



**HAL**  
open science

# Skew-product representations of multidimensional Dunkl Markov processes

Oleksandr Chybiryakov

► **To cite this version:**

Oleksandr Chybiryakov. Skew-product representations of multidimensional Dunkl Markov processes. 2005. hal-00008770

**HAL Id: hal-00008770**

**<https://hal.science/hal-00008770>**

Preprint submitted on 14 Sep 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Skew-product representations of multidimensional Dunkl Markov processes.

Oleksandr Chybiryakov\*

## Abstract

In this note we obtain two different types of skew-product representations of the multidimensional Dunkl Markov processes which generalize the skew-product decomposition in dimension 1 obtained in [GY05d] and the skew-product of an  $n$ -dimensional Brownian motion into its radial and angular parts. We also study the radial part of the Dunkl Markov process, i.e. the projection of the Dunkl Markov process on a Weyl chamber.

**Key Words.** Dunkl processes, Feller processes, Skew-product, Weyl group.

**Mathematics Subject Classification (2000):** 60G25, 60J60, 60J75.

## 1 Introduction.

The study of the multidimensional Dunkl Markov processes was originated in [Rös98] and [RV98]. They were studied further in [GY05c], [GY05d], [GY05b], [GY05a]. These processes share some important properties with Brownian motion: for example, they are martingales and enjoy the chaotic representation property as well as the time-inversion property. To a Dunkl Markov process we can associate two processes: its euclidean norm and its radial part. The euclidean norm of the Dunkl Markov process is in fact a Bessel process and its radial part is a continuous Markov process taking values in a Weyl chamber  $C$ . Note that a particular case of the radial part process - Brownian motion in a Weyl chamber - is studied in [BBO05].

It is well known that Brownian motion (and more generally any rotational invariant diffusion) can be decomposed as the skew-product of a Bessel process and a time-changed spherical Brownian motion (see [Gal63], [PR88]). In general, the following question can be raised: suppose given a group  $G$  acting (not necessarily transitively) on  $\mathbb{R}^n$  and a Markov process on  $\mathbb{R}^n$ , "invariant" by the action of  $G$ . Does this lead to a certain skew-product decomposition of the Markov process? In this paper we answer this question in the case of the Dunkl Markov processes and the groups  $W$  - the group of symmetries of  $\mathbb{R}^n$  and the orthogonal group  $O(\mathbb{R}^n)$ .

---

\*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 175 rue du Chevaleret, F-75013 Paris. chyb@ccr.jussieu.fr

Then we derive some corollaries from the skew-product decompositions we have obtained. We also study  $X^W$ -the radial part of the Dunkl Markov process noting some analogies between  $X^W$  and Bessel processes.

The paper is organized as follows. In Section 3 we give the definition of the Dunkl Markov process and its radial part. We also present the invariance property of the Dunkl Markov process under the action of the group  $O(\mathbb{R}^n)$ .

Section 4 contains the study of the radial part of the Dunkl Markov process. We give the condition on the parameters of  $X^W$  which guarantees that it does not hit the walls of a Weyl chamber  $C$ . This is done by applying a famous argument due to McKean (see Problem 7, p.47 in [McK69]). As a part of the proof we find harmonic functions for  $X^W$  which are analogs of harmonic functions for Bessel processes, i.e. the adequate power or logarithm function. Under the condition that  $X^W$  does not hit the walls of  $C$  it can be characterized as the unique solution of a stochastic integral equation or as the unique solution of the corresponding martingale problem (if  $X^W$  hits the walls of  $C$  it can still be characterized as a unique solution of the stochastic integral equation up to  $T_0$  -the first time it hits the walls of  $C$ ). With the help of these properties, we establish the absolute continuity results between  $X^W$  with different parameters and evaluate conditional Laplace transforms which are multidimensional analogs of Theorem 4.7 in [Yor80].

In Section 5 we generalize the skew-product decomposition given in [GY05d] to multidimensional Dunkl Markov processes. From [GY05b] it is known that the Dunkl Markov process can jump only in the directions given by the roots of an associated root system. In Theorem 23 we construct the Dunkl Markov process from its radial part by adding the jumps in these directions one by one. In order to add the jumps we do the time-change then add jumps in the fixed direction and then do the inverse time-change. In order to add the jumps in a fixed direction we use the perturbation of generators technique from [EK86]. We also see that under a certain invariance condition for the extended infinitesimal generator this perturbation is given by a skew-product with a Poisson process. From Lemma 22 and Theorem 23 we obtain that the Dunkl Markov process can be characterized as the unique solution of a martingale problem. Note a similar result from [RV98], which is recalled in Theorem 4. Our result holds for a more general class of processes - the two-parameter Dunkl processes (see [GY05d] in dimension 1, [GY05a] and [Law05]), but we need to impose the condition which guarantees that the radial part of the Dunkl Markov process does not hit the wall of  $C$ . Using the skew-product decomposition we also obtain the absolute continuity relations for the Dunkl processes which relate two-parameter Dunkl processes with different parameters of jumps.

In Section 6 we define a spherical Dunkl Markov process and its radial part and prove another skew-product decomposition of the Dunkl Markov process which is analogous to the skew-product decomposition of Brownian motion. From [GY05c] the Dunkl Markov process enjoys the time-inversion property. The Markov process obtained after time-inversion is called the Dunkl Markov process "with drift". With help of the obtained skew-product decomposition we prove that the first hitting time of the ball (centered at origin) of the Dunkl Markov process "with drift" is independent from the hitting angle.

## 2 Notations.

We will constantly use martingale problems. Therefore, we need to recall here some definitions from [EK86] which will be used later.

Let  $(S, d)$  be a metric space and  $\mathbf{D}_S[0, \infty)$  ( $\mathbf{C}_S[0, \infty)$ ) the space of right continuous (continuous) functions from  $[0, \infty)$  into  $(S, d)$  having left limits.  $\mathcal{P}(S)$  denotes the set of Borel probability measures on  $S$ . For any  $x \in S$ ,  $\delta_x$  denotes the element of  $\mathcal{P}(S)$  with unit mass at  $x$ . Let  $L$  be the space of all measurable functions on  $S$ .  $\mathcal{A}$  is a linear mapping whose domain  $\mathcal{D}(\mathcal{A})$  is a subspace of  $L$  and whose range  $\mathcal{R}(\mathcal{A})$  lies in  $L$ . Typically  $\mathcal{D}(\mathcal{A})$  will be  $\mathbf{C}_K^\infty(S)$  - the space of infinitely differentiable functions on  $S$  with compact support.

