\Omega)$ , in the sense that truncation operators like the one above are not uniformly bounded in  $\mathcal{H}$  whereas it is always true that for any  $v \in H_0^1(\Omega)$ ,  $\min(v, m) \rightarrow v$  in the  $H^1$  topology.

*Proof of Lemma 7.* We will give a proof using a comparison argument with a suitable function. First let us recall that in a neighborhood of  $\Sigma$ ,

$$\Delta d = \frac{k-1}{d} + g,$$

where  $g$  is a bounded function. Hence

$$Ld^{-\alpha} = -\Delta d^{-\alpha} - \mu \frac{d^{-\alpha}}{d^2} = -d^{-\alpha-2}(\alpha^2 - \alpha(k-2) + \mu - \alpha g d). \quad (58)$$

Let  $\alpha = \alpha(\mu)$  as given by (51). This implies that  $\alpha^2 - \alpha(k-2) + \mu = 0$ . Then

$$L(d^{-\alpha} + C_1 d^{-\alpha+1}) = -d^{-\alpha-1}[-\alpha g + C_1((\alpha-1)^2 - (\alpha-1)(k-2) + \mu - (\alpha-1)gd)].$$

Instead of  $\square$  ing with the operator  $L = -\Delta - a(x)$  consider  $L + M$ , where  $M$  is so large that  $(*)$  holds (with  $C$  replaced by  $M$ , this is the same  $M$  that we use in the definition of the space  $\mathcal{H}$ ). Then, since  $(\alpha-1)^2 - (\alpha-1)(k-2) + \mu > 0$  we conclude that for  $C_1 > 0$  large enough

$$\begin{aligned} (L + M)(d^{-\alpha} + C_1 d^{-\alpha+1}) &= -d^{-\alpha-1}[-\alpha g + C_1((\alpha-1)^2 - (\alpha-1)(k-2) + \mu - (\alpha-1)gd)] \\ &\quad + M(d^{-\alpha} + C_1 d^{-\alpha+1}) \\ &\leq 0 \end{aligned} \quad (59)$$

in some fixed neighborhood  $\Omega_R$ ,  $R > 0$ , of  $\Sigma$ . On the other hand, the first eigenfunction  $\varphi_1$  of  $L$  satisfies

$$(L + M)\varphi_1 = (\lambda_1 + M)\varphi_1 \geq 0. \quad (60)$$

Now, both functions  $\varphi_1$  and  $d^{-\alpha} + C_1 d^{-\alpha+1}$  are smooth away from  $\Sigma$  and near  $\Sigma$  so that one can find  $\varepsilon > 0$  such that  $\varepsilon(d^{-\alpha} + C_1 d^{-\alpha+1}) \leq \varphi_1$  in  $\partial\Omega_R$ . We can use now the same version of the maximum principle as in the previous lemma to deduce that

$$\varepsilon(d^{-\alpha} + C_1 d^{-\alpha+1}) \leq \varphi_1 \quad \text{in } \Omega_R.$$

For the estimate  $\varphi_1 \leq C_2 d^{-\alpha(\mu)}$  we need a result from [DD].

**Theorem 4.** *Let  $\Omega$  be a bounded smooth domain. Assume that  $\tilde{a} \in L^1_{\text{loc}}(\Omega)$ ,  $\tilde{a}$  is bounded below (i.e.  $\inf_{\Omega} \tilde{a} > -\infty$ ) and that it satisfies*

$$\gamma \left( \int_{\Omega} |u|^r \right)^{2/r} + \int_{\Omega} \tilde{a}(x) u^2 \leq \int_{\Omega} |\nabla u|^2$$

for some  $\gamma > 0$  and  $r > 2$ . Let  $\varphi_1 > 0$  denote the first eigenfunction for the operator  $L = -\Delta - \tilde{a}(x)$  with zero Dirichlet boundary condition, normalized by  $\|\varphi_1\|_{L^2(\Omega)} = 1$ , and let  $\zeta_0$  denote the solution of

