



**HAL**  
open science

## A kinetic equation for quantum gases (spin and statistics)

P. Nacher, G. Tastevin, F. Laloë

► **To cite this version:**

P. Nacher, G. Tastevin, F. Laloë. A kinetic equation for quantum gases (spin and statistics). *Journal de Physique I*, 1991, 1 (2), pp.181-209. 10.1051/jp1:1991124 . hal-00008971v2

**HAL Id: hal-00008971**

**<https://hal.science/hal-00008971v2>**

Submitted on 4 Feb 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification  
*Physics Abstracts*  
51.10 — 05.20D — 05.30

## A kinetic equation for quantum gases (spin and statistics)

P.J. Nacher, G. Tastevin and F. Laloë

Département de Physique de l'ENS, 24 rue Lhomond, F 75005 Paris, France

(Received 2 October 1990, accepted 31 October 1990)

**Abstract.** — We generalize our previous work on the compatibility of kinetic equations with second virial corrections to the inclusion of spin and particle indistinguishability; the system is supposed to be sufficiently dilute for higher order virial density corrections (interactions and statistics) to be negligible. We show that the general idea of the “free Wigner transform” can be extended to this situation; the function which appears in the kinetic equation becomes here a matrix which acts in the space of spin states of the particles. Assuming that the collisions are described by a hamiltonian which does not act on the spins (a very good approximation for nuclear spins), we write explicitly a kinetic equation which is valid for this case. The right hand side of the equation is an 18 dimension integral, as for spinless distinguishable particles, but here it contains an additional term due to statistics, which introduces commutators and anticommutators. We discuss the local conservation laws in this formalism and find, as expected, a total number of 8 conserved quantities for spin 1/2 particles (including three components of the magnetization). When the gas is at equilibrium, we obtain a pressure dependence which is in agreement with known calculations on spin polarized gases. We finally study the gradient expansion of the collision integral, and show that the zero-order (local) part is identical with the 4 terms (including identical spin rotation terms) obtained previously by Lhuillier *et al.* The first order (non-local) part contains many terms, which we compare with those obtained by Silin in a context more closely related to the Landau theory.

### Introduction.

In a previous study [1] we have introduced the use of the “free Wigner transform” to write a kinetic equation which remains valid for a semi-dense gas (it is compatible with the two-body correlations which are the origin of the second virial correction to the pressure). For the sake of simplicity, that work was developed only for spinless particle obeying Boltzmann statistics. Here, we wish to generalize it to the inclusion of spin and of particle indistinguishability. This is of course important if one has in mind the application of the theory to spin-polarized helium three [2] or hydrogen [3, 4]; it is also interesting if one wishes to make the connection to the work of Silin [5] or more recent studies on transport in spin-polarized gases [6] to [12]. We will see that the direct continuation of the ideas used in [1] provides results which are in full

agreement with calculations of the effect of spin polarization and statistics on second virial corrections in the pressure of the gas at equilibrium [13]. This was not the case of the pressure obtained from the simpler theory of [12], where collision processes were treated as point processes in space and time (no range, no duration), and where two-body correlations were treated too crudely for obtaining correct virial corrections.

For spinless particles, the free transform is associated with a function  $f(\mathbf{r}, \mathbf{p})$  that depends on the same variables as the one-particle distribution function  $f_I(\mathbf{r}, \mathbf{p})$ . Its physical meaning is nevertheless somewhat different:  $f$  defines the properties of only those particles which are far from all other particles, as opposed to  $f_I$  which describes all of them, including particles in the middle of a collision. Mathematically, because the definition of  $f$  implies some information on the position of all particles (which have to be sufficiently distant from the test particle), this free distribution can be expressed only from the  $\mathcal{N}$ -body distribution function, and not only in terms of  $f_I$  (or even  $f_{II}$ ). Another difference is that the integral over  $d^3p$  of this function does not give the local number density of particles  $n$ , as would be the case for  $f_I$ , but a "free" number density  $n_f$ ; it actually differs from  $n$  by a density correction which is second order in  $n$  (see also the physical discussion of this difference given by Snider [14]). Physically, this integral can be seen as a local characterization of the chemical potential of the system. A way to understand this is to notice that, because  $f$  describes the properties of the ideal gas which exists in between the locations where collisions take place, one can write in terms of  $f$  relations that are valid for an infinitely dilute gas. An analogy can be seen for example with a liquid in equilibrium with its saturated vapour at very low density:  $f$  would then describe the distribution function of only those atoms which are in this dilute gas rather than inside the dense liquid phase. One can therefore use the well known ideal gas relation:

$$\mu = k_B T \text{Log} n_f \lambda^3 \quad (1)$$

(where  $\lambda$  is the thermal wavelength) which, in terms of  $f$ , translates into:

$$\int d^3p f = \lambda^{-3} \exp(\mu/k_B T) \quad (2)$$

Actually, in this article we include spin variables, so that we will have to replace  $f(\mathbf{r}, \mathbf{p})$  by a matrix,  $\rho_S(\mathbf{r}, \mathbf{p})$ , which depends on one position  $\mathbf{r}$  and on one momentum  $\mathbf{p}$ . In relation (2),  $f$  should then be interpreted as the trace of  $\rho_S$  over spin variables.

In [1], two forms of the formalism were developed in succession: the first form introduces a kinetic equation with, in the collision term, a gradient expansion containing 5 dimension collision integrals (the same number of dimensions as in the Boltzmann theory); the second form [1d] makes no gradient expansion but requires the use of 18 dimension integrals. Another difference is the non-local character of the density corrections of [1d]. Here we shall take the second formalism, the main reason being that it gives much more compact equations; this is important when one wants to keep simultaneously spin, statistics and non-local interaction integrals. Indeed, in [12], the inclusion of spin and statistics in the Boltzmann equation already resulted in the introduction of 4 different collision terms in the right hand side (spin anticommutators to account for the changes of the cross sections introduced by statistics, commutators for identical spin rotation effect); the number of terms was even larger in [1b], where correlation and retardation effects in collisions were included (without spin or statistics). Treating both problems simultaneously would therefore lead to about 20 terms in the interaction integral! The use of 18 dimension integrals is indeed much more convenient; if necessary, at the end of the calculation, one can always make a spatial gradient expansion which gives the full detailed structure of the terms.

We wish to stress here that the present work is limited to the study of the effect of binary collisions only, which implies that the system is not too dense and that the inclusion of second

virial corrections (in the pressure at equilibrium for example) is sufficient. In other words, we study particle correlations, but only to the lowest density order at which they appear. Recently, Snider has considered a more general situation where this restriction is not necessary, and shown how the consideration of the distribution function for free particles is also useful in the context of dense gases [15]. Another recent article connected to ours is reference [16], where spin and statistics are included in a kinetic equation which reduces to that of [12] in its local terms (including for example identical-spin-rotation effects), but can also generate non-local terms. The generalization of the collision integral of [12] to degenerate systems has been given recently by Jeon and Mullin [10b], and applied by these authors to the study of spin diffusion [10c]. See the work of Meyerovich [17] for a general study of the combined effects of internal variables and non-localities, as well as [18] and [19] for examples of very recent studies of spin dynamics in quantum gases or dilute systems.

For convenience, the organisation and the notation of the present article have been chosen as close as possible to those of reference [1d]. In section 1, we give the definition of the “free Wigner transform” which is adequate for a collision between two identical particles with spin, and study its evolution in a two-body collision; we also discuss the expression of the difference between the real and the free distribution functions (correction to the distribution function). In section 2, we propose a kinetic equation for a dilute system of many identical particles with spin, in a gas where they undergo only binary collisions; we then study the local conservation laws in this formalism and, finally, the equilibrium situation where we recover the  $M$  dependence of the pressure already obtained in [13].

## 1. Free Wigner transform (two particle system).

1.1 DEFINITION. — We consider here a system of two particles, for which we wish to adapt and generalize the definition of the “free Wigner transform”, given in [1a], in order to include particle indistinguishability and spin. As in [1a], a cancellation of the effects of the potential at short range will be obtained by the adequate use of two unitary transformations  $\Omega$ ; nevertheless, we will see that that this is not sufficient when the particles are indistinguishable: in addition, it is necessary to remove the effects of statistics at short relative distance by another mathematical transformation, which we introduce below in section 1.1.4.

1.1.1 Notation. — Our starting point in this section is the two-particle density operator  $\rho$ , on which we will apply the unitary transformations associated with the two Möller operators  $\Omega^{(+,-)}$ . We assume that the hamiltonian does not act on the spins, so that these operators act only in the space of states of the orbital relative motion of the particles. They transform plane waves into entering or outgoing stationary scattering states respectively (we also assume that there are no bound states so that the  $\Omega$ 's are indeed unitary). The positions of the two interacting particles are noted  $\mathbf{r}_{1,2}$  and their momenta  $\mathbf{p}_{1,2}$ , from which we define as usually the variables of the centre of mass and of the relative motion:

$$\begin{aligned} \mathbf{R} &= \frac{1}{2} (\mathbf{r}_1 + \mathbf{r}_2) & \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2 & \mathbf{p} &= \frac{1}{2} (\mathbf{p}_1 - \mathbf{p}_2) = \hbar \mathbf{k} \end{aligned} \quad (3)$$

The internal state of the particles is described by the quantum number  $m$ , which can take on two values for spin 1/2 particles (such as helium three atoms), but more generally any number of discrete values. The Wigner transforms we shall consider in this section are functions, either of the individual variables  $\mathbf{r}_{1,2}$ 's and  $\mathbf{p}_{1,2}$ 's, or of the variables defined in (3); in addition, because the Wigner transformation acts only inside the orbital space of the two particles, the transforms

remain operators in the space of internal spin states of the two particles; they are therefore described by matrices with elements labelled by pairs of values of quantum numbers  $m$ 's (for spin 1/2 particles,  $4 \times 4$  matrices).

We define the "free entering" and "free outgoing" density operators  $\rho_{\Omega^\dagger}^{(+,-)}$  by:

$$\rho_{\Omega^\dagger}^{(+,-)} = [\Omega^{(+,-)}]^\dagger \rho [\Omega^{(+,-)}] \quad (4)$$

Because the effect of the inverse ( $\Omega^{-1} = \Omega^\dagger$ ) of any of the  $\Omega$  unitary transformations is to change the hamiltonian of the system into that of non-interacting particles, these density operators have a simpler evolution than the real density operator (a free evolution).

We are interested in situations where the particles before collision have the minimum degree of correlation that remains compatible with the symmetrization postulate; mathematically this corresponds to the equation:

$$\rho_{\Omega^\dagger}^{(+)} = \frac{1}{2} [1 + \epsilon P_{ex.}] \rho_+^{(1)}(1) \otimes \rho_+^{(2)}(2) [1 + \epsilon P_{ex.}] \quad (5)$$

where  $P_{ex.}$  is the exchange operator for the two particles, which acts on both orbital and spin variables,  $\epsilon$  is either +1 for bosons or -1 for fermions, and  $\rho_+^{(1,2)}$  define the density operators of each individual particle long before collision (no overlap of the wave packets). These one-particle operators play an important role in the kinetic theory developed in section 2; we denote by  $\rho_{S+}^{(1,2)}(\mathbf{r}, \mathbf{p})$  their Wigner transform (which are still operators in the space of one-particle spin states, or equivalently  $2 \times 2$  matrices) and by  $f_+^{(1,2)}(\mathbf{r}, \mathbf{p})$  their trace over spin variables:

$$\begin{cases} \rho_{S+}^{(1,2)} & = \text{W.T.} \left\{ \rho_+^{(1,2)} \right\} \\ f_+^{(1,2)}(\mathbf{r}, \mathbf{p}) & = \text{Tr}_S \left\{ \rho_{S+}^{(1,2)}(\mathbf{r}, \mathbf{p}) \right\} \end{cases} \quad (6)$$

(where W.T. is for Wigner transform). Because of the presence the exchange terms of (5), the norm of  $\rho_{\Omega^\dagger}^{(+)}$  is not simply the product of the integrals over phase space of  $f_+^{(1)}$  and  $f_+^{(2)}$ .

**1.1.2 Semi-free Wigner transform** — In a first step, we shall introduce the "semi-free" Wigner transform where, as in reference [1a], the short-range effects of the potential are removed by the use of  $\Omega$  transformations; each of them associated with one of the two regions of phase space where the scalar product  $\mathbf{r} \cdot \mathbf{p}$  is either negative (entering region) or positive (outgoing region):

$$\rho_{s f}(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) = Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \text{W.T.} \left\{ \rho_{\Omega^\dagger}^{(+)} \right\} + Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) \text{W.T.} \left\{ \rho_{\Omega^\dagger}^{(-)} \right\} \quad (7)$$

In this equation,  $Y^{(+)}$  is the Heaviside function and  $Y^{(-)}$  is the difference  $1 - Y^{(+)}$ . We now study the properties of this function (operator in the spin-state space); we will see that it still contains short range effects, due to pure statistics, so that another mathematical operation will be needed before obtaining the "completely free" Wigner transform. We evaluate the matrix element:

$$\langle m, m' | \rho_{s f}(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) | m'', m''' \rangle \quad (8a)$$

as a function of the one-particle Wigner transforms  $\rho_{S+}^{(1,2)}(\mathbf{r}, \mathbf{p})$  that define the operators  $\rho_{(+)}^{(1,2)}$ .

