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A momentum conserving model with anomalous thermal conductivity in low dimension

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Anomalous large thermal conductivity has been observed numerically and experimentally in one and two dimensional systems. All explicitly solvable microscopic models proposed to date did not explain this phenomenon and there is an open debate about the role of conservation of momentum. We introduce a model whose thermal conductivity diverges in dimension 1 and 2, while it remains finite in dimension 3. We compute the finite-size thermal conductivity of a system of harmonic oscillators perturbed by a non-linear stochastic dynamics conserving momentum and energy. In the limit as the size N of the system goes to infinity, conductivity diverges like N in dimension 1 and like $\ln N$ in dimension 2. Conductivity remains finite if $d \geq 3$ or if a pinning (on site potential) is present. This result clarifies the role of conservation of momentum in the anomalous thermal conductivity.

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When a small gradient of temperature ∇T is applied to a material, we expect that, in the steady state, the heat current satisfies Fourier's law

$$\langle J \rangle = -\kappa \nabla T$$

where κ is the conductivity of the material. So if the system has length N and the left and right ends are in contact with thermostats at temperature T_ℓ and T_r respectively, the current, in the steady state, should be proportional to $(T_\ell - T_r)/N$.

There has been interest in the question of validity of Fourier's law for low dimensional systems ($d \leq 2$), where standard arguments based on linear response break down (cf. [1] and [2] for a review on the subject). Anomalous large conductivity is observed experimentally in carbon nanotubes and numerically in Fermi-Pasta-Ulam (FPU) systems without pinning (on-site potential), where numerical evidence shows a conductivity diverging like N^α , with $\alpha < 1$ in one dimension, and like $\log N$ in dimension 2 (cf. [2] and references therein). Finite conductivity is observed numerically in all pinned cases, and it is expected in all 3 dimensional cases, as long as some non-linearity is present in the interaction [1, 2]. Consequently it has been suggested that conservation of momentum is an important ingredient for the anomalous conductivity in low dimension [4].

In insulating crystals heat is transported by lattice vibrations, and since the pioneering work of Debye, systems of coupled anharmonic oscillators have been used as microscopic models for heat conduction. Non-linearity is extremely important, in fact in the linear case the average energy current $\langle J \rangle$ is independent of the length N of the system, i.e. the conductivity κ_N diverges like N [5]. In fact in the harmonic crystal the normal modes of the

vibrations (phonons) do not interact and follows ballistic motion. A finite asymptotic conductivity instead should result eventually by the diffusive behavior of phonons due to phonon-phonon interaction caused by anharmonicity. A behavior $\kappa_N \sim N^\alpha$ with $0 < \alpha < 1$ corresponds to a superdiffusive behavior.

Since conductivity in non-linear systems is difficult to compute or estimate analytically, it is natural to model the nonlinearities by stochastic perturbations of the linear dynamics. In some sense these stochastic perturbations simulate (qualitatively) the long time (chaotic) effect of the deterministic non-linear model.

We study in this letter a stochastic model where conductivity can be explicitly computed, and diverges in dimension 1 and 2, while it remains finite in dimension 3. So this is the only explicitly solvable model that has a behavior qualitatively consistent with numerical simulations.

We consider a system of harmonic (linear) coupled oscillators where the Hamiltonian dynamics are perturbed by a random exchange of momentum between nearest neighbor atoms. The random exchange of momentum conserves total momentum and total energy. We construct this noise with a diffusion on the surface of constant kinetic energy and momentum. Because of the conservation laws, this noise introduces a certain non-linearity in the model.

We compute explicitly the conductivity κ_N of the finite closed system of linear size N , by using Green-Kubo formula for the dynamics in the microcanonical ensemble. We find that it has a finite explicit limit as $N \rightarrow \infty$ in $d \geq 3$ or in the pinned cases, while it diverges like N in the unpinned 1-dimensional case, and like $\log N$ in the unpinned 2-dimensional case.

Most of the numerical simulations on FPU models concern open systems with boundaries in contact with thermal reservoirs. In low dimension, since conductivity diverges, the connection with the Green-Kubo formula for the finite system is not always clear in the deterministic cases. So for our model we compute also the thermal conductivity of the open system in contact with thermal reservoirs at different temperatures and we show that it coincides with the Green-Kubo definition.

Other explicitly solvable models have been proposed before as perturbation of the harmonic chain (in [6] and [7] only the number of particles is conserved, in [14] only energy and the number of particles). In all these cases conductivity is always finite.

