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An asymptotic formula for the ranks of the homotopy groups of a finite complex

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Abstract

Let X be a finite simply connected CW complex of dimension n . The loop space homology $H_*(\Omega X; \mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra L_X isomorphic with $\pi_{*-1}(X) \otimes \mathbb{Q}$. Let $Q_X \subset L_X$ be a minimal generating subspace, and set $\alpha = \limsup_i \frac{\log \text{rk} \pi_i(X)}{i}$. Theorem:

If $\dim L_X = \infty$ and $\limsup(\dim(Q_X)_k)^{1/k} < \limsup(\dim(L_X)_k)^{1/k}$, then

$$\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X) = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In particular $\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X)$ grows exponentially in k .

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1 Introduction

Suppose X is a finite simply connected CW complex of dimension n . The homotopy groups of X then have the form

$$\pi_i(X) = \mathbb{Z}^{\rho_i} \oplus T_i,$$

where T_i is a finite abelian group and $\rho_i = \text{rk} \pi_i(X)$ is finite. It is known [6] that either $\pi_i(X) = T_i$, $i \geq 2n$ (X is *rationally elliptic*) or else for all $k \geq 1$, $\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X) > 0$. In this case X is called *rationally hyperbolic*.

In [7] it is shown that in the rationally hyperbolic case $\sum_{i=1}^{n-1} \text{rk } \pi_{k+i}(X)$ grows faster than any polynomial in k . Here we show that with an additional hypothesis this sum grows exponentially in k and, in fact, setting $\alpha = \limsup \frac{\log \text{rk } \pi_i}{i}$ we have

$$\sum_{i=1}^{n-1} \text{rk } \pi_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In subsequent papers we will identify a large class of spaces for which the additional hypothesis holds: in fact it may well hold for all finite simply connected CW complexes.

Note that $\text{rk } \pi_i(X) = \dim \pi_i(X) \otimes \mathbb{Q}$. Thus we work more generally with simply connected spaces X such that each $H_i(X; \mathbb{Q})$ is finite dimensional. In this case $\dim \pi_i(X) \otimes \mathbb{Q}$ is also finite for each i . On the other hand, a theorem of Milnor-Moore-Cartan-Serre asserts that the loop space homology $H_*(\Omega X; \mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra L_X and that the Hurewicz homomorphism is an isomorphism $\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\cong} L_X$. Since there are natural isomorphisms $\pi_i(X) \cong \pi_{i-1}(\Omega X)$ it follows that the results above can be phrased in terms of the integers $\dim(L_X)_i$.

For any graded vector space V concentrated in positive degrees we define the *logarithmic index* of V by

$$\log \text{index } V = \limsup \frac{\log \dim V_k}{k}.$$

In ([5], [6]) it is shown that if X (simply connected) has finite Lusternik-Schnirelmann category (in particular, if X is a finite CW complex) and if X is rationally hyperbolic then

$$\log \text{index } L_X > 0.$$

Now let Q_X denote a minimal generating subspace for the Lie algebra L_X , and let α denote $\log \text{index } L_X$.

Theorem 1. *Let X be a simply connected topological space with finite dimensional rational homology concentrated in degrees $\leq n$. Suppose $\log \text{index } Q_X < \log \text{index } L_X$, then*

$$\sum_{i=1}^{n-1} \dim(L_X)_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In particular this sum grows exponentially in k .

Theorem 2. *Let X be a simply connected topological space with finite dimensional rational homology in each degree, and finite Lusternik-Schnirelmann category. Suppose $\log \text{index } Q_X < \log \text{index } L_X < \infty$. Then for some $d > 0$,*

$$\sum_{i=1}^{d-1} \dim (L_X)_{k+i} = e^{(\alpha+\varepsilon_k)k}, \quad \text{where } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In particular, $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$ grows exponentially in k .

Corollary: The conclusion of Theorem 1 holds for finite simply connected CW complexes X for which L_X is infinite, but finitely generated. The conclusion of Theorem 2 holds for simply connected spaces of finite LS category and finite rational Betti numbers provided that L_X is infinite, but finitely generated, and, $\log \text{index } L_X < \infty$.

Recall that the depth of a graded Lie algebra L is the least m (or ∞) such that $\text{Ext}_{UL}^m(\mathbb{Q}, UL) \neq 0$.

The key ingredients in the proofs of Theorems 1 and 2 are

- A growth condition for L_X established in ([5],[6])
- The fact that $\text{depth } L_X < \infty$, established in ([3],[6])

We shall use Lie algebra arguments in Theorem 3 below to deduce the conclusion of Theorems 1 and 2 from these ingredients, and then deduce Theorems 1 and 2.