Let  $X$  be a measurable stochastic process with values in  $S$  defined on some probability space  $(\Omega, \mathcal{F}, P)$ . For  $\nu \in \mathcal{P}(S)$  we say that  $X$  is a *solution of the  $\mathbf{D}_S[0, \infty)$  martingale problem  $(\mathcal{A}, \nu)$*  if  $X$  is a process with sample paths in  $\mathbf{D}_S[0, \infty)$ ,  $P(X_0 \in \cdot) = \nu(\cdot)$ , and for any  $u \in \mathcal{D}(\mathcal{A})$

$$u(X_t) - u(X_0) - \int_0^t \mathcal{A}u(X_s) ds$$

is an  $(\mathcal{F}_t^X)$ -martingale, where  $\mathcal{F}_t^X := \mathcal{F}\{X_s, s \leq t\}$ . Let  $U$  be an open set of  $S$  and  $X$  be a process with sample paths in  $\mathbf{D}_S[0, \infty)$ . Define the  $(\mathcal{F}_t^X)$ -stopping time

$$\tau := \inf \{t \geq 0 \mid X_t \notin U \text{ or } X_{t-} \notin U\}.$$

Then  $X$  is a *solution of the stopped  $\mathbf{D}_S[0, \infty)$  martingale problem  $(\mathcal{A}, \nu, U)$*  if  $P(X_0 \in \cdot) = \nu(\cdot)$ ,  $X = X_{\cdot \wedge \tau}$  a.s., and for any  $u \in \mathcal{D}(\mathcal{A})$

$$u(X_t) - u(X_0) - \int_0^{t \wedge \tau} \mathcal{A}u(X_s) ds$$

is an  $(\mathcal{F}_t^X)$ -martingale. If there exists a unique solution of a (stopped) martingale problem we will say that the (stopped) martingale problem is well-posed.

In what follows it will be convenient to work with the extended infinitesimal generator (see [Kun69], [RY99], VII). We recall here the definition from ([RY99], VII).

If  $X$  is a Markov process with respect to  $(\mathcal{F}_t)$ , a Borel function  $f$  is said to belong to the domain  $\mathbb{D}_A$  of the *extended infinitesimal generator* (or *extended generator*) if there exists a Borel function  $g$  such that, a.s.,  $\int_0^t |g(X_s)| ds < +\infty$  for every  $t$ , and

$$f(X_t) - f(X_0) - \int_0^t g(X_s) ds$$

is  $(\mathcal{F}_t, P_x)$ -right-continuous martingale for every  $x$ .

## 3 Preliminaries.

Let  $x \cdot y$  denote the usual scalar product for  $x$  and  $y$  on  $\mathbb{R}^n$ . For any  $\alpha \in \mathbb{R}^n \setminus \{0\}$ ,  $\sigma_\alpha$  denotes the reflection with respect to the hyperplane  $H_\alpha \subset \mathbb{R}^n$  orthogonal to  $\alpha$ . For any  $x \in \mathbb{R}^n$ , it is

given by

$$\sigma_\alpha(x) = x - 2\frac{\alpha \cdot x}{\alpha \cdot \alpha}\alpha.$$

For our purposes we need the following definition (see for example [Rös03]):

**Definition 1** *Let  $R \subset \mathbb{R}^n \setminus \{0\}$  be a finite set. Then  $R$  is called a root system, if*

1.  $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$  for all  $\alpha \in R$ ;
2.  $\sigma_\alpha(R) = R$  for all  $\alpha \in R$ .

*The subgroup  $W \subset O(\mathbb{R}^n)$  which is generated by the reflections  $\{\sigma_\alpha | \alpha \in R\}$  is called the Weyl reflection group associated with  $R$ .*

One can prove that for any root system  $R$  in  $\mathbb{R}^n$ , the reflection group  $W$  is finite and the set of reflections contained in  $W$  is exactly  $\{\sigma_\alpha, \alpha \in R\}$  (see [Rös03]).

**Example 2 Root system of type  $A_{n-1}$ .** *Let  $e_1, \dots, e_n$  be the standard basis vectors of  $\mathbb{R}^n$ , then*

$$R = \{\pm(e_i - e_j), 1 \leq i < j \leq n\}$$

*is a root system in  $\mathbb{R}^n$ .*

**Example 3 Root system of type  $B_n$ .**  $R = \{\pm e_i, 1 \leq i \leq n, \pm(e_i \pm e_j), 1 \leq i < j \leq n\}$  *is a root system in  $\mathbb{R}^n$ .*

Without loss of generality we will suppose that if  $R$  is a root system in  $\mathbb{R}^n$  then for any  $\alpha \in R$ ,  $\alpha \cdot \alpha = 2$ , so that for any  $x \in \mathbb{R}^n$

$$\sigma_\alpha(x) = x - (\alpha \cdot x)\alpha. \tag{1}$$

For any given root system  $R$  take  $\beta \in \mathbb{R}^n \setminus \cup_{\alpha \in R} H_\alpha$ , then  $R_+ = \{\alpha \in R | \alpha \cdot \beta > 0\}$  is its positive subsystem. For any  $\alpha \in R$  either  $\alpha \in R_+$  or  $-\alpha \in R_+$ . Of course the choice of  $R_+$  is not unique.

From [RV98], [GY05c] the *Dunkl Markov processes*  $X^{(k)}$  are a family of càdlàg Markov processes with extended generators  $\mathcal{L}_k$  where, for any  $u \in \mathbf{C}_K^2(\mathbb{R}^n)$ ,  $\mathcal{L}_k u$  is given by

$$\mathcal{L}_k u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \left[ \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} - \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2} \right], \tag{2}$$

where  $k$  is a non-negative multiplicity function invariant by the finite reflection group  $W$  associated with  $R$ . It is simple to see that  $\mathcal{L}_k$  does not depend on the choice of  $R_+$ . In what follows suppose that  $X_0 \in \mathbb{R}^n \setminus \cup_{\alpha \in R} H_\alpha$  a.s.

The semi-group densities of the Dunkl Markov process with the generator (2) are given by

$$p_t^{(k)}(x, y) = \frac{1}{c_k t^{\gamma+n/2}} \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \omega_k(y), \quad (x, y \in \mathbb{R}^n), \tag{3}$$

where  $D_k(u, v) > 0$  is the Dunkl kernel (for the properties of the Dunkl kernel see [Rös03]),  $\gamma := \sum_{\alpha \in R_+} k(\alpha)$ ,  $\omega_k(y) = \prod_{\alpha \in R_+} |\alpha \cdot y|^{2k(\alpha)}$  and  $c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2} \omega_k(x) dx$  (see [Rös98]).