In order to compute the conductivity by the Green-Kubo formula, we consider the dynamics of the closed system of length N with periodic boundary conditions. The Hamiltonian is given by

$$\mathcal{H}_N = \frac{1}{2} \sum_{\mathbf{x}} [\mathbf{p}_{\mathbf{x}}^2 + \mathbf{q}_{\mathbf{x}} \cdot (\nu I - \alpha \Delta) \mathbf{q}_{\mathbf{x}}].$$

The atoms are labeled by $\mathbf{x} \in \mathbb{T}_N^d$, the d -dimensional discrete torus of length N . We denote with ∇ and Δ respectively the discrete gradient and the discrete Laplacian on \mathbb{T}_N^d . $\{\mathbf{q}_{\mathbf{x}}\}$ are the displacements of the atoms from their equilibrium positions. The parameter $\alpha > 0$ is the strength of the interparticles springs, and $\nu \geq 0$ is the strength of the pinning (on-site potential).

We consider a stochastic dynamics where the probability density distribution on the phase space at time t , denoted by $P(t, \mathbf{q}, \mathbf{p})$, evolves following the Fokker-Planck equation (cf. [9])

$$\frac{\partial P}{\partial t} = (-A + \gamma S)P = LP.$$

where A is the usual Hamiltonian vector field

$$A = \sum_{\mathbf{x}} \{\mathbf{p}_{\mathbf{x}} \cdot \partial_{\mathbf{q}_{\mathbf{x}}} - [(\nu I - \alpha \Delta) \mathbf{q}_{\mathbf{x}}] \cdot \partial_{\mathbf{p}_{\mathbf{x}}}\}$$

while S is the generator of the stochastic perturbation and $\gamma > 0$ is a positive parameter that regulates its strength. The operator S acts only on the momentums $\{\mathbf{p}_{\mathbf{x}}\}$ and generates a diffusion on the surface of constant kinetic energy and constant momentum. This is defined as follows. For every nearest neighbor atoms \mathbf{x} and \mathbf{z} , consider the $d-1$ dimensional surface of constant kinetic energy and momentum

$$\mathbb{S}_{e, \mathbf{p}} = \{(\mathbf{p}_{\mathbf{x}}, \mathbf{p}_{\mathbf{z}}) \in \mathbb{R}^{2d} : \mathbf{p}_{\mathbf{x}}^2 + \mathbf{p}_{\mathbf{z}}^2 = 2e; \mathbf{p}_{\mathbf{x}} + \mathbf{p}_{\mathbf{z}} = \mathbf{p}\}.$$

The following vector fields are tangent to $\mathbb{S}_{e, \mathbf{p}}$

$$X_{\mathbf{x}, \mathbf{z}}^{i, j} = (p_{\mathbf{z}}^j - p_{\mathbf{x}}^j)(\partial_{p_{\mathbf{z}}^i} - \partial_{p_{\mathbf{x}}^i}) - (p_{\mathbf{x}}^i - p_{\mathbf{z}}^i)(\partial_{p_{\mathbf{z}}^j} - \partial_{p_{\mathbf{x}}^j}).$$

so $\sum_{i, j=1}^d (X_{\mathbf{x}, \mathbf{z}}^{i, j})^2$ generates a diffusion on $\mathbb{S}_{e, \mathbf{p}}$. In $d \geq 2$ we define

$$S = \frac{1}{2(d-1)} \sum_{\mathbf{x}} \sum_{i, j, k} \left(X_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}^{i, j} \right)^2$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the canonical basis of \mathbb{Z}^d . Observe that this noise conserves the total momentum $\sum_{\mathbf{x}} \mathbf{p}_{\mathbf{x}}$ and energy \mathcal{H}_N , i.e.

$$S \sum_{\mathbf{x}} \mathbf{p}_{\mathbf{x}} = 0, \quad S \mathcal{H}_N = 0$$

In dimension 1, in order to conserve total momentum and total kinetic energy, we have to consider a random exchange of momentum between three consecutive atoms, and we define $S = \frac{1}{6} \sum_{x \in \mathbb{T}_N^1} (Y_x)^2$ where

$$Y_x = (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} + (p_{x-1} - p_x)\partial_{p_{x+1}}$$

which is vector field tangent to the surface of constant energy and momentum of the three particles involved.

