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Approximation of rough paths of fractional Brownian motion

Annie Millet and Marta Sanz-Solé

Abstract. We consider a geometric rough path associated with a fractional Brownian motion with Hurst parameter $H \in]\frac{1}{4}, \frac{1}{2}[$. We give an approximation result in a modulus type distance, up to the second order, by means of a sequence of rough paths lying above elements of the reproducing kernel Hilbert space.

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1. Introduction

Consider a d -dimensional fractional Brownian motion W^H with Hurst parameter $H \in]\frac{1}{4}, \frac{1}{2}[\cup]\frac{1}{2}, 1[$ and integral representation

$$W_t^H = \int_0^1 K^H(t, s) dB_s, \quad (1.1)$$

where $K^H(t, s) = 0$, if $s \geq t$ and for $0 < s < t$,

$$K^H(t, s) = c_H (t - s)^{H - \frac{1}{2}} + s^{H - \frac{1}{2}} F_1\left(\frac{t}{s}\right) \quad (1.2)$$

with

$$F_1(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} u^{H - \frac{3}{2}} \left(1 - (u + 1)^{H - \frac{1}{2}}\right) du, \quad (1.3)$$

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for $z > 1$ (see for instance [1], equation (42)). In (1.1), B denotes a standard d -dimensional Brownian motion and in (1.2), (1.3), c_H denotes a positive real constant depending on H .

Let $p \in]1, 4[$ be such that $pH > 1$. In [2], it is proved that the sequence of smooth rough paths based on linear interpolations of W^H converges in the p -variation distance. The limit defines a geometric rough path with roughness p lying above W^H . We will call this object *the enhanced fractional Brownian motion*.

In the recent papers [5], [3], the p -variation distance on rough paths is replaced by a strictly stronger, *modulus type* distance defined as follows:

$$\bar{d}_p(x, y) = \sup_{0 \leq s < t \leq 1} \left(\sum_{i=1}^{[p]} \frac{|x_{s,t}^{(i)} - y_{s,t}^{(i)}|}{(t-s)^{\frac{1}{p}}} \right).$$

In [3], it is proved that the enhanced fractional Brownian motion can actually be obtained by means of the $\bar{d}_p$ distance and also that linear interpolations of W^H define stochastic processes with values in $\mathcal{H}^H$, the reproducing kernel Hilbert space associated with W^H (see Theorem 3.3 in [4] for a description of this space). Then, the authors state a characterization of the topological support of the enhanced fractional Brownian motion among other results.

Our aim in this work is to give a new approximation of the enhanced fractional Brownian motion by means of a sequence of geometric rough paths which, unlike those based on linear interpolations, are not smooth, but also belong to $\mathcal{H}^H$. For the sake of simplicity, we restrict to $[p] = 2$. We are pretty confident that our results extend to $[p] = 3$; however, dealing with higher generality would most likely produce a very technical paper. Our result, as is stated in Theorem 2.1, provides in particular a new approximation of the Lévy area of the fractional Brownian motion.

For any $m \in \mathbb{N}$, we consider the dyadic grid $(t_k^m = k2^{-m}, k = 0, 1, \dots, 2^m)$ and set $\Delta_k^m =]t_{k-1}^m, t_k^m]$ and $\Delta_k^m B = B_{t_k^m} - B_{t_{k-1}^m}$. Define $B(m)_0 = 0$ and for $t \in \Delta_k^m$, $B(m)_t = B_{t_{k-1}^m} + 2^m(t - t_{k-1}^m)\Delta_k^m B$. Our approximation sequence is defined by

$$W(m)_t^H = \int_0^t K^H(t, s) \dot{B}(m)_s ds, \quad (1.4)$$

$m \in \mathbb{N}$, where $\dot{B}(m)_s$ denotes the derivative with respect to s of the path $s \mapsto B(m)_s$. Notice that $W(m)^H \in \mathcal{H}^H$.

Let K_m^H be the orthogonal projection of $K^H(t, \cdot)$ on the σ -field generated by $(\Delta_k^m, k = 1, \dots, m)$. That is, for any $0 < s < t \leq 1$,

$$K_m^H(t, s) = \sum_{k=1}^{2^m} 2^m \left(\int_{\Delta_k^m \cap]0, t]} K^H(t, u) du \right) \mathbf{1}_{\Delta_k^m}(s). \quad (1.5)$$

We clearly have

$$W(m)_t^H = \int_0^1 K_m^H(t, s) dB_s. \quad (1.6)$$

For $H \in]\frac{1}{2}, 1[$, we set $\mathbf{W} = (\mathbf{W}_{s,t} = (W_{s,t}^{(1)}, 0 \leq s \leq t \leq 1)$, $\mathbf{W}(\mathbf{m}) = (\mathbf{W}(\mathbf{m})_{s,t} = (W(m)_{s,t}^{(1)}, 0 \leq s \leq t \leq 1)$, while for $H \in]\frac{1}{4}, \frac{1}{2}[$ we set $\mathbf{W} = (\mathbf{W}_{s,t} = (W_{s,t}^{(1)}, W_{s,t}^{(2)}, 0 \leq s \leq t \leq 1)$ and $\mathbf{W}(\mathbf{m}) = (\mathbf{W}(\mathbf{m})_{s,t} = (W(m)_{s,t}^{(1)}, W(m)_{s,t}^{(2)}, 0 \leq s \leq t \leq 1)$, $m \geq 1$.

The main result of the paper states the convergence of $\mathbf{W}(\mathbf{m})$ to $\mathbf{W}$ in the $\bar{d}_p$ -metric for $p \in]1, 3[$. For $p \in]1, 2[$, the result is an almost trivial consequence of Lemma 3.2 which establishes Hölder continuity in the $L^2[0, 1]$ norm of the kernels K^H , K_m^H , respectively, and a control of the quadratic mean error in the approximation of K^H by K_m^H . For $p \in [2, 3[$, the approximation of the Lévy area relies on representation formulas for the second order multiple integrals by means of the operator K^* given in (2.3) and introduced in [1] (see also [2]). There are two fundamental ingredients. Firstly, Proposition 2.3, giving the rate of convergence of the approximation at the second order level in the $L^q(\Omega)$ -modulus norm; secondly, Lemma 3.5, an extension of the Garsia-Rademich-Rumsey Lemma for geometric rough paths of any roughness p . Other technical results used in the proofs, mainly on the kernels K^H and K_m^H , are given in the Appendix.

For simplicity, in general we shall not write explicitly the dependence on H ; thus W stands for W^H , $K(t, s)$ for $K^H(t, s)$, etc. For any $q \in [1, \infty[$, we denote by $\|\cdot\|_q$ the $L^q(\Omega)$ -norm. We make the convention $\sum_{k=a}^b x_k = 0$ if $b < a$ and denote by C positive constants with possibly different values. For additional notions and notation on rough paths, we refer the reader to [6].

2. Approximation result

For $p \in]1, +\infty[$ we set $\tilde{d}_p = \bar{d}_{p \wedge 2}$, that is

$$\tilde{d}_p(x, y) = \sup_{0 \leq s < t \leq 1} \left(\sum_{i=1}^{[p] \wedge 2} \frac{|x_{s,t}^{(i)} - y_{s,t}^{(i)}|}{(t-s)^{\frac{i}{p}}} \right).$$

The purpose of this section is to prove the following approximation result.

Theorem 2.1. *Let $H \in]\frac{1}{4}, \frac{1}{2}[$, $p \in]2, 4[$ (resp. $H \in]\frac{1}{2}, 1[$, $p \in]1, 2[$), be such that $pH > 1$ and $q \in [1, +\infty[$. The sequence $(\tilde{d}_p(\mathbf{W}(\mathbf{m}), \mathbf{W}), m \geq 1)$, converges to 0 in $L^q(\Omega)$ and a.s. Thus for $H \in]\frac{1}{2}, 1[$ and $p \in]1, 2[$, if $\mathcal{G}_p$ denotes the set of dyadic geometric rough paths endowed with the norm $\tilde{d}_p(0, \cdot)$ and P^H denotes the law of the fractional Brownian motion W^H , then the triple $(X, \mathcal{H}^H, P^H)$ is an abstract Wiener space.*

The next Proposition provides the auxiliary result to state the approximation of the first component of the enhanced fractional Brownian motion.

Proposition 2.2. *Let $0 \leq s < t \leq 1$, $q \in [1, \infty[$.*

(i) For any $H \in]0, \frac{1}{2}[$, $\lambda \in [0, H[$,

$$\left\| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right\|_q \leq C 2^{-m\lambda} |t - s|^{H-\lambda}. \quad (2.1)$$

(ii) For any $H \in]\frac{1}{2}, 1[$, $\varepsilon \in [0, H[$, $\mu \in]0, \frac{\varepsilon}{H(2H+1)}[$,

$$\left\| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right\|_q \leq C 2^{-m\mu} |t - s|^{H-\varepsilon}. \quad (2.2)$$

Proof. By the hypercontractivity inequality, it suffices to prove the results for $q = 2$. In this case, it is an easy consequence of the identity

$$E \left(\left| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right|^2 \right) = \int_0^1 |(K(t, u) - K(s, u)) - (K_m(t, u) - K_m(s, u))|^2 du$$

and of Lemma 3.2. Indeed, by (3.14), we have

$$E \left(\left| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right|^2 \right) \leq C |t - s|^{2H}.$$

Hence, if $t - s < 2^{-m}$, we easily obtain (2.1) and (2.2).

Assume now $H \in]0, \frac{1}{2}[$ and $t - s \geq 2^{-m}$. By (3.15), for $\epsilon \in [0, H]$,

$$E \left(\left| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right|^2 \right) \leq C 2^{-2mH} \leq C 2^{-2m\epsilon} |t - s|^{2(H-\epsilon)}.$$

Hence, (2.1) follows.

Let $H \in]\frac{1}{2}, 1[$ and $t - s \geq 2^{-m}$. Let $\alpha \in]0, 1[$; then (3.14) and (3.16) imply

$$\left\| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right\|_q \leq C |t - s|^{H(1-\alpha)} 2^{-m\lambda\alpha},$$

with $\lambda \in]0, \frac{1}{2H+1}[$. By taking $\alpha = \frac{\varepsilon}{H}$, we obtain (2.2) with $\mu = \lambda \frac{\varepsilon}{H}$. $\square$

Throughout the rest of this section, $H \in]\frac{1}{4}, \frac{1}{2}[$. Following [1], let $\mathcal{H}_K$ denote the set of functions $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that

$$\|\varphi\|_K^2 = \int_0^1 \varphi(s)^2 K(1, s)^2 ds + \int_0^1 ds \left(\int_s^1 |\varphi(t) - \varphi(s)| |K|(dt, s) \right)^2 < +\infty.$$

For any $\varphi \in \mathcal{H}_K$, $0 < s < t$, set

$$\begin{aligned} K^* \left(\mathbb{1}_{[s,t]}(\cdot) (\varphi \cdot - \varphi_s) \right) (u) &= \mathbb{1}_{[0,s]}(u) \int_s^t (\varphi_r - \varphi_s) K(dr, u) \\ &+ \mathbb{1}_{[s,t]}(u) \left(K(t, u) (\varphi_u - \varphi_s) + \int_u^t (\varphi_r - \varphi_u) K(dr, u) \right). \end{aligned} \quad (2.3)$$

Following [2],

$$W_{s,t}^{(2)} = \int_0^1 K^* \left(\mathbb{1}_{[s,t]}(\cdot) (W \cdot - W_s) \right) (u) dB_u + \frac{1}{2} |t - s|^{2H}. \quad (2.4)$$

Moreover, by Theorem 9 in [7], for $W(m)$ defined in (1.4) we have

$$W(m)_{s,t}^{(2)} = \int_0^1 K^* (\mathbf{1}_{[s,t]}(\cdot) (W(m)_\cdot - W(m)_s)) (u) \dot{B}(m)_u du. \quad (2.5)$$

Proposition 2.3. *For each $m \in \mathbb{N}$, $0 < s < t \leq 1$, $q \in [1, \infty[$,*

$$\|W_{s,t}^{(2)} - W(m)_{s,t}^{(2)}\|_q \leq C 2^{-m\mu} |t - s|^{2H-\varepsilon}, \quad (2.6)$$

for some positive constants C and any $\varepsilon \in]0, 2H - \frac{1}{2}[$ and $\mu \in]0, \frac{\varepsilon}{2}[$.