We will need the following result from [RV98]

**Theorem 4** *Let  $(P_t)_{t \geq 0}$  be the semigroup of the Dunkl Markov process given by its density (3) and let  $\mathcal{L}_k$  be given by (2). Let  $(X_t)_{t \geq 0}$  be a càdlàg process on  $\mathbb{R}^n$  such that its euclidean norm process  $(\|X_t\|)_{t \geq 0}$  on  $[0, +\infty[$  is continuous. Then the following statements are equivalent.*

1.  $(X_t)_{t \geq 0}$  is the Dunkl Markov process associated with the semigroup  $(P_t)_{t \geq 0}$ .
2. For any  $f \in \mathbf{C}^2(\mathbb{R}^n)$

$$\left(M_t^f\right)_{t \geq 0} := \left(f(X_t) - f(X_0) - \int_0^t \mathcal{L}_k f(X_s) ds\right)_{t \geq 0}$$

is a local martingale.

3. For any  $f \in \mathbf{C}_K^2(\mathbb{R}^n)$ ,  $\left(M_t^f\right)_{t \geq 0}$  is a martingale.

Consider now a fixed Weyl chamber  $C$  of the root-system  $R$  which is a connected component of  $\mathbb{R}^n \setminus \cup_{\alpha \in R} H_\alpha$ . Since  $\overline{C}$  is a fundamental domain of  $W$ , the space  $\mathbb{R}^n/W$  of  $W$ -orbits in  $\mathbb{R}^n$  can be identified to  $\overline{C}$ , i.e. there exists a homeomorphism  $\phi : \mathbb{R}^n/W \rightarrow \overline{C}$  (for any  $[x] \in \mathbb{R}^n/W$ ,  $\phi([x]) := \overline{C} \cap O_x$ , where  $O_x := \{y \in \mathbb{R}^n \mid [x] = [y]\}$ ). Denote  $X_t^W := \pi \left(X_t^{(k)}\right)$  - a radial part of the Dunkl Markov process (or a radial Dunkl Markov process), where

$$\pi := \phi \circ \pi_1 \tag{4}$$

and  $\pi_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n/W$  denotes the canonical projection. From [GY05c],  $X^W$  is a Markov process with extended generator

$$\mathcal{L}_k^W u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha}, \tag{5}$$

for any  $u \in \mathbf{C}_0^2(\overline{C})$ , such that  $\nabla u(x) \cdot \alpha = 0$  for  $x \in H_\alpha$ ,  $\alpha \in R_+$ . The semigroup densities of  $X^W$  are of the form

$$p_t^W(x, y) = \frac{1}{c_k t^{\gamma+n/2}} \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) D_k^W\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \omega_k(y), \quad (x, y \in C), \tag{6}$$

where

$$D_k^W(u, v) = \sum_{w \in W} D_k(u, wv). \tag{7}$$

From [RV98] the Dunkl Markov process  $(X_t)_{t \geq 0}$  is a Feller process and its euclidean norm  $(\|X_t\|)_{t \geq 0}$  is a Bessel process of index  $\gamma + n/2 - 1$ . It is easy to see that  $(X_t^W)_{t \geq 0}$  defined above is also a Feller process.

In order to state the next result, we denote the trajectory of the Dunkl Markov process starting at  $x$  by  $\left(X_t^{(k, R, x)}\right)_{t \geq 0}$ . The following proposition is an analog for the Dunkl Markov processes of the rotational invariance property of the Brownian motion in  $\mathbb{R}^n$ .

**Proposition 5** Let  $(X_t^{(k, R, x)})_{t \geq 0, x \in \mathbb{R}^n}$  be the Dunkl Markov process,  $O(\mathbb{R}^n)$  - the orthogonal group of  $\mathbb{R}^n$ . Then for any  $\theta \in O(\mathbb{R}^n)$

$$\left( \theta X_t^{(k, R, x)} \right)_{t \geq 0, x \in \mathbb{R}^n} \stackrel{(d)}{=} \left( X_t^{(k_\theta, \theta R, \theta x)} \right)_{t \geq 0, x \in \mathbb{R}^n},$$

where  $k_\theta : \theta R \ni \alpha \rightarrow k({}^t\theta\alpha)$ ,  ${}^t\theta$  is the transpose of  $\theta$ ,  $\theta R = \{\theta\alpha, \alpha \in R\}$ .

**Proof.** It will be convenient to denote  $\mathcal{L}_k^R$  the extended generator of  $(X_t^{(k, R, x)})_{t \geq 0, x \in \mathbb{R}^n}$  given by (2). By Theorem 4 it is enough to prove that for any  $u \in \mathbf{C}^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$

$$\mathcal{L}_{k_\theta}^{\theta R} u(\theta x) = \mathcal{L}_k^R (u \circ \theta)(x).$$

From (1) one has

$$(u \circ \theta)(\sigma_\alpha(x)) = u(\theta x - (\alpha \cdot x)\theta\alpha) = u(\theta x - (\theta\alpha \cdot \theta x)\theta\alpha) = u(\sigma_{\theta\alpha}(\theta x))$$

and

$$\nabla(u \circ \theta)(x) \cdot \alpha = ({}^t\theta \nabla u(\theta x)) \cdot \alpha = \nabla u(\theta x) \cdot \theta\alpha.$$

Then

$$\begin{aligned} \mathcal{L}_k^R (u \circ \theta)(x) &= \frac{1}{2} \Delta u(\theta x) + \sum_{\theta\alpha \in \theta R_+} k(\alpha) \left[ \frac{\nabla u(\theta x) \cdot \theta\alpha}{\theta x \cdot \theta\alpha} - \frac{u(\theta x) - u(\sigma_{\theta\alpha}(\theta x))}{(\theta x \cdot \theta\alpha)^2} \right] \\ &= \mathcal{L}_{k_\theta}^{\theta R} u(\theta x). \end{aligned}$$

■

## 4 Study of the Markov process $X^W$ .

Consider the Weyl chamber  $C = \{x \in \mathbb{R}^n \mid x \cdot \alpha > 0, \alpha \in R_+\}$ . We now study the radial part of the Dunkl Markov process, i.e. the Markov process  $(X_t^W)_{t \geq 0}$  with extended generator given by (5).  $(X_t^W)_{t \geq 0}$  is a continuous Feller process with its values in  $\overline{C}$ .