First, the calculation requires that one relates the Wigner transforms of the two operators  $\rho_{\Omega^\dagger}^{(-)}$  and  $\rho_{\Omega^\dagger}^{(+)}$ , which are connected by a unitary transformation  $S$ :

$$\begin{aligned} \rho_{\Omega^\dagger}^{(-)} &= [\Omega^{(-)}]^\dagger \rho [\Omega^{(-)}] = [\Omega^{(-)}]^\dagger [\Omega^{(+)}] \rho_{\Omega^\dagger}^{(+)} [\Omega^{(+)}]^\dagger [\Omega^{(-)}] \\ &= S \rho_{\Omega^\dagger}^{(+)} S^\dagger \end{aligned} \quad (8b)$$

where the matrix elements of  $S$  are given by:

$$S(\mathbf{k}, \mathbf{k}') = \langle \mathbf{k} | S | \mathbf{k}' \rangle = \langle \mathbf{k} | [\Omega^{(-)}]^\dagger [\Omega^{(+)}] | \mathbf{k}' \rangle \quad (8c)$$

(we assume throughout this article that the hamiltonian, and therefore the operator  $S$ , have no action on the internal variables of the atoms). From (5), we then obtain:

$$\begin{aligned} \rho_{s.f}(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) = & \frac{1}{2} Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \text{W.T.} \left\{ [1 + \epsilon P_{\text{ex}}] \rho_+^{(1)}(1) \otimes \rho_+^{(2)}(2) [1 + \epsilon P_{\text{ex}}] \right\} + \\ & + \frac{1}{2} Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) \text{W.T.} \left\{ S [1 + \epsilon P_{\text{ex}}] \rho_+^{(1)}(1) \otimes \rho_+^{(2)}(2) [1 + \epsilon P_{\text{ex}}] S^\dagger \right\} \end{aligned} \quad (9)$$

In this expression, the direct terms are those which contain no exchange operator  $P_{\text{ex}}$ , the exchange terms those which contain one operator  $P_{\text{ex}}$ , and the double exchange terms those which include one  $P_{\text{ex}}$  on each side of the one-particle operators.

\* direct term: except the factor 1/2 introduced in (5) and the fact that the  $f_+^{(1,2)}$ 's are replaced here by spin operators  $\rho_{S+}^{(1,2)}$ , the calculation is essentially that of reference [1a]. It introduces the same succession of integrals, namely first:

$$\int d^3 \kappa e^{i\kappa \cdot \mathbf{r}} \quad (10a)$$

which allows one to express the Wigner transforms of  $\rho_{\Omega^\dagger}^{(+,-)}$  as functions of their matrix elements (<sup>1</sup>); then, when closure relations are injected in (9) on each side of the one particle operators, the integrals:

$$\int d^3 k'_1 \int d^3 k'_2 S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) \quad (10b)$$

which account for the effect of the unitary transformation  $S$  (which does not act in the spin state space of the atoms); finally, when one comes back from matrix elements of operators to the one-particle Wigner transforms defined in (6), the integral:

$$\int d^3 r' e^{i(\mathbf{k}'_2 - \mathbf{k}'_1) \cdot \mathbf{r}'} \quad (10c)$$

It is convenient at this point to use the shorter notation of [1d] :

$$\int_{12} = \int d^3 \kappa \int d^3 k'_1 \int d^3 k'_2 \int d^3 r' \quad (11a)$$

$$\mathbf{k}_\pm = \frac{\mathbf{p}}{\hbar} \pm \frac{\mathbf{k}}{2} \quad (11b)$$

$$\kappa' = \mathbf{k}'_1 - \mathbf{k}'_2 \quad (11c)$$

and:

$$\begin{cases} \mathbf{r}'_{1,2} = \mathbf{R} \pm \frac{\mathbf{r}'}{2} \\ \mathbf{p}'_{1,2} = \frac{\mathbf{P}}{2} \pm \hbar \frac{\mathbf{k}'_1 + \mathbf{k}'_2}{2} \end{cases} \quad (11d)$$

(<sup>1</sup>) All transforms are made in the space of the relative motion of the two particles only; this is because the variables  $\mathbf{R}$  and  $\mathbf{P}$  remain simply unchanged, see relations (18) and (19) of [1a].

so that the direct term reads:

$$\frac{1}{2} (2\pi)^{-3} \int_{12} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} \langle m | \rho_{S+}^{(1)}(1') | m'' \rangle \langle m' | \rho_{S+}^{(2)}(2') | m''' \rangle \times \\ \times \left[ Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) + Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] \quad (11e)$$

where (1') and (2') summarize the variables  $\mathbf{r}'$  and  $\mathbf{p}'$  written in (11d).

\* double exchange term. The operator  $P_{\text{ex}}$  has two effects: first it changes the matrix elements of  $S$  according to:

$$\langle \mathbf{k} | S P_{\text{ex}} | \mathbf{k}' \rangle = \langle \mathbf{k} | S | -\mathbf{k}' \rangle = S(\mathbf{k}, -\mathbf{k}') \quad (12)$$

second, it changes the spin indices  $m$ . The double exchange term is thus obtained from (11e) by first interchanging the indices  $m$  and  $m'$ , then the indices  $m''$  and  $m'''$ , and finally changing the sign of the vectors  $\mathbf{k}'_{1,2}$  inside the matrix elements of  $S$  and  $S^*$ . Using a parity operation over  $\mathbf{k}'_{1,2}$  and (11d), one can easily see that the double exchange term is also obtained from (11-e) by merely exchanging  $\rho_{S+}^{(1)}$  and  $\rho_{S+}^{(2)}$ ; in other words, it is sufficient to interchange the initial conditions of the two particles.

In the study of two-particle exchange effects, it is well known that the genuinely new effects introduced by particle indistinguishability are generally contained in the interference terms which are linear in the exchange operator; the double exchange terms merely account for the trivial fact that no distinction is made between the two particles (an effect which also occurs in classical physics). Indeed, in the present case, this is the interpretation of the double exchange terms; moreover, because in section 2 we shall assume that the two operators  $\rho_{S+}^{(1,2)}$  are in fact equal, the double exchange term will become merely equal to the direct term, so that it will not require a separate study.

\* exchange terms: in one of the two terms,  $m$  and  $m'$  are interchanged with a change of the sign of  $\mathbf{k}_+$  (or of  $\mathbf{k}'_1$ ); in the other, the interchange occurs for  $m''$  and  $m'''$  and the sign change for the vector  $\mathbf{k}_-$  (or  $\mathbf{k}'_2$ ). We now study these exchange terms in more detail, and we show that they contain a interaction-independent term (pure statistics) which has a limited, microscopic, range.

1.1.3 *Study of the exchange terms.* — In quantum statistical physics, in the calculation of second virial corrections, one usually gets the sum of two terms: one due to the interactions (modified by particle indistinguishability) and one due to pure statistics; the latter accounts for exchange effects which occur when the particles are at a distance of the order of the De Broglie wavelength. Here we have a similar situation. To see why, we can extract from the exchange terms those that are due to pure statistics, which can be done by replacing the elements of the  $S$  matrix by the well-known expression in terms of those of the  $T$  matrix:

$$S(\mathbf{k}', \mathbf{k}) = \delta(\mathbf{k}' - \mathbf{k}) - 2i\pi \frac{\mu}{\hbar^2 k} \delta(k' - k) T(\mathbf{k}', \mathbf{k}) \quad (13)$$

The terms containing two matrix elements of  $T$  (more precisely, one of  $T$ , one of  $T^\dagger$ ) are the lateral scattering terms; those containing one matrix element of  $T$  (or  $T^\dagger$ ), the forward scattering terms; finally, those obtained by substituting delta functions for both coefficients  $S$  and  $S^*$  the non-interacting particle terms due to pure statistics.

Physically, we expect that these terms due to pure statistics play a role only when the two particles are at a distance which is not much larger than the De Broglie wavelength. Indeed, if we

study the term where the exchange operator  $P_{ex}$  acts on the left in (5), we can make the following substitutions:

$$\begin{aligned} \mathbf{k}'_1 &= -\mathbf{k}_+ \\ \mathbf{k}'_2 &= \mathbf{k}_- \end{aligned} \quad (14a)$$

which lead to:

$$\begin{aligned} \boldsymbol{\kappa}' &= -\mathbf{k}_+ - \mathbf{k}_- = -2\mathbf{k} \\ \mathbf{k}'_1 + \mathbf{k}'_2 &= -\boldsymbol{\kappa} \end{aligned} \quad (14b)$$

One then gets:

$$\begin{aligned} &\frac{1}{2} (2\pi)^{-3} \int d^3\boldsymbol{\kappa} \int d^3\mathbf{r}' e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} e^{2i\mathbf{k}\cdot\mathbf{r}'} \times \\ &\times \langle m' | \rho_{S+}^{(1)}(\mathbf{R} + \frac{\mathbf{r}'}{2}, \frac{\mathbf{P}}{2} - \frac{\hbar\boldsymbol{\kappa}}{2}) | m'' \rangle \langle m | \rho_{S+}^{(2)}(\mathbf{R} - \frac{\mathbf{r}'}{2}, \frac{\mathbf{P}}{2} + \frac{\hbar\boldsymbol{\kappa}}{2}) | m''' \rangle \end{aligned} \quad (15)$$

On this expression, one can check that the range of the variable  $\mathbf{r}$  is limited to a distance which is comparable to the inverse of the width of the functions giving the momentum dependence of  $\rho_{S+}^{(1,2)}$ , in other words a microscopic distance. This term is therefore not relevant to the long-distance properties of the system.

The same has no reason to be true of the forward and lateral scattering terms: they account for the effect of statistics on the collision cross sections, a physical effect that clearly has long range consequences on the properties of the system (if a more precise argument is necessary, one can use the results of calculations below to check this point). The exchange terms due to pure statistics in the semi-free Wigner transform have therefore a special short range property, and should thus be treated accordingly.

**1.1.4 Free Wigner transform** — To define a free Wigner transform where all short range effects (potential and statistics) have been discarded, it is therefore natural to remove the pure exchange terms from the semi-free distribution function, keeping only those which arise from the interactions:

$$\rho_{W}^L(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) = \rho_{s,f}(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) - \text{exch.terms} \{ \rho_{s,f}(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}; T=0) \} \quad (16a)$$

We keep the same notation as in [1], so that the letter L for “libre”, free in French, is still used; the second term in the right hand side symbolizes the exchange terms of the semi-free transform where the  $T$  matrix, that is the interaction potential, has been put equal to zero. At the end of section 2.1.2, we come back to the physical reasons for removing the pure exchange terms in this way. This leads to the following definition of the matrix elements of the free distribution function:

$$\begin{aligned} &\langle m, m' | \rho_{W}^L(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) | m'', m''' \rangle = \\ &= \frac{1}{2} (2\pi)^{-3} \int_{12} e^{i(\boldsymbol{\kappa}\cdot\mathbf{r} - \boldsymbol{\kappa}'\cdot\mathbf{r}')} \left\{ \langle m | \rho_{S+}^{(1)}(1') | m'' \rangle \langle m' | \rho_{S+}^{(2)}(2') | m''' \rangle \times \right. \\ &\times \left[ Y^{(+)}(\mathbf{p}\cdot\mathbf{r}) S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) + Y^{(-)}(\mathbf{p}\cdot\mathbf{r}) \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] + \\ &+ \text{id.} \left[ \mathbf{k}_{1,2} \iff -\mathbf{k}'_{1,2}; m \iff m', m'' \iff m''' \right] + \\ &+ \epsilon \langle m' | \rho_{S+}^{(1)}(1') | m'' \rangle \langle m | \rho_{S+}^{(2)}(2') | m''' \rangle \times \\ &\times \left. Y^{(+)}(\mathbf{p}\cdot\mathbf{r}) [S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] + \text{h.c.} \right\} \end{aligned} \quad (16b)$$

In this expression,  $\text{id.} [ \ ]$  symbolizes a double exchange term similar to the direct term, but with the substitutions which are specified inside the bracket; h.c. is for the second exchange term, which is hermitian conjugate of the first [because, in this term, the operator  $P_{\text{ex}}$  acts on the right side in (5), one can show that the same substitutions that transform the first term into the second (double exchange) also interchange the third and the fourth]; the removal of the pure exchange terms has been performed with the help of the equality:

$$\begin{aligned} & \left[ Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) + Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] - \\ & - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) = Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) [S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] \end{aligned} \quad (17a)$$

The consequence of this operation is that, in (16b), the exchange effects now occur only in the "outgoing" region where  $\mathbf{p} \cdot \mathbf{r}$  is positive.