Before proving this proposition, we give an equivalent expression for $W(m)_{s,t}^2$, as follows. The integration by parts formula of Malliavin calculus (see e.g. [8], Equation (1.49)) and (1.6) yield $W(m)_{s,t}^{(2)} = A_{s,t}^1(m) + A_{s,t}^2(m)$, with

$$A_{s,t}^1(m) = \sum_{k=1}^{2^m} \int_0^1 du \mathbf{1}_{\Delta_k^m}(u) 2^m K^* \left(\mathbf{1}_{[s,t]}(\cdot) \int_{\Delta_k^m} dB_r (W(m)_\cdot - W(m)_s) \right) (u), \quad (2.7)$$

$$A_{s,t}^2(m) = \sum_{k=1}^{2^m} \int_0^1 du \mathbf{1}_{\Delta_k^m}(u) 2^m K^* \left(\mathbf{1}_{[s,t]}(\cdot) \int_{\Delta_k^m} dr (K_m(\cdot, r) - K_m(s, r)) \right) (u). \quad (2.8)$$

By definition, for $r \in \Delta_k^m$, $K_m(t, r) = 2^m \int_{\Delta_k^m \cap]0, t]} K(t, u) du = 2^m K(\mathbf{1}_{\Delta_k^m})(t)$. Since $h := K(\mathbf{1}_{\Delta_k^m}) \in \mathcal{H}_K$, the duality relation given in [7], equation (58) and Lemma 3.3 yield

$$\begin{aligned} A_{s,t}^2(m) &= \sum_{k=1}^{2^m} \int_0^1 dr \mathbf{1}_{\Delta_k^m}(r) 2^{2m} \int_0^1 du \mathbf{1}_{\Delta_k^m}(u) K^* \left(\mathbf{1}_{[s,t]}(\cdot) K(\mathbf{1}_{\Delta_k^m})_{s,\cdot} \right) (u) \\ &= \sum_{k=1}^{2^m} \int_0^1 dr \mathbf{1}_{\Delta_k^m}(r) 2^{2m} \int_s^t K(\mathbf{1}_{\Delta_k^m})(du) (K(\mathbf{1}_{\Delta_k^m})(u) - K(\mathbf{1}_{\Delta_k^m})(s)) \\ &= \sum_{k=1}^{2^m} \int_0^1 dr \mathbf{1}_{\Delta_k^m}(r) 2^{2m} \frac{(K(\mathbf{1}_{\Delta_k^m})(t) - K(\mathbf{1}_{\Delta_k^m})(s))^2}{2} \\ &= \frac{1}{2} \int_0^1 dr |K_m(t, r) - K_m(s, r)|^2 = \frac{1}{2} \|W(m)_{s,t}^{(1)}\|_2^2. \end{aligned}$$

Thus, since $E|W_t - W_s|^2 = |t - s|^{2H}$, Schwarz's inequality, (3.14), (3.15) imply

$$\left| A_{s,t}^2(m) - \frac{1}{2} |t - s|^{2H} \right| \leq C 2^{-m\varepsilon} |t - s|^{2H-\varepsilon}, \quad (2.9)$$

for some positive constant C and $\varepsilon \in]0, H[$.

Hence, in order to establish (2.6) it suffices to prove that for any small parameter $\varepsilon \in]0, 4H - 1[$ and $\mu \in]0, \varepsilon[$,

$$E \left(\left| \int_0^1 K^* (\mathbb{1}_{[s,t]}(\cdot) (W_\cdot - W_s)) (u) dB_u - A_{s,t}^1(m) \right|^2 \right) \leq C 2^{-m\mu} |t - s|^{4H-\varepsilon}. \quad (2.10)$$

for all $m \geq 1$. We devote the next lemmas to the proof of this convergence, using the expression of the operator K^* given in (2.3).

Lemma 2.4. *For any $0 \leq s < t \leq 1$, $m \geq 1$, we set*

$$T_1(s, t) = \int_0^s dB_u \left(\int_s^t (W_r - W_s) K(dr, u) \right),$$

$$T_1(s, t, m) = \sum_{k=1}^{2^m} \int_{\Delta_k^m} dB_r 2^m \left(\int_{\Delta_k^m \cap]0, s]} du \left(\int_s^t (W(m)_v - W(m)_s) K(dv, u) \right) \right).$$

Then for any $\varepsilon \in]0, 2H[$ and $\mu \in]0, \varepsilon[$, there exists $C > 0$ such that

$$E \left(|T_1(s, t, m) - T_1(s, t)|^2 \right) \leq C 2^{-m\mu} |t - s|^{4H-\varepsilon}. \quad (2.11)$$

Proof. Assume $s \in \Delta_I^m$, $I \geq 1$; we consider the decomposition

$$E \left(|T_1(s, t, m) - T_1(s, t)|^2 \right) \leq C \sum_{j=1}^3 \tau_{1,j}(s, t, m),$$

with

$$\tau_{1,1}(s, t, m) = \sum_{k \in \{1, I-1, I\}} E \left(\left| \int_{\Delta_k^m} dB_r 2^m \right. \right. \\ \left. \left. \times \left(\int_{\Delta_k^m \cap]0, s]} du \left(\int_s^t (W(m)_v - W(m)_s) K(dv, u) \right) \right) \right|^2 \right), \quad (2.12)$$

$$\tau_{1,2}(s, t, m) = \sum_{k \in \{1, I-1, I\}} E \left(\left| \int_{\Delta_k^m \cap]0, s]} dB_r \left(\int_s^t (W_v - W_s) K(dv, r) \right) \right|^2 \right), \quad (2.13)$$

$$\tau_{1,3}(s, t, m) = E \left(\left| \sum_{k=2}^{I-2} \int_{\Delta_k^m} dB_r 2^m \int_{\Delta_k^m} du \left(\int_s^t (W(m)_v - W(m)_s) K(dv, u) \right. \right. \right. \\ \left. \left. \left. - \int_s^t (W_v - W_s) K(dv, r) \right) \right|^2 \right). \quad (2.14)$$

By Lemma 3.4, (3.4), Schwarz's inequality and (3.14), any term in the right hand-side of (2.12) is bounded as follows. Let $\varepsilon \in]0, 2H[$, $\lambda \in]\frac{1-(2H-\varepsilon)}{2}, \frac{1}{2}[$; then

$2H - 3 + 2\lambda < -1$, $1 - 2\lambda - (2H - \varepsilon) < 0$ and

$$\begin{aligned}
& E \left(\left| \int_{\Delta_k^m} dB_r 2^m \left(\int_{\Delta_k^m \cap]0, s]} du \left(\int_s^t (W(m)_v - W(m)_s) K(dv, u) \right) \right) \right|^2 \right) \\
& \leq C \int_{\Delta_k^m} dr \int_0^1 d\rho \left| 2^m \int_{\Delta_k^m \cap]0, s]} du \int_s^t (K_m(v, \rho) - K_m(s, \rho)) K(dv, u) \right|^2 \\
& \leq C \int_{\Delta_k^m} dr \int_0^1 d\rho 2^m \int_{\Delta_k^m \cap]0, s]} du \left(\int_s^t dv |v - u|^{2H-3+2\lambda} \right) \\
& \quad \times \left(\int_s^t dv |K_m(v, \rho) - K_m(s, \rho)|^2 |v - u|^{-2\lambda} \right) \\
& \leq C \int_{\Delta_k^m \cap]0, s]} du (s - u)^{2H-2+2\lambda} |t - s|^{2H} (|t - u|^{1-2\lambda} - |s - u|^{1-2\lambda}) \\
& \leq C \int_{\Delta_k^m \cap]0, s]} du (s - u)^{2H-2+2\lambda} |t - s|^{2H} |t - s|^{2H-\varepsilon} |s - u|^{1-2\lambda-(2H-\varepsilon)} \\
& \leq C |t - s|^{4H-\varepsilon} \int_{\Delta_k^m \cap]0, s]} du |s - u|^{\varepsilon-1} \leq C 2^{-m\varepsilon} |t - s|^{4H-\varepsilon}.
\end{aligned}$$

Each term of the right hand-side of (2.13) can be studied using a similar strategy. Thus we obtain for $\varepsilon \in]0, 2H[$:

$$\tau_{1,1}(s, t, m) + \tau_{1,2}(s, t, m) \leq C 2^{-m\varepsilon} |t - s|^{4H-\varepsilon}. \quad (2.15)$$

Set for $s \geq 3 \cdot 2^{-m}$, and hence $I \geq 4$,

$$\begin{aligned}
X_r &= \sum_{k=2}^{I-2} \mathbf{1}_{\Delta_k^m}(r) 2^m \int_{\Delta_k^m} du \left(\int_s^t (W(m)_v - W(m)_s) K(dv, u) \right. \\
&\quad \left. - \int_s^t (W_v - W_s) K(dv, r) \right).
\end{aligned}$$

Notice that $X_r = \int_0^1 g(r, \rho) dB_\rho$, with

$$\begin{aligned}
g(r, \rho) &= \sum_{k=2}^{I-2} \mathbf{1}_{\Delta_k^m}(r) 2^m \int_{\Delta_k^m} du \left(\int_s^t K(dv, u) (K_m(v, \rho) - K_m(s, \rho)) \right. \\
&\quad \left. - \int_s^t K(dv, r) (K(v, \rho) - K(s, \rho)) \right).
\end{aligned}$$

Hence, by Lemma 3.4 and Schwarz's inequality, $\tau_{1,3}(s, t, m) \leq C(\tau_{1,3,1}(s, t, m) + \tau_{1,3,2}(s, t, m))$, with

$$\tau_{1,3,1}(s, t, m) = \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho$$

$$\begin{aligned}
& \times \left| \int_s^t (K_m(v, \rho) - K_m(s, \rho)) (K(dv, u) - K(dv, r)) \right|^2, \\
\tau_{1,3,2}(s, t, m) &= \sum_{k=2}^{I-2} \int_{\Delta_k^m} dr \int_0^1 d\rho \left| \int_s^t K(dv, r) \right. \\
& \quad \left. \times (K_m(v, \rho) - K_m(s, \rho) - K(v, \rho) + K(s, \rho)) \right|^2.
\end{aligned}$$