Theorems 1 and 2 may be compared with the results in [7] and in [8] that assert for n -dimensional finite CW complexes (respectively for simply connected spaces with finite type rational homology and finite Lusternik-Schnirelmann category) that $\sum_{i=1}^{n-1} \dim (L_X)_{k+i}$ (resp. $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$) grows faster than any polynomial in k . These results use only the fact that L_X has finite depth, and require no hypothesis on Q_X .

The hypothesis on Q_X in Theorems 1 and 2 may be restated as requiring that the formal series $\sum_q \dim \text{Tor}_{1,q}^{UL_X} z^q$ have a radius of convergence strictly greater than that of the formal series $\sum_q \dim (UL_X)_q z^q$. Lambrechts [9] has proved a much stronger result under the hypothesis that the formal series $\sum_q \left(\sum_p (-1)^p \dim \text{Tor}_{p,q} \right) z^q$ has a radius of convergence strictly larger than that of $\sum_q \dim (UL_X)_q z^q$.

2 Lie algebras

In this section we work over any ground field k of characteristic different from 2; graded Lie algebras L are defined as in [7] and, in particular, are assumed to satisfy $[x, [x, x]] = 0$, $x \in L_{\text{odd}}$ (This follows from the Jacobi identity except when $\text{char } k = 3$).

A graded Lie algebra L is connected and of finite type if

$$L = \{L_i\}_{i \geq 1} \text{ and each } L_i \text{ is finite dimensional.}$$

We shall refer to these as *cft Lie algebras*. The minimal generating subspaces Q of a cft Lie algebra L are those subspaces Q for which $Q \rightarrow L/[L, L]$ is a linear isomorphism.

A *growth sequence* for a cft Lie algebra L is a sequence (r_i) such that $r_i \rightarrow \infty$ and

$$\lim_{i \rightarrow \infty} \frac{\log \dim L_{r_i}}{r_i} = \log \text{index } L.$$

A *quasi-geometric sequence* (ℓ_i) is a sequence such that for some integer m , $\ell_i < \ell_{i+1} \leq m\ell_i$, all i ; if additionally (ℓ_i) is a growth sequence then it is a *quasi-geometric growth sequence*.

Of particular interest here are the growth conditions

$$0 < \log \text{index } L < \infty; \tag{A.1}$$

and

$$\log \text{index } Q < \log \text{index } L. \tag{A.2}$$

Proposition. *Let L be a cft Lie algebra satisfying (A.1) and (A.2), and assume L has a quasi-geometric growth sequence (r_j) . Then any sequence (s_i) such that $s_i \rightarrow \infty$ has a subsequence (s_{i_j}) for which there are growth sequences (t_j) and (p_j) such that*

$$t_j \leq s_{i_j} < p_j \quad \text{and} \quad p_j/t_j \rightarrow 1.$$

Proof. First note that because of (A.2),

$$\log \text{index } L/Q = \log \text{index } L$$

Now adopt the following notation, for $i \geq 1$:

$$\left. \begin{aligned} \log \text{index } L/Q &= \alpha \\ \dim L_i e^{(\alpha+\varepsilon_i)i} & \\ \dim Q_i &= e^{(\alpha+\sigma_i)i} \\ \dim L_i/Q_i &= e^{(\alpha+\tau_i)i} \\ \dim (UL)_i &= e^{(\alpha+\delta_i)i} \end{aligned} \right\} \quad (1)$$

Then because of (A.1) and (A.2), $0 < \alpha < \infty$, and $\limsup \varepsilon_i = 0$. Moreover, by a result of Babenko ([2], [6]), $\log \text{index } UL = \log \text{index } L$, and so $\limsup(\delta_i) = 0$. Finally (A.2) implies that $\limsup(\sigma_i) = \sigma < 0$, and that $\tau_{q_i} \rightarrow 0$ as $i \rightarrow \infty$.

Next, since r_j is a quasi-geometric sequence, for some fixed m we have $r_j < r_{j+1} \leq mr_j$, all j . It follows that for each s_i in our sequence we may choose q_i in the sequence (r_j) so that

$$s_i < q_i \leq ms_i. \quad (2)$$

Thus $q_i \rightarrow \infty$ as $i \rightarrow \infty$. The adjoint representation of UL in L defines surjections

$$\bigoplus_{(\ell,k,t) \in \mathcal{J}_i} (UL)_\ell \otimes Q_k \otimes L_t \twoheadrightarrow \bigoplus_{(\ell,k,t) \in \mathcal{J}_i} (UL)_\ell \otimes [Q_k, L_t] \twoheadrightarrow L_{q_i}/Q_{q_i}, \quad (3)$$

where \mathcal{J}_i consists of those triples for which $\ell + k + t = q_i$ and $t \leq s_i < t + k$. Thus

$$3(q_i + 1) \max_{(\ell,k,t) \in \mathcal{J}_i} \dim (UL)_\ell \dim [Q_k, L_t] \geq \dim L_{q_i}/Q_{q_i}.$$