The next step is to calculate the trace over particle 2 of the free Wigner transform, obtained by summing (16b) over  $m' = m'''$  and integrating over  $d^3 r_2 d^3 p_2$ . The calculation is similar to that of [12] and, in the exchange terms, introduces the products of one-particle spin operators. For simplification, from now on we assume that the one-particle operators  $\rho_+^{(1)}$  and  $\rho_+^{(2)}$  are equal, and we denote by  $\rho_S^{(+)}(\mathbf{r}, \mathbf{p})$  their common Wigner transform as well as  $f_+(\mathbf{r}, \mathbf{p})$  their trace over spins:

$$\begin{aligned} \rho_S^{(+)}(1) &= \rho_{S+}^{(1)}(\mathbf{r}_1, \mathbf{p}_1) = \rho_{S+}^{(2)}(\mathbf{r}_1, \mathbf{p}_1) \\ f_+(1) &= \text{Tr}_S \left\{ \rho_S^{(+)}(\mathbf{r}_1, \mathbf{p}_1) \right\} \end{aligned} \quad (17b)$$

We then obtain:

$$\begin{aligned} & \int d^3 r_2 \int d^3 p_2 \text{Tr}_{S_2} \left\{ \rho_{\text{W}}^{\text{L}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \right\} = \frac{1}{2} (2\pi)^{-3} \int_{18} e^{i(\kappa \mathbf{r} - \kappa' \cdot \mathbf{r}')} \\ & \left\{ \left[ Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) + Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] f_+(2') \rho_S^{(+)}(1') + \right. \\ & + \left[ Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, -\mathbf{k}'_2) + Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- + \mathbf{k}'_2) \right] f_+(1') \rho_S^{(+)}(2') + \\ & + \epsilon Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) [S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') + \\ & \left. + \epsilon Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) [S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, -\mathbf{k}'_2) - \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- + \mathbf{k}'_2)] \rho_S^{(+)}(1') \times \rho_S^{(+)}(2') \right\} \quad (18) \end{aligned}$$

In this expression,  $\text{Tr}_{S_2}$  refers to a trace over the spin states of the collision partner; again, we use the notation of [1d] :

$$\int_{18} = \int d^3 q \int d^3 r \int d^3 \kappa \int d^3 k'_1 \int d^3 k'_2 \int d^3 r' \quad (19a)$$

and we symbolize by (1') and (2') the one-particle variables:

$$\begin{cases} \mathbf{r}'_{1,2} = \mathbf{r}_1 - \frac{\mathbf{r}}{2} \pm \frac{\mathbf{r}'}{2} \\ \mathbf{p}'_{1,2} = \mathbf{p}_1 - \frac{\mathbf{q}}{2} \pm \hbar \frac{\mathbf{k}'_1 + \mathbf{k}'_2}{2} \end{cases} \quad (19b)$$

where  $\mathbf{q}$  is defined by:

$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2 = 2\hbar \mathbf{k} \quad (19c)$$

In (18), it is possible to change the sign of all primed integration variables,  $\mathbf{k}'_1$ ,  $\mathbf{k}'_2$  and  $\mathbf{r}'$ ; this operation shows that the second term in the right hand side of the equation (the double exchange term) is equal to the first, the fourth to the third; see Appendix A for a brief study of the symmetry properties of this type of expression.

1.2 EVOLUTION IN A BINARY COLLISION. — We now wish to calculate the time evolution of the trace over particle 2 of the free Wigner transform:

$$\rho_S(1) = \rho_S(\mathbf{r}_1, \mathbf{p}_1) = \int d^3 r_2 \int d^3 p_2 \text{Tr}_{S_2} \{ \rho_W^L(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \} \quad (20)$$

According to (18) and to the symmetry of exchange of primed variables, this operator is equal to:

$$\begin{aligned} \rho_S(1) = & (2\pi)^{-3} \int_{18} e^{i(\kappa \mathbf{r} - \kappa' \cdot \mathbf{r}')} \\ & \left\{ [Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) + Y^{(-)}(\mathbf{p} \cdot \mathbf{r}) \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] f_+(2') \rho_S^{(+)}(1') + \right. \\ & \left. + \epsilon Y^{(+)}(\mathbf{p} \cdot \mathbf{r}) [S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') \right\} \quad (21) \end{aligned}$$

To evaluate the time variation of this operator, we proceed as in section 2 of reference [1a] by a series of integrations by parts (see also the end of section 1.1 of [1d]). We use the relation:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] \rho_S(1) = \int d^3 r_2 \int d^3 p_2 \text{Tr}_{S_2} \{ D_T \rho_W^L(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \} \quad (22a)$$

where:

$$D_T = \left[ \frac{\partial}{\partial t} + \frac{\mathbf{P}}{2m} \cdot \nabla_{\mathbf{R}} + \frac{2\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} \right] = \left[ \frac{\partial}{\partial t} + \sum_1^2 \frac{\mathbf{p}_i}{m} \cdot \nabla_{\mathbf{r}_i} \right] \quad (22b)$$

(the term in  $\mathbf{p}_2 \cdot \nabla_{\mathbf{r}_2}$  vanishes when traced over particle 2, as can be seen by integrating by parts over  $d^3 r_2$ ). Then using equations (24) and (25) of [1a] to perform the integration over  $d^3 r'$  and, for each of the terms in the right hand side of (21), a reasoning similar to that of this reference, we obtain:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] \rho_S(1) = I_{\text{coll}} + I'_{\text{coll}} \quad (23)$$

with:

$$\begin{aligned} I_{\text{coll}} = & (2\pi)^{-3} \int_{18} e^{i(\kappa \mathbf{r} - \kappa' \cdot \mathbf{r}')} \frac{q}{m} \delta(\hat{\mathbf{q}} \cdot \mathbf{r}) \\ & \left\{ [S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] f_+(2') \rho_S^{(+)}(1') + \right. \\ & \left. + \epsilon [S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') \right\} \quad (24) \end{aligned}$$

where  $\hat{\mathbf{q}}$  is the unit vector parallel to  $\mathbf{q}$  (or  $\mathbf{p}$ ). The collision integral  $I'_{\text{coll}}$  is obtained by inserting, in expression (21), an operator  $D'_T$  in front of the Wigner transforms  $f_+$ 's or  $\rho_S^{(+)}$ 's; that operator  $D'_T$  is defined similarly to  $D_T$  (Eq. (22b)), but in terms of the primed variables written in (19b). Now, for a system of two particles only,  $I'_{\text{coll}}$  merely vanishes. This is because the  $f_+$ 's and  $\rho_S^{(+)}$ 's are the Wigner transforms associated with the free entering density operator  $\rho_{\Omega^+}^{(+)}$ , see relation (5); this operator evolves as if the particles were completely non-interacting, so that the effect of  $D'_T$  on these functions is zero.

### 1.3 CORRECTION TO THE DISTRIBUTION FUNCTION.

1.3.1 *Real distribution function.* — We can also express the real one-particle distribution function  $(^2)$ , which we note  $\rho_S^I(1)$ , as a function of the Wigner transforms of the operators  $\rho_+^{(1,2)}$  appearing in (5). It is sufficient for this purpose to apply a unitary transformation  $\Omega^{(+)}$  to  $\rho_{\Omega^+}^{(+)}$ , in order to obtain the real two-particle density operator  $\rho$ , to take the Wigner transform, and to trace over particle 2. The calculation is very similar to that done in the preceding section, but here instead of the matrix elements of  $S$  we have to use those of  $\Omega$ , defined by:

$$\Omega(\mathbf{k}, \mathbf{k}') = \langle \mathbf{k} | \Omega^{(+)} | \mathbf{k}' \rangle = \langle \mathbf{k} | \Psi_{\mathbf{k}'}^{(+)} \rangle \quad (25)$$

where the  $|\Psi_{\mathbf{k}'}^{(+)}\rangle$  are the outgoing stationary collision states; see also for example the Appendix of [1a] for the effect of a unitary transform on a Wigner distribution. We then obtain:

$$\begin{aligned} \rho_S^I(1) = (2\pi)^{-3} \int_{18} e^{i(\kappa r - \kappa' r')} \left\{ \Omega(\mathbf{k}_+, \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) f_+(1') \rho_S^{(+)}(2') + \right. \\ \left. + \epsilon \Omega(\mathbf{k}_+, -\mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') \right\} \quad (26) \end{aligned}$$

As in the preceding section, we have made use of the symmetry of parity over all primed variables to condense the direct and double exchange terms into one, as well as the two simple exchange terms (see Appendix A of this article).

1.3.2 *Expression of the correction.* — We define the correction  $\delta\rho_S(1)$  as the difference between the real one-particle distribution and the free distribution  $\rho_S(1)$ :

$$\delta\rho_S(1) = \rho_S^I(1) - \rho_S(1) \quad (27)$$

It is equal to:

$$\begin{aligned} \delta\rho_S(1) = (2\pi)^{-3} \int_{18} e^{i(\kappa r - \kappa' r')} \left\{ X(\mathbf{k}_+, \mathbf{k}_-, \mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) f_+(1') \rho_S^{(+)}(2') + \right. \\ \left. + \epsilon [ X(\mathbf{k}_+, \mathbf{k}_-, -\mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) + \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) ] \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') \right\} \quad (28) \end{aligned}$$

where the coefficients  $X$ 's are defined as in relation (33b) of [1d] :

$$\begin{aligned} X(\mathbf{k}_+, \mathbf{k}_-, \mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) = \Omega(\mathbf{k}_+, \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) - \\ - Y^{(+)}(\hat{\mathbf{q}} \cdot \mathbf{r}) S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \\ - Y^{(-)}(\hat{\mathbf{q}} \cdot \mathbf{r}) \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \quad (29) \end{aligned}$$

We notice that the correction to the distribution function contains terms introduced by the interactions (terms in  $X$ ) for distinguishable particles, modified by particle indistinguishability (terms in  $\epsilon X$ ), and terms due to pure statistics which occur even in the absence of the interactions (terms in  $\epsilon$  with two delta functions). This is reminiscent of the Beth-Uhlenbeck formula or, more generally, of usual virial expansions of the partition function.

<sup>(2)</sup> What we call here "distribution functions" are actually still operators (or matrices) in the one-particle spin state space. For simplicity, we shall not use the words "distribution operators" or "distribution matrices".

**1.3.3 Evolution of the real distribution function.** — We now apply the differential operator  $D_T$  to the distribution function  $\rho_I(1)$ . The calculation is not very different from that of section 1.2, but there are a few changes: Heaviside functions occur in (21) but not in (26), which simplifies the calculation; on the other hand the matrix elements of  $\Omega$  do not conserve energy (in opposition to those of  $S$ ), so that the expression  $\mathbf{q} \cdot \boldsymbol{\kappa} - \mathbf{q}' \cdot \boldsymbol{\kappa}'$  appears - see expression (29a) of [1d];  $\mathbf{q}'$  is equal to  $\mathbf{p}'_1 - \mathbf{p}'_2 = \hbar(\mathbf{k}'_1 + \mathbf{k}'_2)$ . One finds, by the same reasoning as in section 1.2 of that reference:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] \rho_I(1) = I_W + I'_W \quad (30)$$

where the "Wigner collision integral"  $I_W$  is defined by:

$$I_W = (2\pi)^{-3} \int_{18} e^{i(\boldsymbol{\kappa} \cdot \mathbf{r} - \boldsymbol{\kappa}' \cdot \mathbf{r}')} \frac{i}{m} [\mathbf{q} \cdot \boldsymbol{\kappa} - \mathbf{q}' \cdot \boldsymbol{\kappa}'] \left\{ \Omega(\mathbf{k}_+, \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) f_+(2') \rho_S^{(+)}(1') + \epsilon \Omega(\mathbf{k}_+, -\mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) \rho_S^{(+)}(2') \times \rho_S^{(+)}(1') \right\} \quad (31)$$

and where the collision integral  $I'_W$  can be obtained by inserting, in the right hand side of (26), differential operators  $D'_T$  in front of the distribution functions ( $I'_W$  vanishes for a system of two isolated particles).

An interesting property of the Wigner collision integral can be obtained by remarking that (Ref. [1d], Eq. (29b)):

$$\Omega(\mathbf{k}_+, \pm \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \pm \mathbf{k}'_2) [\mathbf{q} \cdot \boldsymbol{\kappa} - \mathbf{q}' \cdot \boldsymbol{\kappa}'] = -\frac{m}{\hbar} \langle \mathbf{k}_+ | \left[ V, |\Psi_{\pm \mathbf{k}'_1}^{(+)}\rangle \langle \Psi_{\pm \mathbf{k}'_2}^{(+)}| \right] | \mathbf{k}_- \rangle \quad (32a)$$

When inserted into (31), this relation gives to the Wigner integral a form which is very similar to that of the Snider collision integral [20], but in terms of the free distribution function  $\rho_S$  instead of the one-particle distribution function (and, of course, with spin and statistics in addition); see Appendix C for more details. Now, when (31) is summed over  $d^3 p_1$ , one can as in paragraph 1.2.2 of [1d] use closure relations to reconstruct the diagonal element of this commutator in the position representation (relation (31') of [1d]), which is simply zero because  $V$  is diagonal in this representation. This gives the relation:

$$\int d^3 p_1 I_W(1) = 0 \quad (32b)$$

## 2. Kinetic equation and conservation laws (many particles).

Until now, we have made exact calculations, valid for a system of two particles. We now turn to a system containing a large number,  $\mathcal{N}$ , of particles. Of course, this automatically makes us leave the ground of exact calculations; we will have to resort to some kind of reasonable Ansatz in order to introduce a kinetic equation satisfying our particular requirements: it should reproduce the evolution of the system with a sufficiently accurate treatment of two-body correlations and lead to correct values of the second virial corrections (throughout this article, we assume that the system is supposed sufficiently dilute for higher order density corrections to be negligible). The essence of the Ansatz that we shall introduce is a modified version of Boltzmann's molecular chaos assumption, the major change being that the duration and finite range of collisions are not ignored. It is based on the use of the free distribution function defined above, and exploits the fact that the "free entering" part of the distribution function remains exactly factorized, even during collision, provided it was so long before collision.