These dynamics can also be written in terms of the solutions of the stochastic differential equations

$$d\mathbf{p}_{\mathbf{x}} = -(\nu I - \Delta) \mathbf{q}_{\mathbf{x}} dt + 2\gamma \Delta \mathbf{p}_{\mathbf{x}} dt + \sqrt{\gamma} d\mathbf{n}_{\mathbf{x}}(t) \quad (1)$$

where of course $\dot{\mathbf{q}}_{\mathbf{x}} = \mathbf{p}_{\mathbf{x}}$ and $\mathbf{n}_{\mathbf{x}}(t)$ are defined by the Ito's stochastic integrals

$$\mathbf{n}_{\mathbf{x}}(t) = \frac{1}{2\sqrt{d-1}} \sum_{\|\mathbf{y}-\mathbf{x}\|=1} \sum_{i, j} \int_0^t (X_{\mathbf{x}, \mathbf{y}}^{i, j} \mathbf{p}_{\mathbf{x}})(s) dw_{\mathbf{x}, \mathbf{y}}^{i, j}(s)$$

Here $w_{\mathbf{x}, \mathbf{y}}^{i, j}(t) = w_{\mathbf{y}, \mathbf{x}}^{i, j}(t)$ are independent standard Wiener processes. In $d = 1$ the expression is similar with the term $2\gamma \Delta \mathbf{p}_{\mathbf{x}}$ replaced by $(\gamma/6)\Delta(4p_x + p_{x+1} + p_{x-1})$.

Defining the energy of the atom \mathbf{x} as

$$e_{\mathbf{x}} = \frac{1}{2} \mathbf{p}_{\mathbf{x}}^2 + \frac{\alpha}{4} \sum_{\mathbf{y}: |\mathbf{y}-\mathbf{x}|=1} (\mathbf{q}_{\mathbf{y}} - \mathbf{q}_{\mathbf{x}})^2 + \frac{\nu}{2} \mathbf{q}_{\mathbf{x}}^2,$$

the energy conservation law can be read locally as

$$e_{\mathbf{x}}(t) - e_{\mathbf{x}}(0) = \sum_{k=1}^d (J_{\mathbf{x}-\mathbf{e}_k, \mathbf{x}}(t) - J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t))$$

where $J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t)$ is the total energy current between \mathbf{x} and $\mathbf{x} + \mathbf{e}_k$ up to time t . This can be written as

$$J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t) = \int_0^t j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(s) ds + M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t). \quad (2)$$

In the above $M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t)$ is the Ito's stochastic integral defined by

$$M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t) = \sqrt{\frac{\gamma}{d-1}} \sum_{i, j} \int_0^t \left(X_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}^{i, j} e_{\mathbf{x}} \right)(s) dw_{\mathbf{x}, \mathbf{y}}^{i, j}(s)$$

The instantaneous energy currents $j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}$ satisfy the equation

$$Le_{\mathbf{x}} = \sum_{k=1}^d (j_{\mathbf{x}-\mathbf{e}_k, \mathbf{x}} - j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k})$$

and can be written as

$$j_{\mathbf{x},\mathbf{x}+\mathbf{e}_k} = j_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}^a + \gamma j_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}^s \quad (3)$$

The first term in (3) is the Hamiltonian contribution to the energy current

$$j_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}^a = -\frac{\alpha}{2}(\mathbf{q}_{\mathbf{x}+\mathbf{e}_k} - \mathbf{q}_{\mathbf{x}}) \cdot (\mathbf{p}_{\mathbf{x}+\mathbf{e}_k} + \mathbf{p}_{\mathbf{x}}) \quad (4)$$

while the noise contribution in $d \geq 2$ is

$$\gamma j_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}^s = -\gamma \nabla_{\mathbf{e}_k} \mathbf{p}_{\mathbf{x}}^2 \quad (5)$$

and in $d = 1$ is

$$\begin{aligned} \gamma j_{x,x+1}^s &= -\gamma \nabla \varphi(p_{x-1}, p_x, p_{x+1}) \\ \varphi(p_{x-1}, p_x, p_{x+1}) &= \\ \frac{1}{6} [p_{x+1}^2 + 4p_x^2 + p_{x-1}^2 + p_{x+1}p_{x-1} - 2p_{x+1}p_x - 2p_xp_{x-1}] \end{aligned}$$

