Estimation of the Memory Parameter of the Infinite Source Poisson Proces

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September 16, 2005

Abstract

Long range dependence induced by heavy tails is a widely reported feature of internet traffic. Long range dependence can be defined as the regular variation of the variance of the integrated process, and half the index of regular variation is then refered to as the *Hurst index*. The infinite source Poisson process (a particular case of which is the $M/G/\infty$ queue) is a simple and popular model with this property, when the tail of the service time distribution is regularly varying. The Hurst index of the infinite source Poisson process is then related to the index of regular variation of the service times. In this paper, we present a wavelet based estimator of the Hurst index of this process, when it is observed either continously or discretely over an increasing time interval. Our estimator is shown to be consistent and robust to some form of nonstationarity. Its rate of convergence is investigated.

Keywords: long range dependence, heavy tails, semiparametric estimation, wavelets, Poisson point processes, internet traffic.

1 Introduction

We consider the infinite source Poisson process with random transmission rate defined by

$$\mathbf{X}(t) = \sum_{\ell=1}^{K_0} \tilde{U}_\ell \mathbb{1}_{\{t<\tilde{\eta}_\ell\}} + \sum_{\ell\in\mathbb{N}} U_\ell \, \mathbb{1}_{\{t_\ell \le t < t_\ell+\eta_\ell\}}, \quad t \ge 0 \, , \tag{1.1}$$

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where

- K_0 is the number of users at time zero, and $(\tilde{\eta}_\ell, \tilde{U}_\ell)$, $\ell = 1, \ldots, K_0$ are their departure times and transmission rates; these variables will be referred to as *the initial conditions*;
- the arrival times $\{t_\ell\}_{\ell \geq 0}$ are the points of a unit rate homogeneous Poisson process on the positive half-line, independent of the initial conditions;
- the durations and transmission rates $\{(\eta_\ell, U_\ell)\}$ are i.i.d. random variables with values in $\mathbb{R}_+ \times \mathbb{R}$ and independent of the Poisson process and of the initial conditions.

This process was considered by Resnick and Rootzén [2000], Mikosch et al. [2002] among others. The $M/G/\infty$ queue is a special case, namely, for $U_\ell \equiv 1$. An important motivation for the infinite source Poisson process is to model the instantaneous rate of the workload going though an Internet link. Although too simple models are generally not relevant for the Internet traffic at the packet level, it is generally admitted that rather simple models can be used for higher level (the so called *flow level*) traffic such as TCP or HTTP sessions, one of them being the infinite source Poisson process (see Barakat et al. [2002]). One way to empirically analyze the Internet traffic at the flow level using the infinite source Poisson process would consist in retrieving all the variables $\{t_\ell, \eta_\ell, U_\ell\}$ involved in the observed traffic during a given period of time, but this would require to collect all the relevant information in the packets headers (such as source and destination addresses) for separating the aggregated workload into transmission rates at a pertinent level (see Duffield et al. [2002] for many insights about this problem).

It is well known that heavy tails in the durations $\{\eta_k\}$ result in long range dependence of the process $\mathbf{X}(t)$, cf. for instance Resnick and Samorodnitsky [2001]. Long range dependence can be defined by the regular variation of the autocovariance of the process or more generally by the regular variation of the variance of the integrated process:

$$\mathrm{var}\left(\int_0^t \mathbf{X}(s)\,\mathrm{d}s\right) = L(t)t^{2H},$$

where L is a slowly varying function at infinity and $H > 1/2$ is often refered to as the *Hurst index* of the process. For the infinite source Poisson process, the Hurst index H is related to the tail index α of the durations by the relation $H = (3-\alpha)/2$. The long range dependence property has motivated many empirical studies of Internet traffic and theoretical ones concerning its impact on queuing (these questions are studied in the $M/G/\infty$ case in Parulekar and Makowski [1997]). However, up to the best of our knowledge, no statistical procedure to estimate H has been rigorously justified. It is the aim of this paper to propose an estimator of the Hurst index of the infinite source Poisson process, and to derive its statistical properties. We propose to estimate H (or equivalently α) from a path of the process $\mathbf{X}(t)$ over a finite interval $[0, T]$, observed either continuously or discretely. In practice this can be done by counting all the packets going through some point of the network and then collect local traffic rate measurements. Our estimator is based on the so-called wavelet coefficients of a path. There is a wide literature on this methodology to estimate long range dependence, starting as early as Wornell and Oppenheim [1992] for instance, but we are not aware of rigorous results for non Gaussian or non Stable processes. The main contribution of this paper is thus the proof of the consistency of our estimator. We also investigate the rate of convergence of the estimator in the case $\alpha > 1$. If the process is observed continuously, the rate of convergence is good. In the case of discrete observations, the rate is much smaller. Also, the choice of the tuning parameters of the estimators is much more restricted in the latter case, and practitioners should perhaps be aware of this; see Section 4.3 for details.

Outline of the paper The process $\mathbf{X}$ is formally defined in Section 2. We state our assumptions and, using a point process representation of $\mathbf{X}$, we establish some of its main properties. The wavelet coefficients are defined and the scaling property of their variances is obtained in Section 3. The estimator is defined and its properties are established in section 4. The Appendix contains technical Lemmas. A Monte-Carlo study and a real data analysis are reported in a companion paper Faÿ et al. [2005].

2 Basic properties of the model

2.1 Assumptions

We now introduce the complete assumption on the joint distribution of the transmissions rates and durations.

Assumption 1. (i) The random vectors $\{(\eta, U),\ (\eta_\ell, U_\ell), \ell \in \mathbb{Z}\}$ are i.i.d. with distribution ν on $\mathbb{R}_+^* \times \mathbb{R}$ and independent of the homogeneous Poisson point process on the real line with points $\{t_\ell\}_{\ell \in \mathbb{Z}}$ such that $t_\ell < t_{\ell+1}$ for all ℓ and $t_{-1} < 0 \leq t_0$;

(ii) there exists a positive integer p^* such that $\mathbb{E}[|U|^{p^*}] < \infty$;

(iii) there exist a real number $\alpha \in (0, 2)$ and positive functions $L_0, \ldots, L_{p^*}$ slowly varying at infinity such that for all $t > 0$ and $p = 0, \ldots, p^*$,

$$\mathbb{E}\left[|U|^p \mathbb{1}_{\{\eta > t\}}\right] = L_p(t) t^{-\alpha} . \tag{2.1}$$

For $p = 0, \ldots, p^*$, define

$$H_p(t) = \mathbb{E}\left[|U|^p \mathbb{1}_{\{\eta > t\}}\right] = L_p(t) t^{-\alpha} , \quad t \geq 0,$$

Since $\eta \in \mathbb{R}_+^*$, the functions H_p are continuous at zero and $H_p(0) = \mathbb{E}[|U|^p]$. Condition (2.1) is equivalent to saying that the functions H_p, $p = 0, 1, \ldots, p^*$, are regularly varying with index $-\alpha$. If $\alpha > 1$ and $p^* \geq 2$, Assumption 1 and Karamata Theorem imply the following asymptotic equivalence:

$$\mathbb{E}\left[U^2\{\eta - t\}_+\right] = \int_t^\infty H_2(v)\mathrm{d}v \sim \frac{1}{\alpha - 1} L_p(t) t^{1-\alpha} \; . \tag{2.2}$$

Remark 2.1. We do not assume that U is nonnegative. This allows to consider applications other than teletraffic modeling. For instance, the process $\mathbf{X}$ could be used to model the volatiliy of some financial time series.

Remark 2.2. In the sequel, we will often have to separate the cases $\mathbb{E}[\eta] = \infty$ and $\mathbb{E}[\eta] < \infty$. These cases are respectively implied by $\alpha < 1$ and $\alpha > 1$. If $\alpha = 1$, the finiteness of $\mathbb{E}[\eta]$ depends on the precise behavior of L_0 at infinity.

Example 2.1. Assumption 1 implies in particular that the tail of the distribution of η is regularly varying with index α. This in turns implies Assumption 1 if U and η are independent and $\mathbb{E}[|U|^{p^*}] < \infty$, in which case the functions L_p differ by a multiplicative constant.

Example 2.2. Assumption 1 also holds in the following case which is of interest in teletraffic modeling. In a TCP/IP traffic context, η and U represent respectively the duration of a download session and its intensity (bit rate). Then $W := U\eta$ represents the amount of transmitted data. We assume that there exist two regimes, $U \geq u_0$ (xDSL/LAN/Cable connection), and $U < u_0$ (RTC connection), for some $u_0 > 0$.

- The distribution of W given $U = u \geq u_0$ is heavy tailed and independent of u: $\mathbb{P}(W \geq w | U = u) = L(w) w^{-\alpha}$;
- The distribution of W given $U = u \in (0, u_0)$ is light-tailed and independent of u; for instance, we assume expontially decaying tails:

  $$\mathbb{P}(W \geq w | U = u) \leq \exp(-\beta w^{-\gamma}) \; ,$$

  for some $\beta > 0$ and $\gamma > 0$.

If $\mathbb{E}[|U|^{-\alpha-\epsilon} | U < u_0] < \infty$ for some $\epsilon > 0$ and $\mathbb{E}[|U|^{p^*}] < \infty$, then (2.1) holds. Indeed, for $p \leq p^*$,

$$\begin{aligned}
\mathbb{E}[U^p \mathbb{1}_{\{\eta > t\}} | U \geq u_0] &= \mathbb{E}[U^p \mathbb{1}_{\{W > Ut\}} | U \geq u_0] \\
&= \mathbb{E}[U^p L(Ut)(Ut)^{-\alpha} | U \geq u_0] \\
&= L(t) t^{-\alpha} \mathbb{E}[U^{p-\alpha} L(Ut)/L(t) | U \geq u_0] \; .
\end{aligned}$$

Since L is slowly varying at infinity, $\lim_{t\to\infty} L(ut)/L(t) = 1$, uniformly with respect to u in compact sets of $(0, +\infty)$ and there exists $t_0 > 0$ such that for $u \geq u_0$, $t \geq t_0$

$$\frac{L(ut)}{L(t)} \leq (1+\alpha)u^{\alpha/2} \, .$$

See *e.g.* [Resnick, 1987, Proposition 0.8]. Then by the Dominated Convergence Theorem,

$$\lim_{t\to\infty} \mathbb{E}[U^{p-\alpha} L(Ut)/L(t) | U \geq u_0] = \mathbb{E}[U^{p-\alpha} | U \geq u_0]$$

and thus, the following asymptotic equivalence holds as $t \to \infty$,

$$\mathbb{E}[U^p \mathbb{1}_{\{\eta > t\}} | U \geq u_0] \sim \mathbb{E}[U^{p-\alpha} | U \geq u_0] L(t) t^{-\alpha} \, . \tag{2.3}$$

Consider now the low-bitrate regime. For any $\epsilon > 0$,

$$\mathbb{E}[U^p \mathbb{1}_{\{\eta > t\}} | U < u_0] = \mathbb{E}[U^p \exp\{-\beta(Ut)^\gamma\} | U < u_0] \leq C t^{-\alpha-\epsilon} \mathbb{E}[U^{p-\alpha-\epsilon} | U < u_0]$$

for some positive constant C. Hence

$$\lim_{t\to\infty} t^\alpha L^{-1}(t) \mathbb{E}[U^p \mathbb{1}_{\{\eta > t\}} | U < u_0] = 0.$$

This and (2.3) imply (2.1).

2.2 Point Process representation and stationary version

Let $\mathcal{N}$ denote a Poisson point process on a set E endowed with a σ-field $\mathcal{E}$ with control measure μ, that is a random measure such that for any disjoint $A_1, \ldots, A_p$ in $\mathcal{E}$, $\mathcal{N}(A_1), \ldots, \mathcal{N}(A_p)$ are independent random variables with Poisson law with respective parameters $\mu(A_i)$, $i = 1, \ldots, p$. The main property of Poisson point processes that we will use is the following cumulant formula (see for instance [Resnick, 1987, Chapter 3]). For any positive integer p and functions $f_1, \ldots, f_p$ such that $\int |f_i|^p \mathrm{d}\mu < \infty$ for all $i = 1, \ldots, p$, the p-th order joint cumulant of $\mathcal{N}(f_1), \ldots, \mathcal{N}(f_p)$ exists and is given by

$$\mathrm{cum}(\mathcal{N}(f_1), \ldots, \mathcal{N}(f_p)) = \int f_1 \cdots f_p \, \mathrm{d}\mu \, . \tag{2.4}$$

Let N_S be the point processes on $\mathbb{R}^3$ with points $(t_\ell, \eta_\ell, U_\ell)_{\ell\in\mathbb{Z}}$, *i.e.* $N_S = \sum_{\ell\in\mathbb{Z}} \delta_{t_\ell, \eta_\ell, U_\ell}$.

Under Assumption 1(i), it is a Poisson point process with control measure $\mathrm{Leb} \otimes \nu$, where Leb is Lebesgue's measure on $\mathbb{R}$. For $t, u \in \mathbb{R}$, define

$$\begin{aligned} A_t &= \{(s, v) \in \mathbb{R} \times \mathbb{R}_+ \mid s \leq t < s + v\} \, ; \\ B_u &= \{\lambda u \mid \lambda \in [1, \infty)\} \quad . \end{aligned}$$

We can now show that if $\mathbb{E}[\eta] < \infty$, a suitable choice of the initial conditions makes $\mathbf{X}$ a stationary process.

