Exponential growth of Lie algebras of finite global dimension

Yves Félix, Steve Halperin, Jean-Claude Thomas

An asymptotic formula for the ranks of the homotopy groups of a finite complex

Yves Felix, Steve Halperin and Jean-Claude Thomas

September 23, 2005

Abstract

Let X be a finite simply connected CW complex of dimension n. The loop space homology $H_*(\Omega X;\mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra L_X isomorphic with $\pi_{*-1}(X)\otimes\mathbb{Q}$. Let $Q_X\subset L_X$ be a minimal generating subspace, and set $\alpha = \limsup_i \frac{\log \operatorname{rk}\pi_i(X)}{i}$. Theorem:
If $\dim L_X=\infty$ and $\limsup(\dim\,(Q_X)_k)^{1/k} < \limsup(\dim\,(L_X)_k)^{1/k}$, then

$$\sum_{i=1}^{n-1}\operatorname{rk}\pi_{k+i}(X) = e^{(\alpha+\varepsilon_k)k}\,, \qquad \text{where}\,\varepsilon_k\to 0 \text{ as}\, k\to\infty\,.$$

In particular $\sum_{i=1}^{n-1}\operatorname{rk}\pi_{k+i}(X)$ grows exponentially in k.

AMS Classification : 55P35, 55P62, 17B70

Key words : Homotopy Lie algebra, graded Lie algebra, exponential growth

1 Introduction

Suppose X is a finite simply connected CW complex of dimension n. The homotopy groups of X then have the form

$$\pi_i(X) = \mathbb{Z}^{\rho_i}\oplus T_i\,,$$

where T_i is a finite abelian group and $\rho_i = \operatorname{rk}\pi_i(X)$ is finite. It is known [6] that either $\pi_i(X)=T_i$, $i\geq 2n$ (X is *rationally elliptic*) or else for all $k\geq 1$, $\sum_{i=1}^{n-1}\operatorname{rk}\pi_{k+i}(X)>0$. In this case X is called *rationally hyperbolic.*

In [7] it is shown that in the rationally hyperbolic case $\sum_{i=1}^{n-1} \mathrm{rk}\, \pi_{k+i}(X)$ grows faster than any polynomial in k. Here we show that with an additional hypothesis this sum grows exponentially in k and, in fact, setting $\alpha = \limsup \frac{\log \mathrm{rk}\, \pi_i}{i}$ we have

$$\sum_{i1}^{n-1} \mathrm{rk}\, \pi_{k+i} = e^{(\alpha+\varepsilon_k)k}\,, \qquad \text{where}\, \varepsilon_k \to 0 \text{ as } k \to \infty\,.$$

In subsequent papers we will identify a large class of spaces for which the additional hypothesis holds: in fact it may well hold for all finite simply connected CW complexes.

Note that $\mathrm{rk}\, \pi_i(X) = \dim \pi_i(X) \otimes \mathbb{Q}$. Thus we work more generally with simply connected spaces X such that each $H_i(X;\mathbb{Q})$ is finite dimensional. In this case $\dim \pi_i(X) \otimes \mathbb{Q}$ is also finite for each i. On the other hand, a theorem of Milnor-Moore-Cartan-Serre asserts that the loop space homology $H_*(\Omega X;\mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra L_X and that the Hurewicz homomorphism is an isomorphism $\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\cong} L_X$. Since there are natural isomorphisms $\pi_i(X) \cong \pi_{i-1}(\Omega X)$ it follows that the results above can be phrased in terms of the integers $\dim\, (L_X)_i$.

For any graded vector space V concentrated in positive degrees we define the *logarithmic index* of V by

$$\log\, \mathrm{index}\, V = \limsup \frac{\log \dim V_k}{k}\,.$$

In ([5], [6]) it is shown that if X (simply connected) has finite Lusternik-Schnirelmann category (in particular, if X is a finite CW complex) and if X is rationally hyperbolic then

$$\log\, \mathrm{index}\, L_X > 0\,.$$

Now let Q_X denote a minimal generating subspace for the Lie algebra L_X, and let α denote $\log\, \mathrm{index}\, L_X$.