We define conductivity using the microcanonical version of the Green-Kubo formula. Consider the closed system on \mathbb{T}_N^d in the microcanonical measure. This is usually defined as the uniform measure on the energy surface defined by $\mathcal{H} = N^d e$, for a given $e > 0$. Our dynamics conserve also $(\sum \mathbf{p}_{\mathbf{x}})^2 + \nu(\sum \mathbf{q}_{\mathbf{x}})^2$. Notice that the dynamics is invariant under the change of coordinates $\mathbf{p}'_{\mathbf{x}} = \mathbf{p}_{\mathbf{x}} - \sum_{\mathbf{y}} \mathbf{p}_{\mathbf{y}}$ and $\mathbf{q}'_{\mathbf{x}} = \mathbf{q}_{\mathbf{x}} - \sum_{\mathbf{y}} \mathbf{q}_{\mathbf{y}}$. Consequently, without any loss of generality, we can fix $\sum \mathbf{p}_{\mathbf{x}} = 0$ and $\sum \mathbf{q}_{\mathbf{x}} = 0$ in the microcanonical measure. The conductivity of the finite system in the direction \mathbf{e}_1 is defined as

$$\begin{aligned} \kappa_N^{1,1} &= \lim_{t \rightarrow \infty} \frac{d}{2e^2 t} \sum_{\mathbf{x}} \mathbb{E} (J_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}(t) J_{0,\mathbf{e}_1}(t)) \\ &= \lim_{t \rightarrow \infty} \frac{d}{2e^2 t N^d} \mathbb{E} \left(\left[\sum_{\mathbf{x}} J_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}(t) \right]^2 \right) \end{aligned} \quad (6)$$

where \mathbb{E} is the expectation starting with the microcanonical distribution defined above. We have used in (6) the translation invariance property of the microcanonical measure and of the dynamics. Notice that because of the gradient form of $j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^s$ (cf (5)) and the periodic boundary conditions, we have $\sum_{\mathbf{x}} j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1} = \sum_{\mathbf{x}} j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a$. By explicit calculation one can show (cf. [8] for the details) the decorrelation

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\left[\int_0^t \sum_{\mathbf{x}} j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a \right] \left[\sum_{\mathbf{x}} M_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}(t) \right] \right) = 0$$

and that the contribution to $\kappa_N^{1,1}$ due to the variances of $M_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}$ is

$$\lim_{t \rightarrow \infty} \frac{d}{2e^2 N^d t} \mathbb{E} \left(\left[\sum_{\mathbf{x}} M_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}(t) \right]^2 \right) = \gamma.$$

The contribution of $j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a$ is then

$$\begin{aligned} &\frac{d}{e^2} \sum_{\mathbf{x}} \int_0^\infty \mathbb{E} (j_{0,\mathbf{e}_1}^a(0) j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a(t)) dt \\ &= -\frac{d}{e^2} \left\langle j_{0,\mathbf{e}_1}^a(0) L^{-1} \left(\sum_{\mathbf{x}} j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a \right) \right\rangle_N \end{aligned} \quad (7)$$

where $\langle \cdot \rangle_N$ denotes the microcanonical expectation. By explicitly calculation we can solve the equation

$$L^{-1} \left(\sum_{\mathbf{x}} j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1}^a \right) = \frac{\alpha}{\gamma} \sum_{\mathbf{x},\mathbf{y}} g_N(\mathbf{x} - \mathbf{y}) \mathbf{p}_{\mathbf{x}} \cdot \mathbf{q}_{\mathbf{y}}$$

where $g_N(\mathbf{x})$ is a function on \mathbb{T}_d^N solution of the equation

$$4\Delta g_N(\mathbf{x}) = (\delta(\mathbf{x} - \mathbf{e}_1) - \delta(\mathbf{x} + \mathbf{e}_1)) \quad (8)$$

for $d \geq 2$, or

$$\frac{1}{3} \Delta [4g_N(x) + g_N(x+1) + g_N(x-1)] = (\delta(x-1) - \delta(x+1)) \quad (9)$$

for $d = 1$. Substituting in (7), and using equivalence of ensembles estimates (cf [8] for the details) we have

$$\begin{aligned} \kappa_N^{1,1} &= \gamma + \frac{\alpha^2}{2\gamma} \sum_{\mathbf{x}} (g_N(\mathbf{x} - \mathbf{e}_1) - g_N(\mathbf{x} + \mathbf{e}_1)) \Gamma_N(0, \mathbf{x}) \\ &\quad + o_N \end{aligned} \quad (10)$$

where Γ_N is the kernel of the operator $(\nu I - \alpha \Delta)^{-1}$ on \mathbb{T}_d^N , and o_N is bounded if $d = 1$ and $\nu = 0$, of order $N^{-1} \log N$ if $d = 2$ and $\nu = 0$, and of order $N^{-d/2}$ in all other cases. A simple Fourier analysis shows that, if $\nu = 0$ (i.e. the unpinned case), (10) diverge like N in dimension 1 and like $\log N$ in dimension 2. In all other cases $\kappa_N^{1,1}$ converge to a finite limit $\kappa^{1,1}$ given by