For each s_i we may therefore choose $(\ell_i, k_i, t_i) \in \mathcal{J}_i$ so that

$$3(q_i + 1) \dim (UL)_{\ell_i} \dim [Q_{k_i}, L_{t_i}] \geq \dim L_{q_i}/Q_{q_i}. \quad (4)$$

Lemma 1. $(k_i + t_i) \geq \frac{1}{m} q_i$, and $(k_i + t_i) \geq \frac{1}{m-1} \ell_i$. In particular, $k_i + t_i \rightarrow \infty$ as $i \rightarrow \infty$.

Proof. It follows from (2) that $k_i + t_i \geq s_i \geq \frac{1}{m} q_i$. Since $q_i = k_i + t_i + \ell_i$, $(k_i + t_i) \geq \frac{1}{m-1} \ell_i$. Since $s_i \rightarrow \infty$ so does $k_i + t_i$. \square

Next, define $\varepsilon(k_i, t_i)$ by

$$\dim [Q_{k_i}, L_{t_i}] = e^{[\alpha + \varepsilon(k_i, t_i)](k_i + t_i)}.$$

Lemma 2. $\varepsilon(k_i, t_i) \rightarrow 0$ and $\varepsilon_{k_i+t_i} \rightarrow 0$ as $i \rightarrow \infty$.

Proof. Define $\varepsilon(q_i)$ by $3(q_i + 1) = e^{\varepsilon(q_i)q_i}$. Then (4) reduces to

$$\varepsilon(q_i)q_i + (\alpha + \delta_{\ell_i})\ell_i + [\alpha + \varepsilon(k_i, t_i)](k_i + t_i) \geq (\alpha + \tau_{q_i})q_i.$$

Since $\ell_i + k_i + t_i = q_i$,

$$\varepsilon(q_i)\frac{q_i}{k_i + t_i} + \delta_{\ell_i}\frac{\ell_i}{k_i + t_i} + \varepsilon(k_i, t_i) \geq \tau_{q_i}\frac{q_i}{k_i + t_i}.$$

Now as $i \rightarrow \infty$, $q_i \geq s_i \rightarrow \infty$. Thus $\varepsilon(q_i) \rightarrow 0$. Moreover, since q_i belongs to a growth sequence, $\varepsilon_{q_i} \rightarrow 0$ and $\tau_{q_i} \rightarrow 0$. Use Lemma 1 to conclude that

$$\varepsilon(q_i)\frac{q_i}{k_i + t_i} \rightarrow 0 \quad \text{and} \quad \tau_{q_i}\frac{q_i}{k_i + t_i} \rightarrow 0,$$

and hence

$$\liminf \varepsilon(k_i, t_i) \geq -\limsup \delta_{\ell_i}\frac{\ell_i}{k_i + t_i}.$$

Next, since $\limsup \delta_{\ell_i} = 0$ and $\frac{\ell_i}{k_i+t_i} \leq m-1$, it follows that

$$-\limsup \delta_{\ell_i}\frac{\ell_i}{k_i + t_i} = 0.$$

Finally, $[Q_{k_i}, L_{t_i}]$ embeds in $L_{k_i+t_i}/Q_{k_i+t_i}$, and it follows that

$$\varepsilon_{k_i+t_i} \geq \varepsilon(k_i, t_i).$$

But $k_i+t_i \rightarrow \infty$ and so $\limsup \varepsilon_{k_i+t_i} \leq 0$. This, together with $\liminf \varepsilon(k_i, t_i) \geq 0$, completes the proof of the lemma. \square

Next, since $Q_{k_i} \otimes L_{t_i} \rightarrow [Q_{k_i}, L_{t_i}]$ is surjective we have

$$\sigma_{k_i}k_i + \varepsilon_{t_i}t_i \geq \varepsilon(k_i, t_i)(k_i + t_i) \tag{5}$$

Lemma 3. $t_i/k_i \rightarrow \infty$ and $t_i \rightarrow \infty$ as $i \rightarrow \infty$.