## 2.1 DISTRIBUTION APPEARING IN THE KINETIC EQUATION

**2.1.1 Defining the one-particle free distribution function** — Our first task is to define the distribution function (actually an operator acting in the space of spin states for one particle; see note <sup>(2)</sup>) in terms of which we shall write the kinetic equation. For a system of  $\mathcal{N}$  particles, the Wigner transform of the full density operator is a very complex mathematical object, containing all correlations developed by the system. During its motion inside the gas, every particle constantly creates around it, and leaves behind it, a “wake” of correlations which play an important physical role. This implies rapid variations of the Wigner transform two-particle density operator as a function of the relative distance between the two particles (but not as a function of the position of their centre of mass). On the other hand, this type of variation can not be reproduced by a simple product of functions  $f(1) \times f(2)$ , whatever the definition of  $f$  is; see sections 3.2 and 3.3 of reference [21] for a more detailed discussion of this question. As a consequence, if we intend at some point of the calculation to write a factorisation of a distribution function in terms of a simpler (one-body-like) distribution, that is a distribution which depends on a small number of variables in phase space, we necessarily have to remove the short range two-body correlations by some mathematical technique. We will now reason in two successive steps: first we shall go from  $\mathcal{N}$  particles to only two; then from two to one, with the help of the results obtained in section 1.

Clearly, the complexity of the problem is already strongly reduced if we trace the full distribution function (that is the density operator) over  $\mathcal{N} - 2$  particles. This can be done by integration over the positions and momentums of these particles and trace over their spin variables. If we did simply that, we would obtain the usual Wigner distribution of the two-particle density operator, which among other things includes the effects of perturbations from the  $\mathcal{N} - 2$  collision partners on the two test particles; these perturbations remain present even if the two particles are far apart. This therefore does not correspond to the notion of free distribution: what we need here is the distribution function for pair of particles that are far from all other particles. Consequently, as in section 2 of reference [1d], we proceed in a slightly different way when performing the trace operation: we assume that the range of integration of the positions of the  $\mathcal{N} - 2$  particles is restricted to some minimum distance from the two test particles (the exact value of this arbitrary distance is presumed to be of little importance for a dilute gas). We shall actually just assume that this operation has been already been done, with adequate normalization, and that it has provided the density operator  $\rho$  (or its Wigner transform), which we shall call the “isolated two-particle density operator”, and which can be used as the starting point <sup>(3)</sup> for calculations of the same type as in section 1.

We now have to reduce the number of variables to that of one particle only. Because of the useful factorization property of the entering free density operator - obtained in (4) under the action of the inverse of the unitary transformation  $\Omega^{(+)}$  - one could be tempted to reason in terms of the one-particle operators  $\rho_+(1, 2)$  appearing in (5). The difficulty with this idea, however, is the following: if the entering part reproduces well the Wigner transform of the real density operator in the entering region  $\mathbf{r} \cdot \mathbf{p} < 0$  (almost identically at large distances, that is long before collision, and more approximately inside the potential), it nevertheless completely destroys all effects of collisions in the outgoing region. This is because the inverse of the unitary transformation  $\Omega^{(+)}$  suppresses all the effects of the interactions (the particles become “transparent”) and can therefore completely change the velocities and the positions of the particles after collision. In other words, the unitary transformation does too much: it indeed removes nicely the correlations

---

<sup>(3)</sup> A well-known property of the real two-particle density operator is that it is factorized in two one-particle density operators at large distances, possibly symmetrized by exchange as in (5). Because we start here from a different two-particle density operator, that for isolated particles, this long distance factorization is changed into a different product, that of free density operators  $\rho_s$ .

inside the potential range, which allows preserving an initial factorization, but it does this at the expense of strong, unphysical, modifications occurring at large distances inside the outgoing region of phase space. This is exactly the point where we need the free Wigner transform: while it also preserves the essential factorization property of the entering part (even inside the range of the potential), it uses for the outgoing part another unitary transformation; in this way, one avoids introducing drastic modifications of the physical properties of the particles long after collision.

As the building block of our kinetic equation, we therefore decide to use the free one-particle distribution  $\rho_S$ , which was defined in relation (20) as the trace over particle two of the free two-particle distribution function; the latter is in turn defined through (16b) and (5) from the "isolated two-particle density operator" introduced above in this section.

**2.1.2 Closure Ansatz** — What is now needed is an additional relation in order to obtain a closed kinetic equation. We shall simply write:

$$\rho_S^{(+)}(1) = \rho_S(1) \quad (33)$$

In other words, we take for the density operator of any atom long before collision the trace over the collision partner of the free distribution for the preceding collision; this amounts to ignoring any physical effect of the correlations which appear in the outgoing part of the free distribution. This is a good approximation in a dilute system, for which there is a weak probability of multiple successive collisions between the same partners. Relation (33) is well adapted to a model where more than two-body interactions are ignored: because the outgoing part of the free transform, after it appears in the plane  $\mathbf{p} \cdot \mathbf{r} = 0$  of phase space, propagates completely freely in the region  $\mathbf{p} \cdot \mathbf{r} > 0$  (no effect of the potential, see Ref. [1a]), what we inject in the entering part of the next collision is just the long range effects of the collision; we do not include the possibility of the interaction potential being still active while a particle approaches the next collision partner, which would be a three-particle interaction.

At this point, it is instructive to come back to the choice we have made in section 1 for the definition of the free Wigner transform, more specifically to the removal in section 1.1.4 of the terms due to pure statistics in order to arrive to the "completely free" distribution function. We can now see better how essential this precaution was. A relation like (33), if  $\rho_S$  was derived from the "semi-free" distribution by a trace over the collision partner as in (20), would actually become inconsistent: the effects of exchange are added to a product  $\rho_+$ 's in (9), and would therefore already be contained in a definition of  $\rho_S$  through this modified version of (20), but then injecting (33) into (9) would amount to introduce them again: the whole procedure would become a kind of vicious circle. With our definition of the (completely) free Wigner transform of section 1.1.4, where the pure exchange terms have been removed from the right hand side of (20),  $\rho_S$  contains no pure exchange effect and this redundancy is avoided. For example, for non-interacting particles, it is easy to check that (33) becomes an exact relation when the pure exchange terms are removed from  $\rho_W^L$  as we have done, but not if  $\rho_S$  was obtained by trace of the semi-free distribution.

**2.2 FORM OF THE KINETIC EQUATION.** — Now that we have made contact with the calculations of section 1, we can use them as a guide in order to write explicitly a kinetic equation in terms of the free distribution  $\rho_S$ . We remark that equation (23), with the form of the collision integral that we have given in (24), is in fact rather general: it was obtained in section 1 without any particular assumption concerning the evolution of the system. The only step at which the dynamics of the distribution functions was used in section 1 is when we noticed that the additional collision integrals  $I'_{\text{coll}}$  and  $I'_W$  are zero for a system of two particles only, but we shall not use this particular property in what follows.

We therefore write our kinetic equation in the form:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] \rho_S(1) = I_{\text{coll.}} + \delta I_{\text{coll.}} \quad (34)$$

where  $I_{\text{coll.}}$  is simply obtained <sup>(4)</sup> by substituting (33) into (24):

$$\begin{aligned} I_{\text{coll.}} = & (2\pi)^{-3} \int_{18} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} \frac{q}{m} \delta(\hat{\mathbf{q}} \cdot \mathbf{r}) \\ & \left\{ \left[ S(\mathbf{k}_+, \mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ - \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] f(2') \rho_S(1') + \right. \\ & \left. + \epsilon \left[ S(\mathbf{k}_+, -\mathbf{k}'_1) S^*(\mathbf{k}_-, \mathbf{k}'_2) - \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) \right] \rho_S(2') \times \rho_S(1') \right\} \quad (35) \end{aligned}$$

In this equation,  $f(1)$  is the trace over spin of the free distribution  $\rho_S$ :

$$f(\mathbf{r}_1, \mathbf{p}_1) = \text{Tr}_S \{ \rho_S(\mathbf{r}_1, \mathbf{p}_1) \} \quad (36)$$

For the moment, we do not specify the exact form of the correction  $\delta I_{\text{coll.}}$  to the collision integral, noting only that it will be third order in density. Actually, it will play little role in the following, mostly because this dependence in density makes it similar to a three-body collision term, which is beyond the accuracy that one can expect from our theory.

The Wigner transform  $\rho_S^I(1)$  of the real one-body density operator will be taken as equal to the free distribution  $\rho_S$ , plus a correction  $\delta \rho_S$  given, according to (28), by:

$$\begin{aligned} \delta \rho_S(1) = & (2\pi)^{-3} \int_{18} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} \left\{ X(\mathbf{k}_+, \mathbf{k}_-, \mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) f(1') \rho_S(2') + \right. \\ & \left. + \epsilon [ X(\mathbf{k}_+, \mathbf{k}_-, -\mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) + \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2) ] \rho_S(2') \times \rho_S(1') \right\} \quad (37) \end{aligned}$$

where the definition of the coefficient  $X$  is given in (29). We notice that  $\delta \rho_S$  is the sum of three contributions: that of pure interactions, which is the first term in the right hand side of (37); a crossed term between interaction and statistics, which is the term in  $\epsilon X$ ; finally a term due to pure statistics, containing two delta functions, which can be written:

$$\delta \rho_S^{\text{stat}}(1) = \epsilon (2\pi)^{-3} \int d^3 q \int d^3 r \int d^3 \kappa \int d^3 r' e^{i(\kappa \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}' / \hbar)} \rho_S(2') \times \rho_S(1') \quad (38a)$$

where the  $\mathbf{p}'_{1,2}$  are given by inserting the relation  $\mathbf{k}'_1 + \mathbf{k}'_2 = -\kappa$  inside (11d). If, in addition, we assume that the spatial dependence of the distribution functions is very slow, or even that the gas is in a uniform situation, the integrations over  $d^3 r$  and  $d^3 r'$  introduce delta functions of  $\mathbf{q}$  and  $\kappa$ , and the equation simplifies into:

$$\delta \rho_S^{\text{stat}}(1) = \epsilon h^3 [\rho_S(1)]^2 \quad (38b)$$

<sup>(4)</sup> One should remember that, like  $\rho_S$  itself, the collision integrals  $I_{\text{coll}}$  and  $\delta I_{\text{coll}}$  are in fact operators acting in the space of one-particle spin states ( $2 \times 2$  matrices if the particles have  $1/2$  spins). For simplicity, we nevertheless keep the same notation as in reference[1d].

( $h$  is the Planck constant  $2\pi\hbar$ ); the exchange term is merely proportional to the square of the matrix describing the free distribution. This result fits well with the physical idea of an exchange term: only particles which are in the same cell of phase space ( $\mathbf{r}$ ,  $\mathbf{p}$ ) undergo mutual exchange; the relative correction to the distribution function is of the order of the number of particles in a cell of volume  $h^3$ .

