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Abstract : We consider the two-dimensional problem of recovering globally in time the heat flux on the surface of a layer inside of a heat conducting body from two interior temperature measurements. The problem is ill-posed. The approximation function is represented by a two-dimensional Sinc series and the error estimate is given.

Key words and phrases : heat equation, heat flux, Sinc series, ill-posed problem, regularization, numerical results. **AMS Subjects Classification :** 35K05, 35K20, 35R30, 42A38

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1. Introduction

In this paper, we consider the problem of recovering the heat flux on the surface of a thin layer inside of a heat conducting body from transient temperature measurements. The problem is raised in many applications in Physics and Geology. In fact, in many physical situation (see, e.g., [B]), we cannot attach a temperature sensor at the surface of the body (for example, the skin of a missile). On the other hand, we can easily measure the temperature history at an interior point of the body. Hence, to get the heating history in the body, we want to use temperature measured in the interior of the body. In the simplest model, the heat-conducting body is assumed to have a constant conductivity and represented by the half-line $x > 0$ (see, e.g. [C, EM, LN, TV]),. While giving many useful results, this model is not suitable for the case of a body having a series of superposed layers, each of which has a constant conductivity.

Precisely, we shall consider the problem corresponding to a thin layer of the body represented by the strip $\mathbb{R} \times (0, 3)$, say. Let u be the temperature in the strip. For the uniqueness of solution, we shall have to measure the temperature history at two interior lines $\mathbb{R}\times\{y = 1\}$ and $\mathbb{R}\times\{y = 2\}$. From these measurements, we can identify uniquely the heating history inside of the layer (see, e.g., [B]). However, to make clearly ideas, we limit to the problem of finding the heat flux $u_y(x, 1, t) \equiv w(x, t)$ on the one side of the boundary of the slab $\mathbb{R} \times (1, 2)$. Our problem is classical and can be changed to the one of finding the solution of a convolution equation of Volterra type which can be solved on any finite time interval by the iteration (see, e.g., [F]). However, it is surprised that the problem is ill-posed if we consider the problem on the global time interval and the literature on this way is very scarce. In fact, despite uniqueness, the global solution in $L^2(\mathbb{R}\times\mathbb{R}_+)$ is unstability and hence, in this point of view, a sort of regularization is in order.

As discussed in the latter paragraph, the main purpose of our paper is to present a regularization of the problem. Moreover, an effective way of approximating the heat flux w is also worthy of considering. Indeed, we recall that the problem can be changed into a convolution equation. There are many methods for regularizing the equation (see [TA, B]). In the present paper, using the method of truncated integration, one can approximate the Fourier transform of the solution by a function having the *compact support* in $\mathbb{R}^2$. Therefore, the solution can be represented by an expansion of two dimensional Sinc series (see [AGTV]). The Sinc method is based on the Cardinal functions

$$S(p,d)(z) = \frac{sin[(\pi(z-pd))/d]}{\pi(z-pd)/d}, \;\; p\in\mathbb{Z}, d>0$$

which dates back to the works of many mathematicians (Bohr, de la Vallee Poussin, E. T. Whittaker, ...). The one dimensional version of the method is studied very clearly and completely in [S]. Some primary results related to the two dimensional Sinc approximation were given in [AGTV]. As is known, the Sinc series converges very rapidly at an incredible $0(e^{-cn^{1/2}})$ rate, where $c>0$ and n is the dimension of approximation (see [S]). Hence, this method, which is new in our knowledge, is very effective.

The remainder of the present paper is divided into three sections. In Section 2, we state precisely the problem, change it into an integral equation of convolution type, and state the main result of our paper. In Section 3, we give the proof of the main result and in Section 4 we present some numerical results in terms of two dimensional Sinc functions.

2. Notations and the main result

Consider the problem of determining the heat flux

$$u_y(x,1,t) \equiv w(x,t), \;\; x\in\mathbb{R}, \;\; t>0, \tag{1}$$

where u satisfies

$$\Delta u - \frac{\partial u}{\partial t} = 0 \quad x\in\mathbb{R}, \; 1<y<2, \; t>0, \tag{2}$$

with the boundary conditions

$$u(x,2,t) = g(x,t), \;\; x\in\mathbb{R}, \;\; t>0, \tag{3}$$

$$u(x,1,t) = f(x,t), \;\; x\in\mathbb{R}, \;\; t>0, \tag{4}$$

and the initial condition

$$u(x,y,0)=0, \quad x\in\mathbb{R}, \quad 1<y<2 \tag{5}$$

where f, g are given.

We shall transform Problem (1)-(5) into a convolution equation.