Denote

$$\bar{\omega}_k(x) = \prod_{\alpha \in R_+} (\alpha \cdot x)^{k(\alpha)}, \quad (8)$$

then for any  $u \in \mathbf{C}_0^2(\overline{C})$ , such that  $\nabla u(x) \cdot \alpha = 0$  for  $x \in H_\alpha$ ,  $\alpha \in R_+$ , and  $x \in C$

$$\begin{aligned} \mathcal{L}_k^W u(x) &= \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} \\ &= \frac{1}{2} \Delta u(x) + \nabla u(x) \cdot \nabla \log \bar{\omega}_k(x). \end{aligned}$$

The following proposition gives a condition on the function  $k : R \rightarrow \mathbb{R}_+$  in the definition of the extended generator (5), which ensures that the process  $X^W$ , such that  $X_0^W \in C$  a.s., never touches  $\partial C$ .

**Proposition 6** Let  $X^W$  be the radial Dunkl process,  $X_0^W \in C$  a.s., with extended generator given by (5). Suppose that for any  $\alpha \in R$ ,  $k(\alpha) \geq \frac{1}{2}$ . Define

$$T_0 := \inf \{t > 0 \mid X_t^W \in \partial C\}, \quad (9)$$

then

$$T_0 = +\infty \quad \text{a.s.}$$

In order to prove this proposition we need the following two lemmas

**Lemma 7** For  $x \in C$  define

$$\delta(x) := \prod_{\alpha \in R_+} (\alpha \cdot x)^{1-2k(\alpha)}, \quad (10)$$

then  $\delta$  is harmonic in  $C$  for  $\mathcal{L}_k^W$  given by (5), i.e.  $\mathcal{L}_k^W \delta(x) = 0$  for any  $x \in C$ .

**Remark 8** It is important in the following proof that the function  $k$  in the definition of  $\mathcal{L}_k^W$  is constant on the orbits of  $R$  under the action of the associated Weyl group.

**Proof.** For  $i = 1, \dots, m$  let  $R^i$  be the orbits of  $R$  under the action of the associated Weyl group  $W$ . Denote  $R_+^i = R^i \cap R_+$ . Then for any  $\{i_1, \dots, i_l\} \subset \{1, \dots, m\}$

$$\hat{R}_+ = R_+^{i_1} \cup \dots \cup R_+^{i_l}$$

is also a positive subsystem of the root system  $\hat{R} = R^{i_1} \cup \dots \cup R^{i_l}$  in  $\mathbb{R}^n$  (the two conditions in Definition 1 for  $\hat{R}$  are easily verified). Denote

$$\pi_i(x) = \prod_{\alpha \in R_+^i} (\alpha \cdot x).$$

Since  $k : R \rightarrow \mathbb{R}_+$  is constant on  $R_+^i$  define

$$k_i := k(\alpha), \alpha \in R_+^i,$$

then

$$\bar{\omega}_k(x) = \prod_{i=1}^m \pi_i^{k_i}(x)$$

and

$$\delta(x) = \prod_{i=1}^m \pi_i^{1-2k_i}(x).$$

We want to prove that  $\mathcal{L}_k^W \delta(x) = 0$  for any  $x \in C$ . From Theorem 4.2.6 in ([DX01], p.140), for any root system  $R$ , if  $\pi(x) = \prod_{\alpha \in R_+} (\alpha \cdot x)$ , then

$$\Delta \pi(x) = 0.$$



By induction if  $m = 1$

$$\Delta \pi^{1-2k} + 2 (\nabla \pi^{1-2k} \cdot \nabla \log \pi^k) = 0. \quad (11)$$

Indeed

$$\begin{aligned} \Delta \pi^{1-2k} &= \nabla \cdot \nabla \pi^{1-2k} = \nabla \cdot ((1-2k) \pi^{-2k} \nabla \pi) \\ &= (1-2k) [-2k \pi^{-2k-1} (\nabla \pi \cdot \nabla \pi) + \pi^{-2k} \Delta \pi] \\ &= -2k(1-2k) \pi^{-2k-1} (\nabla \pi \cdot \nabla \pi). \end{aligned}$$

On the other hand

$$2 (\nabla \pi^{1-2k} \cdot \nabla \log \pi^k) = 2k(1-2k) \pi^{-2k-1} (\nabla \pi \cdot \nabla \pi)$$

and the result (11) follows.

For an arbitrary  $m$  denote

$$\bar{\pi} := \pi_1^{1-2k_1} \dots \pi_{m-1}^{1-2k_{m-1}}$$

and

$$\hat{\pi} := \pi_1^{k_1} \dots \pi_{m-1}^{k_{m-1}}.$$

We need to check that

$$A = \Delta \left( \pi_1^{1-2k_1} \dots \pi_{m-1}^{1-2k_{m-1}} \pi_m^{1-2k_m} \right) + 2 \left( \nabla \left( \pi_1^{1-2k_1} \dots \pi_{m-1}^{1-2k_{m-1}} \pi_m^{1-2k_m} \right) \cdot \nabla \log \left( \pi_1^{k_1} \dots \pi_{m-1}^{k_{m-1}} \pi_m^{k_m} \right) \right) = 0$$

or

$$\Delta (\bar{\pi} \pi_m^{1-2k_m}) + 2 (\nabla (\bar{\pi} \pi_m^{1-2k_m}) \cdot \nabla \log (\hat{\pi} \pi_m^{k_m})) = 0.$$