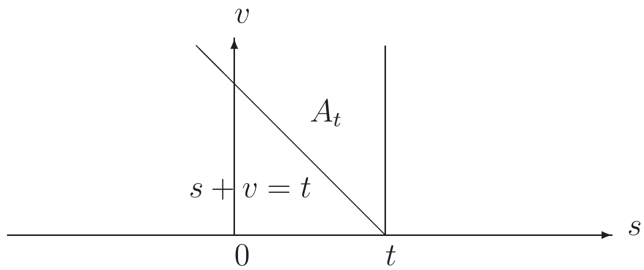

Figure 1: The set A_t

Proposition 2.1. *If Assumption 1(i) holds and $\mathbb{E}[\eta] < \infty$, then the process*

$$X_S(t) = \sum_{\ell \in \mathbb{Z}} U_\ell \mathbb{1}_{\{t_\ell \le t < t_\ell + \eta_\ell\}} \tag{2.5}$$

is well defined and strictly stationary. It has the following point process representation

$$X_S(t) = \int_0^\infty N_S(A_t \times B_u)\,\mathrm{d}u - \int_{-\infty}^0 N_S(A_t \times B_u)\,\mathrm{d}u\,. \tag{2.6}$$

If $K_0 = \sup\{\ell > 0 \mid t_{-\ell} + \eta_{-\ell} > 0\}$, $\tilde{U}_\ell = U_{-\ell}$ and $\tilde{\eta}_\ell = \eta_{-\ell} + t_{-\ell}$, then $\mathbf{X}(t) = X_S(t)$ for all $t \ge 0$.

Proof. The number of non vanishing terms in the sum (2.5) is $N_S(A_t \times \mathbb{R})$ and has a Poisson distribution with mean

$$\int_{\mathbb{R}\times\mathbb{R}_+} \mathbb{1}_{A_t}(s,v)\,\mathrm{d}s\,\mathrm{d}v = \int_{\mathbb{R}_+} v\mu(\mathrm{d}v) = \mathbb{E}[\eta]\,.$$

Thus X_S is well defined and stationary since N_S is stationary. The number of indices $\ell > 0$ such that $t_{-\ell} + \eta_{-\ell} > 0$ is $N_S(A_0 \times \mathbb{R})$, hence if K_0 is the largest of those ℓ's, it is almost surely finite and

$$\sum_{t_\ell < 0} U_\ell \mathbb{1}_{\{t_\ell \le t < t_\ell + \eta_\ell\}} = \sum_{\ell=1}^{K_0} \tilde{U}_\ell \mathbb{1}_{\{t < \tilde{\eta}_\ell\}}\,.$$

□

In (1.1), define $X_0(t) = \sum_{\ell=1}^{K_0} \tilde{U}_\ell \mathbb{1}_{\{t<\tilde{\eta}_\ell\}}$ and

$$X_+(t) = \sum_{\ell \in \mathbb{N}} U_\ell\, \mathbb{1}_{\{t_\ell \le t < t_\ell + \eta_\ell\}}, \quad t \ge 0\,. \tag{2.7}$$

Then, $\mathbf{X} = X_0 + X_+$. If K_0 and $\tilde{\eta}_\ell, \ell > 0$ are finite, then, almost surely, $\mathbf{X}(t) = X_+(t)$, for large enough t. This is true in particular if $\mathbf{X} = X_S$, when X_S is well defined, *i.e.* if $\mathbb{E}[\eta] < \infty$. In practice, the initial conditions are arbitrary. The previous results show that their effect is limited in time. Thus, without loss of meaningful generality, we will henceforth only consider the two cases $\mathbf{X} = X_+$ (zero initial conditions) and $\mathbf{X} = X_S$ (stationary initial conditions), the latter being only defined for $\mathbb{E}[\eta] < \infty$. Finally note that X_+ also admits a point process representation:

$$X_+(t) = \int_0^\infty N_S(A_t^+ \times B_u)\,\mathrm{d}u - \int_{-\infty}^0 N_S(A_t^+ \times B_u)\,\mathrm{d}u \;, \tag{2.8}$$

where $A_t^+ = A_t \cap \mathbb{R}_+^2$.

2.3 Second order properties

Applying the Point process representation (2.6) and (2.8), Formula (2.4) gives the following expressions for the first and second moments of X_+ and X_S.

Proposition 2.2. *Let Assumption 1 hold with $p^* \geq 2$. The process X_+ is nonstationary with expectation and autocovariance function given, for $s \leq t$ by*

$$\mathbb{E}[X_+(t)] = \mathbb{E}[U(\eta \wedge t)] \;,$$
$$\mathrm{cov}(X_+(s), X_+(t)) = \mathbb{E}[U^2\{s - (t-\eta)_+\}_+] = \int_{t-s}^t H_2(v)\mathrm{d}v \;.$$

If $\mathbb{E}[\eta] < \infty$ $p^* \geq 2$, then X_S is weakly stationary with finite variance and expectation and autocovariance function given by

$$\mathbb{E}[X_S(t)] = \mathbb{E}[U\eta] \;,$$
$$\mathrm{cov}(X_S(0), X_S(t)) = \mathbb{E}[U^2(\eta - t)_+] \sim \frac{1}{1-\alpha} L_2(t) t^{1-\alpha} \;, \quad t \to +\infty.$$

If $\alpha \in (0,1)$ and if t and s tend to infinity at the same rate, the following asymptotic equivalent of $\mathrm{cov}(X_+(s), X_+(t))$ holds. For all $t, s > 0$, as $T \to \infty$,

$$\mathrm{cov}(X_+(Ts), X_+(Tt)) \sim \frac{1}{1-\alpha} L_2(T) T^{1-\alpha}\{(s \vee t)^{1-\alpha} - |t-s|^{1-\alpha}\} \;. \tag{2.9}$$

In accordance with the notation in use in the context of long memory processes, we can define the Hurst index of the process $\mathbf{X}$ as $H = (3-\alpha)/2$, because the variance of the integrated process as T^{2H}.

3 Wavelet coefficients

3.1 Continuous observation

Let ψ be a bounded real valued function with compact support in $[0, M]$ and such that

$$\int_0^M \psi(s)\, \mathrm{d}s = 0 \;. \tag{3.1}$$

For integers $j \geq 0$ and $k \in \mathbb{Z}$, define

$$\psi_{j,k}(s) = 2^{-j/2}\psi(2^{-j}s - k). \tag{3.2}$$

The wavelet coefficients of the path are defined as

$$\mathbf{d}_{j,k} = \int_0^\infty \psi_{j,k}(s)\mathbf{X}(s)\, \mathrm{d}s \;. \tag{3.3}$$

Assume that a path of the process $\mathbf{X}$ is observed continuously between times 0 and T. Since $\psi_{j,k}$ has support in $[k2^j, (k+M)2^j]$, the coefficients $\mathbf{d}_{j,k}$ can be computed for all (j,k) such that $T2^{-j} \geq M$ and $k = 0, 1, \ldots, T2^{-j} - M$.

If the stationary version $\mathbf{X} = X_S$ is observed, then we denote $\mathbf{d}_{j,k} = d^S_{j,k}$. Since the process X_S is stationary, for each admissible j, the coefficients $d^S_{j,k}$, $k = 0, \ldots, T2^{-j} - M$ form a stationary sequence.

Recall that the process X_S is well defined only if $\mathbb{E}[\eta] < \infty$, in which case the coefficient $d^S_{j,k}$ can be expressed as

$$d^S_{j,k} = \sum_{\ell \in \mathbb{Z}} U_\ell \int_{t_\ell}^{t_\ell + \eta_l} \psi_{j,k}(s)\, ds \;. \tag{3.4}$$

Nevertheless, Lemma A.1 shows that it is possible to define coefficients $d^S_{j,k}$ as in (3.4), even if $\mathbb{E}[\eta] = \infty$, and that the sequence of coefficients at a given scale j, $\{d^S_{j,k},\ k \in \mathbb{Z}\}$ is stationary and, when $p^* \geq 1$, $\mathbb{E}[d^S_{j,k}] = 0$. Thus, we will take (3.4) as a definition of the coefficients $d^S_{j,k}$, which will be used to prove our results. This representation and (2.4) easily yield an expression of the variance of the stationary wavelet coefficients.

Lemma 3.1. *Let Assumption 1 hold with $p^* \geq 2$. We have*

$$\mathbb{E}[d^S_{j,k}] = 0 \;, \quad \mathrm{var}(d^S_{j,k}) = \mathcal{L}(2^j)\, 2^{(2-\alpha)j} \;, \tag{3.5}$$

where

$$\mathcal{L}(z) := z^\alpha \int_0^\infty \int_{-\infty}^\infty \left(\int_{-\infty}^\infty \left\{ \int_t^{t+vz^{-1}} \psi(s)\, \mathrm{d}s \right\}^2 \mathrm{d}t \right) w^2 \nu(\mathrm{d}v, \mathrm{d}w) \tag{3.6}$$

is slowly varying as $z \to \infty$. More precisely, as $z \to \infty$, we have the asymptotic equivalence $\mathcal{L}(z) \sim CL_2(z)$ with $C = \alpha \int_0^\infty \int_{-\infty}^\infty \left\{ \int_x^{x+y} \psi(s)\,\mathrm{d}s \right\}^2 \mathrm{d}x\, t^{-\alpha-1} \mathrm{d}t > 0$.

3.2 Discrete wavelet coefficients

Let ϕ be a bounded $\mathbb{R} \to \mathbb{R}$ function with compact support included in $[-M+1, 1]$ and such that

$$\sum_{k \in \mathbb{Z}} \phi(t-k) = 1, \quad t \in \mathbb{R}. \tag{3.7}$$

Let I_ϕ denote the operator defined on the set of functions $x : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{I}_\phi[x](t) = \sum_{k \in \mathbb{Z}} x(k)\phi(t-k).$$

The discrete wavelet coefficients of x are then the wavelet coefficients of $\mathrm{I}_\phi[x]$.

From a computational point of view, it is convenient to chose ϕ and ψ to be the so-called father and mother wavelets of a multiresolution analysis, cf. Daubechies [1992] or Meyer [1992]. The simplest choice is to take ϕ and ψ associated to the Haar system, in which case $M = 1$, $\phi = \mathbb{1}_{[0,1)}$ and $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$.

If the process $\mathbf{X}$ is observed discretely, we still denote $\mathbf{d}_{j,k}$ its discrete wavelet coefficients:

$$\mathbf{d}_{j,k} = \int \psi_{j,k}(s) \mathrm{I}_\phi[\mathbf{X}](s)\,\mathrm{d}s\,. \tag{3.8}$$

If we observe $\mathbf{X}(0), \mathbf{X}(1), \ldots, \mathbf{X}(T-1)$ for some positive integer T, we can compute $\mathbf{d}_{j,k}$ for all j, k such that $0 \le k \le 2^{-j}(T-M+1) - M$. Roughly, for $2^j \ge T/M$, no coefficients can be computed and if $2^j < T/M$ the number of computable discrete wavelet coefficients at scale 2^{-j} is of order $T2^{-j} + 1 - M$ for j and T large.

If the stationary version X_S is observed discretely, we will denote $\mathbf{d}_{j,k} = d^{SD}_{j,k}$. If X_+ is observed discretely, we will denote $\mathbf{d}_{j,k} = d^D_{j,k}$. As in the case of continuous observations, $d^{SD}_{j,k}$ has a series representation that can be extended to the case $\mathbb{E}[\eta] = \infty$.

Remark 3.1. Observe that the choice of time units is unimportant here. Indeed, in Assumption 1, changing the time units simply amounts to adapt the slowly varying functions L_k's and the rate of the arrival process $\{t_k\}$. Clearly both adaptations do not modify our results as far as precise multiplicative constants are not considered.

3.3 Averaged observations

We describe now a third observation scheme for which our results can easily be extended. Suppose that T is a positive integer and that we observe local laverages of the trajectory

$$\overline{\mathbf{X}}(k) := \int_k^{k+1} \mathbf{X}(t)\,\mathrm{d}t = \int \mathbf{X}(t)\,\phi_H(t-k)\,\mathrm{d}t, \quad k = 0, 1, \ldots, T-1\ ,$$

where $\phi_H := \mathbb{1}_{[0,1]}$ is the Haar wavelet. Let $\bar{\mathrm{I}}_\phi$ denote the operator on locally integrable functions x defined by

$$\bar{\mathrm{I}}_\phi[x](t) = \sum_{k\in\mathbb{Z}} \left(\int x(s)\phi_H(s-k)\mathrm{d}s \right) \phi(t-k)$$

For this observation scheme, as in Section 3.2, one may compute the wavelet coefficients of the function $\bar{\mathrm{I}}_\phi[\mathbf{X}]$ at all scale and location indices (j,k) such that $0 \le k \le 2^{-j}(T-M+1) - M$. If $\phi = \phi_H$

and ψ is the Haar mother wavelet, $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$, then,

the wavelet coefficients of $\bar{\mathrm{I}}_\phi[\mathbf{X}]$ are precisely the continuous wavelet coefficients defined in (3.3). For any other choice of ϕ and ψ, this is no longer true. We will not treat this case but all our results can be extended at the cost of more technicalities.

4 Estimation

Tail index estimation methods do not seem appropriate here for estimating the parameter α. Indeed, the parameter α is the tail index of the unobserved durations $\{\eta_k\}$, whereas the observed process $\mathbf{X}(t)$ always has finite variance ($\mathbb{E}[|\mathbf{X}(t)|^p] < \infty$ if and only if $\mathbb{E}[|U^p|] < \infty$ and the marginal distribution of $\mathbf{X}(t)$ is Poisson if $U = 1$ a.s.). But as shown by Proposition 2.2, α is related to the second order properties of the process: the coefficient $H = (3-\alpha)/2$ can be viewed as its Hurst index, *i.e.* H governs the rate of decay of the autocovariance function of the process. Therefore it seems natural to use an estimator of the Hurst index.