$$\kappa^{1,1} = \gamma + \frac{\alpha^2}{2\gamma} \sum_{\mathbf{z}} g(\mathbf{z}) (\Gamma(0, \mathbf{z} + \mathbf{e}_1)) - \Gamma(0, \mathbf{z} - \mathbf{e}_1))$$

where Γ is the kernel of the operator $(\nu I - \alpha \Delta)^{-1}$ on \mathbb{Z}^d , while g is the solution of equations (8) or (9) in \mathbb{Z}^d . Observe that this expression diverges for $\gamma \rightarrow 0$, as expected since deterministic harmonic oscillators have infinite conductivity [5].

Consider now the open case where the finite system is in contact with 2 heat baths at different temperatures T_ℓ and T_r . We model these external thermal baths with Langevin-type reservoirs at these given temperatures (cf. [1, 2, 8]). If $T_\ell = T_r = T$ the centered gaussian product measure of variance T is the unique stationary state. When $T_\ell \neq T_r$, this system has still a unique stationary measure, but it cannot be computed explicitly. Let

us denote the expectation with respect to this stationary measure with $\langle \cdot \rangle_{ss,N}$. The conductivity of this finite system can be defined as

$$\kappa_N^{ss} = \frac{N \langle j_{\mathbf{x}, \mathbf{x} + \mathbf{e}_1} \rangle_{ss,N}}{T_\ell - T_r}.$$

By using method similar to the one used in [14], one can compute explicitly κ_N^{ss} and show that $\frac{\kappa_N^{ss}}{\kappa_N^{1,1}} \rightarrow 1$ in all cases.

Discussion. The exact results presented in this letter concerning the stochastic model we introduced give some indications about the role of conservation of momentum and of confinement (pinning) in heat conduction for the nonlinear deterministic Hamiltonian case. In fact the behavior of the conductivity that we proved in our stochastic model is (qualitatively) the same as that indicated by numerical simulation for the deterministic non-linear FPU models, at least if $d \geq 2$ or for pinned models.

In the one-dimensional unpinned model we have a divergence like N . At first sight one can find this fact surprising: in this case we have ballistic transport of energy as for the deterministic harmonic chain [5], while noise dramatically changes the conductivity of the pinned chain. We can explain this with the following argument. In deterministic harmonic systems the energy of each mode is conserved, in both pinned or unpinned chain; so if modes are created by initial or boundary conditions, they cannot be dissipated by the dynamics. This causes ballistic transport and diverging conductivity, in both cases (cf. [5], [10]). The effect of the energy-momentum conservative noise we have introduced is to scatter modes randomly with rate proportional to k^2 , for small wavenumber k . The *velocity* of the k -mode is given by the gradient of the dispersion function $\nabla\omega(k)$. In the unpinned chain $\nabla\omega(k) \sim 1$ for small k , so small wavenumber modes have little probability to be scattered, and their movement results in a ballistic contribution to energy transport. In the pinned chain small wavenumber modes move very slowly ($\nabla\omega(k) \sim k$), so they have high probability to be scattered and then diffused while they cross the system.

In [14] we considered the unpinned 1-dimensional harmonic chain with noise that conserves only energy, and prove that conductivity is finite in any dimension. In this last case all modes are scattered with constant rate (independent of k).

In non-linear FPU type of interaction, a behavior $\kappa_N \sim N^\alpha$, with $0 < \alpha < 1$ is observed numerically. But numerical simulations are not conclusive about the value of α and there is an intense debate in the literature on this value (cf. [2]). As suggested recently by R.

Livi [11], there is probably not universal behavior of the conductivity in one-dimensional systems (unlike the logarithmic behavior of the two-dimensional systems) and the value of α may depend on the specific non-linearity of the interaction. We think that this is due to some super-diffusive behavior of the longwave modes. It remains to understand what causes this superdiffusive behavior in the non-linear case with respect to the ballistic behavior that we have proved for the harmonic chain with conservative noise. The non-linearity seems to create some scattering of the longwave modes, which results in a breaking of the ballistic transport, and in a superdiffusive behavior of these modes. An extreme case is given by the 1-dimensional coupled-rotors model, which is an example of a non-linear chain that conserves momentum and has finite conductivity [12]. In this example, rotobreathers (isolated rotors with high kinetic energy that turn very fast) scatter waves that try to pass through them (cf. [13]).

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