One has

$$\Delta (\bar{\pi} \pi_m^{1-2k_m}) = \pi_m^{1-2k_m} \Delta \bar{\pi} + \bar{\pi} \Delta (\pi_m^{1-2k_m}) + 2 (\nabla \bar{\pi} \cdot \nabla \pi_m^{1-2k_m})$$

and

$$\begin{aligned} \nabla (\bar{\pi} \pi_m^{1-2k_m}) \cdot \nabla \log (\hat{\pi} \pi_m^{k_m}) &= (\bar{\pi} \nabla \pi_m^{1-2k_m} + \pi_m^{1-2k_m} \nabla \bar{\pi}) \cdot (\nabla \log \pi_m^{k_m} + \nabla \log \hat{\pi}) \\ &= (\bar{\pi} \nabla \pi_m^{1-2k_m} \cdot \nabla \log \pi_m^{k_m}) + (\bar{\pi} \nabla \pi_m^{1-2k_m} \cdot \nabla \log \hat{\pi}) + ((\pi_m^{1-2k_m} \nabla \bar{\pi}) \cdot \nabla \log \pi_m^{k_m}) + ((\pi_m^{1-2k_m} \nabla \bar{\pi}) \cdot \nabla \log \hat{\pi}), \end{aligned}$$

then

$$\begin{aligned} A &= \pi_m^{1-2k_m} \Delta \bar{\pi} + 2 \pi_m^{1-2k_m} (\nabla \bar{\pi} \cdot \nabla \log \hat{\pi}) + \bar{\pi} \Delta (\pi_m^{1-2k_m}) + 2 \bar{\pi} (\nabla \pi_m^{1-2k_m} \cdot \nabla \log \pi_m^{k_m}) \\ &\quad + 2 (\nabla \bar{\pi} \cdot \nabla \pi_m^{1-2k_m}) + 2 ((\bar{\pi} \nabla \pi_m^{1-2k_m}) \cdot \nabla \log \hat{\pi}) + 2 ((\pi_m^{1-2k_m} \nabla \bar{\pi}) \cdot \nabla \log \pi_m^{k_m}). \end{aligned}$$

By induction

$$\Delta \bar{\pi} + 2 (\nabla \bar{\pi} \cdot \nabla \log \hat{\pi}) = 0$$

and

$$\Delta (\pi_m^{1-2k_m}) + 2 (\nabla \pi_m^{1-2k_m} \cdot \nabla \log \pi_m^{k_m}) = 0,$$

hence

$$\begin{aligned} A &= 2 (\nabla \bar{\pi} \cdot \nabla \pi_m^{1-2k_m}) + 2 ((\bar{\pi} \nabla \pi_m^{1-2k_m}) \cdot \nabla \log \hat{\pi}) + 2 ((\pi_m^{1-2k_m} \nabla \bar{\pi}) \cdot \nabla \log \pi_m^{k_m}) \\ &= 2 \bar{\pi} \pi_m^{1-2k_m} [(\nabla \log \bar{\pi} \cdot \nabla \log \pi_m^{1-2k_m}) + (\nabla \log \pi_m^{1-2k_m} \cdot \nabla \log \hat{\pi}) + (\nabla \log \bar{\pi} \cdot \nabla \log \pi_m^{k_m})]. \end{aligned}$$

But one has

$$\begin{aligned} (\nabla \log \bar{\pi} \cdot \nabla \log \pi_m^{1-2k_m}) &= \left( \nabla \log \left( \pi_1^{1-2k_1} \dots \pi_{m-1}^{1-2k_{m-1}} \right) \cdot \nabla \log \pi_m^{1-2k_m} \right) \\ &= \sum_{i=1}^{m-1} (1-2k_i)(1-2k_m) (\nabla \log \pi_i \cdot \nabla \log \pi_m), \end{aligned}$$

$$\begin{aligned} (\nabla \log \pi_m^{1-2k_m} \cdot \nabla \log \hat{\pi}) &= \left( \nabla \log \pi_m^{1-2k_m} \cdot \nabla \log \left( \pi_1^{k_1} \dots \pi_{m-1}^{k_{m-1}} \right) \right) \\ &= \sum_{i=1}^{m-1} k_i (1-2k_m) (\nabla \log \pi_m \cdot \nabla \log \pi_i), \end{aligned}$$

$$\begin{aligned} (\nabla \log \bar{\pi} \cdot \nabla \log \pi_m^{k_m}) &= \left( \nabla \log \left( \pi_1^{1-2k_1} \dots \pi_{m-1}^{1-2k_{m-1}} \right) \cdot \nabla \log \pi_m^{k_m} \right) \\ &= \sum_{i=1}^{m-1} k_m (1-2k_i) (\nabla \log \pi_i \cdot \nabla \log \pi_m) \end{aligned}$$

and for  $i \neq j$

$$\begin{aligned} (\nabla \log \pi_i \cdot \nabla \log \pi_j) &= \frac{1}{\pi_i \pi_j} (\nabla \pi_i \cdot \nabla \pi_j) \\ &= \frac{1}{2\pi_i \pi_j} (\Delta(\pi_i \pi_j) - \pi_i \Delta \pi_j - \pi_j \Delta \pi_i) = 0. \end{aligned}$$

Hence  $A = 0$  and the lemma is proven. ■

In the same way one can prove the following

**Lemma 9** For  $x \in C$  define

$$\bar{\delta}(x) := \prod_{\alpha \in R_+, k(\alpha) \neq \frac{1}{2}} (\alpha \cdot x)^{1-2k(\alpha)} \log \prod_{\alpha \in R_+, k(\alpha) = \frac{1}{2}} (\alpha \cdot x), \quad (12)$$

then  $\bar{\delta}$  is harmonic in  $C$  for  $\mathcal{L}_k^W$  given by (5).

**Remark 10** One can equally obtain this result while passing to the limit as  $\varepsilon \rightarrow 0+$  in the equation

$$\mathcal{L}_{k_\varepsilon}^W \left( \prod_{\alpha \in R_+} (\alpha \cdot x)^{1-2k_\varepsilon(\alpha)} \right) = 0,$$

where for any  $\alpha \in R$   $k_\varepsilon(\alpha) = k(\alpha)$  if  $k(\alpha) \neq \frac{1}{2}$  and  $k_\varepsilon(\alpha) = \frac{1}{2} - \frac{\varepsilon}{2}$ , if  $k(\alpha) = \frac{1}{2}$ . Denote

$$\hat{\pi} = \prod_{\alpha \in R_+, k(\alpha) = \frac{1}{2}} (\alpha \cdot x).$$

One has

$$\mathcal{L}_{k_\varepsilon}^W \left( \prod_{\alpha \in R_+} (\alpha \cdot x)^{1-2k_\varepsilon(\alpha)} - \prod_{\alpha \in R_+, k(\alpha) \neq \frac{1}{2}} (\alpha \cdot x)^{1-2k(\alpha)} \right) = 0,$$

then

$$\mathcal{L}_{k_\varepsilon}^W \left( \left( \frac{\hat{\pi}^\varepsilon - 1}{\varepsilon} \right) \prod_{\alpha \in R_+, k(\alpha) \neq \frac{1}{2}} (\alpha \cdot x)^{1-2k_\varepsilon(\alpha)} \right) = 0$$

and one recovers (12) as  $\varepsilon \rightarrow 0 +$ .