4.1 The estimator

Lemma 3.1 provides the rationale for the following minimum contrast estimator of α which is related to the local Whittle estimator, cf. Künsch [1987], Robinson [1995]. Let

(i) $\mathbf{d}_{j,k}$ denote the wavelet coefficients which are actually available; these may be the continous/discrete, stationary/nonstationary coefficients;

(ii) Δ be a set of indices (j,k) of available wavelet coefficients;

Denote the mean scale index over Δ

$$\delta := \frac{1}{\#\Delta} \sum_{(j,k)\in\Delta} j \; .$$

The reduced local Whittle contrast function is

$$\hat{W}(\alpha') = \log\left(\sum_{(j,k)\in\Delta} \frac{\mathbf{d}_{j,k}^2}{2^{(2-\alpha')j}}\right) + \delta \log(2)(2-\alpha') \; . \tag{4.1}$$

The local Whittle estimator of α is then defined as

$$\hat{\alpha} := \arg\min_{\alpha'\in(0,2)} \hat{W}(\alpha') \tag{4.2}$$

In order to simplify the proof of our result, from now on, we take Δ of the form

$$\Delta = \{(j,k)\; ; J_0 < j \leq J_1 \; , \quad 0 \leq k \leq n_j - 1\} \; ,$$

with

$$J = \max\{j; 2^j \leq (T-M+1)/(M+1)\} \; ,$$
$$n_j = 2^{J-j}$$

and integers J_0 and J_1 such that

$$0 < J_0 < J_1 \leq J. \tag{4.3}$$

The sequence of integers J depends on T in such a way that $2^J \asymp T$. Note that the dependence of the sequences J,J_0,J_1,n_j etc. with respect to T is omitted in our notations.

4.2 Consistency

Our first result is valid in the case $\alpha \in (1,2)$.

Theorem 4.1. *Let Assumption 1 hold with $\alpha \in (1,2)$ and $p^* \geq 4$. Assume that J_0 and J_1 depend on T in such a way that*

$$\lim_{T\to\infty} J_0 = \lim_{T\to\infty} (J_1 - J_0) = \infty \ , \tag{4.4}$$

$$\limsup_{T\to\infty} J_0/J < 1/\alpha \ . \tag{4.5}$$

Then $\hat{\alpha}$ is a consistent estimator of α.

Our estimator is also consistent in the potentially unstable case, that is when α is not assumed to be in $(1,2)$, provided that the assumptions on the functions ϕ and ψ are strengthened. We assume that

$$\int_{-\infty}^{\infty} s\ \psi(s)\ \mathrm{d}s = 0 \ , \tag{4.6}$$

and there exist constants a and b such that for all $t \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} k\ \phi(t-k) = a + bt \ . \tag{4.7}$$

These conditions are not satisfied by the Haar wavelet, but hold for any Daubechies wavelets; cf. Cohen [2003].

Theorem 4.2. *Let Assumption 1 hold with $\alpha \in (0,2)$ and $p^* \geq 4$. Assume that J_0 and J_1 depend on T in such a way that (4.4) and (4.5) hold, and*

$$\limsup_{T\to\infty} J_1/J < 1/(2-\alpha) \ . \tag{4.8}$$

Then $\hat{\alpha}$ is a consistent estimator of α.

Remark 4.1. Conditions (4.4), (4.5) and (4.8) are satisfied by the choice $J_0 = \lfloor J/2 \rfloor$ and $J_1 = \lfloor J/2 + \log(J) \rfloor$.

Proof of Theorems 4.1 and 4.2. For clarity of notation, we denote $\sum_j = \sum_{j=J_0+1}^{J_1}$, $\Delta_j := \{k :\ (j,k) \in \Delta\}$ and $\#\Delta_j = n_j$. Elementary computations give

$$\delta = J_0 + 2 + (J_0 - J_1)/(2^{J_1 - J0} - 1) \tag{4.9}$$

so that $\delta - (J_0 + 2) \to 0$ under (4.4). By Karamata's representation theorem, the slowly varying function $\mathcal{L}$ defined in (3.6) can be written as

$$\mathcal{L}(z) = c\,(1 + r(z))\ \exp\left\{\int_1^z \frac{\ell(s)}{s}\mathrm{d}s\right\} \ ,$$

with $c>0$ and $\lim_{z\to\infty}\ell(z)=\lim_{z\to\infty}r(z)=0$. Define $\mathcal{L}_0(z)=c\exp\left\{\int_1^z\frac{\ell(s)}{s}\mathrm{d}s\right\}$, $r^*(z)=\sup_{z'\geq z}|r(z')|$ and $\ell^*(z)=\sup_{z'\geq z}|\ell(z')|$. The functions r^* and ℓ^* are non increasing and tend to zero at infinity. Introduce some notation that will be used throughout the proof.

$$W(\alpha')=\log\left(\sum_j 2^{(\alpha'-\alpha)j}n_j\mathcal{L}(2^j)\right)+\delta\log(2)(2-\alpha')\;,$$

$$W_0(\alpha')=\log\left(\sum_j 2^{(\alpha'-\alpha)j}n_j\mathcal{L}_0(2^j)\right)+\delta\log(2)(2-\alpha')\;,$$

$$w_{j,0}(\alpha'):=\frac{2^{(\alpha'-\alpha)j}n_j\mathcal{L}_0(2^j)}{\sum_{j'}2^{(\alpha'-\alpha)j'}n_{j'}\mathcal{L}_0(2^{j'})}\;,\quad w_j(\alpha'):=\frac{2^{(\alpha'-\alpha)j}n_j\mathcal{L}(2^j)}{\sum_{j'}2^{(\alpha'-\alpha)j'}n_{j'}\mathcal{L}(2^{j'})}\;,$$

$$v_j=\mathcal{L}(2^j)\,2^{(2-\alpha)j}\;,\quad \Lambda_j=n_j^{-1}\sum_{k=0}^{n_j-1}\left\{v_j^{-1}(\mathbf{d}_{j,k})^2-1\right\}\;,$$

$$E(\alpha'):=\sum_j w_j(\alpha')\Lambda_j\;.$$

We have

$$W(\alpha')-W_0(\alpha')=\log\left(1+\frac{\sum_j 2^{(\alpha'-\alpha)j}n_j\,\mathcal{L}_0(2^j)\,r(2^j)}{\sum_j 2^{(\alpha'-\alpha)j}n_j\mathcal{L}_0(2^j)}\right)\;.$$

The fraction inside the logarithm in the last display is bounded by $r^*(2^{J_0})$, thus for J large enough

$$\sup_{\alpha'}|W(\alpha')-W_0(\alpha')|\leq Cr^*(2^{J_0})\;.$$

Standard algebra yields

$$W_0'(\alpha')=\log 2\sum_j w_{j,0}(\alpha')(j-\delta)$$
$$W_0''(\alpha')=\log^2(2)\sum_j w_{j,0}(\alpha')(j-\sum_{j'}w_{j',0}(\alpha')j')^2.$$

By Lemma A.13, under (4.4),

$$\lim_{T\to\infty}W_0'(\alpha)=0\;,\;\lim_{T\to\infty}W_0''(\alpha)=2\;.$$

Thus, there exist $\eta>0$ and $\zeta>0$ such that

$$\liminf_{T\to\infty}\inf_{\alpha'\in(\alpha-\eta,\alpha+\eta)}W_0''(\alpha')>\zeta\;.$$

This implies that for large T, and some positive constant c

$$W(\hat\alpha)-W(\alpha)\geq W_0'(\alpha)\log(2)(\hat\alpha-\alpha)+c(\hat\alpha-\alpha)^2-2r^*(2^{J_0})\tag{4.10}$$

Since $W_0'(\alpha) \to 0$ and $|\hat{\alpha} - \alpha| \leq 2$, this implies that for all $\epsilon > 0$,

$$\limsup_{T\to\infty} \mathbb{P}((\hat{\alpha} - \alpha)^2 > \epsilon) \leq \limsup_{T\to\infty} \mathbb{P}(W(\hat{\alpha}) - W(\alpha) \geq c\epsilon) \; . \tag{4.11}$$

Write

$$\begin{aligned} \hat{W}(\alpha') &= W(\alpha') + \log\left\{1 + E(\alpha')\right\} \; , \\ W(\hat{\alpha}) - W(\alpha) &= \hat{W}(\hat{\alpha}) - \hat{W}(\alpha) - \log\left\{1 + E(\hat{\alpha})\right\} + \log\left\{1 + E(\alpha)\right\} \\ &\leq 2 \sup_{\alpha' \in (0,2)} \left|\log\left\{1 + E(\alpha')\right\}\right| \; . \end{aligned} \tag{4.12}$$

Consistency will follow from (4.11) and (4.12) provided that we can prove that $\sup_{\alpha' \in (0,2)} |E(\alpha')| = o_P(1)$. If $\alpha > 1$, take $\varepsilon \in (0, (\alpha-1)/2)$ such that $\limsup J_0/J < 1/(\alpha+\varepsilon)$ which is possible by assumption (4.5). Define

$$J_2 = \begin{cases} J_1 \text{ if } \alpha \leq 1 \; , \\ J_1 \wedge [J/(\alpha+\varepsilon)] \; \text{ if } \alpha > 1 \; . \end{cases} \tag{4.13}$$

so that, for T large enough, $J_0 < J_2 \leq J_1$. Write

$$E(\alpha') = \sum_{j=J_0+1}^{J_2} w_j(\alpha')\Lambda_j + \sum_{j=J_2+1}^{J_1} w_j(\alpha')\Lambda_j =: E_1(\alpha') + E_2(\alpha') \; ,$$

with the convention $\sum_{j=J_2+1}^{J_1} = 0$ if $J_2 = J_1$. By Lemma A.11,

$$\sup_{\alpha' \in (0,2)} |E_1(\alpha')| = O_P(2^{-\xi_1 J}) \tag{4.14}$$

for some positive ξ_1. Treat now E_2 for $\alpha > 1$ and $J_2 = [J/(\alpha+\varepsilon)] > J_1$. For all $\alpha' \in (0, 2)$, we have $\alpha' - \alpha - 1 < -2\epsilon$. Since $\mathcal{L}$ is slowly varying, we obtain, for some positive constant C, for all $j = J_2 + 1, J_2 + 2, \ldots, J_1$, $w_j(\alpha') \leq C2^{-\epsilon(J_2 - J_0)}$. Using Lemma A.10, it follows that

$$\mathbb{E}\left[\sup_{\alpha' \in (0,2)} |E_2(\alpha')|\right] \leq C\,(J_1 - J_2)\, 2^{-\epsilon(J_2 - J_0)} = O(2^{-\xi_2 J}) \; , \tag{4.15}$$

for some $\xi_2 > 0$ because $\limsup J_0/J < 1/(\alpha+\varepsilon)$. This concludes the proof. $\square$

4.3 Rate of Convergence in the stable case

Theorem 4.3. *Let Assumption 1 hold with $\alpha \in (1,2)$ and $p^* = 4$. Assume moreover that L_4 is bounded and that $\mathcal{L}(z) = c + O(z^{-\beta})$ with $c > 0$ and $\beta > 0$. Assume that the observed process $\mathbf{X}$ is either X_+ or X_S.*

If $\mathbf{X}$ is observed continuously on $[0,T]$, so that $\mathbf{d}_{j,k} = d^S_{j,k}$ or $\mathbf{d}_{j,k} = d_{j,k}$. Then the rate of convergence in probability of $\hat{\alpha}$ is $T^{-\beta/(2\beta+\alpha)}$, obtained for $J_0 = \lfloor J/\{2\beta+\alpha\} \rfloor$ and $J_1 = J$.

If $\mathbf{X}$ is observed at discrete time points $1, 2, \ldots, T$, so that $\mathbf{d}_{j,k} = d^{SD}_{j,k}$ or $\mathbf{d}_{j,k} = d^D_{j,k}$. Then the rate of convergence in probability of $\hat{\alpha}$ is $T^{-\gamma/(2\gamma+\alpha)}$ with $\gamma = \beta \wedge (2-\alpha)$, obtained for $J_0 = \lfloor J/\{2\gamma+\alpha\} \rfloor$ and $J_1 = J$.

Remark 4.2. These rates of convergence are infeasible in practice since the choice of J_0 depends on the parameter to be estimated α and the unknown smoothness parameter β. They do not depend on the stationarity of the observed process but only on the observation scheme. The rate of convergence in the case of discrete observations is at most $T^{-(2-\alpha)/(4-\alpha)}$. This degradation of the rate of convergence is caused by a phenomenon similar to aliasing. It is clear that these rates of convergence are the best possible for our estimator under the assumption on $\mathcal{L}$, since this choice of J_0 makes the squared bias and the variance of the same order of magnitude. But this of course does not prove that this is the best possible rate of convergence for the estimation of α under this observation scheme.

The rate of convergence of our estimator is derived under non primitive assumptions on the function $\mathcal{L}$. The following lemma allows to check them through conditions on the joint distribution of (U, η).

Lemma 4.4. *Let Assumption 1 hold.*

(i) If there exists positive constants c and β such that, as $t \to \infty$,

$$L_2(t) = c + O(t^{-\beta}),$$

then there exists a constant c' such that, as $z \to \infty$,

$$\mathcal{L}(z) = c' + \begin{cases} O(z^{-\beta}) & \text{if } \beta < 2-\alpha, \\ O(z^{\alpha-2}\log z) & \text{if } \beta = 2-\alpha, \\ O(z^{\alpha-2}) & \text{if } \beta > 2-\alpha. \end{cases} \tag{4.16}$$

(ii) If there exists positive constants c and β such that, as $t \to 0$,

$$\mathbb{E}[U^2\{\cos(\eta t) - 1\}] = c\,|t|^{-\alpha}\{1 + O(|t|^{\beta})\},$$

then there exists a constant c' *such that, as* $z \to \infty$,

$$\mathcal{L}(z) = c' + O(z^{-\beta}) \;, \tag{4.17}$$

provided that ψ *belongs to the Sobolev space* $W^{(\alpha+\beta)/2-1}$.