Put

$$\Gamma(x,y,t,\xi,\eta,\tau)=\frac{1}{4\pi(t-\tau)}\exp\left(-\frac{(x-\xi)^2+(y-\eta)^2}{4(t-\tau)}\right)$$

and

$$G(x,y,t,\xi,\eta,\tau)=\Gamma(x,y,t,\xi,\eta,\tau)-\Gamma(x,4-y,t,\xi,\eta,\tau). \tag{6}$$

We have

$$G_{\xi\xi}+G_{\eta\eta}+G_\tau=0.$$

Integrating the identity

$$div(G\nabla u-u\nabla G)-\frac{\partial}{\partial\tau}(uG)=0$$

over the domain $(-n,n)\times(1,2)\times(0,t-\varepsilon)$ and letting $n\to\infty, \varepsilon\to 0$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^t g(\xi,\tau)G_\eta(x,y,t,\xi,2,\tau)d\xi d\tau \quad &+ \int_{-\infty}^{+\infty}\int_0^t G(x,y,t,\xi,1,\tau)w(\xi,\tau)d\xi d\tau \\ &- \int_{-\infty}^{+\infty}\int_0^t f(\xi,\tau)G_\eta(x,y,t,\xi,1,\tau)d\xi d\tau+u(x,y,t)=0.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_0^t G(x,y,t,\xi,1,\tau)w(\xi,\tau)d\xi d\tau=-u(x,y,t)+\\ &\int_{-\infty}^{+\infty}\int_0^t G_\eta(x,y,t,\xi,1,\tau)f(\xi,\tau)d\xi d\tau-\int_{-\infty}^{+\infty}\int_0^t g(\xi,\tau)G_\eta(x,y,t,\xi,2,\tau)d\xi d\tau.\end{aligned} \tag{7}$$

Letting $y\to 1^+$ in (8), we have

$$\int_{-\infty}^{+\infty}\int_0^t\left[\frac{1}{2\pi(t-\tau)}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)-\frac{1}{2\pi(t-\tau)}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)\right]w(\xi,\tau)d\xi d\tau$$

$$= -f(x,t) - \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{t}\frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)f(\xi,\tau)d\xi d\tau$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{t} g(\xi,\tau)\frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+1}{4(t-\tau)}\right)d\xi d\tau. \tag{8}$$

This implies that

$$(P-Q)*w(x,t) = -f(x,t) - R_1 * f(x,t) + R_2 * g(x,t) \tag{9}$$

where we put $w(x,t) = f(x,t) = g(x,t) = 0$ if $(x,t) \in \mathbb{R}\times(-\infty,0)$,

$$P(x,t) = \begin{cases} \frac{1}{t}\exp\left(-\frac{x^2}{4t}\right) & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases},$$

$$Q(x,t) = \begin{cases} \frac{1}{t}\exp\left(-\frac{x^2+4}{4t}\right) & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases},$$

$$R_1(x,t) = \begin{cases} \frac{1}{t^2}\exp\left(-\frac{x^2+4}{4t}\right) & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases},$$

and

$$R_2(x,t) = \begin{cases} \frac{1}{t^2}\exp\left(-\frac{x^2+1}{4t}\right) & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases}.$$

From now on, every function $\varphi = \varphi(x,t)$ defined on $\mathbb{R}\times\mathbb{R}_+$ will be seen as a function defined on $\mathbb{R}^2$ by putting $\varphi(x,t) = 0$ for $t < 0$.

Now, we state our main result.

Theorem

Let $a \in (0,2)$ and $\varepsilon \in (0, 2^{-3/2})$. Assume that $w_0 \in L^2(\mathbb{R}^2)$ is the (unique) solution of (9) corresponding to the exact data $f_0, g_0 \in L^2(\mathbb{R}^2)$ and that $f, g \in L^2(\mathbb{R}^2)$ are measured data satisfying

$$\|f - f_0\|_2 \le \varepsilon,\ \|g - g_0\|_2 \le \varepsilon$$

where $\|.\|_2$ is the $L^2(\mathbb{R}^2)$-norm.

Then, from f, g, we can construct a sequence $\{a_{mn,\varepsilon}\}$ $(m, n \in \mathbb{Z})$ and a $C > 0$ independent of ε such that

$$\left\| w_0 - \sum_{n=-\infty}^{+\infty}\sum_{|m|\le|n|} a_{mn,\varepsilon} S(m,\pi b_\varepsilon^2)(x) S(n,\pi b_\varepsilon^2)(t)\right\|_2 \le \sqrt{C\varepsilon^{2-a} + \eta(\varepsilon)}$$

where $b_\varepsilon = \varepsilon^{a/4}$,

$$S(p,d)(z) = \frac{\sin\left[\pi(z-pd)/d\right]}{\pi(z-pd)/d},\ p\in\mathbb{Z},\ d>0,$$

and $\eta(\varepsilon)\to 0$ *as* $\varepsilon\downarrow 0$.