Theorem 1. *Let X be a simply connected topological space with finite dimensional rational homology concentrated in degrees $\leq n$. Suppose $\log\, \mathrm{index}\, Q_X < \log\, \mathrm{index}\, L_X$, then*

$$\sum_{i=1}^{n-1} \dim\, (L_X)_{k+i} = e^{(\alpha+\varepsilon_k)k}\,, \qquad \text{where}\, \varepsilon_k \to 0 \text{ as}\, k \to \infty\,.$$

In particular this sum grows exponentially in k.

Theorem 2. *Let X be a simply connected topological space with finite dimensional rational homology in each degree, and finite Lusternik-Schnirelmann category. Suppose log index $Q_X <$ log index $L_X < \infty$. Then for some $d > 0$,*

$$\sum_{i=1}^{d-1} \dim (L_X)_{k+i} = e^{(\alpha+\varepsilon_k)k}\,, \qquad \text{where}\, \varepsilon_k \to 0 \text{ as}\, k \to \infty\,.$$

In particular, $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$ grows exponentially in k.

Corollary: The conclusion of Theorem 1 holds for finite simply connected CW complexes X for which L_X is infinite, but finitely generated. The conclusion of Theorem 2 holds for simply connected spaces of finite LS category and finite rational Betti numbers provided that L_X is infinite, but finitely generated, and, log index $L_X < \infty$.

Recall that the depth of a graded Lie algebra L is the least m (or ∞) such that $\mathrm{Ext}^m_{UL}(\mathbb{Q}, UL) \neq 0$.

The key ingredients in the proofs of Theorems 1 and 2 are

- A growth condition for L_X established in ([5],[6])
- The fact that depth $L_X < \infty$, established in ([3],[6])

We shall use Lie algebra arguments in Theorem 3 below to deduce the conclusion of Theorems 1 and 2 from these ingredients, and then deduce Theorems 1 and 2.

Theorems 1 and 2 may be compared with the results in [7] and in [8] that assert for n-dimensional finite CW complexes (respectively for simply connected spaces with finite type rational homology and finite Lusternik-Schnirelmann category) that $\sum_{i=1}^{n-1} \dim (L_X)_{k+i}$ (resp. $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$) grows faster than any polynomial in k. These results use only the fact that L_X has finite depth, and require no hypothesis on Q_X.

The hypothesis on Q_X in Theorems 1 and 2 may be restated as requiring that the formal series $\sum_q \dim \mathrm{Tor}_{1,q}^{UL_X} z^q$ have a radius of convergence strictly greater than that of the formal series $\sum_q \dim (UL_X)_q z^q$. Lambrechts [9] has proved a much stronger result under the hypothesis that the formal series $\sum_q \left(\sum_p (-1)^p \dim \mathrm{Tor}_{p,q}\right) z^q$ has a radius of convergence strictly larger than that of $\sum_q \dim (UL_X)_q z^q$.

2 Lie algebras

In this section we work over any ground field $\Bbbk$ of characteristic different from 2; graded Lie algebras L are defined as in [7] and, in particular, are assumed to satisfy $[x,[x,x]] = 0$, $x \in L_{\mathrm{odd}}$ (This follows from the Jacobi identity except when char $\Bbbk = 3$).

A graded Lie algebra L is connected and of finite type if

$$L = \{L_i\}_{i\geq 1} \text{ and each } L_i \text{ is finite dimensional.}$$

We shall refer to these as *cft Lie algebras.* The minimal generating subspaces Q of a cft Lie algebra L are those subspaces Q for which $Q \to L/[L,L]$ is a linear isomorphism.

