



HAL
open science

A counter example to the Bueler's conjecture.

Gilles Carron

► **To cite this version:**

Gilles Carron. A counter example to the Bueler's conjecture.. Proc. Amer. Math. Soc., 2007, 137 (5), pp.1579–1583. hal-00009000

HAL Id: hal-00009000

<https://hal.science/hal-00009000>

Submitted on 22 Sep 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A COUNTER EXAMPLE TO THE BUELER'S CONJECTURE.

GILLES CARRON

ABSTRACT. We give a counter example to a conjecture of E. Bueler stating the equality between the DeRham cohomology of complete Riemannian manifold and a weighted L^2 cohomology where the weight is the heat kernel.

1. INTRODUCTION

1.1. **Weighted L^2 cohomology :** We first describe weighted L^2 cohomology and the Bueler's conjecture. For more details we refer to E. Bueler's paper ([2] see also [5]).

Let (M, g) be a complete Riemannian manifold and $h \in C^\infty(M)$ be a smooth function, we introduce the measure μ :

$$d\mu(x) = e^{2h(x)} d\text{vol}_g(x)$$

and the space of L^2_μ differential forms :

$$L^2_\mu(\Lambda^k T^*M) = \{\alpha \in L^2_{loc}(\Lambda^k T^*M), \|\alpha\|_\mu^2 := \int_M |\alpha|^2(x) d\mu(x) < \infty\}.$$

Let $d_\mu^* = e^{-2h} d^* e^{2h}$ be the formal adjoint of the operator $d : C_0^\infty(\Lambda^k T^*M) \rightarrow L^2_\mu(\Lambda^{k+1} T^*M)$. The k^{th} space of (reduced) L^2_μ cohomology is defined by :

$$\mathbb{H}_\mu^k(M, g) = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{\overline{dC_0^\infty(\Lambda^{k-1} T^*M)}} = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{\overline{d\mathcal{D}_\mu^{k-1}(d)}}$$

where we take the L^2_μ closure and $\mathcal{D}_\mu^{k-1}(d)$ is the domain of d , that is the space of forms $\alpha \in L^2_\mu(\Lambda^{k-1} T^*M)$ such that $d\alpha \in L^2_\mu(\Lambda^k T^*M)$. Also if $\mathcal{H}_\mu^k(M) = \{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0, d_\mu^* \alpha = 0\}$ then we also have $\mathcal{H}_\mu^k(M) \simeq \mathbb{H}_\mu^k(M)$. Moreover if the manifold is compact (without boundary) then the celebrated Hodge-deRham theorem tells us that these two spaces are isomorphic to $H^k(M, \mathbb{R})$ the real cohomology groups of M . Concerning complete Riemannian manifold, E. Bueler had made the following interesting conjecture [2] :

Conjecture : *Assume that (M, g) is a connected oriented complete Riemannian manifold with Ricci curvature bounded from below. And consider for $t > 0$ and $x_0 \in M$, the heat kernel $\rho_t(x, x_0)$ and the heat kernel measure $d\mu(x) = \rho_t(x, x_0) d\text{vol}_g(x)$, then 0 is an isolated eigenvalue of the self adjoint operator $dd_\mu^* + d_\mu^* d$ and for any k we have an isomorphism :*

$$\mathcal{H}_\mu^k(M) \simeq H^k(M, \mathbb{R}).$$

E. Bueler had verified this conjecture in degree $k = 0$ and according to [3] it also hold in degree $k = \dim M$. About the topological interpretation of some weighted

1991 *Mathematics Subject Classification.* Primary 58A14 ; Secondary 58J10.
Key words and phrases. L^2 cohomology.

L^2 cohomology, there is results of Z.M .Ahmed and D. Strook and more optimal results of N. Yeganefar ([1],[5]). Here we will show that we can not hope more :

Theorem 1.1. *In any dimension n , there is a connected oriented manifold M , such that for any complete Riemannian metric on M and any smooth positive measure μ , the natural map :*

$$\mathcal{H}_\mu^k(M) \rightarrow H^k(M, \mathbb{R})$$

is not surjective for $k \neq 0, n$.

Actually the example is simple take a compact surface S of genus $g \geq 2$ and

$$\Gamma \simeq \mathbb{Z} \rightarrow \hat{S} \rightarrow S$$

be a cyclic cover of S and in dimension n , do consider $M = \mathbb{T}^{n-2} \times \hat{S}$ the product of a $(n-2)$ torus with \hat{S} .

2. AN TECHNICAL POINT : THE GROWTH OF HARMONIC FORMS :

We consider here a complete Riemannian manifold (M^n, g) and a positive smooth measure $d\mu = e^{2h} d\text{vol}_g$ on it.