Owing to (3.4), (3.7), we have for $\lambda \in]0, 1[$, $u, r \in \Delta_k^m$,

$$\begin{aligned}
& \left| \frac{\partial K}{\partial v}(v, u) - \frac{\partial K}{\partial v}(v, r) \right| \\
& \leq C \left| \frac{\partial K}{\partial v}(v, u) - \frac{\partial K}{\partial v}(v, r) \right|^\lambda \left(\left| \frac{\partial K}{\partial v}(v, u) \right|^{1-\lambda} + \left| \frac{\partial K}{\partial v}(v, r) \right|^{1-\lambda} \right) \\
& \leq C 2^{-m\lambda} |v - (u \vee r)|^{H-\frac{3}{2}} [(u \wedge r)^{-1} + |v - (u \vee r)|^{-1}]^\lambda. \tag{2.16}
\end{aligned}$$

Thus, taking $\lambda := H$ yields $\tau_{1,3,1}(s, t, m) \leq C 2^{-2mH} \sum_{j=1}^2 \tau_{1,3,1,j}(s, t, m)$, with

$$\begin{aligned}
\tau_{1,3,1,1}(s, t, m) &= \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho \left(\int_s^t dv |K_m(v, \rho) - K_m(s, \rho)| \right. \\
& \quad \left. \times |v - (u \vee r)|^{H-\frac{3}{2}} (u \wedge r)^{-H} \right)^2, \\
\tau_{1,3,1,2}(s, t, m) &= \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho \left(\int_s^t dv |K_m(v, \rho) - K_m(s, \rho)| \right. \\
& \quad \left. \times |v - (u \vee r)|^{-\frac{3}{2}} \right)^2.
\end{aligned}$$

Let $a = 2 - \epsilon$, with $\epsilon \in]0, 2H[$. Schwarz's inequality along with (3.14) yield

$$\begin{aligned}
\tau_{1,3,1,1}(s, t, m) &\leq C \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \left(\int_s^t dv |v - (u \vee r)|^{-a} dv \right) \\
& \quad \times \left(\int_s^t dv |v - s|^{4H-3+a} |u \wedge r|^{-2H} \right) \\
&\leq C |t - s|^{4H-\epsilon} \int_{t_1^m}^{t_{I-2}^m} du (s - \bar{u}_m)^{\epsilon-1} (\underline{u}_m)^{-2H} \\
&\leq C |t - s|^{4H-\epsilon} s^{\epsilon-2H} \leq |t - s|^{4H-\epsilon} 2^{-m(\epsilon-2H)}. \tag{2.17}
\end{aligned}$$

Indeed, $\int_s^t dv |v - (u \vee r)|^{-2+\epsilon} \leq C(s - \bar{u}_m)^{\epsilon-1}$ for $\bar{u}_m$ defined by (3.13). Let $\epsilon \in]0, 2H[$ using Schwarz's inequality and (3.14), we obtain

$$\tau_{1,3,1,2}(s, t, m) \leq C \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \left(\int_s^t dv |v - (u \vee r)|^{-2-2H+\epsilon} \right)$$

$$\begin{aligned}
& \times \left(\int_s^t dv |v - (u \vee r)|^{2H-\varepsilon-1} |v - s|^{2H} \right) \\
& \leq C |t - s|^{4H-\varepsilon} \int_{t_1^m}^{t_{I-2}^m} du \int_s^t dv (v - \bar{u}_m)^{-2-2H+\varepsilon} \\
& \leq C |t - s|^{4H-\varepsilon} \int_{t_1^m}^{t_{I-2}^m} du (s - \bar{u}_m)^{-1-2H+\varepsilon} \\
& \leq C |t - s|^{4H-\varepsilon} 2^{-m(\varepsilon-2H)}. \tag{2.18}
\end{aligned}$$

From (2.17), (2.18) we deduce that for $\varepsilon \in]0, 2H[$,

$$\tau_{1,3,1}(s, t, m) \leq C |t - s|^{4H-\varepsilon} 2^{-m\varepsilon}. \tag{2.19}$$

Let $\delta \in]0, 2H[$, $\alpha \in]0, 2H[$, $\lambda \in]0, 1[$ and $\mu \in]\frac{1}{2}, 1-H[$. Notice that for these choices, $-2\mu + 1 - 2H + \delta < 0$. Hölder's inequality together with (3.14) and (3.15) yield for any $\lambda \in]0, 1[$,

$$\tau_{1,3,2}(s, t, m) \leq C \tau_{1,3,2,1}(s, t, m)^\lambda \tau_{1,3,2,2}(s, t, m)^{1-\lambda},$$

where

$$\begin{aligned}
\tau_{1,3,2,1}(s, t, m) &= \int_{t_1^m}^{t_{I-2}^m} dr \left(\int_s^t dv (v - r)^{2H-3+2\mu} \right) \left(\int_s^t dv (v - r)^{-2\mu} (v - s)^{2H} \right), \\
\tau_{1,3,2,2}(s, t, m) &= \int_{t_1^m}^{t_{I-2}^m} dr \left(\int_s^t dv (v - r)^{2H-3+2\mu} \right) \left(\int_s^t dv (v - r)^{-2\mu} 2^{-2mH} \right).
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\tau_{1,3,2,1}(s, t, m) &\leq C |t - s|^{4H-\delta} \int_{t_1^m}^{t_{I-2}^m} dr (s - r)^{2H-2+2\mu} (s - r)^{-2\mu+1-2H+\delta} \\
&\leq C |t - s|^{4H-\delta},
\end{aligned}$$

while for the second one, we obtain

$$\tau_{1,3,2,2}(s, t, m) \leq C 2^{-2mH} |t - s|^{2H-\alpha} \int_{t_1^m}^{t_{I-2}^m} dr (s - r)^{2H-2+2\mu} (s - r)^{-2\mu+1-2H+\alpha}.$$

Consequently,

$$\tau_{1,3,2}(s, t, m) \leq C |t - s|^{(4H-\delta)\lambda + (2H-\alpha)(1-\lambda)} 2^{-2mH(1-\lambda)}.$$

Take α, δ arbitrarily small and $1 - \lambda = \frac{\varepsilon - H\delta}{2H + \alpha}$. Then for $\beta < \varepsilon < 2H$, we have proved that

$$\tau_{1,3,2}(s, t, m) \leq C |t - s|^{4H-\varepsilon} 2^{-m\beta}.$$

This inequality, together with (2.15) and (2.19) yields (2.11). $\square$

Lemma 2.5. For any $0 \leq s < t \leq 1$, set

$$T_2(s, t) = \int_s^t dB_u K(t, u) (W_u - W_s),$$

$$T_2(s, t, m) = \sum_{k=1}^{2^m} \int_{\Delta_k^m} dB_r 2^m \left(\int_{\Delta_k^m \cap]s, t]} du K(t, u) (W(m)_u - W(m)_s) \right).$$

Then, for $b \in]0, 2H[$, there exists a constant $C > 0$ such that for each $m \geq 1$

$$E \left(|T_2(s, t, m) - T_2(s, t)|^2 \right) \leq C 2^{-mb} |t - s|^{4H-b}. \quad (2.20)$$

Proof. Let $s \in \Delta_I^m$, $t \in \Delta_J^m$. We have

$$E \left(|T_2(s, t, m) - T_2(s, t)|^2 \right) \leq C \sum_{j=1}^3 T_{2,j}(s, t, m),$$

with for $\mathcal{I} = \{I, I+1, J-2, J-1, J\}$

$$T_{2,1}(s, t, m) = \sum_{k \in \mathcal{I}} E \left(\left| \int_{\Delta_k^m} dB_r 2^m \int_{\Delta_k^m \cap]s, t]} du K(t, u) (W(m)_u - W(m)_s) \right|^2 \right),$$

$$T_{2,2}(s, t, m) = \sum_{k \in \mathcal{I}} E \left(\left| \int_{\Delta_k^m \cap]s, t]} dB_r K(t, r) (W_r - W_s) \right|^2 \right),$$

$$T_{2,3}(s, t, m) = E \left(\left| \sum_{k=I+2}^{J-3} \int_{\Delta_k^m} dB_r \left[2^m \int_{\Delta_k^m \cap]s, t]} du K(t, u) (W(m)_u - W(m)_s) - K(t, r) (W_r - W_s) \right] \right|^2 \right).$$

Owing to Lemma 3.4 applied to the Gaussian process

$$X_r := \mathbf{1}_{\Delta_k^m}(r) \int_0^1 dB_\rho \left(2^m \int_{\Delta_k^m \cap]s, t]} du K(t, u) (K_m(u, \rho) - K_m(s, \rho)) \right)$$

and Schwarz's inequality, we have for any $k = 1, \dots, 2^m$,

$$\begin{aligned} T(s, t, m, k) &:= E \left(\left| \int_{\Delta_k^m} dB_r 2^m \int_{\Delta_k^m \cap]s, t]} du K(t, u) (W(m)_u - W(m)_s) \right|^2 \right) \\ &\leq C 2^{2m} \int_{\Delta_k^m} dr \int_0^1 d\rho \left(\int_{\Delta_k^m \cap]s, t]} du K^2(t, u) \right) \\ &\quad \times \left(\int_{\Delta_k^m \cap]s, t]} du |K_m(u, \rho) - K_m(s, \rho)|^2 \right). \end{aligned}$$

Let $k = I, I+1$; since $\int_{\Delta_k^m \cap]s, t]} du K^2(t, u) \leq \int_{]s, t]} du K^2(t, u) \leq C |t - s|^{2H}$, we have for any $b \in]0, 2H[$,

$$\begin{aligned} T(s, t, m, k) &\leq C 2^m |t - s|^{2H} \left(\int_{\Delta_k^m \cap]s, t]} du |u - s|^{2H} \right) \\ &\leq C 2^m |t - s|^{4H-b} \int_{\Delta_k^m \cap]s, t]} du |u - s|^b \leq C 2^{-mb} |t - s|^{4H-b}. \end{aligned}$$

Let $k = J - 2, J - 1, J$ with $J - 2 > I + 1$ then for $u \in \Delta_k^m$, (3.5) implies $|K(t, u)|^2 \leq C|t - u|^{2H-1}$ and $|t - u| \leq C2^{-m}$; we obtain for $b \in]0, 2H[$,

$$\begin{aligned} T(s, t, m, k) &\leq C2^m \left(\int_{\Delta_k^m \cap]s, t]} du |t - u|^{2H-1-b} 2^{-mb} du \right) \left(\int_{\Delta_k^m \cap]s, t]} du |u - s|^{2H} \right) \\ &\leq C|t - s|^{4H-b} 2^{-mb}. \end{aligned}$$

We therefore have proved that for $b \in]0, 2H[$,

$$T_{2,1}(s, t, m) \leq C2^{-bm}|t - s|^{4H-b}. \quad (2.21)$$

The analysis of the term $T_{2,2}(s, t, m)$ is easier. Indeed, the isometry property of the stochastic integral yields for any $k = 1, \dots, 2^m$,

$$E \left(\left| \int_{\Delta_k^m \cap]s, t]} dB_r K(t, r)(W_r - W_s) \right|^2 \right) = C \int_{\Delta_k^m \cap]s, t]} dr K^2(t, r) |r - s|^{2H}. \quad (2.22)$$