**Proof.** Suppose that  $k(\alpha) = \frac{1}{2}$  for any  $\alpha \in R$  then for any  $u \in \mathbf{C}^2(C)$ ,  $x \in C$

$$\mathcal{L}_{\frac{1}{2}}^W u(x) = \frac{1}{2} (\Delta u(x) + (\nabla u(x) \cdot \nabla \log \pi(x))),$$

where  $\pi(x) = \prod_{\alpha \in R_+} (\alpha \cdot x)$ . Then

$$\Delta \log \pi(x) = -\frac{1}{\pi^2(x)} (\nabla \pi(x) \cdot \nabla \pi(x)) + \frac{1}{\pi(x)} \Delta \pi(x) = -\frac{1}{\pi^2(x)} (\nabla \pi(x) \cdot \nabla \pi(x))$$

and

$$\Delta \log \pi(x) + (\nabla \log \pi(x) \cdot \nabla \log \pi(x)) = 0.$$

Denote

$$\begin{aligned} \bar{\pi}_1 &:= \prod_{\alpha \in R_+, k(\alpha) \neq \frac{1}{2}} (\alpha \cdot x)^{1-2k(\alpha)} = \prod_{1 \leq i \leq m, k_i \neq \frac{1}{2}} \pi_i^{1-2k_i}, \\ \hat{\pi}_1 &:= \prod_{\alpha \in R_+, k(\alpha) \neq \frac{1}{2}} (\alpha \cdot x)^{k(\alpha)} = \prod_{1 \leq i \leq m, k_i \neq \frac{1}{2}} \pi_i^{k_i}, \\ \bar{\pi}_2 &:= \log \prod_{\alpha \in R_+, k(\alpha) = \frac{1}{2}} (\alpha \cdot x), \\ \hat{\pi}_2 &:= \prod_{\alpha \in R_+, k(\alpha) = \frac{1}{2}} (\alpha \cdot x)^{\frac{1}{2}}, \end{aligned}$$

then  $\bar{\omega}_k = \hat{\pi}_1 \hat{\pi}_2$  and  $\bar{\delta} = \bar{\pi}_1 \bar{\pi}_2$ . In order to prove Lemma 9 one needs to check that

$$\Delta (\bar{\pi}_1 \bar{\pi}_2) + 2 (\nabla (\bar{\pi}_1 \bar{\pi}_2) \cdot \nabla \log (\hat{\pi}_1 \hat{\pi}_2)) = 0,$$

but it is equal to

$$\begin{aligned} &\bar{\pi}_2 [\Delta \bar{\pi}_1 + 2 (\nabla \bar{\pi}_1 \cdot \nabla \log (\hat{\pi}_1))] + \bar{\pi}_1 [\Delta \bar{\pi}_2 + 2 (\nabla \bar{\pi}_2 \cdot \nabla \log (\hat{\pi}_2))] \\ &+ 2 (\nabla \bar{\pi}_1 \cdot \nabla \bar{\pi}_2) + 2 \bar{\pi}_2 (\nabla \bar{\pi}_1 \cdot \nabla \log \hat{\pi}_2) + 2 \bar{\pi}_1 (\nabla \bar{\pi}_2 \cdot \nabla \log \hat{\pi}_1) \end{aligned}$$

and in the same way as in the proof of Lemma 7 one can see that it is equal to zero. ■

Using Lemma 7 and Lemma 9 one obtains the following result, which will lead to a presentation of  $X^W$  (see Corollaries 13, 14 below).

**Lemma 11** Let  $\bar{\omega}_k$  be defined by (8) and  $k(\alpha) \geq \frac{1}{2}$  for any  $\alpha \in R$ . Suppose that  $X_0 \in C$  a.s., then there exists a unique solution  $X$  of the stochastic integral equation

$$X_t = X_0 + \beta_t + \int_0^t \nabla \log \bar{\omega}_k(X_s) ds, \quad (13)$$

where  $(\beta_t)_{t \geq 0}$  is a Brownian motion on  $\mathbb{R}^n$ . Furthermore  $\mathbb{P}(\forall t > 0, X_t \in C) = 1$ .

**Proof.** We will follow the proof of the similar argument given in ([ABJ02] Lemma 3.2). As soon as the function  $\nabla \log \bar{\omega}_k \in \mathbf{C}^\infty(C)$  the equation (13) has a unique (strong) maximal solution in  $C$ , defined up to time  $\zeta$ , where  $\zeta$  is an explosion time or the exit time from  $C$ .

**Case 1** ( $k(\alpha) \neq \frac{1}{2}$  for any  $\alpha \in R$ ). From Lemma 7 the function  $\delta$  defined by (10) is harmonic and positive on  $C$ . By Ito's formula one deduces that  $\{\delta(X_t), t < \zeta\}$  is a positive local martingale, thus it converges a.s. when  $t \rightarrow \zeta$ . But  $\delta = +\infty$  on  $\partial C$ , therefore  $\|X_t\| \rightarrow +\infty$  when  $t \rightarrow \zeta$ . On the other hand by Ito's formula for  $t < \zeta$

$$\begin{aligned} \|X_t\|^2 &= \|X_0\|^2 + 2 \int_0^t (X_s \cdot d\beta_s) + 2 \int_0^t \gamma ds + tn \\ &= \|X_0\|^2 + 2 \int_0^t \|X_s\| d\tilde{\beta}_s + t(n + 2\gamma), \end{aligned}$$

where  $\gamma = \sum_{\alpha \in R_+} k(\alpha)$  and  $\tilde{\beta}_t = \int_0^t \|X_s\|^{-1} (X_s \cdot d\beta_s)$ ,  $t < \zeta$  is a real-valued Brownian motion up to time  $\zeta$ . This shows that  $\|X_t\|^2$  is the square of a  $(n + 2\gamma)$ -dimensional Bessel process up to time  $\zeta$  (started at  $\|X_0\|^2 > 0$  a.s.). Since  $\|X_t\|^2 \rightarrow +\infty$ , by standard results (see [RY99], XI.1)  $\zeta = +\infty$  a.s. This implies that  $T_0 = +\infty$  a.s., where  $T_0$  is defined by (9).