Example 4.1. Assume that η has a Pareto distribution, *i.e.* $\mathbb{P}(\eta > t) = (1 \vee t)^{-\alpha}$, and is independent of U. This corresponds to Lemma 4.4(i) with $\beta = \infty$, and we can easily compute an exact expression for the $O(z^{\alpha-2})$ term:

$$\mathcal{L}(z) = c' + \frac{\alpha \mathbb{E}[U^2]}{2-\alpha} z^{\alpha-2} + o(z^{\alpha-2}) \;.$$

The best possible rate of convergence of $\hat{\alpha}$ is thus $T^{-(2-\alpha)/(4-\alpha)}$, irrelevant of the observation scheme.

Example 4.2. Let $\alpha \in (1,2)$ and suppose that η is the absolute value of a symmetric α-stable random variable. Then Assumption 1 holds, say, if U is independent of η and has sufficiently many finite moments, and

$$\mathbb{E}[\cos(\eta t)] = \exp(-\sigma\,|t|^\alpha) = 1 - \sigma\,|t|^\alpha + O(|t|^{2\alpha}).$$

By Lemma 4.4, the best possible rate of convergence of $\hat{\alpha}$ is thus $T^{-\gamma/(2\gamma+\alpha)}$ with $\gamma = \alpha$ for continuous time observations and $\gamma = 2 - \alpha$ for discrete time observations.

Proof of Theorem 4.3. In the following, we give a decomposition of the error valid under the assumption

$$0 < \liminf_{T\to\infty} \frac{J_0}{J} \le \limsup_{T\to\infty} \frac{J_0}{J} < 1.$$

Optimizing J_0 in this decomposition will then give the result. We use the same notations as in the Proof of Theorems 4.1 and 4.2 with $J_1 = J$. We first give a first rough rate of convergence for $\hat{\alpha}$ by adapting the proof of Theorem 4.2. Under the present assumptions, $\mathcal{L}_0(z) = c$, which implies $W_0'(\alpha) = 0$, and $r^*(z) = O(z^{-\beta})$ as $z \to \infty$. Then, (4.10), (4.12), (4.14) and (4.15) yield:

$$(\hat{\alpha} - \alpha)^2 = O_P(2^{-\xi J} + 2^{-\beta J_0}). \tag{4.18}$$

Since $\hat{\alpha}$ is consistent and α is an interior point of the parameter set, the first derivative of the contrast function vanishes at $\hat{\alpha}$ with probability tending to one. Hence

$$0 = \frac{\sum_{(j,k)\in\Delta} j\, 2^{(\hat{\alpha}-2)j} \mathbf{d}_{j,k}^2}{\sum_{(j,k)\in\Delta} 2^{(\hat{\alpha}-2)j} \mathbf{d}_{j,k}^2} \;-\; \delta \log(2) \;.$$

By definition of δ, this yields

$$
\begin{aligned}
0 &= \sum_{(j,k)\in\Delta} (j-\delta)\, 2^{(\hat\alpha-2)j} \mathbf{d}_{j,k}^2 \\
&= \sum_{(j,k)\in\Delta} (j-\delta)\, 2^{(\alpha-2)j} \mathbf{d}_{j,k}^2 + \log(2)(\hat\alpha-\alpha) \sum_{(j,k)\in\Delta} j(j-\delta) 2^{(\tilde\alpha-2)j} \mathbf{d}_{j,k}^2
\end{aligned}
$$

for a random $\tilde\alpha$ between α and $\hat\alpha$. By definition of Λ_j, the last display implies

$$
\hat\alpha - \alpha = -\frac{\sum_j (j-\delta) 2^{-j} \mathcal{L}(2^j)(1+\Lambda_j)}{\log 2 \sum_j j(j-\delta) 2^{(\tilde\alpha-\alpha-1)j} \mathcal{L}(2^j)(1+\Lambda_j)} \; .
$$

Denote the sum in the denominator of the last expression by D, and write

$$
\begin{aligned}
D &= \sum\nolimits_j j(j-\delta) 2^{-j} \mathcal{L}(2^j) + \sum\nolimits_j j(j-\delta) \mathcal{L}(2^j) 2^{-j} (2^{(\tilde\alpha-\alpha)j} - 1)(1+\Lambda_j) \\
&\quad + \sum\nolimits_j j(j-\delta) 2^{-j} \mathcal{L}(2^j) \Lambda_j =: S + R_1 + R_2 \; .
\end{aligned}
$$

Using Lemma A.13 and (4.9), one easily gets that $S \sim 2^{1-J_0}$ as $J \to \infty$.

Using Lemma A.10, and the fact that $|\tilde\alpha - \alpha| \le |\hat\alpha - \alpha| = o_P(J^{-2})$, one gets similarly $R_1 = o_P(2^{-J_0})$. To bound R_2, we proceed as for bounding $E(\alpha')$ in the proof of Theorems 4.1 and 4.2 (here with $\alpha' = \alpha > 1$): we write $\sum_j = \sum_{j=J_0+1}^{J_2} + \sum_{j=J_2+1}^{J}$ and apply Lemmas A.10 and A.11 to obtain $R_2 = o_P(2^{-J_0})$. Hence, we finally get

$$
\hat\alpha - \alpha = \frac{2^{J_0}}{2\log 2} \left\{ \sum\nolimits_j (j-\delta) 2^{-j} \mathcal{L}(2^j) \; + \sum\nolimits_j (j-\delta) 2^{-j} \mathcal{L}(2^j) \Lambda_j \right\} \{1 + o_p(1)\} \; . \tag{4.19}
$$

In (4.19), the terms inside the curly brackets are interpreted as a deterministic bias term and a stochastic fluctuation term. The bias is bounded as follows:

$$
2^{J_0} \sum\nolimits_j (j-\delta) 2^{-j} \mathcal{L}(2^j) = 2^{J_0} \sum\nolimits_j (j-\delta) 2^{-j} (\mathcal{L}(2^j) - c) = O(2^{-\beta J_0}) \; . \tag{4.20}
$$

In the case of continuous time observations, *i.e.* $\mathbf{d}_{j,k} = d_{j,k}^S$ or $\mathbf{d}_{j,k} = d_{j,k}$, we have

$$
\sum\nolimits_j (j-\delta) 2^{-j} \mathcal{L}(2^j) \Lambda_j = O_P(2^{-J/2 + (\alpha/2-1)J_0}) \; . \tag{4.21}
$$

Gathering this bound with (4.19) and (4.20), and setting $J_0 = J/(2\beta+\alpha)$ yields the first claim of Theorem 4.3, *i.e.* $\hat\alpha - \alpha = O_P(2^{-\beta/(2\beta+\alpha)})$. We now prove (4.21). Define $\beta_j = n_j^{-1} \sum_{k=0}^{n_j-1} \{v_j^{-1} (d_{j,k}^S)^2 - 1\}$. Then $\beta_j = \Lambda_j$ if $\mathbf{d}_{j,k} = d_{j,k}^S$. Since $\alpha > 1$, Lemmas 3.1 and A.2 yield, for some positive constant C,

$$
\mathbb{E}[\beta_j^2] = \mathrm{var}(\beta_j) \le C \, \frac{L_4(2^j)}{\mathcal{L}^2(2^j)} 2^{\alpha j - J} \; . \tag{4.22}
$$

Since $\mathcal{L}$ is bounded away from zero and L_4 is bounded, the ratio $L_4/\mathcal{L}^2$ is also bounded. The Minkowski inequality then yields, for some constant $C > 0$,

$$\mathbb{E}\left[\left(\sum_j (j-\delta)2^{-j}\mathcal{L}(2^j)\beta_j\right)^2\right]^{1/2} \leq C\, 2^{-J/2} \sum_j |j-\delta|\, 2^{(\alpha/2-1)j} = O(2^{-J/2+(\alpha/2-1)J_0}) \; .$$

If $\mathbf{d}_{j,k} = d_{j,k}$, we use (A.40) in Lemma A.10, and obtain $\mathbb{E}[|\Lambda_j - \beta_j|] \leq C\, n_j^{-1/2}$ for some constant $C > 0$. Hence, in this case, since $-1/2 < \alpha/2 - 1$,

$$\mathbb{E}\left[\left|\sum_j (j-\delta)2^{-j}\mathcal{L}(2^j)\,(\Lambda_j - \beta_j)\right|\right] \leq C\, 2^{-J/2} \sum_j |j-\delta|\, \mathcal{L}(2^j)\, 2^{-j/2} = o(2^{-J/2+(\alpha/2-1)J_0}) \; .$$

The last two displays imply (4.21).

We now briefly adapt the previous proof to the case of discrete observations. Define $v_j^D = \mathbb{E}[(d_{j,k}^{SD})^2]$. Lemma A.6 [iii] implies $v_j^D = v_j + O(1)$. Thus we have $v_j^D = \mathcal{L}_D(2^j)2^{(2-\alpha)j}$ and

$$\mathcal{L}_D(z) = \mathcal{L}(z) + O(z^{\alpha-2}) = c + O(z^{-\gamma}) \; ,$$

with $\gamma = \beta \wedge (2-\alpha)$. Then, defining

$$\Lambda_j^D = n_j^{-1} \sum_{k=1}^{n_j} \{(v_j^D)^{-1}\mathbf{d}_{j,k}^2 - 1\} \; ,$$

we obtain that (4.19) still holds with $\mathcal{L}_D$ and Λ^D replacing $\mathcal{L}$ and Λ, respectively. Lemma A.12 implies that Λ_j^D has the same order of magnitude as Λ_j, so that the stochastic fulctuation term has the same order of magnitude as in the previous case. The difference comes from the bias term, which is $O(2^{-\gamma J_0})$. Thus, $\hat{\alpha} - \alpha = O_P(2^{-\gamma J_0} + 2^{-\gamma J_0})$, and setting $J_0 = J/(2\gamma + \alpha)$ yields the second claim of Theorem 4.3.

□

5 Concluding remarks

We have showed in this work that wavelet methods for estimating the long memory parameter of a standard traffic model are consistent, robust to non-stationary situations, even when the process do not converge to a stationary state, and robust to discrete data sampling. However the study of the rates raises some questions as it might cast some doubt on the quality of the estimator of the tail index α, the main parameter of the model from a traffic modeling point of view. To make a comparison, let us recall that, if an sample of size n of i.i.d. random variables with survival function of $\bar{F}(t) = t^{-\alpha}\{1 + O(t^{-\beta})\}$, then a rate of convergence $n^{-\beta/(2\beta+\alpha)}$ can be obtained, for instance using the Hill estimator, cf.

Hall and Welsh [1984]. For instance, a sample of size n of i.i.d. Pareto random variable is observed, then a parametric rate can be obtained, while the wavelet estimator has a dramatically deteriorating rate for α close to 2. But this comparison is not completely relevant, especially in the case of discrete observations, since the durations η_k are not observed. Finally, let us draw a practical conclusion from our study. Some precaution should be made on the choice of scales used in the estimation as shown by the conditions on J_0 and J_1. In particular, if only discrete observations are available, the best possible rate of convergence is obtained for a much bigger value of J_0 than if continous observations are available. Too small a value of J_0 will induce an important bias in finite samples. Praticitioners should be aware of this restriction and be careful in the interpretation of the results.

Appendix

A Technical results

Lemma A.1. *Let Assumption 1 hold. Let f be a bounded measurable compactly supported function such that $\int f(s)\,\mathrm{d}s = 0$. Define*

$$\tilde{f}(t,v,w) = w \int_t^{t+v} f(s)\,\mathrm{d}s \; .$$

Then $\int |\tilde{f}(t,v,w)|^p \,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w) < \infty$,

$\mathbb{E}[N_S(\tilde{f})] = 0$ and $\int_0^\infty X_+(s) f(s)\,\mathrm{d}s = N_S(\tilde{f}\mathbb{1}_{\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}})$. If moreover $\mathbb{E}[\eta] < \infty$, then $N_S(\tilde{f}) = \int X_S(s)\, f(s)\,\mathrm{d}s$.

Proof. Assume that f has compact support in $[0, M]$. Then $\int_t^{t+v} f(s)\,\mathrm{d}s = 0$ if $t + v < 0$ or $t > M$. If moreover $\int_0^M f(s)\,\mathrm{d}s = 0$, then the former integral also vanishes if $t < 0$ and $t + v > M$. Then, denoting $\|f\|_1 = \int |f(s)|\,\mathrm{d}s$, we have

$$\int_{-\infty}^{\infty} \left| \int_t^{t+v} f(s)\,\mathrm{d}s \right|^p \,\mathrm{d}t \le \|f\|_1^p \int_{-\infty}^{\infty} \{\mathbb{1}_{[0,M]}(t) + \mathbb{1}_{[0,M]}(t+v)\}^p \,\mathrm{d}t = 2^p M \|f\|_1^p \; .$$

Thus $\int |\tilde{f}(t,v,w)|^p \,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w) \le 2^p M \|f\|_1^p \mathbb{E}[|U|^p]$. The fact that $\mathbb{E}[N_S(\tilde{f})] = 0$ is then a straightforward application of Fubini's Theorem. The other properties follow directly from the definition of N_S. $\square$

For $p = 1, \ldots, p^*$, define the signed measures

$$\nu_p(\mathrm{d}v) := \int u^p \nu(\mathrm{d}v, \mathrm{d}u) \ . \tag{A.1}$$

Proof of Lemma 3.1. Using the above notation, we have $d^S_{j,k} = N_S(\tilde{\psi}_{j,k})$, since (3.1) implies that $\int \psi_{j,k}(s)\,\mathrm{d}s = 0$ for all j, k, hence the wavelet coefficients have zero mean by Lemma A.1. To compute the variance, we apply the cumulants formula (2.4) and a change of variable.