Moreover, if we assume in addition that $w_0 \in L^1(\mathbb{R}^2)\cap H^2(\mathbb{R}^2)$ *then, for* $a=\frac{4}{3}$ *there exists a* $D>0$ *independent of* ε *such that*

$$\left\| w_0 - \sum_{n=-\infty}^{+\infty}\sum_{|m|\le|n|} a_{mn,\varepsilon} S(m,\pi b_\varepsilon^2)(x) S(n,\pi b_\varepsilon^2)(t)\right\|_2 < D\varepsilon^{\frac{1}{3}}.$$

Proof

The proof consists of two steps. In Step 1, we shall construct a regularized solution and in Step 2, we give the error estimate.

Step 1 : Construction of a regularized solution

We have

$$\begin{aligned}\hat{P}(z,r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,t)e^{-i(xz+tr)}dxdt \\ &= \frac{1}{\sqrt{2}\sqrt{z^4+r^2}}\left[\sqrt{\sqrt{z^4+r^2}+z^2} - i\,\mathrm{sgn}(r)\sqrt{\sqrt{z^4+r^2}-z^2}\right]\end{aligned}$$

and

$$\begin{aligned}\hat{Q}(z,r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(x,t)e^{-i(xz+tr)}dxdt \\ &= \frac{1}{\sqrt{z^4+r^2}}e^{-2u}\left[u\cos 2v - v\sin 2v - i\,\mathrm{sgn}(r)(u\sin 2v + v\cos 2v)\right]\end{aligned}$$

where

$$u = \frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}+z^2},\ v = \frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}-z^2}.$$

We put

$$\begin{aligned}F(x,t) &= -f(x,t) + R_1 * f(x,t) + R_2 * g(x,t),\\ F_0(x,t) &= -f_0(x,t) + R_1 * f_0(x,t) + R_2 * g_0(x,t).\end{aligned}$$

Taking the Fourier-transform of both sides of (9), we have

$$\left(\hat{P}(z,r) - \hat{Q}(z,r)\right)\hat{w}(z,r) = \hat{F}(z,r).$$

It call for the following formula of the regularized equation

$$w_\varepsilon(x,t) = \frac{1}{2\pi}\int\limits_{D_\varepsilon} \frac{\hat{F}(z,r)}{\hat{P}(z,r)-\hat{Q}(z,r)} e^{i(xz+tr)} dzdr.$$

where

$$b_\varepsilon = \varepsilon^{\frac{a}{5}} \in (0,1), \;\; D_\varepsilon = \left\{ (z,r)/b_\varepsilon^4 \le z^4 + r^2 \le \frac{1}{b_\varepsilon^4} \right\}.$$

We have

$$supp\ \hat{w}_\varepsilon \subset D_\varepsilon \subset \left[-\frac{1}{b_\varepsilon^2}, \frac{1}{b_\varepsilon^2} \right] \times \left[-\frac{1}{b_\varepsilon^2}, \frac{1}{b_\varepsilon^2} \right].$$

As in [AGLT], p. 121, we have

$$\begin{aligned} w_\varepsilon(x,t) &= \sum_{n=-\infty}^{+\infty} \sum_{|m|\le|n|} w_\varepsilon(m\pi b_\varepsilon^2, n\pi b_\varepsilon^2) S(m,\pi b_\varepsilon^2)(x) S(n,\pi b_\varepsilon^2)(t) \\ &= \sum_{n=-\infty}^{+\infty} \sum_{|m|\le|n|} a_{mn} S(m,\pi b_\varepsilon^2)(x) S(n,\pi b_\varepsilon^2)(t) \end{aligned}$$

where

$$a_{mn} = w_\varepsilon(m\pi b_\varepsilon^2, n\pi b_\varepsilon^2) = \frac{1}{2\pi}\int\limits_{D_\varepsilon} \frac{\hat{F}(z,r)}{\hat{P}(z,r)-\hat{Q}(z,r)} e^{i(mz+nr)\pi b_\varepsilon^2} dzdr. \tag{10}$$

Step 2 : Error estimates

Noting that

$$\left(\hat{P}(z,r) - \hat{Q}(z,r)\right) \widehat{w}_0(z,r) = \widehat{F}_0(z,r),$$

we have

$$\|w_0 - w_\varepsilon\|_2^2 = |\widehat{w}_0 - \widehat{w}_\varepsilon|_2 = \int\limits_{D_\varepsilon} \left| \frac{\hat{F}(z,r) - \hat{F}_0(z,r)}{\hat{P}(z,r)-\hat{Q}(z,r)} \right|^2 dzdr + \int\limits_{\mathbb{R}^2 \setminus D_\varepsilon} |\hat{w}_0(z,r)|^2 dzdr. \tag{11}$$