We now calculate the time evolution of  $\delta\rho_S$ . By reasoning that is directly transposed from that of sections 1.2 and 1.3.3 (one simply has to take the difference), we obtain the following result:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] \delta\rho_S(1) = I_W - I_{\text{coll.}} + \delta I_X \quad (39a)$$

where  $I_W$  is obtained from (31) by simply replacing the  $\rho_S^{(+)}$ 's by  $\rho_S$ 's, and where  $\delta I_X$  is defined by:

$$\begin{aligned} \delta I_X = (2\pi)^{-3} \int_{18} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} \left\{ X(\mathbf{k}_+, \mathbf{k}_-, \mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) D'_T f(1') \rho_S(2') \right. \\ \left. + \epsilon [X(\mathbf{k}_+, \mathbf{k}_-, -\mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) + \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] D'_T \rho_S(2') \times \rho_S(1') \right\} \end{aligned} \quad (39b)$$

(the operator  $D'_T$  acts on the two distributions which follow it; we recall that it is defined as the total derivative (22b), but in terms of the primed variables which appear in these distributions). We therefore have:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] [\rho_S(1) + \delta\rho_S(1)] = I_W + \delta I_X + \delta I_{\text{coll.}} \quad (40)$$

At this point it is natural, and we will see below that this leads to exact conservation laws, to set:

$$\delta I_{\text{coll.}} = -\delta I_X \quad (41)$$

so that we get the simpler equation:

$$\left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}_1} \right] [\rho_S(1) + \delta\rho_S(1)] = I_W \quad (42)$$

Now that its value has been specified, we can check that  $\delta I_{\text{coll.}}$  is third order in density (or more), as mentioned at the beginning of this section. This can be seen by injecting the kinetic equation (34) into the right hand side of (39b), which leads to the following implicit definition of this collision integral:

$$\begin{aligned} \delta I_{\text{coll.}} = -(2\pi)^{-3} \int_{18} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} X(\mathbf{k}_+, \mathbf{k}_-, \mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) \\ \left\{ f(1') \left[ I_{\text{coll.}}[\rho_S(2')] + \delta I_{\text{coll.}}[\rho_S(2')] \right] + \left[ I_{\text{coll.}}[f(1')] + \delta I_{\text{coll.}}[f(1')] \right] \rho_S(2') \right\} \\ - \epsilon (2\pi)^{-3} \int_{18} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} [X(\mathbf{k}_+, \mathbf{k}_-, -\mathbf{k}'_1, \mathbf{k}'_2; \hat{\mathbf{q}} \cdot \mathbf{r}) + \delta(\mathbf{k}_+ + \mathbf{k}'_1) \delta(\mathbf{k}_- - \mathbf{k}'_2)] \\ \left\{ \rho_S(2') \left[ I_{\text{coll.}}[\rho_S(1')] + \delta I_{\text{coll.}}[\rho_S(1')] \right] + (1') \longleftrightarrow (2') \right\} \end{aligned} \quad (43)$$

In this expression,  $I_{\text{coll.}}[\rho_S(2')]$  for example refers to the evolution of  $\rho_S(2')$  under the effect of collisions, which is obtained from (35), but of course with new notation for the variables (1') and (2') under the integral [for instance (1'') and (2'')];  $I_{\text{coll.}}[f(1')]$  is the trace over spin variables of  $I_{\text{coll.}}[\rho_S(1')]$ , and a similar convention is used for  $\delta I_{\text{coll.}}$ . In (43), because the  $\rho_S$ 's are operators in the spin space, one should be careful about not changing the order of the operators  $\rho_S(1')$  and  $\rho_S(2')$  when writing the contribution of the last part of the exchange term, noted (1')  $\leftrightarrow$  (2').

As a first approximation, one may ignore the contribution of  $\delta I_{\text{coll.}}$  in the right hand side of (43), which then becomes an explicit definition; if we do that, the collision integral  $\delta I_{\text{coll.}}$  becomes exactly third order in density. The next approximation is to inject this correction into the right hand side of (43), which then acquires a component fourth order in density, then by iteration a fifth order component, etc. All these corrections, as already noted, are actually beyond the accuracy of our calculations, but keeping them will allow us to obtain exact conservation laws (instead of laws valid up to second order only).

With this definition, and with (37) which gives the real one-particle distribution function  $\rho_S^{\text{I}}$  from  $\rho_S$ , the set of equations that we have introduced is now closed and complete.

### 2.3 CONSERVATION LAWS.

The local number density of particles at the point of space  $\mathbf{r} = \mathbf{r}_1$  is the sum of the "free particle density":

$$N_L(\mathbf{r}) = \int d^3 p_1 \text{Tr}_S \{ \rho_S(1) \} \quad (44a)$$

and of the density correction:

$$\delta N(\mathbf{r}) = \int d^3 p_1 \text{Tr}_S \{ \delta \rho_S(1) \} \quad (44b)$$

We remark that, according to (37), this correction contains contributions of the interactions as well as of pure statistics. Similarly, we introduce the "free particle current" by:

$$\mathbf{J}_L = \int d^3 p_1 \frac{\mathbf{p}_1}{m} \text{Tr}_S \{ \rho_S(1) \} \quad (45a)$$

as well as its correction:

$$\delta \mathbf{J}(\mathbf{r}) = \int d^3 p_1 \frac{\mathbf{p}_1}{m} \text{Tr}_S \{ \delta \rho_S(1) \} \quad (45b)$$

Introducing the total number density and currents:

$$\begin{aligned} N(\mathbf{r}) &= N_L(\mathbf{r}) + \delta N(\mathbf{r}) \\ \mathbf{J}(\mathbf{r}) &= \mathbf{J}_L(\mathbf{r}) + \delta \mathbf{J}(\mathbf{r}) \end{aligned} \quad (46)$$

we can integrate equation (42) over  $d^3 p_1$  and obtain, using (32b):

$$\frac{\partial}{\partial t} N(\mathbf{r}) + \nabla \cdot \mathbf{J}(\mathbf{r}) = \int d^3 p_1 \text{Tr}_S \{ I_w(1) \} = 0 \quad (47)$$

which gives the first local conservation law.

The local density of internal angular momentum is treated in a similar way. To simplify, we assume that the particles are spin 1/2 particles, and we define:

$$\mathcal{M}(\mathbf{r}) = \mathcal{M}_L(\mathbf{r}) + \delta \mathcal{M}(\mathbf{r}) = \int d^3 p_1 \text{Tr}_S \{ \boldsymbol{\sigma} [\rho_S(1) + \delta \rho_S(1)] \} \quad (48a)$$

where  $\sigma$  is the Pauli matrix acting in the space of spin states; as  $N(\mathbf{r})$ , the local density of angular momentum is the sum of a free part plus a correction. The associated current is, for component  $i$  of this internal angular momentum:

$$\mathbf{J}_{\mathcal{M},i}(\mathbf{r}) = \int d^3 p_1 \frac{p_1}{m} \text{Tr}_S \{ \sigma_i [\rho_S(1) + \delta \rho_S(1)] \} \quad (48b)$$

(it is also the sum of a free part and a correction, but we do not write it explicitly). Because relation (32b) is an operatorial relation in spin space, very little is to be changed in the preceding reasoning in order to obtain the local conservation law for internal angular momentum:

$$\frac{\partial}{\partial t} \mathcal{M}_i(\mathbf{r}) + \nabla \cdot \mathbf{J}_{\mathcal{M},i}(\mathbf{r}) = 0 \quad (48c)$$

For particles with more than two internal states (still uncoupled to the collision hamiltonian), one would obtain in addition local conservation laws for higher order operators (alignment, etc).

For the study of the local conservation of momentum, we continue to transpose the results of section 3 of [1d] and write the "free stress tensor" as:

$$\bar{\bar{Q}}_L = m^{-1} \int d^3 p_1 [\mathbf{p}_1] \otimes [\mathbf{p}_1] \text{Tr}_S \{ \rho_S(1) \} \quad (49a)$$

and its density correction:

$$\delta \bar{\bar{Q}}_X = m^{-1} \int d^3 p_1 [\mathbf{p}_1] \otimes [\mathbf{p}_1] \text{Tr}_S \{ \delta \rho_S(1) \} \quad (49b)$$

Now, if we multiply (42) by  $\mathbf{p}_1$ , and sum over  $d^3 p_1$  while taking the trace, we obtain:

$$m \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}) + \nabla \cdot [\bar{\bar{Q}}_L + \delta \bar{\bar{Q}}_X] = \int d^3 p_1 \mathbf{p}_1 \text{Tr}_S \{ I_W(1) \} \quad (50)$$

The study of this conservation law leads naturally to the evaluation of the integral in the right hand side of this equation, which should be put in the form of the divergence of an additional stress tensor  $\delta \bar{\bar{Q}}_W$ . We do not give explicitly the reasoning, because it is very close to that of reference [1d], see equation (49) of this reference, and we only write the result :

$$\begin{aligned} \delta \bar{\bar{Q}}_W = & \frac{1}{4} (2\pi)^{-3} \int d^3 p_1 \int_{18} e^{i(\kappa \mathbf{r} - \kappa' \mathbf{r}')} \frac{i}{m} [\mathbf{q} \cdot \kappa - \mathbf{q}' \cdot \kappa'] [\mathbf{q} \otimes \mathbf{r}] \\ & \left\{ \Omega(\mathbf{k}_+, \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) \sum_{n=0,2,4} \frac{1}{(n+1)!} \left[ \frac{\mathbf{r}}{2} \cdot \nabla_{\mathbf{r}_1} \right]^n f(1'_0) f(2'_0) + \right. \\ & \left. + \epsilon \Omega(\mathbf{k}_+, -\mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) \sum_{n=0,2,4} \frac{1}{(n+1)!} \left[ \frac{\mathbf{r}}{2} \cdot \nabla_{\mathbf{r}_1} \right]^n \text{Tr}_S \{ \rho_S(1'_0) \times \rho_S(2'_0) \} \right\} \quad (51) \end{aligned}$$

where  $(1'_0)$  and  $(2'_0)$  are used instead of  $(1')$  and  $(2')$  to emphasize that all space derivatives in the right hand side are taken at  $\mathbf{r} = 0$ . Since expression (51) contains a factor  $[\mathbf{q} \cdot \kappa - \mathbf{q}' \cdot \kappa']$  which vanishes "on shell" (when energy is conserved), we can if we wish replace the products of  $\Omega$  and  $\Omega^*$  by coefficients  $X$ 's, defined in (29); this adds terms in  $S \times S^*$  and  $\delta \times \delta$  which do not contribute to the result. Because the  $X$ 's are zero in the absence of interactions, this emphasizes

the fact that  $\delta\bar{Q}_W$  is the contribution of the interactions to the pressure tensor; it is second order in density while the other contributions are first order only <sup>(5)</sup>.

When all contributions are summed, we therefore obtain the local momentum conservation law:

$$\frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}) + \nabla_{\mathbf{r}} \cdot \bar{\mathbf{Q}}(\mathbf{r}) = 0 \quad (52a)$$

where the total stress tensor is defined by:

$$\bar{\mathbf{Q}} = \bar{\mathbf{Q}}_L + \delta\bar{\mathbf{Q}}_X + \delta\bar{\mathbf{Q}}_W \quad (52b)$$

As for the local conservation of energy, it will not be studied in great detail here, and we only briefly repeat the essence of the argument given in section 3.3 of [1d]. Because the Wigner collision integral reduces, in homogeneous situations, to the LL collision integral of reference [12], and because that integral conserves locally the density of kinetic energy (see Appendix I of [12]), our theory predicts that the local variation of kinetic energy will be slow in hydrodynamic situations. The density of energy will then be defined as the sum of that of kinetic energy, plus that of potential energy, which is a function of the square of  $N(\mathbf{r})$ . This ensures that the density of energy has only hydrodynamical variations.

**2.4 SYSTEM AT THERMAL EQUILIBRIUM.** — We now wish to study the properties of a system at thermal equilibrium. Nevertheless, we assume that the spin polarization of the atoms is arbitrary; in other words, we assume that the orbital variables of the atoms have reached equilibrium, but not the internal spin variables. This metastable situation may occur when the longitudinal relaxation time of the spins is sufficiently long, and when a method of spin polarization is available, as in spin-polarized helium three or hydrogen [2, 3, 4]. We will see that, in this case, we can recover by a different method the polarization dependence of the pressure already obtained in [13].

Because the free distribution function remains unperturbed by quantum virial corrections [14, 21], it is exactly gaussian at equilibrium and one can write (for the sake of simplicity, we only study the case of spin 1/2 particles, but the generalization is straightforward):

$$\rho_S(\mathbf{p}_1) = f_m(p_1) \frac{1}{2} [1 + \boldsymbol{\sigma} \cdot \mathbf{M}] \quad (53a)$$

with:

$$f_m(p) = \alpha_m \exp(-\beta p^2/2m) \quad (53b)$$

where  $\beta = 1/k_B T$  ( $k_B$  is the Boltzmann constant,  $T$  the temperature), and where the normalization coefficient  $\alpha_m$  has the following expression in function of the free number density  $N_L$ :

$$\alpha_m = \left(\frac{\beta}{2\pi m}\right)^{3/2} N_L \quad (53c)$$

---

<sup>(5)</sup> In order to change the density, let us assume that we multiply the real one-particle density operator,  $\rho_S + \delta\rho_S$ , by a scaling constant  $x$ ; then,  $\rho_S$  changes proportionally to a series of powers of  $x$  of the form

$[1 + \alpha x + \beta x^2 + \dots]$ , while the correction  $\delta\rho_S$  starts only at first order and scales as  $[-\alpha x - \beta x^2 - \dots]$ . Quantities such as  $\mathbf{J}$  or  $\mathcal{M}$  for example are indeed exactly proportional to  $x$ , and the same is true of the sum

$\bar{\mathbf{Q}}_L + \delta\bar{\mathbf{Q}}_X$  (but not of  $\delta\bar{\mathbf{Q}}_X$  only), while  $\delta\bar{\mathbf{Q}}_W$  is second order (or more) in  $x$ .