$$\begin{aligned}\mathrm{var}(d^S_{j,k}) &= \int_0^\infty \nu_2(\mathrm{d}v) \int_{-\infty}^\infty \left\{ \int_t^{t+v} \psi_{j,k}(s)\,\mathrm{d}s \right\}^2 \mathrm{d}t \\ &= 2^{2j} \int_0^\infty \nu_2(\mathrm{d}v) \int_{-\infty}^\infty \left\{ \int_t^{t+2^{-j}v} \psi(s)\,\mathrm{d}s \right\}^2 \mathrm{d}t = 2^{(2-\alpha)j} \mathcal{L}(2^j) \ ,\end{aligned}$$

by definition of $\mathcal{L}$. To conclude the proof, we show the asymptotic equivalence for $\mathcal{L}$. For $p = 2, \ldots,$ define

$$K_p(y) := \int_{-\infty}^\infty \left\{ \int_t^{t+y} \psi(u)\,\mathrm{d}u \right\}^p \mathrm{d}t = \int_{-\infty}^\infty (\Psi(t+y) - \Psi(t))^p\,\mathrm{d}t \ , \tag{A.2}$$

where $\Psi(u) := \int_0^u \psi(y)\,\mathrm{d}y$. By (3.1), Ψ has compact support in $[0, M]$ and,

$$|K_p(y)| \leq \int_{-M}^M |\Psi(t+y) - \Psi(t)|^p\,\mathrm{d}t \leq 2M\, \|\psi\|_\infty^p\, |y|^p, \quad y \in \mathbb{R} \ . \tag{A.3}$$

If $|y| \geq M$, either $\Psi(t+y)$ or $\Psi(t)$ vanishes for any t, hence

$$K_p(y) = (1 + (-1)^p) \|\Psi\|_p^p, \quad |y| \geq M \ . \tag{A.4}$$

From (A.2), the two first derivatives of K_p can be expressed as

$$K_p'(y) = p \int_0^M \psi(u)(\Psi(u) - \Psi(u-y))^{p-1}\,\mathrm{d}u \ , \tag{A.5}$$

$$K_p''(y) = p(p-1) \int_0^M \psi(u)\psi(u-y)(\Psi(u) - \Psi(u-y))^{p-2}\,\mathrm{d}u \ , \tag{A.6}$$

which are continuous by continuity of Ψ, and with support in $[-M, M]$ by (A.4). Inserting $K_2(v/z) = \int_0^{v/z} K_2'(t)\mathrm{d}t$ in (3.6) and inverting the integrals yield

$$\mathcal{L}(z) = z^\alpha \int_0^M K_2'(t) H_2(z\,t)\,\mathrm{d}t = L_2(z) \int_0^M \frac{L_2(zt)}{L_2(z)} K_2'(t) t^{-\alpha}\,\mathrm{d}t \sim_{z\to\infty} C\, L_2(z) \ , \tag{A.7}$$

Indeed, for any $\epsilon \in (0, M)$

$$L_2(z) \int_\epsilon^M \frac{L_2(zt)}{L_2(z)} K_2'(t) t^{-\alpha} \, \mathrm{d}t \sim L_2(z) \int_\epsilon^M K_2'(t) t^{-\alpha} \, \mathrm{d}t$$

as $z \to \infty$, by the Uniform Convergence Theorem for slowly varying functions. Since $|K_2'(y)| \le c|y|$, we have for $z \ge 1/\epsilon$

$$\begin{aligned}\int_0^\epsilon L_2(zt) K_2'(t) t^{-\alpha} \, \mathrm{d}t &= z^{\alpha-1} \int_0^{z\epsilon} L_2(u) K_2'(u/z) u^{-\alpha} \mathrm{d}u \\ &\le C z^{\alpha-2} \{ \int_0^1 L_2(u) u^{-\alpha+1} \mathrm{d}u + \int_1^{z\epsilon} L_2(u) u^{-\alpha+1} \mathrm{d}u \}\end{aligned}$$

Using $u^{-\alpha} L_2(u) \sim \mathbb{E}|U|^2$ as $u \to 0$ and Karamata's Theorem, we obtain the bound

$$L_2(z)^{-1} \int_0^\epsilon L_2(zt) K_2'(t) t^{-\alpha} \, \mathrm{d}t \le C L_2(z)^{-1} z^{\alpha-2} + \epsilon^{2-\alpha} \frac{L_2(z\epsilon)}{L_2(z)} \le C \epsilon^{2-\alpha} \, .$$

(A.7) follows by letting ϵ go to zero and noting that since $K_2'(0) = 0$, $C := \int_0^M K_2'(t) t^{-\alpha} \, \mathrm{d}t = \alpha \int_0^M K_2(t) t^{-\alpha-1} \, \mathrm{d}t$ is a positive constant. $\square$

Proof of Lemma 4.4. From (A.7), we have

$$\mathcal{L}(z) = \int_0^M K_2'(t) \, t^{-\alpha} \{ c + r(zt) \} \, \mathrm{d}t = c' + \int_0^M K_2'(t) \, t^{-\alpha} \, r(zt) \, \mathrm{d}t \, ,$$

where $L_2(t) = c + r(t)$. Since $r(t) = O(t^{-\beta})$ as $t \to \infty$ and $r(t) = H_2(t) t^\alpha - c$, there exists $C > 0$ such that $|r(t)| \le C t^{-\beta}$. By (A.5) and (A.6), we have $K_2'(t) \sim \|K_2\|_2 \, t$ as $t \to 0$. If $\beta < 2 - \alpha$, Eq (4.16) follows immediately. For $\beta \ge 2 - \alpha$, let us write

$$\begin{aligned}\left| \int_0^M K_2'(t) \, t^{-\alpha} \, r(zt) \, \mathrm{d}t \right| &= \left| \int_0^{z^{-1}} K_2'(t) \, t^{-\alpha} \, r(zt) \, \mathrm{d}t + \int_{z^{-1}}^M K_2'(t) \, t^{-\alpha} \, r(zt) \, \mathrm{d}t \right| \\ &\le C_1 \int_0^{z^{-1}} t^{1-\alpha} \, \mathrm{d}t + C_2 \, z^{-\beta} \int_{z^{-1}}^M t^{1-\alpha-\beta} \, \mathrm{d}t \, ,\end{aligned}$$

where we used that $r(zt) = H_2(zt)(zt)^\alpha - c$ is bounded for $zt \le 1$ for the first integral, and $|r(t)| \le C t^{-\beta}$ for the second one. This yields (4.16).

Let us now show (4.17). Let ψ^* denotes the Fourier transform of ψ,

$$\psi^*(\xi) = \int_0^M \psi(t) \mathrm{e}^{-\mathrm{i}\xi t} \, \mathrm{d}t \, .$$

The Sobolev condition on ψ and (3.1) imply that

$$\int |\xi|^{\alpha+\beta-2} |\psi^*(\xi)|^2 \,\mathrm{d}\xi < \infty$$

so that the lemma is a direct consequence of the following formula; for all $z > 0$,

$$\mathcal{L}(z) = \frac{z^\alpha}{\pi} \int_{-\infty}^{\infty} \xi^{-2} |\psi^*(\xi)|^2 \,\mathbb{E}[U^2 \{\cos(\eta\xi/z) - 1\}] \,\mathrm{d}\xi \,. \tag{A.8}$$

Prove now (A.8). For $p = 2$, (A.5) gives, since $\psi\Psi$ integrates to $\Psi^2/2$ and $\Psi(0) = \Psi(M) = 0$,

$$K_2'(y) = 2 \int_0^M \psi(u)(\Psi(u) - \Psi(u-y)) \,\mathrm{d}u = -2 \int_{-\infty}^{\infty} \psi(u)\, \Psi(u-y) \,\mathrm{d}u \,.$$

Using standard calculations, one gets

$$\int_{-\infty}^{\infty} K_2'(y)\, \mathrm{e}^{-\mathrm{i}y\xi} \,\mathrm{d}y = \frac{2\mathrm{i}}{\xi} |\psi^*(\xi)|^2 \,.$$

The function K_2' is bounded and compactly supported (see the proof of Lemma 3.1), hence square integrable. By Parseval's formula, we find, for all $y \geq 0$,

$$K_2(y) = \int_{-\infty}^{\infty} \mathbb{1}_{[0,y]}(u)\, K_2'(u) \,\mathrm{d}u = \pi^{-1} \int_{\xi=-\infty}^{\infty} \xi^{-2} |\psi^*(\xi)|^2 (\mathrm{e}^{\mathrm{i}\xi y} - 1) \,\mathrm{d}\xi \,.$$

From (A.7), we have

$$\mathcal{L}(z) = z^\alpha \int_0^\infty K_2'(t) H_2(z\,t) \,\mathrm{d}t = z^\alpha \mathbb{E}[U^2 K_2(\eta/z)] \,,$$

Since $\xi^{-2} |\psi^*(\xi)|^2$ is integrable, we may replace K_2 by the above formula and exchange expectation and integration signs, giving

$$\mathcal{L}(z) = \frac{z^\alpha}{\pi} \int_{\xi=-\infty}^{\infty} \xi^{-2} |\psi^*(\xi)|^2 \,\mathbb{E}[U^2 (\mathrm{e}^{\mathrm{i}\xi t} - 1)] \,\mathrm{d}\xi \,.$$

Observing that $\xi^{-2} |\psi^*(\xi)|^2$ is an even function of ξ and that the imaginary part of the expectation is an odd function of ξ, we get (A.8). $\square$

Lemma A.2. *Let Assumption 1 hold with $p^* \geq 4$. Then, there exists a positive constant $C > 0$ such that*

$$\mathrm{var}\left(\sum_{k=0}^{n-1} (d_{j,k}^S)^2 \right) \leq Cn \left\{ L_2^2(2^j)\, 2^{(4-2\alpha)j} + L_4(2^j)\, 2^{(3-\alpha)j} \right\} \,.$$

Note that the first term dominates for $\alpha < 1$ and the second one dominates for $\alpha > 1$.

Proof. Write

$$\mathrm{var}\left(\sum_{k=0}^{n-1}(d^S_{j,k})^2\right) = 2\sum_{k,k'=0}^{n-1}\mathrm{cov}^2(d^S_{j,k},d^S_{j,k'}) + \sum_{k,k'=0}^{n-1}\mathrm{cum}(d^S_{j,k},d^S_{j,k},d^S_{j,k'},d^S_{j,k'})\ . \tag{A.9}$$

For $p \in \mathbb{N}^*$, $k \in \mathbb{Z}$ and $v \geq 0$, define

$$K_p(v,k) := \int_{-\infty}^{\infty}\left\{\int_t^{t+v}\psi(s-k)\,\mathrm{d}s\right\}^p\left\{\int_t^{t+v}\psi(s')\,\mathrm{d}s'\right\}^p\ \mathrm{d}t\ .$$

We have the following properties: $K_p(v,-k) = K_p(v,k)$, by Cauchy Schwartz Inequality, for all $k \in \mathbb{Z}$ and $v > 0$, $|K_p(v,k)| \leq K_p(v,0) = K_{2p}(v)$, where we use the notation (A.2); if $k > v+M$, then the support $[0,M]$ of ψ cannot intersect both $[t-k,t-k+v]$ and $[t,t+M]$ so that $K_p(v,k)=0$; if $k > M$ and $v > k+M$, then the support of ψ cannot intersect both $[t-k,t-k+v]$ and $[t,t+M]$ without being included in one of them, so that, by (3.1), $K_p(v,k)=0$. Gathering the above facts gives, for all $k \in \mathbb{Z}$ and $v > 0$,

$$|K_p(v,k)| \leq K_{2p}(v)\left\{\mathbb{1}(|k|\leq M) + \mathbb{1}(v \in [|k|-M,|k|+M])\right\}\ . \tag{A.10}$$

Recall that $d^S_{j,k} = N_S(\tilde{\psi}_{j,k})$. Thus Formula (2.4) implies

$$\mathrm{cov}(d^S_{j,k},d^S_{j,k'}) = 2^{2j}\int_0^{\infty}K_1(v,|k-k'|)\,\nu_2(2^j\mathrm{d}v)\ , \tag{A.11}$$

$$\mathrm{cum}(d^S_{j,k},d^S_{j,k},d^S_{j,k'},d^S_{j,k'}) = 2^{3j}\int_0^{\infty}K_2(v,|k-k'|)\,\nu_4(2^j\mathrm{d}v)\ . \tag{A.12}$$

Thus, separating the cases $|k-k'| \leq M$ and $|k-k'| > M$ in the sums of the RHS in (A.9) and using (A.10), we get

$$n^{-1}\mathrm{var}\left(\sum_{k=0}^{n-1}(d^S_{j,k})^2\right) \leq\ 4\,(2M+1)\,2^{4j}\ \left(\int_0^{\infty}K_2(v)\ \nu_2(2^j\mathrm{d}v)\right)^2 \tag{A.13}$$

$$+\,4\,2^{4j}\sum_{l=M+1}^{n-1}\left(\int_{l-M}^{l+M}K_2(v)\ \nu_2(2^j\mathrm{d}v)\right)^2 \tag{A.14}$$

$$+\,2\,(2M+1)\,2^{3j}\int_0^{\infty}K_4(v)\,\nu_4(2^j\mathrm{d}v) \tag{A.15}$$

$$+\,2\,2^{3j}\sum_{l=M+1}^{n-1}\int_{l-M}^{l+M}K_4(v)\,\nu_4(2^j\mathrm{d}v) \tag{A.16}$$

The RHS in (A.13) is $4\,(2M+1)\,(\mathrm{var}(d^S_{j,k})$, cf. the proof of Lemma 3.1. In this proof, the properties of K_4 are similar to those of K_2 (see (A.3) and (A.5)) and this yields similarly that (A.15) is asymptotically equivalent to $2(2M+1)L_4(2^j)2^{(3-\alpha)}$. By elementary computations, the term (A.16) is less than

$$2\,2^{3j}\,(2M+1)\int_1^{n+M-1} K_4(v)\,\nu_4(2^j\mathrm{d}v)$$

which is less than (A.15). The term (A.14) is treated similarly by noting that

$$\begin{aligned}\sum_{l=M+1}^{n-1}\left(\int_{l-M}^{l+M} K_2(v)\ \nu_2(2^j\mathrm{d}v)\right)^2 &\leq \int_0^{\infty} K_2(v)\ \nu_2(2^j\mathrm{d}v)\sum_{l=M+1}^{n-1}\int_{l-M}^{l+M} K_2(v)\ \nu_2(2^j\mathrm{d}v)\\ &\leq (2M+1)\left(\int_0^{\infty} K_2(v)\ \nu_2(2^j\mathrm{d}v)\right)^2\end{aligned}$$

so that it is of the same order as (A.13). This concludes the proof. □

Lemma A.3. *Let Assumption 1 hold with $p^* \geq 2$. Then the following assertions hold.*