$$\begin{cases} -\Delta \zeta_0 - \tilde{a}(x) \zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists  $C = C(\Omega, \gamma(a), r) > 0$  such that

$$C^{-1} \zeta_0 \leq \varphi_1 \leq C \zeta_0.$$

*Proof of Lemma 7 continued.* We use the above theorem with  $\tilde{a} = a - M$ . In view of this result it suffices to show that

$$\zeta_0 \leq C d^{-\alpha(\mu)}.$$

Using (58) and taking  $\alpha = \alpha(\mu)$  we have

$$\begin{aligned} (L + M)(d^{-\alpha} - C d^{-\alpha+1}) &= -d^{-\alpha-1}[-\alpha g - C((\alpha - 1)^2 - (\alpha - 1)(k - 2) + \mu - (\alpha - 1)gd)] \\ &\quad + M(d^{-\alpha} - C d^{-\alpha+1}) \\ &\geq 1 \end{aligned}$$

in  $\Omega_R$  if we choose  $R > 0$  small and  $C > 0$  large enough. Now take  $C_1$  large enough that  $\zeta_0 \leq C_1(d^{-\alpha} - C d^{-\alpha+1})$  in  $\partial\Omega_R$ . Using the maximum principle as before we deduce that  $\zeta_0 \leq C_1(d^{-\alpha} - C d^{-\alpha+1})$ , which finishes the proof.  $\square$

**Remark 5.** The fact that  $d^{1-k/2} \in \mathcal{H}$  for  $\mu = H^2$  was used in the proof above at the point where the maximum principle was applied. That argument requires that both functions that one would like to compare are in  $\mathcal{H}$ . In general, if one of these functions does not belong to  $\mathcal{H}$  then the maximum principle cannot be applied; see [DD] for an example.

## 4. Some applications

### 4.1. Minimizers for the Hardy inequality

We start this section by extending a result of Brezis and Marcus [BM] regarding the quantity

$$J_\lambda = \inf_{u \in C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 - \lambda u^2}{\int_\Omega u^2 / d(x)^2}, \quad (61)$$

where as usual  $d(x) = \text{dist}(x, \Sigma)$ .

The case studied in [BM] corresponds to  $\Sigma = \partial\Omega$ , and an interesting feature that the authors found in that work is the following, which we state in our situation:

**Theorem 5.** *Fix  $\lambda \in \mathbb{R}$ . Then the infimum in (61) is achieved (in  $H_0^1(\Omega)$ ) if and only if*

$$J_\lambda < H^2.$$

*Proof.* To prove that the condition  $J_\lambda < H^2$  is sufficient for the infimum in (61) to be achieved, one just needs to mimic the arguments in [BM] so we skip this step.

We prove the converse, that is, the claim that if  $J_\lambda = H^2$  then the infimum is not achieved, with an argument similar in spirit to that of [BM]. Suppose that the infimum is achieved by a function  $u \in H_0^1(\Omega)$ , which we can assume to be nonnegative and not identically zero. Assume also that  $J_\lambda = H^2$ . Then  $u$  satisfies

$$-\Delta u - H^2 \frac{u}{d(x)^2} = \lambda u.$$

It follows that  $\lambda$  is the first eigenvalue for the operator  $-\Delta - H^2/d^2$  and that  $u > 0$ . Moreover  $u$  has to be a multiple of  $\varphi_1$  (for this result see e.g. [DD, Lemma 2.3]). But by (50) we know that  $\varphi_1 \sim d^{1-k/2}$ . This shows on the one hand that  $\int_\Omega u^2/d^2 = \infty$ . But Hardy's inequality (4) implies on the other hand that  $\int_\Omega u^2/d^2 < \infty$ .  $\square$