In (53a), the spin polarization is fixed by the length of the vector  $\mathbf{M}$ ; for instance, this length is zero for an unpolarized gas, one for a full polarization. We can separate the effects related to the centre of mass from those related to the relative motion by writing:

$$f_m(1') \times f_m(2') = f_{2m}(\mathbf{p}'_1 + \mathbf{p}'_2) \times f_{m/2}\left(\frac{\mathbf{p}'_1 - \mathbf{p}'_2}{2}\right) \quad (54)$$

To obtain the corrections to the number density of particles and to the stress tensor, we now have to insert these equalities into the definition (37) of  $\delta\rho_s$ . The result is the sum of three terms, a direct term and two exchange terms (interaction and pure statistics). For the direct term, the calculation is perfectly similar to that of section 4 of reference [1d] and we do not need to reproduce it here; we get a first contribution to  $\delta N$  in the form:

$$\delta N_{\text{int}}^{(+)} = 2^{3/2} [\lambda_T]^3 (N_L)^2 \Sigma^{(+)} \quad (55)$$

where, for convenience, we now denote by  $\Sigma^{(+)}$  the dimensionless quantity that was called  $\Sigma$  in [1d]:

$$\Sigma^{(+)} = \int d^3r \langle \mathbf{r} | e^{-\beta H_{\text{rel}}} - e^{-\beta H_0} | \mathbf{r} \rangle = \int_0^\infty dk e^{-\beta \hbar^2 k^2 / m} \sum_l (2l+1) \frac{1}{\pi} \delta'_l(k) \quad (56)$$

In (55),  $\lambda_T$  is the thermal wavelength of the atoms:

$$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}} = \hbar \sqrt{\frac{2\pi\beta}{m}} = h \sqrt{\frac{\beta}{2\pi m}} \quad (57)$$

The value of  $\delta N_{\text{int}}^{(+)}$  gives the contribution to  $\delta N$  for distinguishable particles. For the crossed terms between interactions and statistics, the calculation is also similar, but nevertheless with two differences; the first is that there is a minus sign affecting the variable  $k'_1$  in  $X$ , which corresponds to the action of the parity operator  $P_r$  reversing the relative position of the two particles; the second difference is that one has to take traces over the spin states, which introduces an extra factor:

$$\frac{1}{4} \text{Tr}_S \left\{ [1 + \boldsymbol{\sigma} \cdot \mathbf{M}] [1 + \boldsymbol{\sigma} \cdot \mathbf{M}] \right\} = \frac{1}{2} [1 + M^2] \quad (58)$$

This contribution is therefore equal to:

$$\delta N_{\text{int}}^{\text{ex}} = \epsilon 2^{3/2} \left[ \frac{1 + M^2}{2} \right] [\lambda_T]^3 (N_L)^2 \Sigma^{(-)} \quad (59a)$$

where  $\Sigma^{(-)}$  differs from  $\Sigma^{(+)}$  by the effect of the parity operator  $P_r$ :

$$\Sigma^{(-)} = \int d^3r \langle -\mathbf{r} | e^{-\beta H_{\text{rel}}} - e^{-\beta H_0} | \mathbf{r} \rangle = \int_0^\infty dk e^{-\beta \hbar^2 k^2 / m} \sum_l (-1)^l (2l+1) \frac{1}{\pi} \delta'_l(k) \quad (59b)$$

Finally, the term due to pure statistics may be obtained by integration of (38b) over momentum. One notices that the square of the equilibrium distribution function  $f_m(p)$  is equal to

$2^{-3/2}\alpha_m f_{m/2}$ , so that its norm is  $2^{-3/2}\alpha_m$ , or  $2^{-3/2}N_L(\lambda_T/h)^3$ ; the contribution of the trace over spins is the same as in the preceding term, and one obtains:

$$\delta N_{\text{stat}} = \epsilon 2^{-3/2} \left[ \frac{1+M^2}{2} \right] [\lambda_T]^3 (N_L)^2 \quad (60)$$

The total value of  $\delta N$  at equilibrium is then the sum:

$$\delta N = \delta N_{\text{int}}^{(+)} + \delta N_{\text{int}}^{\text{ex}} + \delta N_{\text{stat}} = 2^{3/2} [\lambda_T]^3 (N_L)^2 \left[ \Sigma^{(+)} + \epsilon \frac{1+M^2}{2} [\Sigma^{(-)} + \frac{1}{8}] \right] \quad (61)$$

We now have to study the pressure tensor and to evaluate the sum of the three terms in the right hand side of (52b). The major difference with the calculation of section 4.2 of [1d] is that  $\delta \rho_S$  contains three terms in the right hand side of (37), instead of only one for  $\delta f$ . Moreover,  $\delta \bar{Q}_W$  does not contain the same function  $X$  as  $\delta \rho$ , but a product of  $\Omega$ 's. But we have already seen (end of Sect. 2.3) that this product of  $\Omega$ 's may be replaced by factors  $X$ 's without affecting the result; the same argument based on the cancellation of the factor  $[\mathbf{q} \cdot \boldsymbol{\kappa} - \mathbf{q}' \cdot \boldsymbol{\kappa}']$  also permits to add pure exchange terms with the same delta functions as in (37). Then, because the arguments of section 4.2 of [1d] do not depend on the particular expression of the function  $X$ , one can follow the same reasoning term by term. We just recall the main steps without details.

In the evaluation of  $\delta \bar{Q}_X$ , one can distinguish between two contributions, that which is bilinear in the total momentum  $\mathbf{P}$ , and that which is bilinear in  $\mathbf{q}$  (for parity reasons, the crossed terms vanish). The calculation of the former is completely similar to that of  $\delta N$ , since the only difference appears in an integral over the total momentum  $\mathbf{P}$ , which does not play any role other than being a normalisation factor in front of the result. One then obtains a contribution which is similar to (61):

$$2^{3/2} [\lambda_T]^3 (N_L)^2 \frac{k_B T}{2} \left[ \Sigma^{(+)} + \epsilon \frac{1+M^2}{2} [\Sigma^{(-)} + \frac{1}{8}] \right] \quad (62a)$$

As for the term which is quadratic in  $\mathbf{q}$ , it exactly cancels the contribution of  $\delta \bar{Q}_W$  at equilibrium, as can be seen by considering the term  $n = 0$  of (51), and using an integration by parts over  $d^3 r'$  as in reference [1d]. Consequently, one finds that (62a) is the only contribution to the correction  $\delta P$  to the pressure, and we obtain again relation (72) of [1d]:

$$\frac{\delta P}{N_L k_B T} = \frac{1}{2} \frac{\delta N}{N_L} \quad (62b)$$

Finally, the virial coefficient for the pressure is given by:

$$B_2(T) = -\frac{1}{2} \frac{\delta N}{N} = -2^{1/2} [\lambda_T]^3 N \left[ \Sigma^{(+)} + \epsilon \frac{1+M^2}{2} [\Sigma^{(-)} + \frac{1}{8}] \right] \quad (63)$$

(one can ignore the difference between  $N_L$  and  $N$ , in the right hand side of this expression). This equation generalizes to any value of  $M$  the result obtained in [13] for  $M$  equal to one or zero, and by a different method.

2.5 GRADIENT EXPANSION OF THE COLLISION INTEGRAL. — We now assume that the spatial dependence of  $\rho_S$  is slow, more precisely that it is sufficiently weak in relative value on any microscopic distance. This allows the use of Taylor expansions for the distribution functions appearing inside the collision integral  $I_{\text{coll}}$ , written in (35). One obtains in this way several kinds of terms; first, those which are zero order in gradients, and will be called “local terms” as usual in the context of kinetic theory; then, terms which are first order in gradients and which, as we will see, have a more complicated structure. We shall not go beyond first order; this is sufficient for comparing our results to those obtained by Silin [5].

Before we proceed, it is useful to examine the nature of collision terms in general. When, into (35), we insert expression (13) to replace the matrix elements of the  $S$  matrix by those of  $T$ , we obtain two kinds of terms in  $I_{\text{coll}}$ . The first are linear in  $T$  or  $T^*$ , and correspond to forward scattering terms; they contain one three-dimensional delta function of vectors and one of vector length only (conservation of energy in the matrix elements of  $S$ ). The second include the product  $T \times T^*$ , and they correspond to the effect of lateral scattering; they contain only one-dimensional delta functions. It will be convenient to distinguish between forward and lateral scattering terms in the calculations. Actually, depending where we wish to put the emphasis, we have altogether three independent ways for classifying the various collision terms: direct or exchange, forward or lateral scattering, local or non-local.

Two remarks will be helpful for the calculations below. The first is that the forward scattering terms in  $T$  and  $T^*$  are simply hermitian conjugate of each other; see Appendix A, symmetry ( $\epsilon$ ); it will then be sufficient to calculate the first of these terms. The second remark is that the direct term of (35) has exactly the same form as the collision term studied in reference [1a] and [1b]; one can for example see it by integrating equation (22) of [1a] over  $d^3q$  and replacing the entering free Wigner transform by the product  $f(2') \rho_S(1')$ . We can therefore immediately transpose to the study of our direct term the results of [1b] and write that the direct term of  $I_{\text{coll}}$  is the sum of three contributions:

$$I_{\text{coll}}^{\text{dir.}} = I_B + G_{\xi}^{\text{dir.}} + G_{\Delta}^{\text{dir.}} \quad (64)$$

where  $I_B$  is the Boltzmann collision integral and  $G_{\xi}^{\text{dir.}}$  and  $G_{\Delta}^{\text{dir.}}$  are the first-order gradient corrections in the absence of statistics; their precise expressions can be obtained by making the following substitutions in equations (23c) and (23d) of [1b]:

$$f(1) \implies \rho_S(1) \quad f(1') \implies \rho_S(1') \quad (65)$$

while the  $f(2)$  and  $f(2')$  remain unchanged.

We now study the gradient expansion of the whole collision integral  $I_{\text{coll}}$ , with of course more emphasis on exchange terms since the others are already known from (64).

2.5.1 *Local terms.* — The local terms are obtained by replacing  $\mathbf{r}$  and  $\mathbf{r}'$  by zero in the positions associated with (1') and (2'). They are the sum of the Boltzmann integral  $I_B$  of (64) with exchange terms, which we now study, starting with the forward scattering term in  $T$ . The only difference with the reasoning of section 3.2 of [1a] is that  $T(\mathbf{k}_+, k_+ \hat{\mathbf{k}})$  is replaced by  $T(\mathbf{k}_+, -k_+ \hat{\mathbf{k}})$ , while the distribution function becomes of course now a product of spin operators as in (35). The adaptation of formula (44) of [1a] therefore gives a contribution:

$$-8\epsilon(i\pi^3\hbar) T_{p/\hbar}(-1) \rho_S(\mathbf{r}_1, \mathbf{p}_1 - \mathbf{q}) \times \rho_S(\mathbf{r}_1, \mathbf{p}_1) \quad (66a)$$

which, as that equation, should be integrated over  $d^3q$ ; the variable made explicit in  $T$  is the cosine of the collision angle:  $T(-1)$  refers to the value of  $T$  when the values of the momentum before and after collision are opposed, as usual for exchange terms in collision theory. Now, we have to

add to (66a) the term is  $T^*$ , which is hermitian conjugate, so that we obtain the following value<sup>(6)</sup>:

$$8\epsilon(i\pi^3\hbar) [T_{p/\hbar}(-1)]^* \rho_S(\mathbf{r}_1, \mathbf{p}_1) \times \rho_S(\mathbf{r}_1, \mathbf{p}_1 - \mathbf{q}) \quad (66b)$$

Putting these equations together gives the result:

$$-\frac{\epsilon}{2} \frac{q}{m} \left\{ \sigma_{\text{fwd.}}^{\text{ex.}}(k) [\rho_S(1), \rho_S(2)]_+ + i\tau_{\text{fwd.}}^{\text{ex.}}(k) [\rho_S(1), \rho_S(2)] \right\} \quad (66c)$$

with the definition already introduced in [12]:

$$\sigma_{\text{fwd.}}^{\text{ex.}}(k) - i\tau_{\text{fwd.}}^{\text{ex.}}(k) = \frac{8\pi^3 m}{\hbar^2 k} i T_k(\hat{\mathbf{k}}, -\hat{\mathbf{k}}) \quad (66d)$$

where, in (66c), the brackets with a + mean anticommutators, otherwise commutators.

The lateral scattering terms are treated as in section 3.3 of [1a], with very little change: just a sign in the cosine of the collision angle for  $T$  (not for  $T^*$ ). Formulas (51) and (52) of [1a] become here (with an additional integration over the momentum of collision partner):

$$\epsilon \frac{4\pi^4 m^2}{\hbar^4} \int d^3 q \int d^2 \hat{\mathbf{k}}' \left[ T_{p/\hbar}(-\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}') T_{p/\hbar}^*(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}') \right] \rho_S(2') \times \rho_S(1') \quad (67a)$$

where (1') and (2') have a new definition, which is actually nothing but the usual definition for the Boltzmann integral, where the two particles are located at the same point as particle (1):

$$\begin{cases} \mathbf{r}'_{1,2} = \mathbf{r} \\ \mathbf{p}'_{1,2} = \mathbf{p} - \frac{\mathbf{q}}{2} \pm q \frac{\hat{\mathbf{k}}'}{2} \end{cases} \quad (67b)$$

It is convenient to introduce the following (real) "generalized cross sections" (relation (16d) of [12]):

$$\sigma_k^{\text{ex}}(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') - i\tau_k^{\text{ex}}(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') = \frac{4\pi^4 m^2}{\hbar^4} T_k(-\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') T_k^*(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \quad (67c)$$

We can again make use of symmetry ( $\alpha$ ) of Appendix A, which here becomes merely a parity over  $\hat{\mathbf{k}}'$ . This interchanges (1') with (2'), while it is easy to see on (67c) that  $\sigma_k^{\text{ex}}$  is even and  $\tau_k^{\text{ex}}$  is odd. Finally, we get the following result<sup>(7)</sup> for the local and lateral part of the exchange term of  $I_{\text{coll}}$ .

$$\frac{\epsilon}{2} \int d^3 q \frac{q}{m} \int d^2 \hat{\mathbf{k}}' \left\{ \sigma_k^{\text{ex}}(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}') [\rho_S(1'), \rho_S(2')]_+ + i\tau_k^{\text{ex}}(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}') [\rho_S(1'), \rho_S(2')] \right\} \quad (67d)$$

We have now made contact with the calculations of [12], but there is still one last step to make for obtaining exactly the collision integral  $I_{\text{LL}}$  of this reference: for the anticommutators, one has

<sup>(6)</sup> we could also adapt the reasoning of section 3.2 of [1a] and calculate the value of this term without invoking the property of Hermitian conjugation. One should then be careful about the sign change inside the three dimensional delta function which, for this second exchange term, modifies the exponential under

the integral; consequently, after integrating over  $d^3 x'$  and  $d^3 y'$ , which leads to two determinations of  $\hat{\mathbf{k}}'$ ,

$\hat{\mathbf{k}}' = \pm \hat{\mathbf{k}}$ , one then finds when summing over  $d^3 z'$  that the only contribution comes from the opposite values of the two vectors.