A *growth sequence* for a cft Lie algebra L is a sequence (r_i) such that $r_i \to \infty$ and

$$\lim_{i\to\infty} \frac{\log \dim L_{r_i}}{r_i} = \log \operatorname{index} L\,.$$

A *quasi-geometric sequence* (ℓ_i) is a sequence such that for some integer m, $\ell_i < \ell_{i+1} \leq m\ell_i$, all i; if additionally (ℓ_i) is a growth sequence then it is a *quasi-geometric growth sequence.*

Of particular interest here are the growth conditions

$$0 < \log \operatorname{index} L < \infty\,; \tag{A.1}$$

and

$$\log \operatorname{index} Q < \log \operatorname{index} L\,. \tag{A.2}$$

Proposition. *Let L be a cft Lie algebra satisfying (A.1) and (A.2), and assume L has a quasi-geometric growth sequence (r_j). Then any sequence (s_i) such that $s_i \to \infty$ has a subsequence (s_{i_j}) for which there are growth sequences (t_j) and (p_j) such that*

$$t_j \leq s_{i_j} < p_j \qquad \text{and} \quad p_j/t_j \to 1\,.$$

Proof. First note that because of (A.2),

$$\log \operatorname{index} L/Q = \log \operatorname{index} L$$

Now adopt the following notation, for $i \geq 1$:

$$\left.\begin{array}{l}\log \operatorname{index} L/Q = \alpha \\ \dim L_i e^{(\alpha+\varepsilon_i)i} \\ \dim Q_i = e^{(\alpha+\sigma_i)i} \\ \dim L_i/Q_i = e^{(\alpha+\tau_i)i} \\ \dim (UL)_i = e^{(\alpha+\delta_i)i}\end{array}\right\} \tag{1}$$

Then because of (A.1) and (A.2), $0 < \alpha < \infty$, and $\limsup \varepsilon_i = 0$. Moreover, by a result of Babenko ([2], [6]), $\log \operatorname{index} UL = \log \operatorname{index} L$, and so $\limsup(\delta_i) = 0$. Finally (A.2) implies that $\limsup(\sigma_i) = \sigma < 0$, and that $\tau_{q_i} \to 0$ as $i \to \infty$.

Next, since r_j is a quasi-geometric sequence, for some fixed m we have $r_j < r_{j+1} \leq m r_j$, all j. It follows that for each s_i in our sequence we may choose q_i in the sequence (r_j) so that

$$s_i < q_i \leq m s_i\,. \tag{2}$$

Thus $q_i \to \infty$ as $i \to \infty$. The adjoint representation of UL in L defines surjections

$$\bigoplus_{(\ell,k,t)\in\mathcal{J}_i} (UL)_\ell \otimes Q_k \otimes L_t \twoheadrightarrow \bigoplus_{(\ell,k,t)\in\mathcal{J}_i} (UL)_\ell \otimes [Q_k, L_t] \twoheadrightarrow L_{q_i}/Q_{q_i}\,, \tag{3}$$

where $\mathcal{J}_i$ consists of those triples for which $\ell + k + t = q_i$ and $t \leq s_i < t + k$. Thus

$$3(q_i+1) \max_{(\ell,k,t)\in\mathcal{J}_i} \dim (UL)_\ell \,\dim [Q_k, L_t] \geq \dim L_{q_i}/Q_{q_i}\,.$$

For each s_i we may therefore choose $(\ell_i, k_i, t_i) \in \mathcal{J}_i$ so that

$$3(q_i+1) \dim (UL)_{\ell_i} \dim [Q_{k_i}, L_{t_i}] \geq \dim L_{q_i}/Q_{q_i}\,. \tag{4}$$

Lemma 1. *$(k_i+t_i) \geq \frac{1}{m}\, q_i$, and $(k_i+t_i) \geq \frac{1}{m-1}\ell_i$. In particular, $k_i+t_i \to \infty$ as $i \to \infty$.*

Proof. It follows from (2) that $k_i + t_i \geq s_i \geq \frac{1}{m}\, q_i$. Since $q_i = k_i + t_i + \ell_i$, $(k_i + t_i) \geq \frac{1}{m-1}\ell_i$. Since $s_i \to \infty$ so does $k_i + t_i$. □

Next, define $\varepsilon(k_i, t_i)$ by

$$\dim [Q_{k_i}, L_{t_i}] = e^{[\alpha+\varepsilon(k_i,t_i)](k_i+t_i)}\,.$$

Lemma 2. *$\varepsilon(k_i, t_i) \to 0$ and $\varepsilon_{k_i+t_i} \to 0$ as $i \to \infty$.*