### 4.2. Study of a semilinear problem

In this section, we return to the study of a semilinear problem studied in [DN]. For  $p > 1$ ,  $0 < \mu \leq H^2$  and  $\lambda > 0$  consider the equation

$$\begin{cases} -\Delta u - \frac{\mu}{d(x)^2} u = u^p + \lambda & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (62)$$

where as usual  $d(x) = \text{dist}(x, \Sigma)$ . We showed in [DN] that (at least for small values of  $\mu > 0$ ) there exists a critical exponent

$$p_0 = 1 + \frac{2}{\alpha(\mu)} \quad \text{with} \quad \alpha(\mu) = H - \sqrt{H^2 - \mu}$$



such that (62) admits no solution (in any reasonable sense) for  $p > p_0$  and  $\lambda > 0$ , whereas for some  $\lambda^* = \lambda^*(p)$  solutions exist when  $p < p_0$  and  $0 < \lambda \leq \lambda^*$  (and again no solution exists when  $\lambda > \lambda^*$ ). However, the critical case  $p = p_0$  remained open. Using Lemma 7 in combination with Theorem 4, and following the proof of Proposition 6.1 of [DN], one can prove the following:

→ **Theorem 6.** Given any  $\lambda > 0$ , ~~problem~~ (62) with  $p = p_0$  admits no solution.

## 5. Estimate for solutions of some singular equations

In what follows we will use the method developed in [DD] to prove Theorem 3. The idea is to work with  $w = u/\varphi_1$ , which satisfies an elliptic equation to which Moser's iteration technique can be applied. In the argument it is desirable to approximate the potential  $a(x)$  by bounded ones. In order to get the convergence of the corresponding solutions, it is convenient to rewrite the equation (7) as

$$-\Delta u - \tilde{a}(x)u = C_0u + f,$$

where

$$\tilde{a} = a - C_0$$

→ and  $C_0$  is chosen large enough, larger than  ~~$C$  in (7)~~ (although it will be taken even larger at one point below). We observe that now for any  $h \in \mathcal{H}^*$  the equation

$$\begin{cases} -\Delta v - \tilde{a}v = h & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (63)$$

has a unique solution  $v \in \mathcal{H}$ . Let us also note that the first eigenfunction for the operator  $-\Delta - \tilde{a}$  is still  $\varphi_1$ .

Let us state a result which is a kind of Sobolev inequality with weight (see a proof in [DD]).

**Lemma 9.** Assume that  $a$  satisfies (8). Then for any  $2 \leq q \leq r$  there is a constant  $C$  depending only  $\Omega$ ,  $r$  and  $\gamma(a)$  such that

$$\left( \int_{\Omega} \varphi_1^s |w|^q \right)^{2/q} \leq C \int_{\Omega} \varphi_1^2 (|\nabla w|^2 + w^2) \quad (64)$$

for all  $w \in C^1(\overline{\Omega})$ , where  $s$  is given by the relation

$$\frac{s}{r} = \frac{q-2}{r-2}.$$

**Lemma 10.** *Let  $0 < m < r$  and suppose that*

$$p > \frac{2r}{m(r-2)} \quad (65)$$

and

$$p \geq \frac{r}{r-m}. \quad (66)$$

Then for  $f \in \mathcal{H}^*$ , the unique solution  $v$  to (63) satisfies

$$|v(x)| \leq C \|\varphi_1^{1-m} h\|_p \varphi_1(x), \quad a.e. x \in \Omega.$$

**Remark 6.** If  $m \geq 1$ , the assumption  $h \in \mathcal{H}^*$  can be dropped since one can prove that  $\|h\|_{\mathcal{H}^*} \leq C \|\varphi_1^{1-m} h\|_p$ .