(i) If $\alpha \in (1/2, 2)$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1}(\mathbb{E}[d_{j,k}])^2 \leq C2^{(3-2\alpha)j}L_1^2(2^j)\ . \tag{A.17}$$

(ii) If $\alpha \in (1, 2)$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1}\mathrm{var}(d_{j,k} - d_{j,k}^S) \leq C\,2^{(2-\alpha)j}\,L_2(2^j)\ . \tag{A.18}$$

(iii) If $\alpha \in (0, 1]$, for all $\epsilon > 0$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1}\mathrm{var}(d_{j,k} - d_{j,k}^S) \leq C\,2^{(2-\alpha)j}\,L_2(2^j)\,n^{1-\alpha+\epsilon}\ . \tag{A.19}$$

(iv) If (4.6) hold then, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1}(\mathbb{E}[d_{j,k}])^2 \leq C2^{(3-2\alpha)j}L_1^2(2^j)\ . \tag{A.20}$$

Proof. By Lemma A.1 we have that $d_{j,k} = N_S(\tilde{\psi}_{j,k} \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}})$. Thus

$$\begin{aligned} \mathbb{E}[d_{j,k}] &= \int_0^\infty \nu_1(\mathrm{d}v) \int_0^\infty \mathrm{d}u \int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w \\ &= -\int_0^\infty \nu_1(\mathrm{d}v) \int_{-\infty}^0 \mathrm{d}u \int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w \;, \end{aligned} \tag{A.21}$$

since (3.1) implies that $\int_{-\infty}^\infty \mathrm{d}u \int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w = 0$. Define $\Psi(t) = \int_0^t \psi(s)\,\mathrm{d}s$. Again, (3.1) implies that Ψ also has compact support in $[0, M]$. Thus

$$\begin{aligned} \int_{-\infty}^0 \mathrm{d}u \int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w &= 2^{3j/2} \int_{-\infty}^{-k} (\Psi(x + 2^{-j}v) - \Psi(x))\,\mathrm{d}x \\ &= 2^{3j/2} \int_0^{(2^{-j}v-k)_+ \wedge M} \Psi(x)\,\mathrm{d}x \;, \end{aligned} \tag{A.22}$$

Inserting (A.22) in (A.21) and exchanging the two integrals give that

$$\begin{aligned} |\mathbb{E}[d_{j,k}]| &\le 2^{3j/2} \int_0^M |\Psi(x)|\, H_1(2^j(x+k))\,\mathrm{d}x \\ &= 2^{(3/2-\alpha)j}\, L_1(2^j) \int_0^M |\Psi(x)|\,(x+k)^{-\alpha}\, \frac{L_1(2^j(x+k))}{L_1(2^j)}\,\mathrm{d}x \;. \end{aligned}$$

Along the lines that lead to (A.7), we get $|\mathbb{E}[d_{j,0}]| \le C 2^{(3/2-\alpha)j}\, L_1(2^j) \int_0^M |\Psi(x)|\, x^{-\alpha}\,\mathrm{d}x$. The integral in this RHS is finite for since $\Psi(x) = O(x)$ at $x = 0$. Note now that for any $\epsilon > 0$, there exists some constant C_ϵ such that $\forall j, t \ge 1$, $L_1(tx)/L_1(t) \le C_\epsilon x^\epsilon$. Then, for $k \ge 0$

$$|\mathbb{E}[d_{j,k}]| \le C\, 2^{(3/2-\alpha)j}\, L_1(2^j) \int_0^M |\Psi(x)|\,(x+k)^{\epsilon-\alpha}\,\mathrm{d}x \;\le C 2^{(3/2-\alpha)j} L_1(2^j)(k \vee 1)^{\epsilon-\alpha}. \tag{A.23}$$

Hence (A.17) for $\alpha > 1/2$ by choosing ϵ small enough.

We now prove (A.18) and (A.19). By Lemma A.1, $d^S_{j,k} - d_{j,k} = N_S(\tilde{\psi}_{j,k} \mathbb{1}_{\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}})$. Applying (2.4) yields

$$\mathrm{var}(d^S_{j,k} - d_{j,k}) = \int_0^\infty \nu_2(\mathrm{d}v) \int_{-\infty}^0 \mathrm{d}u \left(\int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w \right)^2 \;. \tag{A.24}$$

Proceeding as for (A.22), one obtains

$$\begin{aligned} \int_{-\infty}^0 \mathrm{d}u \left(\int_u^{u+v} \psi_{j,k}(w)\,\mathrm{d}w \right)^2 &= 2^{2j} \int_{-\infty}^{-k} (\Psi(x + 2^{-j}v) - \Psi(x))^2\,\mathrm{d}x \\ &= 2^{2j} \int_0^{(2^{-j}v-k)_+ \wedge M} \Psi^2(y)\,\mathrm{d}y \;. \end{aligned} \tag{A.25}$$

The case $\alpha > 1$ is then similar as for establishing (A.17) from (A.22) and (A.21). The case $\alpha \in (0, 1]$ is also similar except that the sum $\sum_{k=0}^{n-1}$ is no longer bounded independently of n.

We now show (A.20). If the function ψ satisfies (4.6) then $\int_0^M \Psi(s)\,\mathrm{d}s = 0$. In (A.22), it gives

$$\left|\int_{-\infty}^{0} \mathrm{d}u \int_{u}^{u+v} \psi_{j,k}(w)\,\mathrm{d}w\right| \leq M\,\|\Psi\|_\infty\,\mathbb{1}(v \in [k2^j, (k+M)2^j])\ . \tag{A.26}$$

Hence, by inserting this in (A.21), for $k \geq 1$

$$|\mathbb{E}[d_{j,k}]| \leq C\,2^{(3/2-\alpha)j}\,L_1(2^j)\{\tilde{H}_1(k,j) - \tilde{H}_1(k+M,j)\}\ , \tag{A.27}$$

where

$$\tilde{H}_1(k,j) := \frac{2^{\alpha j} H_1(k2^j)}{L_1(2^j)} = k^{-\alpha}\frac{L_1(k2^j)}{L_1(2^j)}\ .$$

Since $L_1(k2^j)/L_1(2^j)$ is less than any positive power of k, there exists $K > 0$ such that, for all $k \geq K$ and $j \geq 0$, $\tilde{H}_1(k,j) \leq 1$. Note also that by definition of $\tilde{H}_1(k,j)$, it is nonincreasing with respect to k. Thus, for all $n > K + M$,

$$\begin{aligned}\sum_{k=K}^{n-1}\{\tilde{H}_1(k,j) - \tilde{H}_1(k+M,j)\}^2 &\leq \sum_{k=K}^{n-1}\{\tilde{H}_1(k,j) - \tilde{H}_1(k+M,j)\}\\ &= \sum_{k=K}^{K+M-1}\tilde{H}_1(k,j) - \sum_{k=n}^{n+M-1}\tilde{H}_1(k,j) \leq M\ .\end{aligned}$$

Inserting this in (A.27) and using (A.23) for bounding $\sum_{k=0}^{M-1}(\mathbb{E}[d_{j,k}])^2$ yield (A.20). $\square$

Lemma A.4. *Define, for all $t \in \mathbb{R}$ and $v > 0$,*

$$g_{t,v} = \mathrm{I}_\phi[\mathbb{1}_{[t,t+v)}] \quad \textit{and} \quad h_{t,v} = \mathbb{1}_{[t,t+v)} - g_{t,v}. \tag{A.28}$$

We have the following properties for all $t \in \mathbb{R}$ and all $v > 0$.

(i) $\int g_{s,v}(t)\,\mathrm{d}s = v$,

(ii) $\|g_{t,v}\|_\infty \leq M\,\|\phi\|_\infty$ *and the support of* $g_{t,v}$ *is included in* $[\lceil t\rceil - M + 1, \lceil t+v\rceil]$,

(iii) *the support of* $1 - g_{t,v}$ *is included in* $(-\infty, \lceil t\rceil] \cup [\lceil t+v\rceil - M + 1, \infty)$,

(iv) $\|h_{t,v}\|_\infty \leq 1 + M\,\|\phi\|_\infty$ *and the support of* $h_{t,v}$ *is included in* $[t - M + 1, \lceil t\rceil] \cup [t + v - M + 1, \lceil t+v\rceil]$.

Proof. By definition of $g_{s,v}$ in (A.28), we have, using that ϕ has a compact support and then (3.7),

$$\int g_{s,v}(t)\mathrm{d}s = \int \sum_{k\in\mathbb{Z}} \mathbb{1}_{(k-v,k]}(s)\phi(t-k)\mathrm{d}s = v\sum_{k\in\mathbb{Z}}\phi(t_k) = v \;.$$

We have, for all non-integer $s \in \mathbb{R}$,

$$|g_{t,v}(s)| \le \|\phi\|_\infty \sum_{k\in\mathbb{Z}} \mathbb{1}_{(-M+1,1)}(s) \le M\,\|\phi\|_\infty.$$

Hence the bound $\|h_{t,v}\|_\infty \le 1 + \|g_{t,v}\|_\infty$. Denote by $\lceil s\rceil$ the smallest integer larger than or equal to s. We have

$$g_{t,v}(s) = \sum_{k=\lceil t\rceil}^{\lceil t+v\rceil-1} \phi(s-k), \quad s\in\mathbb{R}, \tag{A.29}$$

so that $g_{t,v}$ has its support included in the segment $[\lceil t\rceil - M + 1, \lceil t+v\rceil]$. It follows that the support of $h_{t,v}$ is included in $[t-M+1, \lceil t+v\rceil]$. Now, by (3.7), we have

$$\begin{aligned} 1 - g_{t,v} &= \mathrm{I}_\phi[1 - \mathbb{1}_{[t,t+v)}] = -\mathrm{I}_\phi[\mathbb{1}_{[t,t+v)^c}], \\ h_{t,v} &= \mathbb{1}_{[t,t+v)^c} - \mathrm{I}_\phi[\mathbb{1}_{[t,t+v)^c}], \end{aligned}$$

where A^c denotes the complementary set of A in $\mathbb{R}$. The support of $\mathrm{I}_\phi[\mathbb{1}_{[t,t+v)^c}]$ is included in $(-\infty, \lceil t\rceil] \cup [\lceil t+v\rceil - M + 1, \infty)$; thus $h_{t,v}$'s one is in $(-\infty, \lceil t\rceil] \cup [t+v-M+1, \infty)$. Hence the result. □

Lemma A.5. *Let Assumption 1 hold. Let f be a bounded measurable compactly supported function such that $\int f(s)\,\mathrm{d}s = 0$. Define*

$$\hat f(t,v,w) = w\int_{-\infty}^{\infty} g_{t,v}(s)f(s)\,\mathrm{d}s\;, \quad \check f(t,v,w) = w\int_{-\infty}^{\infty} h_{t,v}(s)f(s)\,\mathrm{d}s\;.$$

Then, for $p = 1,\dots,p^*$, $\int |\hat f(t,v,w)|^p\,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w) < \infty$, $\int |\check f(t,v,w)|^p\,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w) < \infty$, $\int \mathrm{I}_\phi[X_+](s)\,f(s)\,\mathrm{d}s = N_S(\hat f \mathbb{1}_{\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}})$, and $\mathbb{E}[N_S(\hat f)] = \mathbb{E}[N_S(\check f)] = 0$. If moreover $\mathbb{E}[\eta] < \infty$, then $N_S(\hat f) = \int \mathrm{I}_\phi[X_S](s)\,f(s)\,\mathrm{d}s$.