For the particular values of $k \in \mathcal{I}$, the right hand-side of (2.22) can be analyzed following similar ideas as for $T_{2,1}(s, t, m)$, which yields for $b \in]0, 2H[$

$$T_{2,2}(s, t, m) \leq C2^{-mb}|t - s|^{4H-b}. \quad (2.23)$$

We now study $T_{2,3}(s, t, m)$ and note that $T_{2,3}(s, t, m) = 0$ if $|t - s| \leq 2^{-m}$. Thus, we may assume that $t - s \geq 2^{-m}$. First, we apply Lemma 3.4 and obtain

$$T_{2,3}(s, t, m) \leq C(T_{2,3,1}(s, t, m) + T_{2,3,2}(s, t, m)),$$

where

$$\begin{aligned} T_{2,3,1}(s, t, m) &= \int_{\bar{s}_m}^{\bar{t}_m - 2^{1-m}} dr \int_0^1 d\rho \left| 2^m \int_{\underline{x}_m}^{\bar{r}_m} du (K(t, u) - K(t, r)) \right. \\ &\quad \left. \times (K_m(u, \rho) - K_m(s, \rho)) \right|^2, \\ T_{2,3,2}(s, t, m) &= \int_{\bar{s}_m}^{\bar{t}_m - 2^{1-m}} dr \int_0^1 d\rho \left| 2^m \int_{\underline{x}_m}^{\bar{r}_m} du K(t, r) \right. \\ &\quad \left. \times ([K_m(u, \rho) - K_m(s, \rho)] - [K(r, \rho) - K(s, \rho)]) \right|^2. \end{aligned}$$

By Schwarz's inequality and (3.14), for $b \in]0, 2H[$,

$$\begin{aligned} T_{2,3,1}(s, t, m) &\leq \int_{\bar{s}_m}^{\bar{t}_m - 2^{1-m}} dr 2^m \int_{\underline{x}_m}^{\bar{r}_m} du |K(t, u) - K(t, r)|^2 |u - s|^{2H} \\ &\leq C|t - s|^{2H} \int_{\bar{s}_m}^{\bar{t}_m - 2^{1-m}} dr 2^m \int_{\underline{x}_m}^{\bar{r}_m} du |K(t, u) - K(t, r)|^2 \\ &\leq C2^{-2mH}|t - s|^{2H} \leq C2^{-mb}|t - s|^{4H-b} \end{aligned}$$

where the last inequalities follow from (3.19) and $|t - s| \geq 2^{-m}$.

Owing to (3.15), we have for $u \in [\underline{r}_m, \bar{r}_m]$

$$\begin{aligned} \int_0^1 d\rho |K_m(s, \rho) - K(s, \rho)|^2 &\leq C 2^{-2mH}, \\ \int_0^1 d\rho |K_m(u, \rho) - K(r, \rho)|^2 &\leq C \int_0^1 d\rho \left(|K_m(u, \rho) - K(u, \rho)|^2 \right. \\ &\quad \left. + |K(u, \rho) - K(r, \rho)|^2 \right) \leq C 2^{-2mH}. \end{aligned}$$

Schwarz's inequality, along with (3.5) and the above estimates yield

$$\begin{aligned} T_{2,3,2}(s, t, m) &\leq C \int_{\bar{s}_m}^{\underline{t}_m - 2^{1-m}} dr 2^{-2mH} (|r|^{2H-1} + |t-r|^{2H-1}) \\ &\leq C 2^{-2mH} (t^{2H} - s^{2H} + |t-s|^{2H} + 2^{-2mH}) \\ &\leq C 2^{-2mH} |t-s|^{2H} \leq C 2^{-mb} |t-s|^{4H-b} \end{aligned}$$

for $b \in]0, 2H[$. Indeed, for each $H \in]0, \frac{1}{2}[$, and $s < t$, $t^{2H} - s^{2H} \leq (t-s)^{2H}$ and we are assuming that $2^{-m} < |t-s|$. Thus, (2.20) is proved. $\square$

Lemma 2.6. *For any $0 \leq s < t \leq 1$, set*

$$\begin{aligned} T_3(s, t) &= \int_s^t dB_u \int_u^t K(dr, u) (W_r - W_u) \\ T_3(s, t, m) &= \sum_{k=1}^{2^m} 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m \cap]s, t]} du \int_u^t K(dv, u) (W(m)_v - W(m)_u). \end{aligned}$$

There exists a positive constant C such that, for any $\epsilon \in]0, 4H-1[$

$$E \left(|T_3(s, t, m) - T_3(s, t)|^2 \right) \leq C 2^{-m\epsilon} |t-s|^{4H-\epsilon}, \quad (2.24)$$

for each $m \geq 1$.

Proof. Assume $s \in \Delta_I^m$, $t \in \Delta_J^m$; we consider the upper bound

$$E \left(|T_3(s, t, m) - T_3(s, t)|^2 \right) \leq C \sum_{j=1}^3 T_{3,j}(s, t, m),$$

where for $\mathcal{J} = \{I, I+1, J-1, J\}$

$$\begin{aligned} T_{3,1}(s, t, m) &= \sum_{k \in \mathcal{J}} E \left(\left| 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m \cap]s, t]} du \int_u^t K(dv, u) \right. \right. \\ &\quad \left. \left. \times (W(m)_v - W(m)_u) \right|^2 \right), \end{aligned} \quad (2.25)$$

$$T_{3,2}(s, t, m) = \sum_{k \in \mathcal{J}} E \left(\left| \int_{\Delta_k^m \cap]s, t]} dB_r \int_r^t K(dv, r) (W_v - W_r) \right|^2 \right), \quad (2.26)$$

$$T_{3,3}(s, t, m) = E \left(\left| \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m} du \right. \right. \\ \left. \left. \times \left(\int_u^t K(dv, u)(W(m)_v - W(m)_u) - \int_r^t K(dv, r)(W_v - W_r) \right) \right|^2 \right).$$

Lemma 3.4 along with Schwarz's inequality yield for each term of the sum in the right hand side of (2.25) the upper bound

$$C \int_{\Delta_k^m} dr \int_0^1 d\rho 2^m \int_{\Delta_k^m \cap]s, t]} du \left(\int_u^t K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right)^2.$$

Fix $a \in]2 - 4H, 1]$. From Schwarz's inequality, (3.4) and (3.14) we deduce the following estimates for this integral:

$$C \int_{\Delta_k^m} dr 2^m \int_{\Delta_k^m \cap]s, t]} du \left(\int_u^t dv |v - u|^{-a} \right) \left(\int_u^t |v - u|^{4H-3+a} \right) \\ \leq C (2^{-m} \wedge |t - s|) |t - s|^{4H-1}.$$

A similar analysis yields the same result for each term in the right hand-side of (2.26). Consequently,

$$T_{3,1}(s, t, m) + T_{3,2}(s, t, m) \leq C (2^{-m} \wedge |t - s|) |t - s|^{4H-1}. \quad (2.27)$$

If $|t - s| \leq 2^{-m}$ then $T_{3,3}(s, t, m) = 0$. Hence, let us assume that $t - s \geq 2^{-m}$; in this case $T_{3,3}(s, t, m)$ is equal to $E \left(\int_0^1 dB_r X_r \right)^2$, with $X_r = \int_0^1 dB_\rho g(r, \rho)$, and

$$g(r, \rho) = \sum_{k=I+2}^{J-2} \mathbf{1}_{\Delta_k^m}(r) 2^m \int_{\Delta_k^m} du \left[\int_u^t K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right. \\ \left. - \int_r^t K(dv, r)(K(v, \rho) - K(r, \rho)) \right].$$

We at first study the contribution to $T_{3,3}(s, t, m)$ of the integrands

$$g_1(r, \rho) = \sum_{k=I+2}^{J-2} \mathbf{1}_{\Delta_k^m}(r) 2^m \int_{\Delta_k^m} du \int_u^{u \vee r} K(dv, u)(K_m(v, \rho) - K_m(u, \rho)), \\ g_2(r, \rho) = \sum_{k=I+2}^{J-2} \mathbf{1}_{\Delta_k^m}(r) 2^m \int_{\Delta_k^m} du \int_r^{u \vee r} K(dv, r)(K(v, \rho) - K(r, \rho)),$$

which we denote by $T_{3,3,j}(s, t, m)$, $j = 1, 2$. Actually, both are similar and therefore we only study the first one. Lemma 3.4, (3.4), (3.14) and Schwarz's inequality imply, for each $a \in]2 - 4H, 1]$,

$$T_{3,3,1}(s, t, m) \leq C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_u^{u \vee r} dv |v - u|^{-a} \int_u^{u \vee r} dv |v - u|^{4H-3+a} \\ \leq C 2^{-m(4H-1)} |t - s|. \quad (2.28)$$

We end the analysis of the term $T_{3,3}(s, t, m)$ by studying the contribution of $T_{3,3,3}((s, t, m))$ defined in terms of the integrand

$$g_3(r, \rho) = \sum_{k=I+2}^{J-2} \int_{\Delta_k^m} dr 2^m \int_{\Delta_k^m} du \int_{u \vee r}^t \left[K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right. \\ \left. - K(dv, r)(K(v, \rho) - K(r, \rho)) \right].$$

Notice that $g_3(r, \rho)$ is the sum of two analogous terms where the set Δ_k^m of the integral with respect to the variable u is replaced by $[\underline{r}_m, r[$, $[r, \bar{r}_m[$, respectively. Again, the contribution of both terms is similar, so that we concentrate on the first one. That is, we consider

$$T_{3,3,3}^+(s, t, m) := E \left(\left| \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dB_r \int_{[\underline{r}_m, r[} du \int_r^t \left[K(dv, u) \right. \right. \right. \\ \left. \left. \left. \times (W(m)_v - W(m)_u) - K(dv, r)(W_v - W_r) \right] \right|^2 \right).$$

As before, all the arguments rely on Lemma 3.4, (3.4), (3.14), a suitable factorization of the integrands along with Schwarz's inequality. In order to deal with the singularity at $v = r$, we first replace the integral with respect to the variable v by $\int_r^{\bar{r}_m + 2^{-m}}$. Given $a \in]2 - 4H, 1[$, the corresponding contribution to $T_{3,3,3}^+(s, t, m)$ is bounded by

$$C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \int_0^1 d\rho \left(\left| \int_r^{\bar{r}_m + 2^{-m}} K(dv, u) \right. \right. \\ \left. \left. \times (K_m(v, \rho) - K_m(u, \rho)) \right|^2 + \left| \int_r^{\bar{r}_m + 2^{-m}} K(dv, r)(K(v, \rho) - K(r, \rho)) \right|^2 \right) \\ \leq C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \int_r^{\bar{r}_m + 2^{-m}} dv |v - r|^{-a} \int_r^{\bar{r}_m + 2^{-m}} dv |v - r|^{4H-3+a} \\ \leq C 2^{-m(4H-1)} |t - s|. \quad (2.29)$$