**Case 2** (There exists  $\alpha \in R$  such that  $k(\alpha) = \frac{1}{2}$ ). From Lemma 9 and Ito's formula,  $\{\bar{\delta}(X_t), t < \zeta\}$  is a continuous local martingale. Let  $A_t := \langle \bar{\delta}(X), \bar{\delta}(X) \rangle_t$  for  $t < \zeta$  and  $\tau_t := \inf \{s \geq 0 \mid A_s = t\}$ , then by Theorem 1.7 in ([RY99], Chapter V)  $B_t = \bar{\delta}(X_{\tau_t}) - \bar{\delta}(X_0)$  will be a real-valued Brownian motion up to time  $A_\zeta$ . If  $\zeta < +\infty$  we have already seen that  $\|X_t\|$  can not tend to infinity. Hence, if  $t \rightarrow \zeta$ , either  $\bar{\delta}(X_t)$  tends to  $+\infty$  or to  $-\infty$ . Therefore either  $B_t$  tends to  $+\infty$  or to  $-\infty$ , when  $t \rightarrow A_\zeta$ , but that is impossible for Brownian motion, either because  $A_\zeta < +\infty$ , or because  $A_\zeta = +\infty$  and  $B_t \geq 0$  ( $B_t \leq 0$ ) infinitely often when  $t \rightarrow +\infty$ . ■

**Proof of Proposition 6.** Let

$$C_m := \left\{ x \in C \mid x \cdot \alpha > \frac{1}{m}, \alpha \in R, \|x\| < m \right\}. \quad (14)$$

Take  $g_m \in \mathbf{C}^\infty(\mathbb{R}^n)$  such that  $g_m \equiv \log \bar{\omega}_k$  on  $C_{m+1}$  and  $g_m \equiv 0$  on  $\mathbb{R}^n \setminus C_{m+2}$ . For any  $u \in \mathbf{C}_K^\infty(\mathbb{R}^n)$ , define  $\mathcal{L}_m$  by

$$\mathcal{L}_m u(x) = \frac{1}{2} \Delta u(x) + (\nabla u(x) \cdot \nabla g_m(x)).$$

Let  $(X_t^W)_{t \geq 0}$  be the radial part of the Dunkl Markov process and  $\hat{\tau}_m := \inf \{s > 0 \mid X_s^W \notin C_m\}$ .

For any  $f \in \mathbf{C}_K^\infty(\mathbb{R}^n)$  there exists  $\tilde{f} \in \mathbf{C}_K^\infty(C)$  such that  $\tilde{f} \equiv f$  on  $C_{m+1}$ . Then from (5)

$$\begin{aligned} & \tilde{f}(X_{t \wedge \hat{\tau}_m}^W) - \tilde{f}(X_0^W) - \int_0^{t \wedge \hat{\tau}_m} \mathcal{L}_k^W \tilde{f}(X_s^W) ds \\ &= f(X_{t \wedge \hat{\tau}_m}^W) - f(X_0^W) - \int_0^{t \wedge \hat{\tau}_m} \mathcal{L}_m f(X_s^W) ds \end{aligned} \quad (15)$$

is a martingale.

Let  $X$  be the solution of (13) such that

$$\mathbb{P}(X_0 \in \cdot) = \mathbb{P}(X_0^W \in \cdot) = \bar{\nu}(\cdot) \quad (16)$$

for  $\bar{\nu} \in \mathcal{P}(C)$ . Let  $\bar{\tau}_m := \inf \{s > 0 \mid X_s \notin C_m\}$ . Then from Lemma 11 one deduces that

$$\bar{\tau}_m \rightarrow +\infty \text{ a.s.} \quad (17)$$

Furthermore from (13) and Ito's formula one deduces that

$$f(X_{t \wedge \bar{\tau}_m}) - f(X_0) - \int_0^{t \wedge \bar{\tau}_m} \mathcal{L}_m f(X_s) ds \quad (18)$$

is a martingale for  $f \in \mathbf{C}_K^\infty(\mathbb{R}^n)$ .

By Theorem 3.3 in ([EK86], p.379) for any  $\nu \in \mathcal{P}(\mathbb{R}^n)$  the  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{L}_m, \nu)$  is well-posed. Then by Theorem 6.1 in ([EK86], p.216) for each  $\nu \in \mathcal{P}(\mathbb{R}^n)$  the stopped  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{L}_m, \nu, C_m)$  is well-posed. But from (15), (18) and (16)  $X_{\cdot \wedge \hat{\tau}_m}^W$  and  $X_{\cdot \wedge \bar{\tau}_m}$  are solutions of  $(\mathcal{L}_m, \bar{\nu}, C_m)$ . Hence

$$(X_{t \wedge \hat{\tau}_m}^W)_{t \geq 0} \stackrel{(d)}{=} (X_{t \wedge \bar{\tau}_m})_{t \geq 0} \quad (19)$$

and  $\mathbb{P}(\hat{\tau}_m < t) = \mathbb{P}(\bar{\tau}_m < t)$ , for any  $t > 0$ . Hence from (17)

$$\hat{\tau}_m \rightarrow +\infty \text{ a.s.}$$

and passing to the limit as  $m \rightarrow \infty$  in (19) one obtains that

$$(X_t^W)_{t \geq 0} \stackrel{(d)}{=} (X_t)_{t \geq 0}$$

and

$$T_0 = +\infty \text{ a.s.},$$

where  $T_0$  is defined by (9). ■

Since  $X_t^W = \pi(X_t)$ , where  $X_t$  is the Dunkl Markov process one obtains

**Corollary 12** *Let  $(X_t)_{t \geq 0}$  be the Dunkl Markov process, such that  $X_0 \in \mathbb{R}^n \setminus \cup_{\alpha \in R_+} H_\alpha$  a.s., with extended generator given by (2). Suppose that, for any  $\alpha \in R$ ,  $k(\alpha) \geq \frac{1}{2}$ . Define*

$$T_0 := \inf \{t > 0 \mid X_t \in \cup_{\alpha \in R_+} H_\alpha\},$$

then

$$T_0 = +\infty \text{ a.s.}$$

The proof of Proposition 6 leads to

**Corollary 13** *Let  $(X_t^W)_{t \geq 0}$  be the radial Dunkl process, such that  $X_0 \in C$  a.s., with extended generator given by (5). Suppose that for any  $\alpha \in R$   $k(\alpha) \geq \frac{1}{2}$ . Then  $X^W$  is the unique solution to the stochastic integral equation*

$$X_t = X_0 + \beta_t + \int_0^t \nabla \log \bar{\omega}_k(X_s) ds,$$

where  $(\beta_t)_{t \geq 0}$  is a Brownian motion in  $\mathbb{R}^n$  and  $\bar{\omega}_k$  is given by (8). This solution is strong in the sense of ([IW81], IV).