For $(z,r) \in D_\varepsilon$, we have

$$\begin{aligned} \left|\hat{P}(z,r) - \hat{Q}(z,r)\right| &= \frac{1}{\sqrt{u^2+v^2}} \sqrt{1 - 2e^{-2u}\cos 2v + e^{-4u}} \\ &\ge \frac{1}{\sqrt[4]{z^4+r^2}} \left(1 - e^{-\sqrt{2}\sqrt{\sqrt{z^4+r^2}+z^2}} \right) \\ &\ge b_\varepsilon \left(1 - e^{-\sqrt{2}b_\varepsilon} \right) \ge \frac{1-e^{-\sqrt{2}}}{\sqrt{2}} b_\varepsilon^2. \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}\left\|\hat{F}-\hat{F}_0\right\|_2 &= \|F-F_0\|_2\\ &\le [\|R_1\|_1+1]\,\|f-f_0\|_2+\|R_2\|_1\,\|g-g_0\|_2\\ &\le (\|R_1\|_1+\|R_2\|_1+1)\,\varepsilon.\end{aligned}$$

From (11), (12) and the latter inequality, we get

$$\|w_\varepsilon - w_0\|_2^2 \le \frac{2\varepsilon^2\left(\|R_1\|_1+\|R_2\|_1+1\right)^2}{\left(1-e^{-\sqrt{2}}\right)^2 b_\varepsilon^4}+\eta(\varepsilon)=\frac{2\left(\|R_1\|_1+\|R_2\|_1+1\right)^2}{\left(1-e^{-\sqrt{2}}\right)^2}\varepsilon^{2-a}+\eta(\varepsilon)$$

where $\eta(\varepsilon)=\int\limits_{\mathbb{R}^2\setminus D_\varepsilon}|\hat{w}_0(z,r)|^2\,dzdr$.

Moreover, if $w_0\in H^2(\mathbb{R}^2)$, then choosing $b_\varepsilon=\varepsilon^{\frac{1}{3}}<\frac{1}{\sqrt{2}}$.

We have

$$\|w_0-w_\varepsilon\|_2^2\le\frac{2(\|R_1\|_1+\|R_2\|_1+1)^2}{(1-e^{-\sqrt{2}})^2 b_\varepsilon^4}\varepsilon^2+\int\limits_{z^4+r^2<b_\varepsilon^4}|\hat{w}_0(z,r)|^2\,dzdr+\int\limits_{z^4+r^2>\frac{1}{b_\varepsilon^4}}|\hat{w}_0(z,r)|^2\,dzdr. \quad (13)$$

Note that if $b_\varepsilon<\frac{1}{2}$ and $z^4+r^2<b_\varepsilon^4$, then $z^2+r^2<b_\varepsilon^2\sqrt{2}$, hence

$$\int\limits_{z^4+r^2<b_\varepsilon^2}|\hat{w}_0(z,r)|^2\,dzdr\le 2\,\|w_0\|_1^2\,\pi b_\varepsilon^2. \quad (14)$$

On the other hand if $b_\varepsilon<\frac{1}{2}$ and $z^4+r^2>\frac{1}{b_\varepsilon^4}$, then $z^2+r^2>\frac{1}{b_\varepsilon^2}$, therefore

$$\int\limits_{z^4+r^2>\frac{1}{b_\varepsilon^4}}|\hat{w}_0(z,r)|^2\,dzdr\le\int\limits_{z^4+r^2>\frac{1}{b_\varepsilon^4}}\frac{(z^2+r^2)^2\,|\hat{w}_0(z,r)|^2}{(z^2+r^2)^2}dzdr\le b_\varepsilon^4 E \quad (15)$$

where $E=\left\|(z^2+r^2)\hat{w}_0(z,r)\right\|_2^2$.

By (13)-(15), we have

$$\|w_0-w_\varepsilon\|_2^2\le\frac{2(\|R_1\|_1+\|R_2\|_1+1)^2}{(1-e^{-\sqrt{2}})^2 b_\varepsilon^4}\varepsilon^2+\left(2\|w_0\|_1^2\,\pi+E\right)b_\varepsilon^2\le D^2\varepsilon^{\frac{2}{3}}.$$

where

$$D=\max\left\{\frac{\sqrt{2}\,(\|R_1\|_1+\|R_2\|_1+1)}{1-e^{-\sqrt{2}}},\sqrt{2\,\|w_0\|_1^2\,\pi+E}\right\}.$$

The proof of theorem is completed.

4. Numerical results

We present some results of numerical comparison of the regularized representation of the solution given by (10) and the corresponding exact solution of the problem.