Proof. Assume that f has compact support in $[0, M]$. Applying Lemma A.4(ii) and proceeding as in the proof of Lemma A.1, we have, for some numerical constant C whose face value may change upon each appearance,

$$\left|\int_{-\infty}^{\infty} g_{t,v}(s)f(s)\,\mathrm{d}s\right| \le C\|f\|_1\{\mathbb{1}_{[M-1,2M-1]}(\lceil t\rceil) + \mathbb{1}_{[0,M]}(\lceil t+v\rceil)\}\;,$$

$$\int |\hat f(t,v,w)|^p\,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w) \le C\,\|f\|_1^p\,\mathbb{E}[|U|^p]\;.$$

Applying Lemma A.4(iv), we obtain a similar bound for $\int \check{f}(t,v,w)\,\mathrm{d}t\,\nu(\mathrm{d}v,\mathrm{d}w)$. Thus we can apply Fubini's Theorem and Lemma A.4(i).

$$\mathbb{E}[N_S(\hat{f})] = \int_{\mathbb{R}_+\times\mathbb{R}} \left(\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g_{t,v}(s) f(s)\,\mathrm{d}s \right\} \mathrm{d}t \right) w\,\nu(\mathrm{d}v,\mathrm{d}w) \;,$$

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g_{t,v}(s) f(s)\,\mathrm{d}s \right\} \mathrm{d}t = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g_{t,v}(s)\,\mathrm{d}t \right\} f(s)\,\mathrm{d}s = v \int_{-\infty}^{\infty} f(s)\,\mathrm{d}s = 0 \;.$$

Finally, $\check{f} = \tilde{f} - \hat{f}$, thus $\mathbb{E}[N_S(\check{f})] = \mathbb{E}[N_S(\tilde{f})] - \mathbb{E}[N_S(\hat{f})] = 0$. □

Applying Lemma A.5, we can extend the definition of $d^{SD}_{j,k}$ in (3.8) to the case $\mathbb{E}[\eta] = \infty$ by

$$d^{SD}_{j,k} = N_S(\hat{\psi}_{j,k}) \;. \tag{A.30}$$

Lemma A.6. *(i) Let Assumption 1 holds with $p^* \geq 1$ and $\alpha \in (0,2)$. Then $\mathbb{E}[d^{SD}_{j,k}] = 0$ for all $j \geq 0$ and $k \in \mathbb{Z}$.*

(ii) Let Assumption 1 hold with $p^* \geq 2$ and $\alpha \in (0,2)$. Then $\mathrm{var}(d^S_{j,k} - d^{SD}_{j,k})$ is bounded independently of $j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

(iii) Let Assumption 1 hold with $p^* \geq 2$ and $\alpha \in (1,2)$. Then $|\mathrm{var}(d^S_{j,k}) - \mathrm{var}(d^{SD}_{j,k})|$ is bounded independently of $j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Proof. The first claim is given by Lemma A.5. By Lemma A.5, we have $d^S_{j,k} - d^{SD}_{j,k} = N_S(\check{\psi}_{j,k})$. Hence, applying (2.4),

$$\mathrm{var}(d^S_{j,k} - d^{SD}_{j,k}) = \int_0^{\infty} \nu_2(dv) \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h_{t,v}(s)\,\psi_{j,k}(s)\,ds \right)^2 \mathrm{d}t.$$

By Lemma A.4 (iv) and since the support of ψ is in $[0,M]$, we have, for all $t \in \mathbb{R}$ and $v \geq 0$,

$$\left| \int_{-\infty}^{\infty} h_{t,v}(s)\,\psi_{j,k}(s)\,ds \right| \leq 2^{-j/2}\,(1 + M\,\|\phi\|_\infty)\,\|\psi\|_\infty\,(B(t+1) + B(t+v+1)) \,, \tag{A.31}$$

where, for all $u \in \mathbb{R}$,

$$B(u) = \mathrm{Leb}([u-M,u] \cap 2^j[k,k+M]) \leq M\,\mathbb{1}_{\{2^j k \leq u \leq 2^j k + (2^j+1)M\}} \;. \tag{A.32}$$

This implies that $\int B(u)^2\,du$ is bounded by 2^j up to a multiplicative constant and it follows that $\mathrm{var}(d^S_{j,k} - d^{SD}_{j,k}) = O(1)$. We now suppose that $\alpha \in (1,2)$. Using that

$\mathrm{var}(d^{SD}_{j,k}) = \mathrm{var}(d^S_{j,k}) + \mathrm{var}(d^{SD}_{j,k} - d^S_{j,k}) + 2\mathrm{cov}(d^S_{j,k}, d^{SD}_{j,k} - d^S_{j,k})$, it only remains to show that $\mathrm{cov}(d^S_{j,k}, d^{SD}_{j,k} - d^S_{j,k}) = O(1)$. Applying (2.4), we have

$$\begin{aligned}\mathrm{cov}(d^S_{j,k}, d^S_{j,k} - d^{SD}_{j,k}) &= \mathrm{cov}\big(N_S(\psi_{j,k}), N_S(\check{\psi}_{j,k})\big) \\ &= \int_k^\infty \nu_2(\mathrm{d}v) \int_{-\infty}^\infty \left(\int_{-\infty}^\infty h_{t,v}(s)\, \psi_{j,k}(s)\, \mathrm{d}s\right) \left(\int_t^{t+v} \psi_{j,k}(s)\, \mathrm{d}s\right) \mathrm{d}t \;.\end{aligned}$$

By (A.31), (A.32) and

$$\left|\int_t^{t+v} \psi_{j,k}(s)\, ds\right| \leq \|\psi\|_\infty\, 2^{-j/2}\, v \;;$$

we get $\mathrm{cov}(d^S_{j,k}, d^{SD}_{j,k} - d^S_{j,k}) = O(1)$. □

Lemma A.7. *For all $\alpha \in (1/2, 2)$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,*

$$\sum_{k=0}^{n-1} (\mathbb{E}[d^D_{j,k}])^2 \leq C 2^{(3-2\alpha)j} L_1^2(2^j) \;. \tag{A.33}$$

For all $\alpha \in (1, 2)$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1} \mathrm{var}(d^D_{j,k} - d^{SD}_{j,k}) \leq C\, 2^{(2-\alpha)j}\, L_2(2^j) \;. \tag{A.34}$$

For all $\alpha \in (0, 1]$ and all $\epsilon > 0$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1} \mathrm{var}(d^D_{j,k} - d^{SD}_{j,k}) \leq C\, 2^{(2-\alpha)j}\, L_2(2^j)\, n^{1-\alpha+\epsilon} \;. \tag{A.35}$$

If moreover (3.1), (3.7), (4.6) and (4.7) hold, then, for all $\alpha \in (0, 2)$, there exists $C > 0$ such that, for all $j \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{n-1} (\mathbb{E}[d^D_{j,k}])^2 \leq C 2^{(3-2\alpha)j} L_1^2(2^j) \;. \tag{A.36}$$

Proof. Using the previous notation, we can write $d^D_{j,k} = N_S(\hat{\psi}_{j,k} \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}})$. Thus $d^{SD}_{j,k} - d^D_{j,k} = N_S(\hat{\psi}_{j,k} \mathbb{1}_{\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}})$ and

$$\begin{aligned}\mathbb{E}[d^D_{j,k}] &= -\int_0^\infty \nu_1(\mathrm{d}v) \int_{-\infty}^0 \left\{\int_{-\infty}^\infty g_{t,v}(s)\, \psi_{j,k}(s)\, \mathrm{d}s\right\} \mathrm{d}t \;, \\ \mathrm{var}(d^{SD}_{j,k} - d^D_{j,k}) &= \int_0^\infty \nu_2(\mathrm{d}v) \int_{-\infty}^0 \left(\int_{-\infty}^\infty g_{t,v}(s)\, \psi_{j,k}(s)\, \mathrm{d}s\right)^2 \mathrm{d}t.\end{aligned}$$

By (3.1) $\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s = \int \{1 - g_{t,v}(s)\}\,\psi_{j,k}(s)\,\mathrm{d}s$. Hence this integral vanishes when the support of $\psi_{j,k}$ does not intersect the support of $g_{t,v}$ or when it does not intersect the support of $1 - g_{t,v}$. The support of $\psi_{j,k}$ is included in $2^j[k, k+M]$ and Lemma A.4 gives the supports of $g_{t,v}$ and $1 - g_{t,v}$. We thus obtain

$$2^j k > t + v \text{ or } 2^j(k+M) < t - M + 1 \Rightarrow \int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s = 0$$

$$2^j k > t \text{ and } 2^j(k+M) < t + v - M + 1 \Rightarrow \int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s = 0\ .$$

From Lemma A.4, we know that $\|g_{t,v}\|_\infty \leq M\|\phi\|_\infty$; hence

$$\left|\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right| \leq M\,\|\phi\|_\infty\,\|\psi_{j,k}\|_1 = M\,\|\phi\|_\infty\,\|\psi\|_1 2^{j/2}\ .$$

The three last displays give that, for all $t \leq 0$,

$$\left|\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right| \leq C\,2^{j/2}\,\mathbb{1}(0 < t - 2^j k + v < 2^j M + M - 1)$$

and, finally, for all $k, j \geq 0$,

$$\left|\int_{-\infty}^0 \left\{\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right\}\mathrm{d}t\right| \leq C\,2^{3j/2}\,\{(k - 2^{-j}v)_+ \wedge (2M-1)\} \tag{A.37}$$

$$\int_{-\infty}^0 \left\{\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right\}^2 \mathrm{d}t \leq C\,2^{2j}\,\{(k - 2^{-j}v)_+ \wedge (2M-1)\}\ ,$$

which are similar bounds as those given by (A.22) and (A.25). The proofs of (A.33), (A.34) and (A.35) then follow as in the proof of Lemma A.3. It only remains to show (A.36). Again the proof is similar to the proof of (A.20) in Lemma A.3. One only needs to provide a bound similar as (A.26), namely,

$$\left|\int_{-\infty}^0 \left\{\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right\}\mathrm{d}t\right| \leq C\,2^{3j/2}\,\mathbb{1}(v \in [2^j\,k, 2^j(k+M')])\ ,$$

for some constant M'. As we already have (A.37) at hand, it is now sufficient to show that, for all $v \geq 2^j(k+M) + M - 1$,

$$\int_{-\infty}^0 \left\{\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right\}\mathrm{d}t = -\int_0^\infty \left\{\int g_{t,v}(s)\,\psi_{j,k}(s)\,\mathrm{d}s\right\}\mathrm{d}t = 0\ . \tag{A.38}$$

Since by Lemma A.4 $g_{t+v,\infty}$ has support in $[\lceil t+v\rceil - M + 1, \infty)$, we have, for all $v \geq 2^j(k+M) + M - 1$ and $t > 0$,

$$\int g_{t+v,\infty}(s)\,\psi_{j,k}(s)\,\mathrm{d}s = 0\ .$$

By (A.29), we have $g_{t,v} + g_{t+v,\infty} = g_{t,\infty}$, hence we only need to show that (A.38) holds for $v = \infty$. Observe that, by (A.29) and (4.7),

$$\int_0^\infty g_{t,\infty}(s)\,\mathrm{d}t = \int_0^\infty \sum_{k\in\mathbb{Z}} \mathbb{1}(k \geq t)\,\phi(t-k)\,\mathrm{d}t = \sum_{k\in\mathbb{Z}} k\,\phi(t-k)$$

is linear in t. By Fubini's theorem and using (3.1) and (4.6), we obtain (A.38) with $v = \infty$, which concludes the proof. □

Lemma A.8. *Let Assumption 1 hold with $p^* \geq 2$ and $\alpha > 1$. Then, there exists $C > 0$ such that, for all $n \geq 1$ and $j \geq 0$,*

$$\sum_{k=0}^{n-1} \mathbb{E}[(d_{j,k} - d^S_{j,k})^2] \leq C v_j\ ;$$

$$\sum_{k=0}^{n-1} \mathbb{E}[(\mathbf{d}_{j,k} - d^S_{j,k})^2] \leq C(v_j + n)\ .$$

Lemma A.9. *Let Assumption 1 hold with $p^* \geq 2$ and $\alpha \in (0,1]$. Assume that ϕ and ψ satisfy (4.6) and (4.7). Then, for all $\epsilon > 0$, then there exists $C > 0$ such that, for all $n \geq 1$ and $j \geq 0$,*

$$\sum_{k=0}^{n-1} \mathbb{E}[(\mathbf{d}_{j,k} - d^S_{j,k})^2] \leq C(L_1^2(2^j)\,2^{(3-2\alpha)j} + v_j n^{1-\alpha+\epsilon} + n)\ ,$$

Proof of Lemmas A.8 and A.9. Write $\mathbb{E}[(\mathbf{d}_{j,k} - d^S_{j,k})^2] = (\mathbb{E}[\mathbf{d}_{j,k}])^2 + \mathrm{var}(\mathbf{d}_{j,k} - d^S_{j,k})$ and apply Lemmas 3.1, A.3, A.6 and A.7. □

Lemma A.10. *Let Assumption 1 hold with $\alpha \in (1,2)$ and $p^* \geq 2$. Then*

$$\sup_{0\leq j\leq J} \mathbb{E}|\Lambda_j| = O(1)\ ; \tag{A.39}$$

$$\sup_{n\geq 1, j\geq 0} n^{-1/2}\,\mathbb{E}\left[\left|v_j^{-1}\sum_{k=0}^{n-1}\{(d^S_{j,k})^2 - d^2_{j,k}\}\right|\right] < \infty\ . \tag{A.40}$$

Lemma A.11. *Let Assumption 1 hold with $\alpha \in (0,2)$ and $p^* \geq 4$. If $\alpha \leq 1/2$, assume moreover (4.6) and (4.7).*

Let J^* be a sequence depending on J such that $\limsup J^*/J < (1/\alpha) \wedge (1/(2-\alpha))$. Then, there exists $\epsilon > 0$ such that

$$\sup_{u\in\mathcal{S}} \left|\sum_{j=J_0+1}^{J^*} u_j \Lambda_j\right| = O_P(2^{-\epsilon J}) \tag{A.41}$$

where $\mathcal{S}$ is the set of sequences $u = (u_0, \dots)$ satisfying $\sum_{j\in\mathbb{N}} |u_j| \leq 1$.