Let us finally consider the range $]r_m + 2^{-m}, t[$ for the variable v . We have to study two terms:

$$M_1(s, t, m) = \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \int_0^1 d\rho \left(\int_{\bar{r}_m + 2^{-m}}^t dv \right. \\ \left. \times |K_m(v, \rho) - K_m(u, \rho)| \left| \frac{\partial K}{\partial v}(v, u) - \frac{\partial K}{\partial v}(v, r) \right| \right)^2, \\ M_2(s, t, m) = \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \int_0^1 d\rho \left(\int_{\bar{r}_m + 2^{-m}}^t dv \left| \frac{\partial K}{\partial v}(v, r) \right| \right.$$

$$\times \left[(K_m(v, \rho) - K_m(u, \rho)) - (K(v, \rho) - K(r, \rho)) \right]^2.$$

For $M_1(s, t, m)$, we proceed in a similar way as for the term $\tau_{1,3,1}(s, t, m)$ in Lemma 2.4, as follows. By means of (2.16) we obtain for $\lambda \in]0, 1[$ $M_1(s, t, m) \leq C2^{-2m\lambda} (M_{1,1}(s, t, m) + M_{1,2}(s, t, m))$, with

$$\begin{aligned} M_{1,1}(s, t, m) &= \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du u^{-2\lambda} \int_0^1 d\rho \left(\int_{\bar{r}_m+2^{-m}}^t dv \right. \\ &\quad \left. |K_m(v, \rho) - K_m(u, \rho)| |v - r|^{H-\frac{3}{2}} \right)^2, \\ M_{1,2}(s, t, m) &= \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \int_0^1 d\rho \left(\int_{\bar{r}_m+2^{-m}}^t dv \right. \\ &\quad \left. |K_m(v, \rho) - K_m(u, \rho)| |v - r|^{H-\frac{3}{2}-\lambda} \right)^2. \end{aligned}$$

Let $a \in]2 - 4H, 1[$, $\lambda \in]0, \frac{1}{2}[$. Since $t - s \geq 2^{-m}$, for $u \in [\underline{r}_m, r[$, we have

$$\int_{\bar{r}_m+2^{-m}}^t dv |v - r|^{2H-3+a} |v - u|^{2H} \leq C |t - r|^{4H+a-2}.$$

Consequently, since $r \geq u \geq \underline{r}_m \geq t_{I+1}$ implies $u \geq \frac{r}{2}$

$$\begin{aligned} M_{1,1}(s, t, m) &\leq C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du u^{-2\lambda} \left(\int_{\bar{r}_m+2^{-m}}^t dv |v - r|^{-a} \right) \\ &\quad \times \left(\int_{\bar{r}_m+2^{-m}}^t dv |v - r|^{2H-3+a} |v - u|^{2H} \right) \\ &\leq C \int_s^t r^{-2\lambda} |t - s|^{4H-1} dr \leq C |t - s|^{4H-2\lambda}. \end{aligned} \tag{2.30}$$

Analogously, for $b \in]2 + 2\lambda - 4H, 1[$, $\lambda \in]0, 2H - \frac{1}{2}[$ and $|t - s| \geq 2^{-m}$

$$\begin{aligned} M_{1,2}(s, t, m) &\leq C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \left(\int_{\bar{r}_m+2^{-m}}^t dv |v - r|^{-b} \right) \\ &\quad \times \left(\int_{\bar{r}_m+2^{-m}}^t dv |v - r|^{2H-3-2\lambda+b} |v - u|^{2H} \right) \\ &\leq C \int_s^t |t - r|^{4H-1-2\lambda} dr = C |t - s|^{4H-2\lambda}. \end{aligned} \tag{2.31}$$

Finally, if we additionally use (3.15), we obtain for $a \in]2 - 4H, 1[$

$$M_2(s, t, m) \leq C \sum_{k=I+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{[\underline{r}_m, r[} du \left(\int_r^t dv |v - r|^{-a} \right)$$

$$\begin{aligned}
& \times \left(\int_{\underline{r}_m + 2^{-m}}^t dv |v - r|^{2H-3+a} 2^{-2mH} \right) \\
& \leq C \int_s^t |t - r|^{1-a} 2^{-m(4H-2+a)} dr \leq C 2^{-mb} |t - s|^{4H-b} \quad (2.32)
\end{aligned}$$

for $b \in]0, 4H - 1[$. We easily check that (2.24) follows from (2.27)–(2.32). $\square$

Proof of Proposition 2.3: We remark that Lemmas 2.4 to 2.6 yield the upper bound (2.10). Therefore, for $q = 2$, (2.6) follows from (2.9) and (2.10). The hypercontractivity inequality yields the validity of the same inequality for any $q \in]2, \infty[$. $\square$

Proof of Theorem 2.1:

Let $H \in]\frac{1}{2}, 1[$ and $p \in]\frac{1}{H}, 2[$. The convergence of $\tilde{d}_p(\mathbf{W}(\mathbf{m}), \mathbf{W})$ to zero in $L^q(\Omega)$ is a consequence of (2.2) and the usual version of the Garsia-Rademich-Rumsey lemma (see e.g. [9], Theorem 2.1.3).

Consider the metric space $(\mathcal{G}_p, \tilde{d}_p)$. The canonical embedding $\mathcal{H}^H \hookrightarrow \mathcal{G}_p$ is continuous. Indeed, let $\dot{h}_i$, $i = 1, 2$, belong to $L^2([0, 1])$. Then for $h_i(\cdot) = \int_0^\cdot K(\cdot, r) \dot{h}_i(r) dr$ and $0 \leq s < t \leq 1$,

$$|(h_1)_{s,t}^{(1)} - (h_2)_{s,t}^{(1)}| \leq |t - s|^H \|\dot{h}_1 - \dot{h}_2\|_2 \leq |t - s|^{\frac{1}{p}} \|h_1 - h_2\|_{\mathcal{H}^H}.$$

Consequently, the preceding convergence shows that $(\mathcal{G}_p, \mathcal{H}^H, P^H)$ is an abstract Wiener space.

Let now $H \in]\frac{1}{4}, \frac{1}{2}[$. We follow the outline of the proof of Lemma 3 in [3], but refer to the extension of the Garsia-Rademich-Rumsey lemma stated in the Lemma 3.5.

Fix $p \in]2, 4[$ such that $pH > 1$. We shall prove that there exists $\theta > 0$ such that for every $q \in [1, \infty[$,

$$E \left(\left| \tilde{d}_p(\mathbf{W}, \mathbf{W}(\mathbf{m})) \right|^q \right) \leq C_q 2^{-m\theta q}. \quad (2.33)$$

Indeed, for a fixed $q \in [1, \infty[$, let $M > q$ and $N = 2M$ satisfy $N > \frac{p}{2(Hp-1)}$. Let $\alpha, \beta > 0$ defined by $\alpha = \frac{2}{p} + \frac{1}{M}$, $\beta = \frac{1}{p} + \frac{1}{N}$.

By virtue of (2.1) and (2.6), we easily check that the random variables

$$\begin{aligned}
A_1(m) &:= \int_0^1 \int_0^1 ds dt 1_{\{s < t\}} \frac{|W_{s,t}^{(1)} - W(m)_{s,t}^{(1)}|^{2N}}{|t - s|^{2N\beta}}, \\
A_2(m) &:= \int_0^1 \int_0^1 ds dt 1_{\{s < t\}} \frac{|W_{s,t}^{(2)} - W(m)_{s,t}^{(2)}|^{2M}}{|t - s|^{2M\alpha}},
\end{aligned}$$

satisfy

$$E(A_1(m)) \leq C 2^{-m\mu 2N}, \quad E(A_2(m)) \leq C 2^{-m\mu 2M}, \quad (2.34)$$

for some $\mu > 0$.

Furthermore, the hypercontractivity property and the inequality (3.14) imply that for $0 \leq s < t \leq 1$ and $q \in [1, \infty]$,

$$\sup_m (\|W_{s,t}^{(1)}\|_q + \|W(m)_{s,t}^{(1)}\|_q) \leq C |t - s|^H.$$

This yields

$$\sup_m E(\eta(m)) \leq C, \quad (2.35)$$

where

$$\eta(m) := \int_0^1 \int_0^1 ds dt 1_{\{s < t\}} \frac{|W_{s,t}^{(1)}|^{2N} + |W(m)_{s,t}^{(1)}|^{2N}}{|t - s|^{2N\beta}}.$$

By Lemma 3.5, we deduce that for any $0 \leq s < t \leq 1$,

$$|W_{s,t}^{(1)} - W(m)_{s,t}^{(1)}| \leq C A_1(m)^{\frac{1}{2N}} |t - s|^{\frac{1}{p}}, \quad (2.36)$$

$$|W_{s,t}^{(2)} - W(m)_{s,t}^{(2)}| \leq C \left[A_2(m)^{\frac{1}{2M}} + A_1(m)^{\frac{1}{2N}} \eta(m)^{\frac{1}{2N}} \right] |t - s|^{\frac{2}{p}}. \quad (2.37)$$

Finally, Schwarz's and Hölder's inequalities together with (2.34)-(2.37) conclude the proof of the theorem. $\square$

3. Appendix

Let $W^H = (W_t^H, t \in [0, 1])$ be a d -dimensional fractional Brownian motion with Hurst parameter $H \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$ and integral representation given in (1.1).

Assume $H \in]\frac{1}{2}, 1[$; by computing the integral of the right hand-side of (1.3), we obtain the following expression for the kernel K^H defined in (1.2):

$$K^H(t, s) = c_H \left(H - \frac{1}{2} \right) s^{H-\frac{1}{2}} F_2 \left(\frac{t}{s} \right), \quad (3.1)$$

where for $z > 1$,

$$F_2(z) = \int_0^{z-1} u^{H-\frac{3}{2}} (u+1)^{H-\frac{1}{2}} du. \quad (3.2)$$

From (1.2), it follows that

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left(H - \frac{1}{2} \right) \left(\frac{s}{t} \right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \quad (3.3)$$

holds for any $H \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$ and $0 < s < t < 1$. Consequently, for $H \in]0, \frac{1}{2}[$,

$$\left| \frac{\partial K^H}{\partial t}(t, s) \right| \leq C |t - s|^{H-\frac{3}{2}}. \quad (3.4)$$

The next Lemma collects some technical estimates on the kernel $K^H(t, s)$.