Let the problem

$$\Delta u - \frac{\partial u}{\partial t} = 0, \;\; (x, y) \in \mathbb{R} \times (0, 3), \; t > 0 \tag{16}$$

$$u(x, 1, t) = \frac{1}{t} \mathrm{e}^{\frac{-x^2-1}{4t}}; \;\; u(x, 2, t) = \frac{1}{t} \mathrm{e}^{\frac{-x^2-4}{4t}}; \; u(x, y, 0 = 0 \tag{17}$$

whose the unknown is

$$w(x, t) = u_y(x, 1, t) \tag{18}$$

The exact solution of this problem is

$$w(x, t) = \frac{-1}{2t^2} \mathrm{e}^{\frac{-x^2-1}{4t}}.$$

The approximated solution is calculated from the expansion of two-dimensional Sinc series given by (10) in which

$\hat{F} = \frac{\mathrm{e}^{-u}}{u^2 + v^2} \Big[(\alpha u - \beta v \Big] - i(\alpha v + \beta u) \Big] + \frac{\varepsilon}{\pi \sqrt{r^2 + 1} \sqrt{z^2 + 1}}$

$\alpha = \mathrm{e}^{-2u} \cos 3v - \cos v \; ; \; \beta = \mathrm{e}^{-2u} \sin 3v - \sin v$

$\hat{P} - \hat{Q} = a_1 + i b_1$ where

$a_1 = u - \mathrm{e}^{-2u}(u \cos 2v - v \sin v); \;\; b_1 = \mathrm{sgn}(r)[-v + \mathrm{e}^{-2u}(v \cos 2v + u \sin 2v)].$

Thus we have

$$\left\| \hat{F} - \hat{F}_0 \right\|_{L^2(\mathbb{R}^2)} = \varepsilon$$

which is a perturbation similar to the one operated on the two given functions f and g.

With $\varepsilon = \frac{1}{50}$, $N = 50$ (the size of the double series) and for $(x, t) \in [0, 2] \times [0, 2]$ we have drawn the corresponding approximate surface solution $(x, t) \longrightarrow w_\varepsilon(x, t)$ in Fig.1.

To calculate the double integral in (10) we have used the rectangle rule which gives good accuracy if one integrates on the interval $[\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon]$. The time of calculus with a good computer is very long: 2 hours for 900 points $M = (x, t) \in [0, 2] \times [0, 2]$. It is the reason for which we are limited ourselves to a relatively small size of the double series ($N = 20$). For comparison in Fig.2 we have drawn the exact solution $(x, t) \longrightarrow w(x, t)$.