Proof. Define $\epsilon(q_i)$ by $3(q_i+1) = e^{\varepsilon(q_i)q_i}$. Then (4) reduces to

$$\varepsilon(q_i)q_i + (\alpha + \delta_{\ell_i})\ell_i + [\alpha + \varepsilon(k_i, t_i)](k_i + t_i) \geq (\alpha + \tau_{q_i})q_i\,.$$

Since $\ell_i + k_i + t_i = q_i$,

$$\varepsilon(q_i)\frac{q_i}{k_i + t_i} + \delta_{\ell_i}\frac{\ell_i}{k_i + t_i} + \varepsilon(k_i, t_i) \geq \tau_{q_i}\frac{q_i}{k_i + t_i}\,.$$

Now as $i \to \infty$, $q_i \geq s_i \to \infty$. Thus $\varepsilon(q_i) \to 0$. Moreover, since q_i belongs to a growth sequence, $\varepsilon_{q_i} \to 0$ and $\tau_{q_i} \to 0$. Use Lemma 1 to conclude that

$$\varepsilon(q_i)\frac{q_i}{k_i + t_i} \to 0 \qquad \text{and} \qquad \tau_{q_i}\frac{q_i}{k_i + t_i} \to 0\,,$$

and hence

$$\liminf \varepsilon(k_i, t_i) \geq -\limsup \delta_{\ell_i}\frac{\ell_i}{k_i + t_i}\,.$$

Next, since $\limsup\, \delta_{\ell_i} = 0$ and $\frac{\ell_i}{k_i+t_i} \leq m - 1$, it follows that

$$-\limsup \delta_{\ell_i}\frac{\ell_i}{k_i + t_i} = 0\,.$$

Finally, $[Q_{k_i}, L_{t_i}]$ embeds in $L_{k_i+t_i}/Q_{k_i+t_i}$, and it follows that

$$\varepsilon_{k_i+t_i} \geq \varepsilon(k_i, t_i)\,.$$

But $k_i+t_i \to \infty$ and so $\limsup\, \varepsilon_{k_i+t_i} \leq 0$. This, together with $\liminf\, \varepsilon(k_i, t_i) \geq 0$, completes the proof of the lemma. $\square$

Next, since $Q_{k_i} \otimes L_{t_i} \to [Q_{k_i}, L_{t_i}]$ is surjective we have

$$\sigma_{k_i}k_i + \varepsilon_{t_i}t_i \geq \varepsilon(k_i, t_i)(k_i + t_i) \tag{5}$$

Lemma 3. *$t_i/k_i \to \infty$ and $t_i \to \infty$ as $i \to \infty$.*

Proof. If t_i/k_i does not converge to ∞ then we would have $t_{i_\nu}/k_{i_\nu} \leq T$ for some subsequence (s_{i_ν}). But

$$\sigma_{k_{i_\nu}} \geq -\varepsilon_{t_{i_\nu}}\frac{t_{i_\nu}}{k_{i_\nu}} + \varepsilon(k_{i_\nu}, t_{i_\nu})(1 + t_{i_\nu}/k_{i_\nu})\,.$$

Since (Lemma 1) $k_{i_\nu}+t_{i_\nu}\to\infty$ and (Lemma 2) $\varepsilon(k_{i_\nu},t_{i_\nu})\to 0$, the lim inf of the right hand side of this equation would be ≥ 0. Hence $\limsup \sigma_{k_{i_\nu}}\geq 0$, which would contradict $\limsup\sigma_i<0$. Finally, since $t_i+k_i\to\infty$ it follows that $t_i\to\infty$. $\square$

Lemma 4. *Write $k_i\lambda_i t_i$. Then,*

$$\lambda_i\to 0 \quad \textit{and} \quad \varepsilon_{t_i}\to 0 \qquad\qquad \textit{as} \quad i\to\infty\,.$$

Proof. Lemma 3 asserts that $\lambda_i\to 0$. Rewrite equation (5) as

$$\varepsilon_{t_i}\geq(\varepsilon(k_i,t_i)-\sigma_{k_i})\lambda_i+\varepsilon(k_i,t_i)\,. \tag{6}$$

Since $\limsup\sigma_j$ is finite, the σ_j are bounded above. Thus $-\sigma_j\geq A$, some constant A. Since $\lambda_i\to 0$, $\liminf(-\sigma_{k_i}\lambda_i)\geq 0$. Since (Lemma 2) $\varepsilon(k_i,t_i)\to 0$ it follows from (6) that $\liminf\varepsilon_{t_i}=0$. But $\limsup\varepsilon_{t_i}\leq\limsup\varepsilon_i=0$ and so $\varepsilon_{t_i}\to 0$. $\square$.