<sup>(7)</sup> we recall that  $\mathbf{q} = 2\mathbf{p} = 2\hbar\mathbf{k}$ , see relation (19c), so that all three unit vectors  $\hat{\mathbf{q}}$ ,  $\hat{\mathbf{p}}$  and  $\hat{\mathbf{k}}$  are simply equal.

to use sum rules which are analogous to those used in the Boltzmann collision integral (relation (21b) of [12]), so that one can group in the same integral the forward and lateral scattering terms. Finally, as in [12], the only forward scattering term which remains unpaired with any lateral scattering term is the spin-rotation commutator, and we obtain:

$$I_{\text{coll.}} = I_{\text{LL}} + \dots \quad (68)$$

(where the dots symbolize the non-local corrections). See also the recent work of De Haan [16] for another derivation of the local part of the interaction integral with spin and statistics.

**2.5.2 Non-local corrections** — Again, we do not have to worry about direct terms, which are already known from (64); we just study the exchange terms, and we start with the forward scattering terms. What we have to do is to adapt the calculations of section 4.1 of [1a] to the present case. As in the preceding section, there is only one major difference: each time that a term was associated in [1a] with its complex conjugate, resulting in the introduction of real parts of  $T$ , here we have to reverse the order of spin operators which do not necessarily commute. This is analogous to (66a) and (66b) and results in commutators and anticommutators as in (66c).

The terms in  $\xi_F$  in equation (86) of [1a] then become (including the term for distinguishable particles, already contained in (64):

$$G_{\xi_F} = - \int d^3q \frac{q}{m} \hat{\mathbf{p}} \cdot \left\{ \xi_F(k) f(2) \nabla \rho_S(1) + \epsilon \xi'_F(k) [\nabla \rho_S(1), \rho_S(2)]_+ + i \epsilon \xi''_F(k) [\nabla \rho_S(1), \rho_S(2)] \right. \\ \left. - \xi_F(k) \rho_S(1) \nabla f(2) - \epsilon \xi'_F(k) [\rho_S(1), \nabla \rho_S(2)]_+ - i \epsilon \xi''_F(k) [\rho_S(1), \nabla \rho_S(2)] \right\} \quad (69)$$

where  $\nabla$  is a gradient with respect to the space coordinate, while  $\xi_F$  is given in [1a] and the real numbers  $\xi'_F$  and  $\xi''_F$  are defined by:

$$\xi'_F(k) - i \xi''_F(k) = \frac{\pi^3 m}{\hbar^2} \frac{d}{dk} \left[ \frac{T_k(-1)}{k} \right] \quad (70)$$

Similarly, we can write the terms in  $\Delta_F$  as:

$$G_{\Delta_F} = \int d^3q \frac{q}{m} [\nabla_{\mathbf{p}_1} - \nabla_{\mathbf{p}_2}] \cdot \left\{ \Delta_F(k) \rho_S(1) (\nabla - \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \nabla)) f(2) \right. \\ \left. + \epsilon \Delta'_F(k) [\rho_S(1), (\nabla - \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \nabla)) \rho_S(2)]_+ + i \epsilon \Delta''_F(k) [\rho_S(1), (\nabla - \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \nabla)) \rho_S(2)] \right\} \quad (71)$$

where  $\Delta_F(k)$  is defined in [1a] while the real coefficients  $\Delta'_F(k)$  and  $\Delta''_F(k)$  are given by:

$$\Delta'_F(k) - i \Delta''_F(k) = \frac{2\pi^3 m}{\hbar k} T_k(-1) \quad (72)$$

As for the lateral scattering terms, they are obtained by a procedure which is similar to that of the preceding section: a parity over  $\hat{\mathbf{k}}'$  is used to introduce real quantities with commutators and anticommutators. The term in  $\xi$  is ( $\theta$  is the angle of collision, and  $\cos \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}'$ ):

$$G_{\xi_k(\theta)} = \int d^3q \frac{q}{m} \int d^2\hat{\mathbf{k}}' \\ \hat{\mathbf{k}}' \cdot \left\{ \xi_k(\theta) f(2') \nabla \rho_S(1') + \epsilon \xi'_k(\theta) [\nabla \rho_S(1'), \rho_S(2')]_+ + i \epsilon \xi''_k(\theta) [\nabla \rho_S(1'), \rho_S(2')] \right. \\ \left. - \xi_k(\theta) \rho_S(1') \nabla f(2') - \epsilon \xi'_k(\theta) [\rho_S(1'), \nabla \rho_S(2')]_+ - i \epsilon \xi''_k(\theta) [\rho_S(1'), \nabla \rho_S(2')] \right\} \quad (73)$$

where  $\xi_k(\theta)$  has the same definition as in [1a], while  $\xi'_k(\theta)$  and  $\xi''_k(\theta)$  are given by:

$$\xi'_k(\theta) - i\xi''_k(\theta) = \frac{\pi^4 m^2}{\hbar^4} \left[ i T_k(-\cos \theta) \frac{d}{dk} T_k^*(\cos \theta) \right] \quad (74)$$

The term in  $\Delta_k(\theta)$  is obtained by a similar procedure from equation (86) of [1a] and we get:

$$\begin{aligned} G_{\Delta_k(\theta)} = & \int d^3 q \frac{q}{m} \int d^2 \hat{k}' (1 - \hat{k}' \cdot \hat{p}) \times \\ & (\hat{k}' + \hat{p}) \cdot \{ \Delta_k(\theta) f(2') \nabla \rho_S(1') + \epsilon \Delta'_k(\theta) [\nabla \rho_S(1'), \rho_S(2')]_+ + i \epsilon \Delta''_k(\theta) [\nabla \rho_S(1'), \rho_S(2')] \} \\ & + \int d^3 q \frac{q}{m} \int d^2 \hat{k}' (1 + \hat{k}' \cdot \hat{p}) \times \\ & (\hat{k}' - \hat{p}) \cdot \{ \Delta_k(\theta) \rho_S(1') \nabla f(2') + \epsilon \Delta'_k(\theta) [\rho_S(1'), \nabla \rho_S(2')]_+ + i \epsilon \Delta''_k(\theta) [\rho_S(1'), \nabla \rho_S(2')] \} \end{aligned} \quad (75)$$

where  $\Delta_k(\theta)$  is defined in [1a] while  $\Delta'_k(\theta)$  and  $\Delta''_k(\theta)$  are the real numbers given by:

$$\Delta'_k(\theta) - i \Delta''_k(\theta) = \frac{\pi^4 m^2}{\hbar^4 k} \left[ i T_k(-\cos \theta) \frac{\partial}{\partial \cos \theta} T_k^*(\cos \theta) \right] \quad (76)$$

Finally we find:

$$I_{\text{coll}} = I_{\text{LL}} + G_{\xi_F} + G_{\Delta_F} + G_{\xi_k(\theta)} + G_{\Delta_k(\theta)} + \dots \quad (77)$$

where the terms  $G$  in the right hand side have been written explicitly above, and where the dots symbolize terms which are at least second order in gradients (we shall not study them here).

**2.5.3 Discussion** — It is interesting at this stage to discuss the physical meaning of the various terms that we have written, and to compare them with those contained in the kinetic equation given by Silin [5]. For a general discussion of the limits of applicability of Fermi-like approaches to the study of quantum gases, see the recent article of A.E. Meyerovich [17], in particular the discussion at the end of section IV. Here, we limit ourselves to a comparison between our collision integral and that of reference [5]. Its results are based on a mean field theory where each quasiparticle moves in the average field created by all the others. The motion is therefore Hamiltonian (even if the Hamiltonian is a functional of the distribution function); as a consequence, when the Schrödinger (or Von-Neumann) equation of the density operator is written in terms of the Wigner transform of this operator, one obtains the evolution in the form of a 12 dimensional integral. Mathematically, this is because no trace operation is made over the collision partner. Moreover, when the collision integral is expanded in powers of  $\hbar$ , which amounts to a gradient expansion, one obtains a local term which has the form of a commutator, followed by first order terms which are anticommutators, etc. This is an inherent property of the mean-field Hamiltonian formalism: it does not introduce commutators in first order gradient terms. Another mathematical difference is that Silin does not study the dissipative effects of collisions, but simply adds them by putting a phenomenological collision integral  $\hat{J}$  into the right hand side of the kinetic equation. Generally speaking, the Silin theory makes no attempt to treat the transient short range correlations which occur during binary collisions; it is therefore natural to have significant differences between the two formalisms, so that one can hardly expect to recover exactly the same results. Let us nevertheless compare those particular terms which are similar in the two approaches:

\* our local commutator in  $\tau_{\text{fwd.}}^{\text{ex}}$  in (66c) corresponds to the local term (the commutator) in Silin's equation. They are precisely the molecular field terms which give rise to spin waves in

gases. We refer to [22] for a discussion of the differences and similarities of the results given by a mean-field theory, or an atomic collision approach, for the study of molecular field terms.

\* the next term in Silin's kinetic equation is a "drift term", which contains a space derivative of the distribution function and a momentum derivative of the quasi-particle energy. They correspond to the terms in  $\xi_F$  and  $\xi'_F$  in the first line of equation (69). We have obtained them from the study of retardation effects in collisions; it is natural that they should be associated with changes in the dispersion relation of the quasiparticles in the Landau theory.

\* the last term in the left hand side of Silin's equation is again an anticommutator, containing space derivative of the energy and momentum derivative of the distribution function; in other words, it corresponds to a mean-field force acting on the quasiparticles. In our point of view, these terms have been found as a consequence of the "refraction effects" occurring during atomic collisions discussed in [1a]; they have introduced the terms in  $\Delta_F$  and  $\Delta'_F$  in equation (71). In the latter equation, we nevertheless note the limitation of spatial derivatives to directions that are perpendicular to the relative velocity (defined by the unit vector  $\hat{p}$ ); this is, as discussed in [1a], a direct consequence of energy conservation in atomic collisions. It is clear that, in a mean field-theory, this conservation law, or recoil effects in atomic collisions, are not taken into account: such a theory is better adapted to long range forces and many-particle interactions, where the recoil effects are spread over several collision partners.

We find no correspondence in [5] for many of our collision terms, for example for the commutators terms in  $\xi''_F$  in (69). The first of these terms (that in the first line of the equation) can be described as a direct coupling of spatial variations (rotations) of the magnetization (but not of the magnetization itself) to the magnetization of the collision partner; it would add a correction to the local commutator in  $\tau_{\text{fwd.}}^{\text{ex.}}$  in the calculation of spin wave oscillations. The second, on the second line, is a consequence of the non-local character of the molecular field itself: the spin rotation effect does not depend only on the spin orientation of the collision partner at the same point of space, but also on its value in a neighbourhood of this point. Similarly, the terms in  $\xi_F$  in this second line can be seen as a non-local modification of the cross section which occur in the Boltzmann integral. It would be too long to discuss all collision terms that we have obtained, and we shall just summarize the differences with the results of a mean-field theory:

(i) commutators are also obtained in the non-local terms.

(ii) the lateral scattering terms are treated on the same footing as forward scattering; they include dissipative terms, but also commutators corresponding to identical spin rotation effects, as discussed in [22].

(iii) the conservation of energy in binary collision affects the form of the force terms; it restricts the spatial derivatives to directions which are orthogonal to the relative momentum of the collision.