Proof. If t_i/k_i does not converge to ∞ then we would have $t_{i_\nu}/k_{i_\nu} \leq T$ for some subsequence (s_{i_ν}) . But

$$\sigma_{k_{i_\nu}} \geq -\varepsilon_{t_{i_\nu}}\frac{t_{i_\nu}}{k_{i_\nu}} + \varepsilon(k_{i_\nu}, t_{i_\nu})(1 + t_{i_\nu}/k_{i_\nu}).$$

Since (Lemma 1) $k_{i\nu} + t_{i\nu} \rightarrow \infty$ and (Lemma 2) $\varepsilon(k_{i\nu}, t_{i\nu}) \rightarrow 0$, the lim inf of the right hand side of this equation would be ≥ 0 . Hence $\limsup \sigma_{k_{i\nu}} \geq 0$, which would contradict $\limsup \sigma_i < 0$. Finally, since $t_i + k_i \rightarrow \infty$ it follows that $t_i \rightarrow \infty$. \square

Lemma 4. Write $k_i \lambda_i t_i$. Then,

$$\lambda_i \rightarrow 0 \quad \text{and} \quad \varepsilon_{t_i} \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty.$$

Proof. Lemma 3 asserts that $\lambda_i \rightarrow 0$. Rewrite equation (5) as

$$\varepsilon_{t_i} \geq (\varepsilon(k_i, t_i) - \sigma_{k_i}) \lambda_i + \varepsilon(k_i, t_i). \quad (6)$$

Since $\limsup \sigma_j$ is finite, the σ_j are bounded above. Thus $-\sigma_j \geq A$, some constant A . Since $\lambda_i \rightarrow 0$, $\liminf(-\sigma_{k_i} \lambda_i) \geq 0$. Since (Lemma 2) $\varepsilon(k_i, t_i) \rightarrow 0$ it follows from (6) that $\liminf \varepsilon_{t_i} = 0$. But $\limsup \varepsilon_{t_i} \leq \limsup \varepsilon_i = 0$ and so $\varepsilon_{t_i} \rightarrow 0$. \square

The lemmas above establish the Proposition. Simply set $p_i = t_i + k_i$ and note that $t_i \rightarrow \infty$ (Lemma 3), $p_i \rightarrow \infty$ (Lemma 1), $p_i/t_i = 1 + \lambda_i \rightarrow 1$ (Lemma 4). Furthermore $\varepsilon_{t_i} \rightarrow 0$ (Lemma 4) and $\varepsilon_{p_i} \rightarrow 0$ (Lemma 2). Thus (t_i) and (p_i) are growth sequences. \square

Theorem 3. Let L be a cft Lie algebra of finite depth and satisfying the growth conditions (A.1) and (A.2). Set $\alpha = \log \text{index } L$. Then for some d ,

$$\sum_{i=1}^{d-1} \dim L_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In particular, this sum grows exponentially in k .

Proof. According to [4] there is a finitely generated sub Lie algebra $E \subset L$ such that $\text{Ext}_{UL}^r(\mathbb{k}, UL) \rightarrow \text{Ext}_{UE}^r(\mathbb{k}, UL)$ is non-zero.

Lemma 5. The centralizer, Z , of E in L is finite dimensional.

Proof. Since E has finite depth, $Z \cap E$ is finite dimensional [3]. Choose k so $Z \cap E$ is concentrated in degrees $< k$. Suppose $x \in Z_{\geq k}$ has even degree, and put $F = \mathbb{k}x \oplus E$. Then $\text{Ext}_{UF}^r(\mathbb{k}, UF) \rightarrow \text{Ext}_{UE}^r(\mathbb{k}, UF)$ is zero, contradicting the hypothesis that the composite

$$\text{Ext}_{UL}^r(\mathbb{k}, UL) \rightarrow \text{Ext}_{UF}^r(\mathbb{k}, UL) \rightarrow \text{Ext}_{UE}^r(\mathbb{k}, UL)$$

is non-zero.

It follows that $Z_{\geq k}$ is concentrated in odd degrees, hence an abelian ideal in $Z + E$. Again

$$\text{Ext}_{UL}^r(\mathbb{k}, UL) \rightarrow \text{Ext}_{U(Z+E)}^r(\mathbb{k}, UL) \rightarrow \text{Ext}_{UE}^r(\mathbb{k}, UL)$$

is non-zero. Thus $\text{Ext}_{U(Z+E)}^r(\mathbb{k}, UL) \neq 0$, $Z + E$ has finite depth and every abelian ideal in $Z + E$ is finite dimensional. \square

Choose d so that E is generated in degrees $\leq d-1$. As in the Proposition, set $\log \text{index } L = \log \text{index } L/Q = \alpha$. If the theorem fails we can find a sequence $s_i \rightarrow \infty$ such that

$$\sum_{j=1}^{d-1} \dim L_{s_i+j} \leq e^{(\alpha-\beta)s_i}, \quad (7)$$

some $\beta > 0$. Apply the Proposition to find growth sequences t_i and p_i such that $t_i \leq s_i < p_i$ and $p_i/t_i \rightarrow 1$.