*Proof of Remark 6.* If  $\|\varphi_1^{1-m} h\|_p = +\infty$ , there is nothing to prove. Otherwise  $h$  is locally integrable and for  $\varphi \in C_c^\infty(\Omega)$ ,

$$\left| \int_{\Omega} h\varphi \right| \leq \|h\varphi_1^{1-m}\|_p \|\varphi\varphi_1^{m-1}\|_{p'} \leq \|h\varphi_1^{1-m}\|_p \|\varphi_1\|_{m p'}^{m/m'} \|\varphi\|_{m p'},$$

where we used Hölder's inequality twice. Now (66) implies that  $m p' \leq r$ , so we end up with

$$\left| \int_{\Omega} h\varphi \right| \leq C \|h\varphi_1^{1-m}\|_p \|\varphi\|_{\mathcal{H}} \quad \forall \varphi \in C_c^\infty(\Omega),$$

which is the desired result.  $\square$

*Proof of Lemma 10.* First we note that it is sufficient to prove this result for a bounded potential  $a$ , as long as the constants that appear in the estimates only depend on the constants  $r, \gamma, C$  appearing in (7) and  $\Omega$ . This is the same argument employed in [DD] and we will just sketch it here. Consider  $\tilde{a}_k = \min(\tilde{a}, k)$ , and the first eigenfunction  $\varphi_1^k$  and solution  $v_k$  of (63) with the potential  $a$  replaced by  $a_k$ . Then  $\varphi_1^k \rightarrow \varphi_1$  in  $\mathcal{H}$  and  $v_k \rightarrow v$ . Furthermore,  $\tilde{a}_k$  satisfies

$$\gamma \left( \int_{\Omega} |u|^r \right)^{2/r} + \int_{\Omega} \tilde{a}_k |u|^2 \leq \int_{\Omega} |\nabla u|^2, \quad \forall u \in C_c^\infty(\Omega).$$

So it is enough to establish the results for  $\tilde{a}_k$ . We will assume then that  $\tilde{a}$  is bounded. Then all functions involved belong to  $C^{1,\alpha}(\bar{\Omega})$ .

By working with  $h^+$  and  $h^-$  we can assume that  $h \geq 0$  and hence  $v \geq 0$ . Set

$$w = v/\varphi_1.$$

Then  $w$  satisfies the equation

$$-\nabla \cdot (\varphi_1^2 \nabla w) = \varphi_1 h - (C_0 + \lambda_1) \varphi_1 v.$$

Multiplying the equation by  $w^{2j-1}$  and integrating in  $\Omega$  we find

$$\begin{aligned} \frac{2j-1}{j^2} \int_{\Omega} \varphi_1^2 |\nabla w^j|^2 &= \int_{\Omega} \varphi_1 w^{2j-1} h - (C_0 + \lambda_1) \int_{\Omega} \varphi_1 v w^{2j-1} \\ &= \int_{\Omega} \varphi_1 w^{2j-1} h - (C_0 + \lambda_1) \int_{\Omega} \varphi_1^2 w^{2j}. \end{aligned} \quad (67)$$

Using the variant of Sobolev's inequality (64) applied to  $w^j$  with  $s = mp'$  we obtain

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj} \right)^{2/q} \leq C \int_{\Omega} \varphi_1^2 (|\nabla w^j|^2 + w^{2j}), \quad (68)$$

where  $q$  is given by

$$q = 2 + mp' \frac{r-2}{r}.$$

We note that by (66) we have  $mp' \leq r$  and therefore we can indeed apply Lemma 9. Combining (67) with (68) we get

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj} \right)^{2/q} \leq \frac{Cj^2}{2j-1} \int_{\Omega} \varphi_1 w^{2j-1} h + C \left( 1 - \frac{j^2}{2j-1} (C_0 + \lambda_1) \right) \int_{\Omega} \varphi_1^2 w^{2j}.$$

We make  $C_0$  larger if necessary, so that for  $j \geq 1$  the second term on the right hand side is negative. Therefore

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj} \right)^{2/q} \leq Cj \int_{\Omega} \varphi_1 w^{2j-1} h.$$