## Conclusion.

The formalism of the free Wigner transform adapts well to the inclusion of spin and statistics, and leads to results which are in agreement with those already known for a system at equilibrium; the inconsistency between [12] and [13] is therefore lifted. Writing the collision term of a kinetic equation in terms of the free Wigner transform instead of the real Wigner transform has advantages: all terms which appear can be expressed in terms of phase shifts only; this is because only the elements of the  $S$  matrix appear in (35). Of course, the "off-shell effects", which involve matrix elements of the  $T$  matrix between wave vectors of different lengths, do not disappear completely: they have to play a role because, physically, the system always contains particles which are in the

middle of a collision so that they are sensitive to off-shell effects. Nevertheless, they are only contained in the correction to the distribution function, a property which allows one to study each problem separately. In general, there might be technical difficulties in calculating the correction to the distribution function; fortunately, we have seen in section 2.4 that this is not the case at thermal equilibrium, mostly because the entering and the outgoing parts of the free distribution become identical, so that the discontinuity in the plane  $\mathbf{r} \cdot \mathbf{p} = 0$  vanishes.

When compared to a mean-field theory, our model gives the feeling that a more precise treatment of collisions is obtained, at least for short range binary collisions. Recoil effects are included, and more symmetry is gained in the treatment of forward and lateral scattering. All terms can be expressed exactly in terms of the  $T$  matrix, with no phenomenological coefficient. Moreover, bosons and fermions are treated equivalently, as in [12].

Altogether, the present work remains limited to the study of binary effects, for both collisional effects and degeneracy. This weakness is the price that we have paid for a more detailed treatment of collisions. The recent work of Jeon and Mullin based on the kinetic equation that they have introduced in [10b], or approaches based on the Landau kinetic equation [23, 24], do not have this limitation. A correct second virial correction to the pressure from a Landau-Silin type theory has been obtained by Miyake *et al.* (see the appendix of [9]). It would be useful to make a more detailed term-by-term comparison with the results of these references, with a special emphasis on the various contribution to the virial corrections. Using techniques similar to those developed by the authors of reference [25], it might be possible to extend the validity of our theory to systems with a higher degree of degeneracy, but still dilute in terms of interactions. A recent general discussion of perspectives in the field can be found in reference [24c].

### Acknowledgements.

The authors are grateful to W.J. Mullin, A.E. Meyerovich, R.F. Snider and M. de Haan for numerous useful discussions. The authors belong to the "Laboratoire de Spectroscopie Hertzienne" which is a Laboratoire associé au CNRS (UA 18), à l'Université P. et M. Curie, et à l'Ecole Normale Supérieure.

### Appendix A

In this appendix, we briefly study the symmetries of expressions such as (18), which repeatedly occur in the right hand side of the equations of this article. Equations (11b) and (11c), as well as (19b), give the definitions of the variables inside integral.

( $\alpha$ ) Parity over the primed variables:  $\mathbf{r}'$ ,  $\mathbf{k}'_1$ ,  $\mathbf{k}'_2$ ; this operation exchanges (1') with (2'), the first (direct) term with the second (double exchange), and the third with the fourth (the two simple exchange terms). This is why expressions like (21), (24), and (26) in section 1, or (35) and (37) in section 2, have been simplified and contain in their right hand side two terms instead of four.

( $\beta$ ) General parity over  $\mathbf{r}$ ,  $\mathbf{k}_+$ ,  $\mathbf{k}_-$  (and therefore  $\kappa$ ) as well as  $\mathbf{r}'$ ,  $\mathbf{k}'_1$  and  $\mathbf{k}'_2$  (and therefore  $\kappa$ ); because of rotational invariance:

$$S(\mathbf{k}, \mathbf{k}') = S(-\mathbf{k}, -\mathbf{k}') \quad (\text{A1})$$

nothing is changed in the 18 dimension integral, except the sign of  $\mathbf{r}$  as well as an exchange of (1') and (2'). This is equivalent to exchanging the numbering of the two particles.

( $\gamma$ ) The combination of the two preceding operations gives the parity over unprimed variables, with no exchange of (1') and (2').

(δ) Parity over  $\kappa$  and  $\kappa'$ , which corresponds to:

$$\mathbf{k}_+ \longleftrightarrow \mathbf{k}_- \quad \mathbf{k}'_1 \longleftrightarrow \mathbf{k}'_2 \tag{A2}$$

This operation does not exchange (1') and (2'); it changes the first term into its own hermitian conjugate, the second also, but the third into the hermitian conjugate of the fourth (and conversely). This operation can be used to show the hermiticity of the direct term of expression (35) of the collision integral.

(ε) Combining (α) and (δ) amounts to a parity of  $\mathbf{r}'$  and of  $\kappa$  (but not of  $\kappa'$ ) with the substitutions:

$$\mathbf{k}_+ \longleftrightarrow \mathbf{k}_- \quad \mathbf{k}'_{1,2} \Rightarrow -\mathbf{k}'_{2,1} \tag{A3}$$

which interchanges (1') and (2'). The operation turns each exchange term into its hermitian conjugate. It can be used to show the hermitian character of the exchange term of  $I_{\text{coll}}$  or of similar expressions, such as (37) for example.

### Appendix B

The real two-particle distribution function  $\rho_{\text{SS}}^{\text{II}}(1, 2)$  can also be obtained by a calculation which is similar to that of section 1.3.1; the only difference is that a trace over particle 2 is not necessary. One gets in this way:

$$\begin{aligned} \rho_{\text{SS}}^{\text{II}}(1, 2) = (2\pi)^{-3} \int_{12} e^{i(\kappa \cdot \mathbf{r} - \kappa' \cdot \mathbf{r}')} \left\{ \Omega(\mathbf{k}_+, \mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) \rho_{\text{S}}^{(+)}(1') \otimes \rho_{\text{S}}^{(+)}(2') \right. \\ \left. + \epsilon \Omega(\mathbf{k}_+, -\mathbf{k}'_1) \Omega^*(\mathbf{k}_-, \mathbf{k}'_2) P_{\text{S}}^{\text{ex}} \rho_{\text{S}}^{(+)}(2') \otimes \rho_{\text{S}}^{(+)}(1') \right\} \tag{B1} \end{aligned}$$

where the integration variables are defined in (11a), while the variables (1') and (2') in the distribution function are those of (11d). The direct term is a tensor product of operators, as emphasized by the sign  $\otimes$ ; in the absence of trace over the spin states of the collision partner, one can not as in (26) or (37) write the exchange term with an ordinary product of operators; this is why the exchange operator  $P_{\text{S}}^{\text{ex}}$  has to be present in the equation.

This formula is the generalization of equation (B25) of reference [1b] ; see also the results obtained by de Haan in section 5 and equation (5.2) of [11b] .

### Appendix C

Relation (32a), when inserted into (31), can be used to make the connection between the Snider collision integral [20] and our form of  $I_{\text{w}}$ . In the 18 dimension integral, the sum over  $d^3r'$  with the exponential  $e^{-i\kappa' \cdot \mathbf{r}'}$  changes the Wigner transform of the free entering part into its matrix elements between  $\langle \mathbf{k}'_1 |$  and  $| \mathbf{k}'_2 \rangle$  (with a coefficient  $\hbar^3$ , see for example formula (A6) of Ref. [1a] ); as already noted, the transforms occur only in the space of the relative distance between the two particles, while the variables associated with the centre of mass remain unchanged. Then, writing the relation:

$$\left[ V, |\Psi_{\pm \mathbf{k}'_1}^{(+)} \rangle \langle \Psi_{\pm \mathbf{k}'_2}^{(+)} | \right] = \left[ V, \Omega^{(+)} | \mathbf{k}'_1 \rangle \langle \mathbf{k}'_2 | [\Omega^{(+)}]^\dagger \right] \tag{C1}$$

we can introduce two closure relations over the bases  $|k'_1\rangle$  and  $|k'_2\rangle$ , which absorbs the integrations over the variables  $k'_1$  and  $k'_2$ . The integration over  $d^3\kappa$  is then used to come back to a Wigner transform; this introduces a coefficient  $(2\pi\hbar)^{-3}$ , see for example formula (A1) of [1a], which, with the preceding coefficient  $\hbar^3$ , cancels the factor  $(2\pi)^{-3}$  of equation (31). Finally the sums over  $d^3r$  and  $d^3q$  introduce a trace over the orbital variables of the collision partner, so that one obtains:

$$I_W = \frac{1}{i\hbar} T.W. \left\{ \text{Tr}_2 \left[ V, \Omega^{(+)} [1 + \epsilon P_{ex.}] [\rho_{(+)}(1) \otimes \rho_{(+)}(2)] [\Omega^{(+)}]^\dagger \right] \right\} \quad (C2)$$

where  $\text{Tr}_2$  implies a trace over orbital and spin variables. This relation is very reminiscent of the Snider collision integral [20], but here the right hand side contains the free entering part of the two-body density operator (equation (4)), instead of the product of one-body operators; in a more recent article [14], Snider has also used free distributions in order to extend the validity of the collision integral.

### References

- [1] (a) LALOË F., *J. Phys. France* **50** (1989) 1851;  
 (b) TASTEVIN G., NACHER P.J. et LALOË F., *J. Phys. France* **50** (1989) 1979.  
 (c) NACHER P.J., TASTEVIN G. et LALOË F., *J. Phys. France* **50** (1989) 1907.  
 (d) NACHER P.J., TASTEVIN G. et LALOË F., to be published in *Annalen der Physik*
- [2] BETTS D.S., LALOË F. and LEDUC M., *Progress in Low Temperature Physics*, vol. XII, chap. 2, D.F. Brewer Ed. (Elsevier) 1989.
- [3] GREYTAK T. and KLEPPNER D., *Lectures on spin-polarized hydrogen*, in *New Trends in Atomic Physics*, G. Grynberg and R. Stora Ed. Les Houches Summer School (Elsevier) 1984.
- [4] SILVERA I.F. and WALRAVEN J.T.M., *Spin polarized atomic hydrogen*, in *Progress in Low Temperature Physics*, Vol. X, D.F. Brewer Ed. (Elsevier) 1986.
- [5] SILIN V.P., *J. Eksp. Teor. Phys.* **33**, (1957) 1227 *Sov. Phys. JETP* **6**, (1958) 945.
- [6] MEYEROVICH A.E. in *Helium Three*, chap. 14, W.P. Halperin and L.P. Pitaevskii Eds. (North Holland) 1990; *J. Low Temp. Phys.*, **53**, (1983) 487.
- [7] RUCKENSTEIN A.E. and LÉVY L.P., *Phys. Rev.* **39B**, (1989) 183.
- [8] BASHKIN E.P., *Pis'ma Zh. Teor. Fiz.* **33**, (1981) 8; *JETP Lett.* **33**, (1981) 8; *Zh. Eksp. Teor. Fiz.* **86**, (1984) 937; (*Sov. Phys. JETP* **59**, (1984) 547); *Usp. Fiz. Nauk.* **148**, (1986) 433; *Sov. Phys. Usp.* **29**, (1986) 238.
- [9] MIYAKE K., MULLIN W.J. and STAMP P.C.E., *J. Phys. France* **46**, (1965) 663.
- [10] JEON J.W. and MULLIN W.J., (a) *J. Low Temp. Phys.* **47**, (1987) 421;  
 (b) *J. Phys. France*, **49**, (1988) 1691;  
 (c) *Phys. Rev. Lett.* **62**, (1989) 2691.
- [11] DE HAAN M., (a) *Physica A* **158**, (1989) 559;  
 (b) *Physica A* **164**, (1990) 373.  
 (c) A comparison between approaches for deriving kinetic equations for a dilute inhomogeneous quantum gas, preprint.
- [12] LHUILLIER C. and LALOË F., *J. Phys. France* **43**, (1982) 197 and 225.
- [13] LHUILLIER C. et LALOË F., *J. Phys. France* **40**, (1979) 239.
- [14] SNIDER R.F., *J. Stat. Phys.*, **61**, (1990) 439.
- [15] SNIDER R.F., A reinterpretation of dense gas kinetic theory, preprint.
- [16] de HAAN M., *Physica A* **165**, (1990) 224.
- [17] MEYEROVICH A.E., *Phys. Rev. B*, **39**, (1989) 9318.
- [18] JEON J.W. and GLYDE H., Spin dynamics in a polarized Bose gas, preprint.
- [19] JEON J.W. and MULLIN W., Spin diffusion in dilute, polarized helium three-four solutions, submitted to *Phys. Rev. Lett.*
- [20] SNIDER R.F., *J. Chem. Phys.* **32**, (1960) 1051.

- [21] LALOË F. and MULLIN W.J., *J. Stat. Phys.* **59**, (1990) 725.  
[22] LHUILLIER C. and LALOË F., *Phys. Rev. Lett.* **54**, (1985) 1207.  
[23] BASHKIN E.P. and MEYEROVICH A.E., *Adv. Phys.* **30**, (1981) 1.  
[24] MEYEROVICH A.E., (a) Spin polarized helium three helium four solutions, *Prog. Low. Temp. Phys.*, **11**, (1987) 1;  
(b) Spin polarized phases of helium three, chap. 14 in *Helium Three*, W.P. Halperin and L.P. Pitaevski Eds. (North Holland) 1990.  
(c) Perspectives in the theory of spin-polarized quantum systems: beyond the standard Boltzmann equation, contribution to the LT19 conference in Brighton (1990).  
[25] BOERCKER D.B. and DUFTY J.W., *Ann. Phys.* **119**, (1979) 43; *J. Stat. Phys.* **57**, (1989) 827.

Cet article a été imprimé avec le Macro Package "Editions de Physique Avril 1990".