Proposition 2.1. *Let $o \in M$ be a fixed point, for $x \in M$, let $r(x) = d(o, x)$ be the geodesic distance between o and x , $R(x)$ be the maximum of the absolute value of sectional curvature of planes in $T_x M$ and define $m(R) = \max_{r(x) \leq R} \{|\nabla dh|(x) + R(x)\}$. There is a constant C_n depending only of the dimension such that if $\alpha \in \mathcal{H}_\mu^k(M)$ then on the ball $r(x) \leq R$:*

$$e^{h(x)} |\alpha|(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\sqrt{\text{vol}(B(o, 2R))}} \|\alpha\|_\mu.$$

Proof. – If we let $\theta(x) = e^{h(x)} \alpha(x)$ then θ satisfies the equation :

$$(dd^* + d^*d)\theta + |dh|^2\theta + 2\nabla dh(\theta) - (\Delta h)\theta = 0.$$

where the Hessian of h acts on k forms by :

$$\nabla dh(\theta) = \sum_{i,j} \theta_j \wedge \nabla dh(e_i, e_j) \text{int}_{e_j} \theta,$$

where $\{e_i\}_i$ is a local orthonormal frame and $\{\theta_i\}_i$ is the dual frame. If \mathcal{R} is the curvature operator of (M, g) , the Bochner-Weitzenböck formula tells us that $(dd^* + d^*d)\theta = \nabla^* \nabla \theta + \mathcal{R}(\theta)$. Hence, the function $u(x) = |\theta|(x)$ satisfies (in the distribution sense) the subharmonic estimate :

$$(1) \quad \Delta u(x) \leq C_n (R(x) + |\nabla dh|(x)) u(x).$$

Now according to L. Saloff-Coste (theorem 10.4 in [4]), on $B(o, 2R) = \{r(x) < 2R\}$ the ball of radius $2R$, we have a Sobolev inequality : $\forall f \in C_0^\infty(B(o, 2R))$

$$(2) \quad \|f\|_{L^{\frac{2\nu}{\nu-2}}}^2 \leq C_n \frac{R^2 e^{c_n \sqrt{k_R} R}}{(\text{vol}(B(o, 2R)))^{2/\nu}} \|df\|_{L^2}^2$$

where $-k_R < 0$ is a lower bound for the Ricci curvature on the ball $B(o, 4R)$ and $\nu = \max(3, n)$. With (1) and (2), the Moser iteration scheme implies that for $x \in B(o, R)$,

$$u(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\sqrt{\text{vol}(B(o, 2R))}} \|u\|_{L^2(B(o, 2R))}.$$

From which we easily infer the desired estimate. \square

3. JUSTIFICATION OF THE EXAMPLE AND FURTHER COMMENTS

3.1. Justification. Now, we consider the manifold $M = \mathbb{T}^{n-2} \times \hat{S}$ which is a cyclic cover of $\mathbb{T}^{n-2} \times S$; let γ be a generator of the covering group. We assume M is endowed with a complete Riemannian metric and a smooth measure $d\mu = e^{2h} d\text{vol}_g$. For every $k \in \{1, \dots, n-1\}$ we have a k -cycle c such that $\gamma^l(c) \cap c = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$ and a closed k -form ψ with compact support such that $\int_c \psi = 1$ and such that $(\text{support } \psi) \cap (\text{support } (\gamma^l)^* \psi) = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$. Let $a = (a_p)_{p \in \mathbb{N}}$ be a non zero sequence of real number : then the k -form $\psi_a = \sum_{p \in \mathbb{N}} a_p (\gamma^p)^* \psi$ represents a non zero k cohomology class of M , indeed $\int_{\gamma^p c} \psi_a = a_p$. We define $R_p = \max\{r(\gamma^l(x)), x \in c, l = 0, \dots, p\}$, then if the deRham cohomology class of ψ_a contains $\alpha \in \mathcal{H}_\mu^k(M)$, then according to (2.1), we will have $|a_p| = \left| \int_{\gamma^p c} \alpha \right| \leq M_p \|\alpha\|_\mu$; where

$$M_p = \text{vol}_g(\gamma^p(c)) C_n \frac{e^{C_n m(2R_p)R_p^2}}{\sqrt{\text{vol}(B(o, 2R_p))}} \max_{r(x) \leq R_p} e^{-h(x)}.$$

As a consequence, for the sequence defined by $a_p = (M_p + 1)2^p$, ψ_a can not be represented by a element of $\mathcal{H}_\mu^k(M)$.

3.2. Further comments. Our counter example doesn't exclude that this conjecture hold for a complete Riemannian metric with bounded curvature, positive injectivity radius on the interior of a compact manifold with boundary.

REFERENCES

- [1] Z.M. Ahmed and D.W. Stroock, A Hodge theory for some non-compact manifolds, *J. Differential Geom.*, **54**,(2000), n° 1, 177–225.
- [2] E. Bueler, *The heat kernel weighted Hodge Laplacian on non compact manifolds*, Trans. A.M.S. **351**, **2** (1999), 683–713.
- [3] F.-Z. Gong, and F.Y. Wang, Heat kernel estimates with application to compactness of manifolds, *Q. J. Math.*, **52**, (2001), n° 2, 171–180.
- [4] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds. *J. Diff. Geometry*, **36**, 1992, 417-450.
- [5] N. Yeganefar, Sur la L^2 -cohomologie des variétés à courbure négative, *Duke Math. J.*, **122**, (2004), n°1, 145–180. .

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY (UMR 6629), UNIVERSITÉ DE NANTES, 2, RUE DE LA HOUSSINIÈRE, B.P. 92208, 44322 NANTES CEDEX 3, FRANCE

E-mail address: Gilles.Carron@math.univ-nantes.fr