By Hölder's inequality

$$\int_{\Omega} \varphi_1 w^{2j-1} h \leq \left( \int_{\Omega} \varphi_1^{mp'} w^{(2j-1)p'} \right)^{1/p'} \|\varphi_1^{1-m} h\|_p$$

and therefore

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj} \right)^{1/(qj)} \leq (Cj)^{1/(2j)} \left( \int_{\Omega} \varphi_1^{mp'} w^{(2j-1)p'} \right)^{1/(2jp')} \|\varphi_1^{1-m} h\|_p^{1/(2j)}. \quad (69)$$

Observe now that condition (65) is equivalent to  $q > 2p'$  and therefore a standard iteration argument yields the result. Let indeed  $j_0 = 1/2 + m/2$  and for  $k \geq 1$ , define  $j_k$  inductively by

$$(2j_k - 1)p' = qj_{k-1}. \quad (70)$$

One can easily show that  $\{j_k\}$  is increasing and converges to  $+\infty$  as  $k \rightarrow +\infty$ , so that if

$$\theta_k = \left( \int_{\Omega} \varphi_1^{mp'} w^{qj_k} \right)^{1/(qj_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{(-1/qj_k)},$$

then  $\{\theta_k\}_k$  is increasing and converges to  $\|w\|_\infty$  as  $k \rightarrow \infty$ . Observe in passing that since  $\varphi_1 \in \mathcal{H}$  and  $mp' \leq r$ , we have

$$\left( \int_{\Omega} \varphi_1^{mp'} \right) < \infty.$$

Equation (69) then yields

$$\theta_k \leq (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{-1/(qj_k)+1/(2j_k p')} \theta_{k-1}^{qj_{k-1}/(2j_k p')} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_k p)}. \quad (71)$$

Now, either  $\{\theta_k\}_k$  remains bounded by  $(\int_{\Omega} \varphi_1^{(1-m)p} h^p)^{1/p}$  for all  $k$ , in which case passing to the limit provides the desired inequality, or there exists a smallest integer  $k_0$  such that for  $k \geq k_0 - 1$ ,

$$\theta_k \geq \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/p}.$$

Using this inequality in (71), we obtain for  $k \geq k_0$ ,

$$\theta_k \leq (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{\frac{1}{j_k} \left( -\frac{1}{q} + \frac{1}{2p'} \right)} \theta_{k-1}. \quad (72)$$

Applying (72) inductively, it follows that

$$\|w\|_\infty \leq \prod_{k=k_0}^{\infty} \left[ (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{\frac{1}{j_k} \left( -\frac{1}{q} + \frac{1}{2p'} \right)} \right] \theta_{k_0-1}. \quad (73)$$

Starting from (70), a straightforward computation shows that for some  $c > 0$ ,

$$j_k = \left( \frac{q}{2p'} \right)^k j_0 + \frac{\left( \frac{q}{2p'} \right)^k - 1}{\frac{q}{2p'} - 1} \sim c \left( \frac{q}{2p'} \right)^k \quad \text{as } k \rightarrow \infty.$$

Since  $q/(2p') > 1$ , we then conclude that the infinite product on the right hand side of (73) converges to some finite constant.

If  $k_0 \geq 2$ , applying again (71) for  $k = k_0 - 1$ , we also have

$$\theta_{k_0-1} \leq C \theta_{k_0-2}^{qj_{k_0-2}/(2j_{k_0-1} p')} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_{k_0-1} p')} \leq C \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/p}, \quad (74)$$

where we used the minimality of  $k_0$  in the last inequality. Combining (74) and (73) yields the desired result.