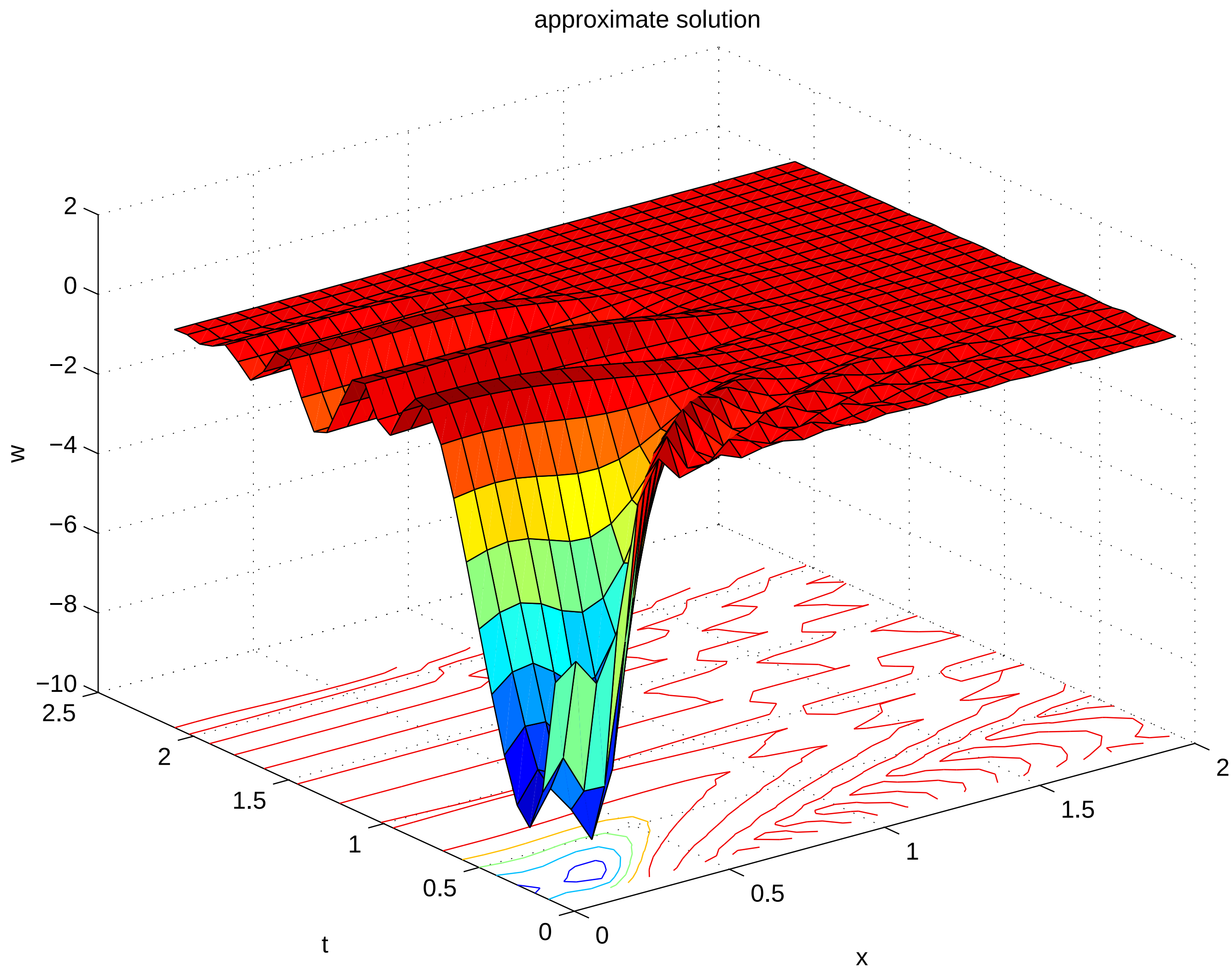

Fig.1

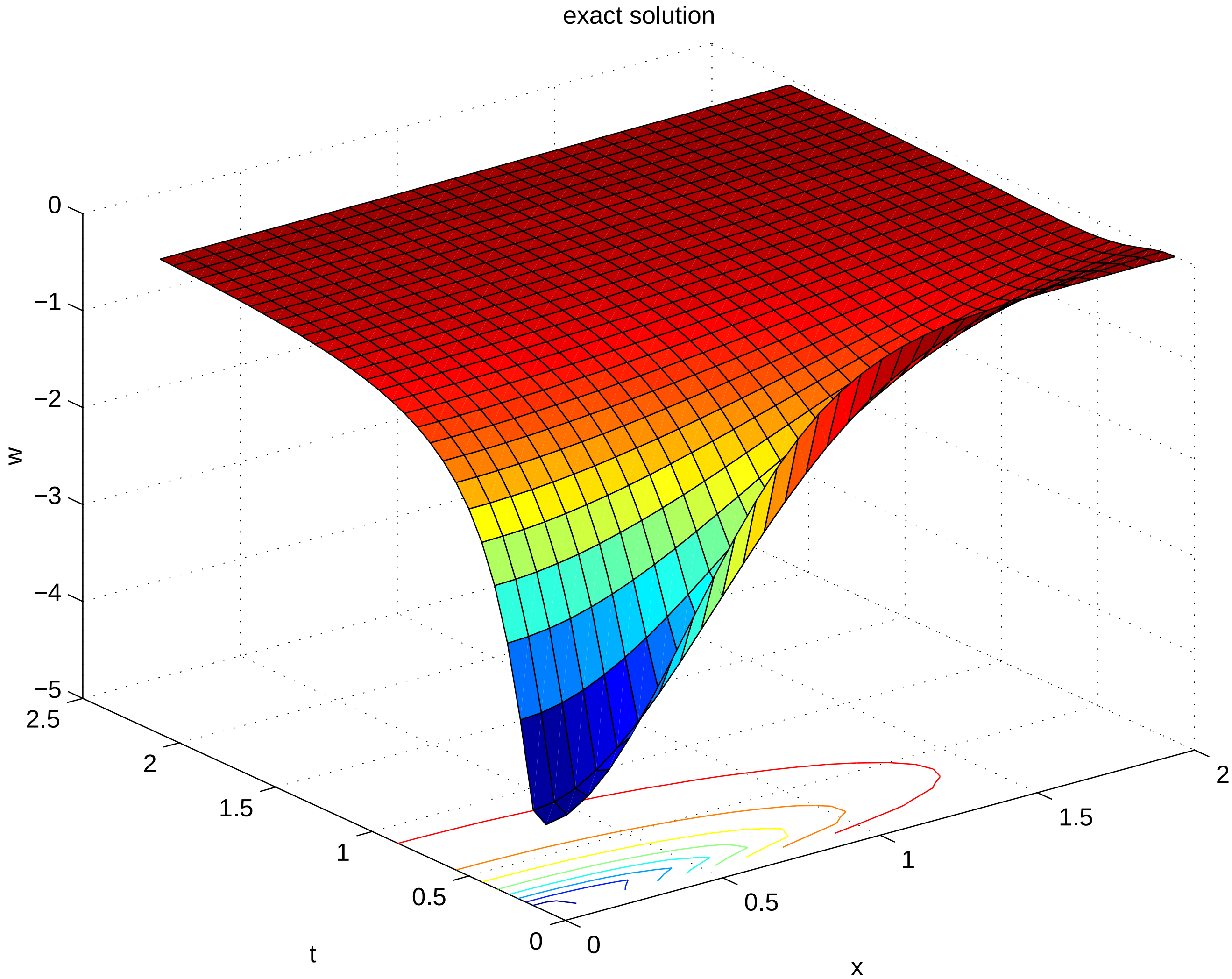

Fig.2

Using the same method as previously we have drawn in Fig.3 the surface $(x,t) \in [0,4]\times[0,4] \longrightarrow w_\varepsilon(x,t)$ which is the regularization of the following problem

$$\Delta u - \frac{\partial u}{\partial t} = 0, \;\; (x,y) \in \mathbb{R} \times (0,3), \; t > 0$$

$$u(x,1,t) = 0; \;\; u(x,2,t) = \frac{1}{t}\mathrm{e}^{\frac{-x^2-4}{4t}}(1-\mathrm{e}^{\frac{-1}{t}}); \; u(x,y,0 = 0 \tag{19}$$

the unknown being $w(x,t) = u_y(x,1,t)$. The exact solution

$$w(x,t) = \frac{1}{t^2}\mathrm{e}^{\frac{-x^2-1}{4t}}.$$

is represented in Fig.4.