We now use (7) to prove that

$$\dim (UE)_{(s_i-t_i, s_i-t_i+d)} \dim L_{(s_i, s_i+d)} < \dim L_{t_i}, \quad i \text{ large}. \quad (8)$$

In fact since $\limsup (\dim (UL)_i)^{1/i} = e^\alpha$ it follows that for some $\gamma > 0$,

$$\sum_{j=1}^{d-1} \dim (UE)_{j+k} \leq e^{\gamma(k+1)}, \quad \text{all } k.$$

Thus it is sufficient to show that

$$\gamma(s_i - t_i + 1) + (\alpha - \beta)s_i < (\alpha + \varepsilon_{t_i})t_i, \quad \text{large } i,$$

where $\varepsilon_{t_i} \rightarrow 0$ as $i \rightarrow \infty$. Write $s_i = \mu_i t_i$; then $\mu_i \rightarrow 1$ and the inequality reduces to the obvious

$$\gamma/t_i + \gamma(\mu_i - 1) + (\alpha - \beta)\mu_i < (\alpha + \varepsilon_{t_i}), \quad \text{large } i.$$

Thus (8) is established.

Choose $s = s_i$, $t = t_i$ so that (8) holds and so that $Z_j = 0$, $j \geq t$. Write $s - t = k$. The adjoint action of UL in L restricts to a linear map

$$\left[\bigoplus_{j=1}^{d-1} (UE)_{k+j} \right] \otimes L_t \rightarrow \bigoplus_{j=1}^{d-1} L_{s+j}$$

and it follows from (8) that for some non-zero $x \in L_t$,

$$(\text{ad } a)x = 0, \quad a \in UE_{(k,k+d)}.$$

On the other hand, since E is generated in degrees $\leq d-1$, $(UE)_{>k} = UE \cdot (UE)_{(k,k+d)}$. Thus $(UE)_{>k} \cdot x = 0$ and so $(UE) \cdot x$ is finite dimensional. A non-zero element y of maximal degree in $UE \cdot x$ satisfies

$$[a, y] = (\text{ad } a)(y) = 0, \quad a \in E,$$

i.e. $y \in Z$ in contradiction to $Z_{\geq t} = 0$. This completes the proof of the Theorem. \square

3 Proof of Theorems 1 and 2

Proof of Theorem 2: We show that L_X satisfies the hypothesis of Theorem 3. Since $\text{depth } L_X < \infty$ ([3]) and (A.2) holds by hypothesis we have only to construct a quasi-geometric growth sequence (r_i) .

Let $\alpha = \log \text{index } L_X$. Then $\alpha > 0$ by [5]. Choose a sequence

$$u_1 < u_2 < \dots$$

such that $(\dim(L_X)_{u_i})^{1/u_i} \rightarrow e^\alpha$.

Next, suppose $\text{cat } X = m$ and put $a \left(\frac{1}{2(m+1)} \right)^{m+1}$. By starting the sequence at some u_j we may assume $\dim(L_X)_{u_i} > \frac{1}{a}$, all i . Thus the formula in ([5], top of page 189) gives a sequence

$$u_i = v_0 < v_1 \dots < v_k u_{i+1}$$

such that $v_{i+1} \leq 2(m+1)v_i$ and

$$\left(\dim(L_X)_{v_j} \right)^{\frac{1}{v_j+1}} \geq [a \dim(L_X)_{v_0}]^{\frac{1}{v_0+1}}, \quad j < k.$$

Since $v_0 = u_i$ and $u_i \rightarrow \infty$ it follows that $a \frac{1}{v_0+1} \rightarrow 1$ as $i \rightarrow \infty$. Hence interpolating the sequences u_i with the sequences v_j gives a quasi-geometric growth sequence (r_j) . \square

Proof of Theorem 1: A theorem of Adams-Hilton [1] shows that

$$UL_X = H_*(\Omega X; \mathbb{Q}) = H(TV, d)$$

where TV is the tensor algebra on V and $V_i \cong H_{i+1}(X; \mathbb{Q})$. Thus V is finite dimensional. Since TV has a strictly positive radius of convergence so do $H(TV, d)$ and L_X :

$$\log \text{index } L_X < \infty .$$

Thus X satisfies the hypotheses of Theorem 2. The fact that d can be replaced by n is proved by Lambrechts in [10]. \square

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