Proof of Lemmas A.10 and A.11. Recall that $d^S_{j,k}$ is always well defined, though unobservable in the case $\mathbb{E}[\eta] = \infty$. As previously, we denote $\beta_j = n_j^{-1} \sum_{k=0}^{n_j-1} \{v_j^{-1}(d^S_{j,k})^2 - 1\}$ and $r_j = \Lambda_j - \beta_j$. By Jensen's inequality, we have

$$\sup_{u \in \mathcal{S}} \left| \sum_{j=J_0+1}^{J^*} u_j \beta_j \right|^2 \leq \sum_{j=J_0+1}^{J^*} \beta_j^2 \ ,$$

hence, applying Lemma A.2, we obtain, for some slowly varying function $\tilde{\mathcal{L}}$,

$$\mathbb{E}\left[\sup_{u \in \mathcal{S}} \left| \sum_{j=J_0+1}^{J^*} u_j \beta_j \right|^2 \right] \leq 2^{-J} \sum_{j=J_0+1}^{J^*} \tilde{\mathcal{L}}(2^j)\, 2^{(\alpha \vee 1)j} \leq C \tilde{\mathcal{L}}(2^{J^*}) 2^{(\alpha \vee 1)J^* - J} = o(1) \ .$$

To bound the remainder terms, the cases $\alpha > 1$ and $\alpha \leq 1$ must be considered separately.

$$\begin{aligned} \mathbb{E}[|r_j|] &\leq (n_j v_j)^{-1} \sum_{k=0}^{n_j-1} \mathbb{E}[|(d^S_{j,k})^2 - \mathbf{d}^2_{j,k}|] \\ &\leq (n_j v_j)^{-1} \sum_{k=0}^{n_j-1} \mathbb{E}[(d^S_{j,k} - \mathbf{d}_{j,k})^2] + 2(n_j v_j)^{-1/2} \left\{ \sum_{k=0}^{n_j-1} \mathbb{E}[(d^S_{j,k} - \mathbf{d}_{j,k})^2] \right\}^{1/2} \ . \end{aligned}$$

If $\alpha > 1$, we can apply Lemma A.8 and obtain

$$\mathbb{E}[|r_j|] = O\left(1/\sqrt{n_j} \ + \ 1/\sqrt{v_j}\right) \ . \tag{A.42}$$

Since $\mathbb{E}[\beta_j] = 0$, this proves (A.39). In Lemma A.8, for $\mathbf{d}_{j,k} = d_{j,k}$, (A.42) can be made more precise, giving (A.40). Now, noting that $\alpha > 1$ implies $J^* < J/\alpha$, we obtain

$$\mathbb{E}\left[\sup_{u \in \mathcal{S}} \left| \sum_{j=J_0+1}^{J^*} u_j r_j \right| \right] \leq \sum_{j=J_0+1}^{J^*} \mathbb{E}[|r_j|] = O(2^{(\alpha/2-1)J_0}) \ .$$

If $\alpha \leq 1$, we apply Lemma A.9 and obtain, for some arbitrarily small $\epsilon > 0$,

$$\mathbb{E}[|r_j|] \leq C\left(2^{(1-\alpha/2)j - J/2} + n_j^{(\epsilon-\alpha)/2} + 1/\sqrt{v_j}\right) \ , \tag{A.43}$$

$$\mathbb{E}\left[\sup_{u \in \mathcal{S}_J} \left| \sum\nolimits_j u_j r_j \right| \right] = O\left(2^{\{(2-\alpha)J^* - J\}/2} + 2^{(J^*-J)(\epsilon-\alpha)/2} + 2^{(\alpha/2-1)J_0}\right) \ .$$

□

Lemma A.12. *Let Assumption 1 hold with $p^* \geq 4$ and $\alpha \in (1,2)$. Then, there exists a positive constant $C > 0$,*

$$\operatorname{var}\left(\sum_{k=0}^{n-1} (d_{j,k}^{SD})^2\right) \leq C L_4(2^j) n 2^{(3-\alpha)j}. \tag{A.44}$$

Proof. The proof is along the same line as the proof of Lemma A.2 except that (A.11) and (A.12) are replaced by

$$\operatorname{cov}(d_{j,k}^{SD}, d_{j,k'}^{SD}) = \int_0^\infty K_1(v, j, |k-k'|)\, \nu_2(\mathrm{d}v)\ , \tag{A.45}$$

$$\operatorname{cum}(d_{j,k}^{SD}, d_{j,k}^{SD}, d_{j,k'}^{SD}, d_{j,k'}^{SD}) = \int_0^\infty K_2(v, j, |k-k'|)\, \nu_4(\mathrm{d}v)\ . \tag{A.46}$$

where

$$K_p^D(v,j,k) := 2^{-jp} \int_{-\infty}^\infty \left\{\int_{-\infty}^\infty g_{t,v}(s) \psi(2^{-j}s-k)\, \mathrm{d}s\right\}^p \left\{\int_{-\infty}^\infty g_{t,v}(s')\, \psi(2^{-j}s')\, \mathrm{d}s'\right\}^p \mathrm{d}t$$

Denote

$$K_p^D(v,j) := 2^{-jp/2} \int_{-\infty}^\infty \left\{\int_{-\infty}^\infty g_{t,v}(u)\, \psi(2^{-j}u)\, \mathrm{d}u\right\}^p \mathrm{d}t$$

It first check that $K_p^D(v,j,k)$ and $K_p^D(v,j)$ satisfy similar properties as the ones needed for $K_p(v,k)$ and $K_p(v)$ in the proof of Lemma A.2. Since, for any integer k, $g_{t,v}(s+k2^j) = g_{t-k2^j,v}(s)$, we have $K_p^D(v,j,k) = K_p^D(v,j,-k)$ and, by Cauchy Schwartz Inequality, $|K_p^D(v,j,k)| \leq |K_{2p}^D(v,j)|$. Recall that Lemma A.4 implies that the support of $g_{t,v}$ is included in $[t-M+1, t+v+1]$. We thus have that, if $2^j k > v + M(1+2^j)$, the support of $\psi(2^{-j}\cdot)$ being included in $[0, 2^j M]$ cannot intersect the one of $g_{t-2^j k,v}$ and the one of $g_{t,v}$ at the same time, implying $K_p^D(v,k) = 0$. If $k > M$ and $v > 2^j k + M(2^j-1)$, the support of ψ cannot intersect both the one of $g_{t-2^j k,v}$ and the one of $g_{t,v}$ without being included in one of them, so that, by (3.1), $K_p^D(v,j,k) = 0$. Hence we obtain the property corresponding to (A.10)

$$|K_p^D(v,j,k)| \leq K_{2p}^D(v,j) \left\{\mathbb{1}(|k| \leq M) + \mathbb{1}(v \in [2^j|k| - M(1+2^j), 2^j|k| + M(2^j-1)]\right)\ . \tag{A.47}$$

Proceeding as in the proof of Lemma A.2, this bound yields

$$n^{-1}\operatorname{var}\left(\sum_{k=0}^{n-1} (d_{j,k}^{SD})^2\right) \leq C \left\{\left(\int_0^\infty K_2^D(v,j)\, \nu_2(\mathrm{d}v)\right)^2 + \int_0^\infty K_4^D(v,j)\, \nu_4(\mathrm{d}v)\right\}\ . \tag{A.48}$$

From (A.28), we have

$$\int_{-\infty}^{\infty} g_{t,v}(u)\,\psi(2^{-j}u)\,\mathrm{d}u = \int_t^{t+v} \psi(2^{-j}u)\,\mathrm{d}u - R(t,v,j),$$

where

$$R(t,v,j) := \int_{-\infty}^{\infty} h_{t,v}(u)\,\psi(2^{-j}u)\,\mathrm{d}u = 2^{j/2}\int_{-\infty}^{\infty} h_{t,v}(u)\,\psi_{j,0}(u)\,\mathrm{d}u\ .$$

Hence, since $\int_t^{t+v}\psi(2^{-j}u)\,\mathrm{d}u \le M\,(v\wedge 2^j)\,\|\psi\|_\infty$, using that, for any real numbers a and b,$|(a+b)^p - a^p| \le p\,|b|\,(|b|^{p-1}\vee |a|^{p-1})$, we find

$$K_p^D(v,j) = 2^{j(1+p/2)}\,K_p(2^{-j}v) + O\left(2^{-jp/2}\int_{t=-\infty}^{\infty} |R(t,v,j)|\,(|R(t,v,j)|^{p-1}\vee v^{(p-1)})\,\mathrm{d}t\right)\ .$$

Using (A.31) and (A.32) gives that $R(t,v,j) = O(1)$ and that $\int R(t,v,j)\mathrm{d}t = O(2^j)$; hence the above O-term is bounded by $2^{j(1-p/2)}(v\wedge 2^j)^{p-1}$ up to a multiplicative constant independent of v. This in turn gives

$$\int_0^{\infty} K_p^D(v,j)\,\nu_p(\mathrm{d}v) = 2^{j(1+p/2)}\int_0^{\infty} K_p(2^{-j}v)\,\nu_p(\mathrm{d}v) + O\left(2^{j(1-p/2)}\int_0^{\infty}(v\wedge 2^j)^{p-1}\,\nu_p(\mathrm{d}v)\right)\ .$$

We know that (A.3) and (A.5) yield $\int_0^\infty K_p(v)\,\nu_2(2^j\mathrm{d}v) \sim L_p(2^j)\,2^{-\alpha j}$; hence, since $|K_p^D(v) - K_p(v)|$ is bounded independently of v,

$$\int_0^{\infty} K_p^D(v)\,\nu_2(2^j\mathrm{d}v) \sim L_p(2^j)\,2^{-\alpha j}$$

Applying the two last displays to the cases $p=2$ and $p=4$ with $\alpha>1$ successively yield

$$\int_0^{\infty} K_2^D(v,j)\,\nu_2(\mathrm{d}v) \le C\,(L_2(2^j)\,2^{(2-\alpha)j} + 1)\ ,$$

$$\int_0^{\infty} K_4^D(v,j)\,\nu_4(\mathrm{d}v) \le C\,(L_4(2^j)\,2^{(3-\alpha)j} + L_4(2^j)\,2^{(2-\alpha)j}\ .$$

This, with (A.48) concludes the result. □

Lemma A.13. *Let ρ be a positive real and $\rho' := (2^\rho - 1)^{-1}$. Let ℓ^* be a non increasing function on $[1,\infty)$ such that $\lim_{s\to\infty}\ell^*(s) = 0$, and let ℓ be a function on $[1,\infty)$ such that $|\ell(s)| \le \ell^*(s)$ for all $s\in[1,\infty)$. Define*

$$L(x) = c\exp\left\{\int_1^x \frac{\ell(s)}{s}\,\mathrm{d}s\right\} \quad \textit{and} \quad \omega_j = \frac{2^{-\rho j}L(2^j)}{\sum_{j'=J_0+1}^{J_1} 2^{-\rho j'}L(2^{j'})}\ .$$

Then, as $J_0 \to \infty$ and for any $\epsilon > 0$

$$\sum_{j=J_0+1}^{J_1} \omega_j j = J_0 + 1 + \rho'(1 + O(\ell^*(2^{J_0}))) + O(J_1(2-\epsilon)^{J_0-J_1})) \tag{A.49}$$

$$\sum_{j=J_0+1}^{J_1} \omega_j j^2 = J_0^2 + 2J_0(1+\rho') + 2\rho'^2 + 3\rho' + 1 + \rho' O(\ell^*(2^{J_0})) + O(J_1^2(2-\epsilon)^{J_0-J_1})) \tag{A.50}$$

Proof. Denote $S_k = \sum_{j=k+1}^{\infty} 2^{-\rho j} L(2^j)$. Then,

$$S_k^{-1} \sum_{j=k+1}^{\infty} jL(2^j)2^{-\rho j} = k + S_k^{-1} \sum_{j=k+1}^{\infty} (j-k)L(2^j)2^{-\rho j} = k + 1 + S_k^{-1} \sum_{j=k+1}^{\infty} S_j \ . \tag{A.51}$$

Note now that $2^{-\rho j} = \rho'(2^{-\rho(j-1)} - 2^{-\rho j})$. Hence

$$S_j = \rho' \sum_{j'=j+1}^{\infty} (2^{-\rho(j'-1)} - 2^{-\rho j'})L(2^{j'}) = \rho' 2^{-\rho j} L(2^{j+1}) + R_j \ ,$$

with

$$R_j = \rho' \sum_{j'=j+1}^{\infty} 2^{-\rho j'} \{L(2^{j'+1}) - L(2^{j'})\} = \rho' \sum_{j'=j+1}^{\infty} 2^{-\rho j'} L(2^{j'}) \left\{ \exp \int_{2^{j'}}^{2^{j'+1}} \frac{\ell(s)}{s} \mathrm{d}s - 1 \right\} \ .$$

Thus $S_j = 2^{-\rho j} L(2^{j+1})\{1 - R_j/S_j\}^{-1}$. Since $|\ell| \leq \ell^*$ and ℓ^* is non increasing, we obtain:

$$\left| \frac{L(2^{j+1})}{L(2^j)} - 1 \right| = \left| \exp \int_{2^j}^{2^{j+1}} \frac{\ell(s)}{s} \mathrm{d}s - 1 \right| \leq C\ell^*(2^j) \ .$$

Hence $|R_j| \leq C\rho' S_j \ell^*(2^j)$ and for any $j \geq k$,

$$S_j = \rho' 2^{-\rho j} L(2^{j+1})\{1 + O(\ell^*(2^j))\} = \rho' 2^{-\rho j} L(2^j)\{1 + O(\ell^*(2^k))\}. \tag{A.52}$$

Using this expression in (A.51) yields

$$S_k^{-1} \sum_{j=k+1}^{\infty} jL(2^j)2^{-\rho j} = k + 1 + \rho'(1 + O(\ell^*(2^k)) \ . \tag{A.53}$$

The left-hand side of (A.49) writes

$$S_{J_0}{}^{-1} \sum_{j>J_0} j2^{-\rho j} L(2^j) + \frac{U}{1-U} \{S_{J_0}{}^{-1} \sum_{j>J_0} j2^{-\rho j} L(2^j) - S_{J_1}{}^{-1} \sum_{j>J_1} j2^{-\rho j} L(2^j)\}.$$

with $U := S_{J_1}/S_{J_0}$. By (A.52), $U = 2^{J_0-J_1}\frac{L(2^{J_1})}{L(2^{J_0})}\{1+O(\ell^*(2^{J_0}))\}$. Noting that $|\frac{L(2^{J_1})}{L(2^{J_0})}| \leq \exp\{l^*(2^{J_0})(J_1-J_0)\}$, we have $U \leq (2e^{\ell^*(2^{J_0})})^{J_0-J_1}\{1+O(\ell^*(2^{J_0}))\}$. Eq. (A.49) follows. Similarly to (A.53), we have

$$S_k^{-1}\sum_{j=k+1}^{\infty} j^2 L(2^j)2^{-\rho j} = k^2 + 2k(1+\rho') + 2(1+\rho')^2 - (1+\rho') + \rho' O(\ell^*(2^k)) .$$

and (A.50) is derived along the same lines.

□

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