Lemma 3.1. *Let $0 < s < t < 1$.*

(1) *Assume $H \in]0, \frac{1}{2}[$. Then,*

$$|K^H(t, s)| \leq C \left(s^{H-\frac{1}{2}} \mathbf{1}_{]0, \frac{t}{2}[}(s) + (t-s)^{H-\frac{1}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right), \quad (3.5)$$

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left(s^{H-\frac{3}{2}} \mathbf{1}_{]0, \frac{t}{2}[}(s) + (t-s)^{H-\frac{3}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right), \quad (3.6)$$

$$\left| \frac{\partial^2 K^H}{\partial t \partial s}(t, s) \right| \leq C(t-s)^{H-\frac{3}{2}} \left(s^{-1} \mathbf{1}_{]0, \frac{t}{2}[}(s) + (t-s)^{-1} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right). \quad (3.7)$$

(2) *For $H \in]\frac{1}{2}, 1[$,*

$$|K^H(t, s)| \leq C \left((t-s)^{H-\frac{1}{2}} \mathbf{1}_{]0, \frac{t}{2}[}(s) + s^{H-\frac{1}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right), \quad (3.8)$$

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C(t-s)^{2H-1} \left(s^{-(H+\frac{1}{2})} \mathbf{1}_{]0, \frac{t}{2}[}(s) + (t-s)^{-(H+\frac{1}{2})} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right). \quad (3.9)$$

Proof. Assume first $H \in]0, \frac{1}{2}[$. It is easy to check that, for any $u > 0$,

$$0 < 1 - (u+1)^{H-\frac{1}{2}} \leq \left(\left(\frac{1}{2} - H \right) u \right) \wedge 1.$$

Hence, for $0 < s < t$, $0 < u < \frac{t}{s} - 1$,

$$\begin{aligned} u^{H-\frac{3}{2}} \left(1 - (u+1)^{H-\frac{1}{2}} \right) &\leq C u^{H-\frac{1}{2}} \mathbf{1}_{]0, 1 \wedge (\frac{t}{s}-1)[}(u) \\ &\quad + C u^{H-\frac{3}{2}} \mathbf{1}_{]1 \wedge (\frac{t}{s}-1), \frac{t}{s}-1[}(u). \end{aligned} \quad (3.10)$$

Thus, from (1.3), (3.10), it follows that

$$\left| F_1 \left(\frac{t}{s} \right) \right| \leq C \int_0^{\frac{t}{s}-1} u^{H-\frac{1}{2}} du \leq C,$$

for $\frac{t}{2} \leq s < t$, while for $0 < s < \frac{t}{2}$,

$$\left| F_1 \left(\frac{t}{s} \right) \right| \leq C \int_0^1 u^{H-\frac{1}{2}} du + C \int_1^\infty u^{H-\frac{3}{2}} du \leq C.$$

Consequently

$$\sup_{0 \leq s < t} \left| F_1 \left(\frac{t}{s} \right) \right| \leq C \quad (3.11)$$

and the identity (1.2) yields (3.5).

By differentiating with respect to the variable s in (1.2) and using (3.11), we obtain

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left(|t-s|^{H-\frac{3}{2}} + s^{H-\frac{3}{2}} + s^{-1} t |t-s|^{H-\frac{3}{2}} \right),$$

which yields (3.6). The inequality (3.7) follows by differentiating with respect to the variable s in (3.3).

Suppose now $H \in]\frac{1}{2}, 1[$. Consider the function F_2 given in (3.2). Clearly, if $\frac{t}{s} - 1 \leq 1$, that is, if $\frac{t}{2} \leq s < t$,

$$\left| F_2 \left(\frac{t}{s} \right) \right| \leq C.$$

Assume $\frac{t}{s} - 1 > 1$. For any $u \in]1, \frac{t}{s} - 1[$, $(1+u)^{H-\frac{1}{2}} \leq Cu^{H-\frac{1}{2}}$. Consequently,

$$\left| F_2 \left(\frac{t}{s} \right) \right| \leq C \left(\int_0^1 u^{H-\frac{3}{2}} du + \int_1^{\frac{t}{s}-1} u^{2H-2} du \right) \leq C \left(\frac{t}{s} \right)^{2H-1}.$$

The previous upper bounds, together with the representation of the kernel K^H given in (3.1), imply

$$\begin{aligned} |K^H(t, s)| &\leq C \left(s^{H-\frac{1}{2}} \left(\frac{t}{s} \right)^{2H-1} \mathbf{1}_{]0, \frac{t}{2}[}(s) + s^{H-\frac{1}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right) \\ &\leq \left(s^{H-\frac{1}{2}} \mathbf{1}_{]0, \frac{t}{2}[}(s) + s^{-H+\frac{1}{2}} (t-s)^{2H-1} \mathbf{1}_{]0, \frac{t}{2}[}(s) + s^{H-\frac{1}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right) \end{aligned}$$

and (3.8) follows.

Differentiating with respect to the variable s in (3.1) yields

$$\begin{aligned} \left| \frac{\partial K^H}{\partial s}(t, s) \right| &\leq C \left(s^{H-\frac{3}{2}} F_2 \left(\frac{t}{s} \right) + s^{H-\frac{1}{2}} \frac{t}{s^2} \left(\frac{t}{s} - 1 \right)^{H-\frac{3}{2}} \left(\frac{t}{s} \right)^{H-\frac{1}{2}} \right) \\ &\leq C \left(s^{H-\frac{3}{2}} \left(\frac{t}{s} \right)^{2H-1} \mathbf{1}_{]0, \frac{t}{2}[}(s) + s^{-(H+\frac{1}{2})} t^{H+\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \right. \\ &\quad \left. + s^{H-\frac{3}{2}} \mathbf{1}_{[\frac{t}{2}, t[}(s) \right), \end{aligned}$$

where in the last inequality we have applied the upper bounds for F_2 obtained before. Replacing in the last expression t^{2H-1} by $C(s^{2H-1} + (t-s)^{2H-1})$ and $t^{H+\frac{1}{2}}$ by $C(s^{H+\frac{1}{2}} + (t-s)^{H+\frac{1}{2}})$, respectively, yields

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left(s^{H-\frac{3}{2}} + (t-s)^{H-\frac{3}{2}} + s^{-(H+\frac{1}{2})} (t-s)^{2H-1} \right). \quad (3.12)$$

If $0 < s < \frac{t}{2}$ then, $s < t-s$ and $(t-s)^{H-\frac{3}{2}} < s^{H-\frac{3}{2}} < s^{-(H+\frac{1}{2})} (t-s)^{2H-1}$, while for $\frac{t}{2} \leq s < t$, the previous inequalities are reversed accordingly. Hence (3.9) clearly follows from (3.12). $\square$

We introduce the notation

$$\underline{t}_m = [2^m t] 2^{-m} \quad \text{and} \quad \bar{t}_m = \underline{t}_m + 2^{-m}, \quad (3.13)$$

for any $m \in \mathbb{N}$. Notice that, K_m^H given in (1.5) satisfies $K_m^H(t, s) = 0$ if $s \geq \bar{t}_m$.

In the next result, we give a bound for the approximation in quadratic mean of the kernel K^H by its projection K_m^H .

Lemma 3.2. (1) Let $H \in]0, \frac{1}{2}[\cup] \frac{1}{2}, 1[$. There exists a positive constant C such that for any $0 < s < t \leq 1$,

$$\sup_{m \geq 1} \int_0^1 \left(|K_m^H(t, u) - K_m^H(s, u)|^2 + |K^H(t, u) - K^H(s, u)|^2 \right) du \leq C |t - s|^{2H}. \quad (3.14)$$

(2) For $H \in]0, \frac{1}{2}[$,

$$\int_0^1 |K^H(t, u) - K_m^H(t, u)|^2 du \leq C (t \wedge 2^{-m})^{2H}. \quad (3.15)$$

(3) For $H \in]\frac{1}{2}, 1[$ and any $\lambda \in]0, \frac{1}{2H+1}[$,

$$\int_0^1 |K^H(t, u) - K_m^H(t, u)|^2 du \leq C 2^{-2m\lambda} t^{2(H-\lambda)}. \quad (3.16)$$

Proof. The operator π_m is a contraction on $L^2[0, 1]$. Thus,

$$\begin{aligned} & \sup_{m \geq 1} \int_0^1 \left(|K_m^H(t, u) - K_m^H(s, u)|^2 + |K^H(t, u) - K^H(s, u)|^2 \right) du \\ & \leq 2 \int_0^1 |K^H(t, u) - K^H(s, u)|^2 du = 2E(|W_t^H - W_s^H|^2) = 2|t - s|^{2H}, \end{aligned}$$

proving (3.14).

By the same argument,

$$\int_0^1 |K^H(t, u) - K_m^H(t, u)|^2 du \leq 4 \int_0^1 |K^H(t, u)|^2 du = 4t^{2H}. \quad (3.17)$$

Therefore (3.15) holds for $t \leq C 2^{-m}$.

Fix $t \in \Delta_I^m$ with $I > 7$. We assume first $H \in]0, \frac{1}{2}[$. Consider the decomposition

$$\int_0^1 |K^H(t, u) - K_m^H(t, u)|^2 du \leq C \sum_{i=1}^5 T_i(t), \quad (3.18)$$

with

$$\begin{aligned} T_1(t) &= \int_0^{t_2^m} |K^H(t, u) - K_m^H(t, u)|^2 du, \\ T_2(t) &= \int_{t_{I-3}^m}^{t_I^m} |K^H(t, u) - K_m^H(t, u)|^2 du, \\ T_3(t) &= \sum_{k=3}^{[2^{m-1}t]} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du, \\ T_4(t) &= \sum_{k=[2^{m-1}t]+2}^{I-3} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du, \end{aligned}$$

$$T_5(t) = \int_{\Delta_{[2^{m-1}t]+1}^m} |K^H(t, u) - K_m^H(t, u)|^2 du.$$

Schwarz's inequality and (3.5) imply

$$T_1(t) \leq 4 \int_0^{t_2^m} |K^H(t, u)|^2 du \leq C \int_0^{t_2^m} u^{2H-1} du = C 2^{-2mH}.$$

Similarly,

$$T_2(t) \leq 4 \int_{t_{I-3}^m}^{t_I^m} |K^H(t, u)|^2 du \leq C \int_{t_{I-3}^m}^t |t - u|^{2H-1} du = C 2^{-2mH}.$$

Let $\lambda \in]H, 1[$ and $k = 3, \dots, [2^{m-1}t]$, which implies $\Delta_k^m \subset]0, \frac{t}{2}[$. By Schwarz's inequality, the mean value theorem and (3.5), (3.6), we obtain

$$\begin{aligned} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du &\leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^2 \\ &\leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^{2\lambda} ||K^H(t, u)| + |K^H(t, v)|||^{2(1-\lambda)} \\ &\leq C 2^{-m(2\lambda-1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv ((u \wedge v)^{2H-1-2\lambda}). \end{aligned}$$

For $u, v \in \Delta_k^m$, $u \wedge v \geq u - 2^{-m}$; thus,

$$T_3(t) \leq C 2^{-2m\lambda} \int_{t_2^m}^{t_{[2^{m-1}t]}^m} du (u - 2^{-m})^{2H-1-2\lambda} \leq C 2^{-2mH}.$$

Fix now $k = [2^{m-1}t] + 2, \dots, I - 3$, so that $\Delta_k^m \subset [\frac{t}{2}, t[$. In this case

$$\begin{aligned} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du &\leq C 2^{-m(2\lambda-1)} \\ &\times \int_{\Delta_k^m} du \int_{\Delta_k^m} dv (t - (u \vee v))^{2H-1-2\lambda}. \end{aligned}$$

Since for $u, v \in \Delta_k^m$, $t - (u \vee v) \geq t - u - 2^{-m} \geq t_{I-2}^m - u$, the previous estimate implies

$$T_4(t) \leq C 2^{-2m\lambda} \int_{t_{[2^{m-1}t]}^m}^{t_{I-3}^m} du (t_{I-2}^m - u)^{2H-1-2\lambda} \leq C 2^{-2mH}.$$

We study the term $T_5(t)$ using the same method as for $T_3(t)$, $T_4(t)$, as follows:

$$\begin{aligned} T_5(t) &\leq 2^m \int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv |K^H(t, u) - K^H(t, v)|^2 \\ &\leq C 2^{-m(2\lambda-1)} \int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv \left((u \wedge v)^{H-\frac{3}{2}} + (t - (u \vee v))^{H-\frac{3}{2}} \right)^{2\lambda} \end{aligned}$$

$$\times \left((u \wedge v)^{H-\frac{1}{2}} + (t - (u \vee v))^{H-\frac{1}{2}} \right)^{2(1-\lambda)}.$$

For $u, v \in \Delta_{[2^{m-1}t]+1}^m$, $u \wedge v > \frac{t}{2} - 2^{-m}$, $u \vee v < \frac{t}{2} + 2^{-m}$ and $t - (u \vee v) > \frac{t}{2} - 2^{-m}$. Thus, the last integral is bounded by

$$\int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv \left(\frac{t}{2} - 2^{-m} \right)^{2H-1-2\lambda}.$$

Moreover, since we are assuming that $t \in \Delta_I^m$, with $I > 7$, $\frac{t}{2} - 2^{-m} \geq 2^{-m+1}$. Thus, we finally obtain for $\lambda = \frac{1}{2}$,

$$T_5(t) \leq C 2^{-2mH}.$$

Then (3.15) follows from the upper bounds obtained so far for $T_i(t)$, $i = 1, \dots, 5$.