The lemmas above establish the Proposition. Simply set $p_i=t_i+k_i$ and note that $t_i\to\infty$ (Lemma 3), $p_i\to\infty$ (Lemma 1), $p_i/t_i=1+\lambda_i\to 1$ (Lemma 4). Furthermore $\varepsilon_{t_i}\to 0$ (Lemma 4) and $\varepsilon_{p_i}\to 0$ (Lemma 2). Thus (t_i) and (p_i) are growth sequences. $\square$

Theorem 3. *Let L be a cft Lie algebra of finite depth and satisfying the growth conditions (A.1) and (A.2). Set $\alpha=\log\operatorname{index}L$. Then for some d,*

$$\sum_{i=1}^{d-1}\dim L_{k+i}=e^{(\alpha+\varepsilon_k)k}\,,\qquad\quad \text{where}\,\varepsilon_k\to 0\,\text{as}\,k\to\infty\,.$$

In particular, this sum grows exponentially in k.

Proof. According to [4] there is a finitely generated sub Lie algebra $E\subset L$ such that $\mathrm{Ext}^r_{UL}(\Bbbk,UL)\to\mathrm{Ext}^r_{UE}(\Bbbk,UL)$ is non-zero.

Lemma 5. *The centralizer, Z, of E in L is finite dimensional.*

Proof. Since E has finite depth, $Z\cap E$ is finite dimensional [3]. Choose k so $Z\cap E$ is concentrated in degrees $<k$. Suppose $x\in Z_{\geq k}$ has even degree, and put $F=\Bbbk x\oplus E$. Then $\mathrm{Ext}_{UF}(\Bbbk,UF)\to\mathrm{Ext}_{UE}(\Bbbk,UF)$ is zero, contradicting the hypothesis that the composite

$$\mathrm{Ext}^r_{UL}(\Bbbk,UL)\to\mathrm{Ext}^r_{UF}(\Bbbk,UL)\to\mathrm{Ext}^r_{UE}(\Bbbk,UL)$$

is non-zero.

It follows that $Z_{\geq k}$ is concentrated in odd degrees, hence an abelian ideal in $Z+E$. Again

$$\mathrm{Ext}^r_{UL}(\Bbbk, UL) \to \mathrm{Ext}^r_{U(Z+E)}(\Bbbk, UL) \to \mathrm{Ext}^r_{UE}(\Bbbk, UL)$$

is non-zero. Thus $\mathrm{Ext}^r_{U(Z+E)}(\Bbbk, UL) \neq 0$, $Z+E$ has finite depth and every abelian ideal in $Z+E$ is finite dimensional. $\square$

Choose d so that E is generated in degrees $\leq d-1$. As in the Proposition, set $\log \operatorname{index} L = \log \operatorname{index} L/Q = \alpha$. If the theorem fails we can find a sequence $s_i \to \infty$ such that

$$\sum_{j=1}^{d-1} \dim L_{s_i+j} \leq e^{(\alpha-\beta)s_i}\,, \tag{7}$$

some $\beta > 0$. Apply the Proposition to find growth sequences t_i and p_i such that $t_i \leq s_i < p_i$ and $p_i/t_i \to 1$.