If  $k_0 = 1$  then by (69),

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj_0} \right)^{1/(qj_0)} \leq C \left( \int_{\Omega} v^r \right)^{m/(2j_0 r)} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_0 p)}, \quad (75)$$

where we used Hölder's inequality and the fact that  $mp' \leq r$ , which follows from (66). Now,

$$\begin{aligned} \gamma \|v\|_r^2 &\leq \|v\|_{\mathcal{H}}^2 = \int_{\Omega} hv \leq \|h\varphi_1^{1-m}\|_p \|\varphi_1^{m-1}v\|_{p'} \\ &\leq \|h\varphi_1^{1-m}\|_p \|w\|_{\infty} \|\varphi_1^m\|_{p'} \leq C \|h\varphi_1^{1-m}\|_p \|w\|_{\infty}. \end{aligned} \quad (76)$$

Using (73), (75) and (76), we obtain

$$\|w\|_{\infty} \leq C \|h\varphi_1^{1-m}\|_p^{\frac{m}{2(m+1)} + \frac{1}{m+1}} \|w\|_{\infty}^{\frac{m}{2(m+1)}},$$

which after simplification yields the desired result.  $\square$

**Lemma 11.** *Let  $0 < m < r$  and suppose that*

$$p < \frac{2r}{m(r-2)} \quad \text{and} \quad p \geq \frac{r}{r-m}.$$

*Then given  $h \in \mathcal{H}^*$ , the unique solution  $v$  to*

$$\begin{cases} -\Delta v - \tilde{a}v = h & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (77)$$

*satisfies*

$$\left( \int_{\Omega} \varphi_1^{mp'} \left| \frac{v}{\varphi_1} \right|^{\alpha} \right)^{1/\alpha} \leq C \|h\varphi_1^{1-m}\|_p. \quad (78)$$

*for any  $\alpha \geq 1$  such that*

$$\frac{1}{\alpha} \geq \frac{1}{p} - \left(1 - \frac{2}{q}\right), \quad (79)$$

*where*

$$q = 2 + mp' \frac{r-2}{r}.$$

*Proof.* The computations of the previous lemma are valid up to (69). Now observe that (69) yields an estimate for

$$\left( \int_{\Omega} \varphi_1^{mp'} w^{qj} \right)^{1/(qj)}$$

if  $qj \geq (2j-1)p'$ , which is equivalent to

$$j \leq \frac{p'}{2p' - q}.$$

Take  $\alpha$  satisfying (79) and  $j = \alpha/q$ . By Hölder's inequality

$$\int_{\Omega} \varphi_1^{mp'} w^{(2j-1)p'} \leq \left( \int_{\Omega} \varphi_1^{mp'} w^{\alpha} \right)^{(2j-1)p'/\alpha} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{1-(2j-1)p'/\alpha},$$

but observe that  $\int_{\Omega} \varphi_1^{mp'} < \infty$  because  $mp' \leq r$  and  $\varphi_1 \in L^r$ . The previous inequality together with (69) yields the result.  $\square$

**Remark 7.** A direct consequence of the above lemma is that if instead of assuming

$$p < \frac{2r}{m(r-2)}$$

we assume that

$$p = \frac{2r}{m(r-2)},$$

then the conclusion is that for all  $1 \leq \alpha < \infty$ ,

$$\left( \int_{\Omega} \varphi_1^{mp'} \left| \frac{v}{\varphi_1} \right|^{\alpha} \right)^{1/\alpha} \leq C \|\varphi_1^{1-m} h\|_p,$$

where the constant  $C$  may depend on  $\alpha$ .

**Remark 8.** In contrast with what we observed in Remark 6,  $h$  need not be in  $\mathcal{H}^*$  for  $\|h\varphi_1^{1-m}\|_p$  to be finite. Hence, in light of inequality (78), one can define by density an operator

$$T = (-\Delta - \tilde{a}(x))^{-1} : L^p(\Omega, \varphi_1^{1-m} dx) \rightarrow L^p(\Omega, \varphi_1^{1-m} dx),$$

which restricted to  $h \in \mathcal{H}^*$  assigns the corresponding solution  $v =: T(h) \in \mathcal{H}$  of (77).