Notice that we have also proved that for $H \in]0, \frac{1}{2}[$,

$$\sum_{k=3}^{I-3} 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^2 \leq C 2^{-2mH}. \quad (3.19)$$

Assume now $H \in]\frac{1}{2}, 1[$ and fix $\lambda \in]0, \frac{1}{2H+1}[$, so that $H - \lambda > 0$. Since the inequality (3.17) holds for any $H \in]0, \frac{1}{2}[\cap]\frac{1}{2}, 1[$, (3.16) holds for any $t \leq C 2^{-m}$. Let now $t \in \Delta_I^m$, with $I > 7$. We apply a similar method as we used in the case $H \in]0, \frac{1}{2}[$, using the decomposition (3.18). In fact, owing to (3.8),

$$\begin{aligned} T_1(t) &\leq C \int_0^{t_2^m} (t - u)^{2H-1} du \leq C 2^{-m} t^{2H-1}, \\ T_2(t) &\leq C \int_{t_{I-3}^m}^{t_I^m} u^{2H-1} du \leq C 2^{-m} t^{2H-1}. \end{aligned}$$

Fix $k = 3, \dots, [2^{m-1}t]$. Schwarz's inequality, along with the mean value theorem and (3.8), (3.9), imply

$$\begin{aligned} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du &\leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^{2\lambda} \\ &\quad \times ||K^H(t, u)| + |K^H(t, v)|||^{2(1-\lambda)} \\ &\leq C 2^{-m(2\lambda-1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv ((t - (u \wedge v))^{(\lambda+1)(2H-1)} (u \wedge v)^{-\lambda(2H+1)} \\ &\leq C 2^{-2m\lambda} t^{(\lambda+1)(2H-1)} \int_{\Delta_k^m} du (u - 2^{-m})^{-\lambda(2H+1)}. \end{aligned}$$

Since $\lambda < \frac{1}{2H+1}$, we have

$$T_3(t) \leq C 2^{-2m\lambda} t^{2(H-\lambda)}.$$

Let now $k = [2^{m-1}t] + 2, \dots, I - 3$. With similar arguments as before, we deduce

$$\begin{aligned} \int_{\Delta_k^m} |K^H(t, u) - K_m^H(t, u)|^2 du &\leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^{2\lambda} \\ &\quad \times ||K^H(t, u)| + |K^H(t, v)|||^{2(1-\lambda)} \\ &\leq C 2^{-m(2\lambda-1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv (t - (u \vee v))^{\lambda(2H-3)} (u \vee v)^{(1-\lambda)(2H-1)} \\ &\leq C 2^{-2m\lambda} t^{(1-\lambda)(2H-1)} \int_{\Delta_k^m} du (t - u - 2^{-m})^{\lambda(2H-3)}. \end{aligned}$$

For $\lambda < \frac{1}{2H+1}$, $\lambda(2H-3) + 1 > 0$. Hence,

$$T_4(t) \leq C 2^{-2m\lambda} t^{(1-\lambda)(2H-1)} \int_{\frac{t}{2}}^{t_{I-3}^m} (t - u - 2^{-m})^{\lambda(2H-3)} \leq C 2^{-2m\lambda} t^{2(H-\lambda)}.$$

Finally, we study the contribution of $T_5(t)$ as follows.

$$\begin{aligned} T_5(t) &\leq 2^m \int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv |K^H(t, u) - K^H(t, v)|^2 \\ &\leq C 2^{-m(2\lambda-1)} \int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv \left((t - (u \wedge v))^{2H-1} \right. \\ &\quad \times \left((u \wedge v)^{-(H+\frac{1}{2})} + (t - (u \vee v))^{-(H+\frac{1}{2})} \right)^{2\lambda} \\ &\quad \times \left((t - (u \wedge v))^{H-\frac{1}{2}} + (u \vee v)^{H-\frac{1}{2}} \right)^{2(1-\lambda)}. \end{aligned}$$

For $u, v \in \Delta_{[2^{m-1}t]+1}^m$, $u \wedge v > C_1 t$, $u \vee v < C_2 t$, $t - (u \wedge v) < C_3 t$ and $t - (u \vee v) > C_4 t$. Thus,

$$T_5(t) \leq C 2^{-m(2\lambda-1)} 2^{-2m} t^{2(H-\lambda)-1} \leq C 2^{-2m\lambda} t^{2(H-\lambda)}$$

The estimates obtained so far imply (3.16). $\square$

In the next Lemma we prove a simple extension of a well-known integration formula for bounded variation functions.

Lemma 3.3. *For any $h \in \mathcal{H}$, $t \geq 0$,*

$$\int_0^t h(u) h(du) = \frac{h^2(t)}{2}, \quad (3.20)$$

where the integral is understood in the sense of Proposition 5 in [7].

Proof. Let $n \geq 1$ and let $h(n)$ be the function obtained by linear interpolation on the n -th dyadic grid of h . We have proved in [7], Theorem 9 that

$$\lim_{n \rightarrow \infty} \int_0^t h(n)(u) h(n)(du) = \int_0^t h(u) h(du),$$

for any $t \geq 0$. Since (3.20) is true with h replaced by $h(n)$, the result follows. $\square$

The following result gives an upper bound for the L^2 norm of a Skorohod integral of a Gaussian process.

Lemma 3.4. *Let $X_t = \int_0^1 g(t, s) dB_s$, $t \in [0, 1]$, with g a deterministic function belonging to $L^2([0, 1]^2)$. Then, the Skorohod integral $\int_0^1 X_s dB_s$ satisfies*

$$E \left(\int_0^1 X_s dB_s \right)^2 \leq C \int_0^1 ds \int_0^1 dr |g(s, r)|^2. \quad (3.21)$$

Proof. The isometry property of the Skorohod integral ([8], Equation (1.48)) yields

$$E \left(\int_0^1 X_s dB_s \right)^2 \leq C \int_0^1 E(X_s)^2 ds + \int_0^1 ds \int_0^1 dr E(|D_r X_s|^2).$$

Since $E(X_s)^2 = \int_0^1 |g(s, r)|^2 dr$ and the Malliavin derivative $D_r X_s$ is equal to $g(s, r)$, (3.21) follows. $\square$

We conclude this section by proving an extension of the Garsia-Rademich-Rumsey lemma used to estimate $\tilde{d}_p(X, Y)$ when X and Y are geometric rough paths with roughness $p \in [2, \infty[$ (see [6], Definition 3.3.3).

Lemma 3.5. *Let X and Y be geometric rough paths with the same roughness $p \in [2, +\infty[$. Set $k = [p]$. For $i = 1, \dots, k$, let $M_i \geq 1$, $\alpha_i = \frac{i}{p} + \frac{1}{M_i}$. Suppose that*

$$\int_0^1 \int_0^1 ds dt 1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)}|^{2M_i} + |Y_{s,t}^{(i)}|^{2M_i}}{|t-s|^{2M_i \alpha_i}} \leq A_i, \quad 1 \leq i \leq k-1, \quad (3.22)$$

$$\int_0^1 \int_0^1 ds dt 1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t-s|^{2M_i \alpha_i}} \leq B_i, \quad 1 \leq i \leq k. \quad (3.23)$$

Then, there exists a constant $C > 0$ such that for any $0 \leq s < t \leq 1$,

$$|X_{s,t}^{(i)}| + |Y_{s,t}^{(i)}| \leq C F_i |t-s|^{\frac{i}{p}}, \quad 1 \leq i \leq k-1, \quad (3.24)$$

$$|X_{s,t}^{(i)} - Y_{s,t}^{(i)}| \leq C G_i |t-s|^{\frac{i}{p}}, \quad 1 \leq i \leq k. \quad (3.25)$$

where F_i and G_i are defined recursively by

$$F_i = A_i^{\frac{1}{2M_i}} + \sum_{j=1}^{i-1} F_j F_{i-j}, \quad 1 \leq i \leq k-1, \quad (3.26)$$

$$G_i = B_i^{\frac{1}{2M_i}} + \sum_{j=1}^{i-1} G_j F_{i-j}, \quad 1 \leq i \leq k. \quad (3.27)$$

Remark: For rough paths X, Y of roughness $p \in [1, \infty[$, $X_{s,t}^{(1)} - Y_{s,t}^{(1)} = (X - Y)_{s,t}^{(1)}$. The usual version of the Garsia-Rademich-Rumsey lemma yields the following. If

$$\int_0^1 \int_0^1 ds dt 1_{\{s \leq t\}} \frac{|X_{s,t}^{(1)} - Y_{s,t}^{(1)}|^{2M_1}}{|t-s|^{2M_1 \alpha_1}} \leq B_1,$$

then $|X_{s,t}^{(1)} - Y_{s,t}^{(1)}| \leq C B_1^{\frac{1}{2M_1}} |t-s|^{\frac{1}{p}}$. Similarly, if

$$\int_0^1 \int_0^1 ds dt 1_{\{s \leq t\}} \frac{|X_{s,t}^{(1)}|^{2M_1} + |Y_{s,t}^{(1)}|^{2M_1}}{|t-s|^{2M_1\alpha_1}} \leq A_1,$$

then $|X_{s,t}^{(1)}| + |Y_{s,t}^{(1)}| \leq C A_1^{\frac{1}{2M_1}} |t-s|^{\frac{1}{p}}$.

Proof of Lemma 3.5: Throughout the proof, the constants F_i , $1 \leq i \leq k-1$ and G_i , $1 \leq i \leq k$ are defined by (3.26), (3.27), respectively. We introduce the following assumption:

(H_i)

$$\begin{aligned} \int_0^1 \int_0^1 ds dt 1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t-s|^{2M_i\alpha_i}} &\leq B_i, \\ |X_{s,t}^{(j)}| + |Y_{s,t}^{(j)}| &\leq C F_j |t-s|^{\frac{j}{p}}, \quad 1 \leq j \leq i-1, \\ |X_{s,t}^{(j)} - Y_{s,t}^{(j)}| &\leq C G_j |t-s|^{\frac{j}{p}}, \quad 1 \leq j \leq i-1, \end{aligned}$$

$i \in \{2, \dots, k\}$, and we prove that (H_i) implies

$$|X_{s,t}^{(i)} - Y_{s,t}^{(i)}| \leq C G_i |t-s|^{\frac{i}{p}}. \quad (3.28)$$

For this, we use an argument similar to the proof of Theorem 2.1.3 in [9].