We now use (7) to prove that

$$\dim (UE)_{(s_i-t_i, s_i-t_i+d)} \dim L_{(s_i,s_i+d)} < \dim L_{t_i}\,, \qquad i \text{ large}\,. \tag{8}$$

In fact since $\limsup (\dim (UL)_i)^{1/i} = e^\alpha$ it follows that for some $\gamma > 0$,

$$\sum_{j=1}^{d-1} \dim (UE)_{j+k} \leq e^{\gamma(k+1)}\,, \qquad \text{all } k\,.$$

Thus it is sufficient to show that

$$\gamma(s_i - t_i + 1) + (\alpha-\beta)s_i < (\alpha + \varepsilon_{t_i})t_i\,, \qquad \text{large } i\,,$$

where $\varepsilon_{t_i} \to 0$ as $i \to \infty$. Write $s_i = \mu_i t_i$; then $\mu_i \to 1$ and the inequality reduces to the obvious

$$\gamma/t_i + \gamma(\mu_i - 1) + (\alpha-\beta)\mu_i < (\alpha + \varepsilon_{t_i})\,, \qquad \text{large}\, i\,.$$

Thus (8) is established.

Choose $s = s_i$, $t = t_i$ so that (8) holds and so that $Z_j = 0$, $j \geq t$. Write $s - t = k$. The adjoint action of UL in L restricts to a linear map

$$\left[\oplus_{j=1}^{d-1}(UE)_{k+j}\right] \otimes L_t \to \oplus_{j=1}^{d-1} L_{s+j}$$

and it follows from (8) that for some non-zero $x \in L_t$,

$$(\mathrm{ad}\, a)x = 0\,, \qquad a \in UE_{(k,k+d)}\,.$$

On the other hand, since E is generated in degrees $\leq d-1$, $(UE)_{>k} = UE \cdot (UE)_{(k,k+d)}$. Thus $(UE)_{>k} \cdot x = 0$ and so $(UE) \cdot x$ is finite dimensional. A non-zero element y of maximal degree in $UE \cdot x$ satisfies

$$[a, y] = (\mathrm{ad}\, a)(y) = 0\,, \qquad a \in E\,,$$

i.e. $y \in Z$ in contradiction to $Z_{\geq t} = 0$. This completes the proof of the Theorem. □

3 Proof of Theorems 1 and 2

Proof of Theorem 2: We show that L_X satisfies the hypothesis of Theorem 3. Since depth $L_X < \infty$ ([3]) and (A.2) holds by hypothesis we have only to construct a quasi-geometric growth sequence (r_i).

Let $\alpha = \log \mathrm{index}\, L_X$. Then $\alpha > 0$ by [5]. Choose a sequence

$$u_1 < u_2 < \cdots$$

such that $(\dim (L_X)_{u_i})^{1/u_i} \to e^\alpha$.

Next, suppose $\mathrm{cat}\, X = m$ and put $a \left(\dfrac{1}{2(m+1)}\right)^{m+1}$. By starting the sequence at some u_j we may assume $\dim (L_X)_{u_i} > \frac{1}{a}$, all i. Thus the formula in ([5], top of page 189) gives a sequence

$$u_i = v_0 < v_1 \cdots < v_k u_{i+1}$$

such that $v_{i+1} \leq 2(m+1)v_i$ and

$$\left(\dim (L_X)_{v_j}\right)^{\frac{1}{v_j+1}} \geq [a \dim (L_X)_{v_0}]^{\frac{1}{v_0+1}}\,, \qquad j < k\,.$$

Since $v_0 = u_i$ and $u_i \to \infty$ it follows that $a \frac{1}{v_0+1} \to 1$ as $i \to \infty$. Hence interpolating the sequences u_i with the sequences v_j gives a quasi-geometric growth sequence (r_j). □

Proof of Theorem 1: A theorem of Adams-Hilton [1] shows that

$$UL_X = H_*(\Omega X; \mathbb{Q}) = H(TV, d)$$

where TV is the tensor algebra on V and $V_i \cong H_{i+1}(X;\mathbb{Q})$. Thus V is finite dimensional. Since TV has a strictly positive radius of convergence so do $H(TV,d)$ and L_X :

$$\log \operatorname{index} L_X < \infty\,.$$

Thus X satisfies the hypotheses of Theorem 2. The fact that d can be replaced by n is proved by Lambrechts in [10]. $\square$

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Université Catholique de Louvain, 1348, Louvain-La-Neuve, Belgium
University of Maryland, College Park, MD 20742-3281,USA
Université d'Angers, 49045 Bd Lavoisier, Angers, France