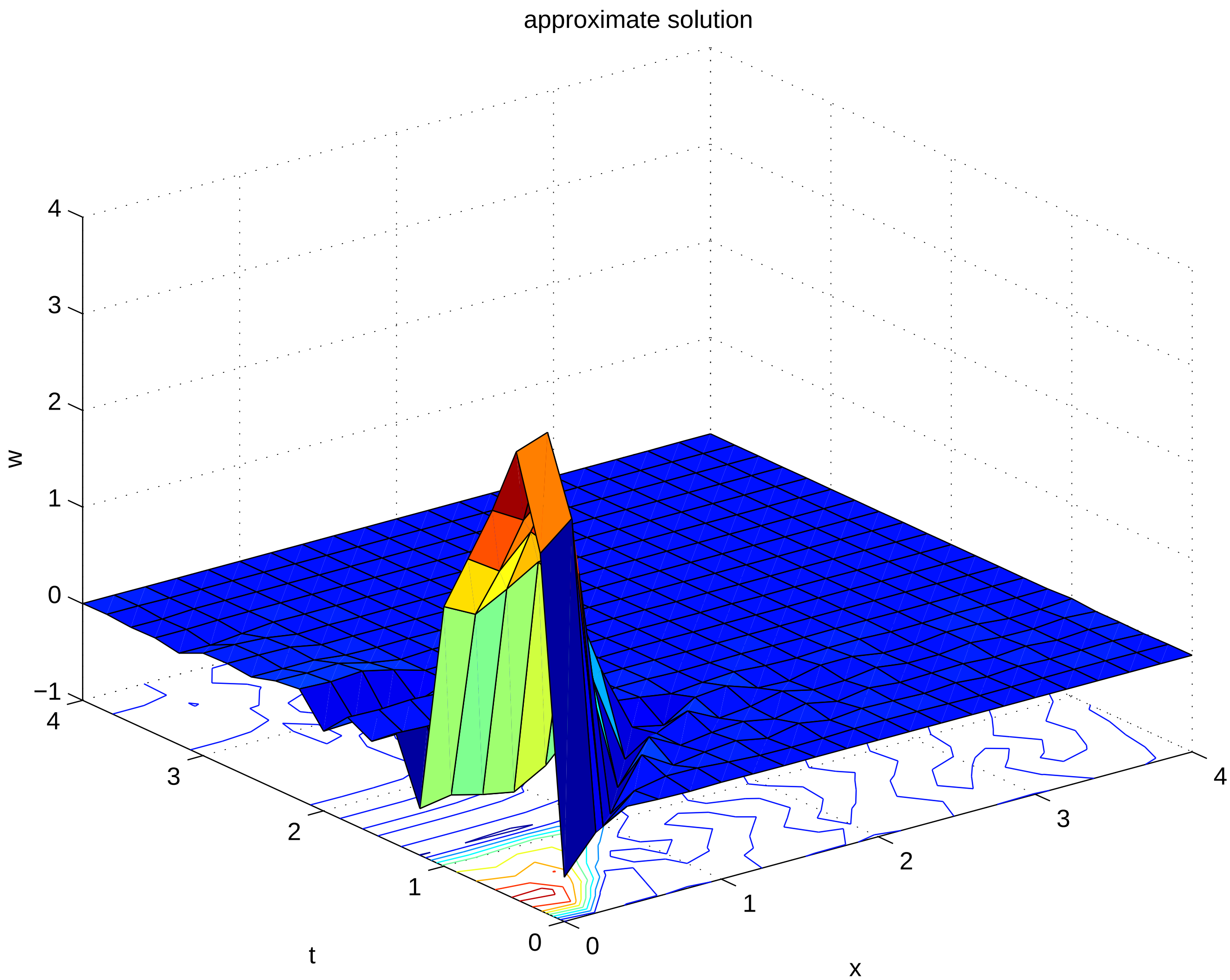

Fig.3

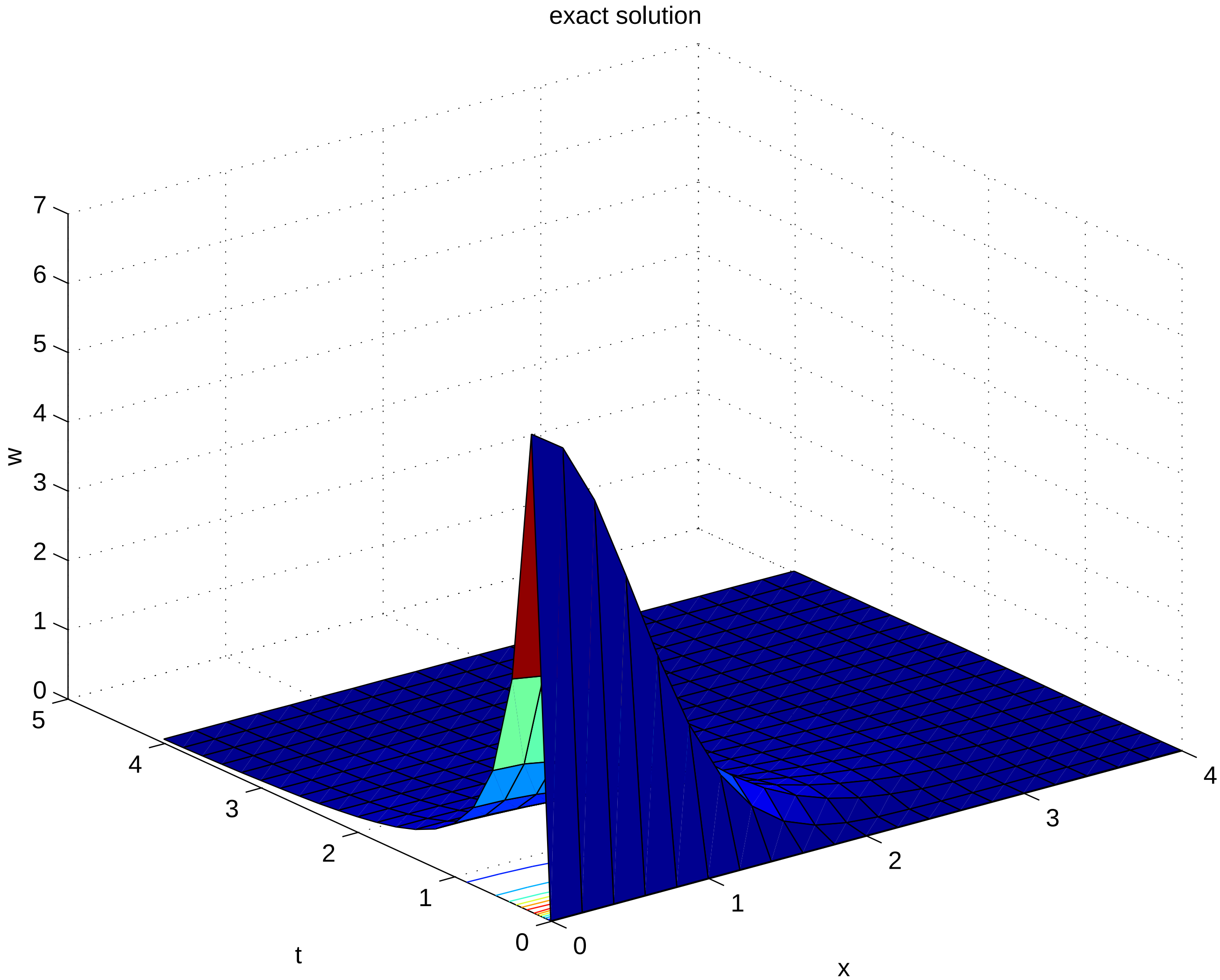

Fig.4

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