Indeed, for every $t \in [0, 1]$, set

$$I(t) = \int_0^t \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t-s|^{2M_i\alpha_i}} ds, \quad J(t) = \int_t^1 \frac{|X_{t,u}^{(i)} - Y_{t,u}^{(i)}|^{2M_i}}{|u-t|^{2M_i\alpha_i}} du.$$

Then $\int_0^1 I(t) dt = \int_0^1 J(t) dt \leq B_i$ and there exists $t_0 > 0$ such that $I(t_0) + J(t_0) \leq 2A_i$. We construct by induction a decreasing sequence $(t_n, n \geq 0)$ such that $\lim_n t_n = 0$ and an increasing sequence $(s_n, n \geq 0)$ such that $s_0 = t_0$, $\lim_n s_n = 1$, and such that there exists $C > 0$ such that for every $n \geq 1$,

$$\left| X_{t_n, t_0}^{(i)} - Y_{t_n, t_0}^{(i)} \right| \leq C \int_0^1 |8 B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p}-1} du + C \sum_{j=1}^{i-1} F_j G_{i-j}, \quad (3.29)$$

$$\left| X_{s_0, s_n}^{(i)} - Y_{s_0, s_n}^{(i)} \right| \leq C \int_0^1 |8 B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p}-1} du + C \sum_{j=1}^{i-1} F_j G_{i-j}. \quad (3.30)$$

Then Chen's identity implies as $n \rightarrow +\infty$,

$$\begin{aligned} |X_{0,1}^{(i)} - Y_{0,1}^{(i)}| &\leq |X_{0,t_0}^{(i)} - Y_{0,t_0}^{(i)}| + |X_{t_0,1}^{(i)} - Y_{t_0,1}^{(i)}| \\ &\quad + \sum_{j=1}^{i-1} \left(|X_{0,t_0}^{(j)} - Y_{0,t_0}^{(j)}| |X_{t_0,1}^{(i-j)}| + |Y_{0,t_0}^{(j)}| |X_{t_0,1}^{(i-j)} - Y_{t_0,1}^{(i-j)}| \right). \end{aligned} \quad (3.31)$$

With the hypothesis (H_i) , we obtain (3.28) with $s = 0$ and $t = 1$.

To construct (t_n) , we suppose that t_{n-1} has been chosen. Let d_{n-1} be defined by $d_{n-1}^{\alpha_i} = \frac{1}{2} t_{n-1}^{\alpha_i}$. Then there exists $t_n \in]0, d_{n-1}[$ such that

$$I(t_n) \leq \frac{4 B_i}{d_{n-1}} \quad \text{and} \quad \frac{|X_{t_n, t_{n-1}}^{(i)} - Y_{t_n, t_{n-1}}^{(i)}|^{2M_i}}{|t_{n-1} - t_n|^{2M_i \alpha_i}} \leq \frac{2I(t_{n-1})}{d_{n-1}}.$$

Indeed, the sets where each one of these inequalities may fail has Lebesgue measure less than $\frac{d_{n-1}}{2}$. Furthermore, for every $n \geq 0$, $2d_{n+1}^{\alpha_i} = t_{n+1}^{\alpha_i} \leq d_n^{\alpha_i} = \frac{1}{2}t_n^{\alpha_i}$ and $|t_n - t_{n+1}|^{\alpha_i} \leq t_n^{\alpha_i} = 2d_n^{\alpha_i} \leq 4(d_n^{\alpha_i} - d_{n+1}^{\alpha_i})$. Hence there exists $a \in]0, 1[$ such that $t_{n+1} \leq a t_n$, so that $\lim_n t_n = 0$ and more precisely,

$$t_n \leq a^n t_0, \quad (3.32)$$

while for any $n \geq 1$,

$$\begin{aligned} |X_{t_{n+1}, t_n}^{(i)} - Y_{t_{n+1}, t_n}^{(i)}| &\leq |2I(t_n)|^{\frac{1}{2M_i}} d_n^{-\frac{1}{2M_i}} |t_n - t_{n+1}|^{\alpha_i} \\ &\leq |8B_i|^{\frac{1}{2M_i}} |d_n d_{n-1}|^{-\frac{1}{2M_i}} 4 |d_n^{\alpha_i} - d_{n+1}^{\alpha_i}| \\ &\leq 4\alpha_i \int_{d_{n+1}}^{d_n} |8B_i|^{\frac{1}{2M_i}} u^{-\frac{1}{M_i} + \alpha_i - 1} du. \end{aligned} \quad (3.33)$$

Let $b = a^{\frac{1}{p}} < 1$; Chen's identity, (H_i) and (3.33) imply that for any $n \geq 1$,

$$\begin{aligned} |X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)}| &\leq |X_{t_n, t_0}^{(i)} - Y_{t_n, t_0}^{(i)}| + |X_{t_{n+1}, t_n}^{(i)} - Y_{t_{n+1}, t_n}^{(i)}| \\ &\quad + \sum_{j=1}^{i-1} \left(|X_{t_{n+1}, t_n}^{(j)} - Y_{t_{n+1}, t_n}^{(j)}| |X_{t_n, t_0}^{(i-j)}| + |Y_{t_{n+1}, t_n}^{(j)}| |X_{t_n, t_0}^{(i-j)} - Y_{t_n, t_0}^{(i-j)}| \right) \\ &\leq |X_{t_n, t_0}^{(i)} - Y_{t_n, t_0}^{(i)}| + C \int_{d_{n+1}}^{d_n} |8B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p} - 1} du \\ &\quad + C \sum_{j=1}^{i-1} (G_j F_{i-j} + F_j G_{i-j}) |t_n - t_{n+1}|^{\frac{j}{p}} |t_0 - t_n|^{\frac{i-j}{p}} \end{aligned}$$

Since $\sup_{1 \leq j \leq i-1} |t_n - t_{n+1}|^{\frac{j}{p}} \leq t_n^{\frac{1}{p}} \leq Cb^n < 1$, an easy induction on n implies that for any $n \geq 1$,

$$|X_{t_n, t_0}^{(i)} - Y_{t_n, t_0}^{(i)}| \leq C \int_0^1 |8B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p} - 1} du + C \left(\sum_{j=1}^{i-1} G_j F_{i-j} \right) \left(\sum_{l=0}^{n-2} b^l \right),$$

which implies (3.29). To prove (3.30), we proceed in a similar way, exchanging the endpoints of the interval $[0, 1]$. Recall that $s_0 = t_0$; suppose that s_{n-1} has been defined and let δ_{n-1} be such that $|1 - \delta_{n-1}|^{\alpha_i} = \frac{1}{2} |1 - s_{n-1}|^{\alpha_i}$. There exists $s_n \in]\delta_{n-1}, 1[$ such that

$$J(s_n) \leq \frac{4B_i}{1 - \delta_{n-1}} \quad \text{and} \quad \frac{|X_{s_{n-1}, s_n}^{(i)} - Y_{s_{n-1}, s_n}^{(i)}|^{2M_i}}{|s_n - s_{n-1}|^{\alpha_i}} \leq \frac{2J(s_{n-1})}{1 - \delta_{n-1}}.$$

Then for every $n \geq 1$, $2|1 - \delta_{n+1}|^{\alpha_i} = |1 - s_{n+1}|^{\alpha_i} \leq |1 - \delta_n|^{\alpha_i} = \frac{1}{2}|1 - t_n|^{\alpha_i}$, so that $s_n \leq \delta_n \leq s_{n+1} \leq \delta_{n+1}$ and for some $\bar{a} \in]0, 1[$

$$1 - s_n \leq \bar{a}^n (1 - t_0), \quad (3.34)$$

so that $\lim_n s_n = 1$ and computations similar to those proving (3.33) yield

$$|X_{s_n, s_{n+1}}^{(i)} - Y_{s_n, s_{n+1}}^{(i)}| \leq 4 \alpha_i \int_{\delta_n}^{\delta_{n+1}} |8 B_i|^{\frac{1}{2M_i}} u^{-\frac{1}{M_i} + \alpha_i - 1} du.$$

Thus if $\bar{b} = \bar{a}^{\frac{1}{p}} < 1$, Chen's identity and (H_i) imply

$$\left| X_{t_0, s_n}^{(i)} - Y_{t_0, s_n}^{(i)} \right| \leq C \int_{t_0}^{s_n} |8 B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p} - 1} du + C \left(\sum_{j=1}^{i-1} F_j G_{i-j} \right) \left(\sum_{l=0}^{n-1} \bar{b}^l \right),$$

which completes the proof of (3.30) and hence that of (3.28) for $s = 0, t = 1$.

To deduce (3.28), for any $s, t \in [0, 1]$ with $s < t$, define $\bar{X}_u = X_{s+(t-s)u}$, $\bar{Y}_u = Y_{s+(t-s)u}$ for $u \in [0, 1]$. Then $\bar{X}$ and $\bar{Y}$ are geometric rough paths with the same roughness p . Moreover, for $0 \leq u < v \leq 1, j = 1, \dots, k$, $\bar{X}_{u,v}^{(j)} = X_{s+(t-s)u, s+(t-s)v}^{(j)}$. In fact, by a change of variables, we see that this identity is obvious for smooth rough paths and therefore it is trivially extended to geometric rough paths.

Furthermore,

$$\begin{aligned} & \int_0^1 \int_0^1 dudv 1_{\{u < v\}} \frac{|\bar{X}_{u,v}^{(i)} - \bar{Y}_{u,v}^{(i)}|^{2M_i}}{|v - u|^{2M_i \alpha_i}} \\ &= (t - s)^{-2+2\alpha_i M_i} \int_s^t \int_s^t dudv 1_{\{u < v\}} \frac{|X_{u,v}^{(i)} - Y_{u,v}^{(i)}|^{2M_i}}{|v - u|^{2M_i \alpha_i}} \\ &\leq (t - s)^{-2+2\alpha_i M_i} B_i = (t - s)^{2M_i \frac{i}{p}} B_i. \end{aligned}$$

Hence, if the pair (X, Y) satisfies (H_i) then $(\bar{X}, \bar{Y})$ satisfies a similar property with constants $\bar{A}_j = (t - s)^{2M_j \frac{j}{p}} A_j$, $\bar{F}_j = |t - s|^{\frac{j}{p}} F_j, 1 \leq j \leq i - 1$, $\bar{B}_j = (t - s)^{2M_j \frac{j}{p}} B_j$, $\bar{G}_j = |t - s|^{\frac{j}{p}}, 1 \leq j \leq i$. This finishes the proof of (3.28).

Taking in the preceding arguments first $X \equiv 0$ and then $Y \equiv 0$, we see recursively that (3.22) implies (H_i) for any $i = 1, \dots, k - 1$, with $B_i = A_i$. Hence we obtain (3.24). Moreover, we also see that (H_i) holds true for any $i = 1, \dots, k$, whenever (3.22), (3.23) are satisfied. This concludes the proof. $\square$

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