On the other hand, given  $h \in L^1(\Omega)$ , one can consider a weak solution  $u \in L^1(\Omega)$  of equation (77) in the sense that  $\int_{\Omega} a(x)|u|\text{dist}(x, \partial\Omega) < \infty$  and

$$\int_{\Omega} u(-\Delta\varphi - \tilde{a}(x)\varphi) = \int_{\Omega} f\varphi$$

for all  $\varphi \in C^2(\bar{\Omega})$  with  $\varphi|_{\partial\Omega} \equiv 0$ . If  $h \in L^p(\Omega, \varphi_1^{1-m} dx)$  and  $u \in L^p(\Omega, \varphi_1^{1-m} dx)$ , is it true that  $u = T(h)$ ?

*Proof of Theorem 3.* Consider now  $u \in H$  satisfying (7) and let  $u_1$  be the solution of

$$\begin{cases} -\Delta u_1 - \tilde{a}u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We remark that Lemma 10 implies that  $\|u_1/\varphi_1\|_{\infty} \leq C\|\varphi_1^{1-m}f\|_p$ . Thus

$$\left( \int_{\Omega} \varphi_1^2 \left| \frac{u_1}{\varphi_1} \right|^l \right)^{1/l} \leq C\|\varphi_1^{1-m}f\|_p \quad (80)$$

for any  $l \geq 1$ . Define  $u_2 = u - u_1$  so that  $u = u_1 + u_2$  and  $u_2 \in H$  is the unique solution of

$$\begin{cases} -\Delta u_2 - \tilde{a}u_2 = C_0 u & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Starting with  $p_1 = 2$  we shall construct a finite increasing sequence  $p_k$  which will stop at some  $\bar{k}$  such that

$$p_{\bar{k}} \geq \frac{r}{r-2}$$

and such that for each  $k = 1, \dots, \bar{k}$  the following inequality holds:

$$\left( \int_{\Omega} \varphi_1^2 \left| \frac{u}{\varphi_1} \right|^{p_k} \right)^{1/p_k} \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2). \quad (81)$$

Indeed, Lemma 11 applied to  $u_2$  with  $p = p_1 = 2$  and  $m_1 = 1$  implies

$$\left( \int_{\Omega} \varphi_1^2 \left| \frac{u_2}{\varphi_1} \right|^{p_2} \right)^{1/p_2} \leq C\|u\|_{L^2}, \quad (82)$$

where  $p_2$  is given by

$$\frac{1}{p_2} = \frac{1}{p_1} - \left(1 - \frac{2}{q}\right)$$

and  $q = 2 + 2(r-2)/r$ . Inequality (80) combined with (82) shows that (81) holds for  $p_2$ .

We continue this process using Lemma 11 repeatedly with  $p$  and  $m$  in that lemma given by

$$\frac{1}{p_{k+1}} = \frac{1}{p_k} - \left(1 - \frac{2}{q}\right), \quad m_k = \frac{2(p_k - 1)}{p_k}$$

and  $q = 2 + 2(r-2)/r$ . At each step we obtain (inductively)

$$\left( \int_{\Omega} \varphi_1^2 \left| \frac{u_2}{\varphi_1} \right|^{p_{k+1}} \right)^{1/p_{k+1}} \leq C \left( \int_{\Omega} \varphi_1^2 \left| \frac{u}{\varphi_1} \right|^{p_k} \right)^{1/p_k} \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2).$$

This together with (80) proves that (81) holds for  $p_{k+1}$ . We can continue in this way provided

$$\frac{1}{p_k} - \left(1 - \frac{2}{q}\right) > 0,$$

or equivalently

$$p_k < \frac{r}{r-2}.$$

Let  $\bar{k}$  be the first time that we find

$$p_{\bar{k}} \geq \frac{r}{r-2}$$

so that (81) still holds for  $\bar{k}$ . If  $p_{\bar{k}} > r/(r-2)$  then we can apply Lemma 10 directly and conclude that

$$\|u_2/\varphi_1\|_{\infty} \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2),$$

which would finish the proof of the theorem.

In the case  $p_{\bar{k}} = r/(r-2)$  we first use Remark 7 and then Lemma 10.  $\square$

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