Denote  $\nu(\cdot) := \mathbb{P}(X_0^W \in \cdot)$ , then  $X^W$  is a unique solution to the  $\mathbf{C}_C[0, \infty)$  martingale problem  $(\hat{\mathcal{L}}_k^W, \nu)$ , where  $\hat{\mathcal{L}}_k^W$  is the restriction of  $\mathcal{L}_k^W$  on  $\mathbf{C}_K^\infty(C)$ .

Suppose now that there exists  $\alpha \in R$  such that  $k(\alpha) < \frac{1}{2}$  and consider the radial Dunkl process. The preceding results extend to this case if one works till  $T_0$  - the first time the process hits the walls of the Weyl chamber. Loosely speaking, here,  $T_0$  is considered as if it were an "explosion time". One gets the following

**Corollary 14** *Let  $(X_t^W)_{t \geq 0}$  be the radial Dunkl process, such that  $X_0 \in C$  a.s., with extended generator given by (5). Suppose that there exists  $\alpha \in R$  such that  $k(\alpha) < \frac{1}{2}$ . Let  $T_0$  be defined by (9). Then  $(X_t^W, t < T_0)$  is the unique solution to the stochastic integral equation*

$$X_t = X_0 + \beta_t + \int_0^t \nabla \log \bar{\omega}_k(X_s) ds, \quad t < T_0$$

where  $(\beta_t)_{t \geq 0}$  is a Brownian motion in  $\mathbb{R}^n$  and  $\bar{\omega}_k$  is given by (8). This solution is strong in the sense of ([IW81], IV).

**Proof.** Is analogous to the proof of Lemma 11 and Proposition 6. ■

We now turn to absolute continuity relationships between different radial Dunkl Markov processes. For convenience define  $\nu : R \rightarrow \mathbb{R}$  by  $\nu(\alpha) = k(\alpha) - \frac{1}{2}$  for  $\alpha \in R$ .  $\nu$  is called the index of the radial Dunkl Markov process,  $P_x^{(\nu)}$  is the law of  $(X_t^W)_{t \geq 0}$  with index  $\nu$ , started at  $x \in C$ . We will also use notations  $p_t^{(\nu)}(x, y)$  for the semigroup density of the radial Dunkl process with index  $\nu$ ,  $D_{(\nu)}^W(x, y) := D_k^W(x, y)$ ,  $\gamma_{(\nu)} = \sum_{\alpha \in R_+} k(\alpha)$ ,  $c_{(\nu)} = c_k$  etc. One obtains the following (see also [Law05])

**Proposition 15** *Let  $X$  be the radial Dunkl Markov process with index  $\nu$ , started at  $x \in C$ , and  $T_0 := \inf \{s \geq 0 \mid X_s \notin C\}$ . Then there are the following absolute continuity relationships:*

a) *Let  $\nu(\alpha) \geq 0$  for any  $\alpha \in R$ , then*

$$P_x^{(0)}|_{\mathcal{F}_t} = \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\nu(\alpha)} \exp \left( \frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds \right) P_x^{(\nu)}|_{\mathcal{F}_t}.$$

b) Let  $\nu(\alpha) < 0$  for at least one  $\alpha \in R$ , then

$$P_x^{(0)}|_{\mathcal{F}_t} = \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\nu(\alpha)} \exp \left( \frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds \right) P_x^{(\nu)}|_{\mathcal{F}_t \cap \{t < T_0\}}.$$

c) Let  $\nu(\alpha) \geq 0$  for any  $\alpha \in R$ , then

$$P_x^{(-\nu)}|_{\mathcal{F}_t \cap \{t < T_0\}} = \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-2\nu(\alpha)} P_x^{(\nu)}|_{\mathcal{F}_t}.$$

**Proof.** For a continuous local martingale  $M$  let  $\mathcal{E}(M)_t := \exp \{M_t - \frac{1}{2} \langle M, M \rangle_t\}$ . Denote

$$D_t := \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\nu(\alpha)} \exp \left( \frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds \right).$$

Take  $T_m := \inf \{s > 0 | X_s \notin C_m\}$ , where  $C_m$  is defined by (14). For any  $t < T_{m+1}$  one has

$$X_t = X_0 + W_t + \sum_{\beta \in R_+} k(\beta) \int_0^t \frac{ds}{\beta \cdot X_s} \beta, \quad (20)$$

where  $(W_t)_{t \geq 0}$  is a Brownian motion in  $\mathbb{R}^n$ , then by Itô's formula

$$\log(\alpha \cdot X_t) = \log(\alpha \cdot X_0) + \int_0^t \frac{d(\alpha \cdot W_s)}{(\alpha \cdot X_s)} + \sum_{\beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) k(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds - \int_0^t \frac{ds}{(\alpha \cdot X_s)^2}$$

for any  $\alpha \in R_+$ . Then one obtains

$$\begin{aligned} \sum_{\alpha \in R_+} -\nu(\alpha) \log(\alpha \cdot X_t) &= \sum_{\alpha \in R_+} -\nu(\alpha) \log(\alpha \cdot X_0) + \sum_{\alpha \in R_+} -\nu(\alpha) \int_0^t \frac{d(\alpha \cdot W_s)}{\alpha \cdot X_s} \\ &\quad - \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds. \end{aligned}$$

Note that

$$\left\langle \sum_{\alpha \in R_+} -\nu(\alpha) \int_0^\cdot \frac{d(\alpha \cdot W_s)}{(\alpha \cdot X_s)} \right\rangle_t = \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s) (\beta \cdot X_s)} ds.$$

Finally

$$D_t = \mathcal{E} \left( \sum_{\alpha \in R_+} -\nu(\alpha) \int_0^\cdot \frac{d(\alpha \cdot W_s)}{(\alpha \cdot X_s)} \right)_t.$$

One has

$$\sum_{\alpha \in R_+} -\nu(\alpha) \int_0^t \frac{d(\alpha \cdot W_s)}{(\alpha \cdot X_s)} = \sum_{i=1}^n \int_0^t \left\{ \sum_{\alpha \in R_+} -\nu(\alpha) \frac{\alpha_i}{(\alpha \cdot X_s)} \right\} dW_s^i,$$

where  $W_t := (W_t^1, \dots, W_t^n)$ . Now one can find continuous bounded functions  $f_i$ , such that for any  $x \in C_{m+1}$

$$f_i(x) = \sum_{\alpha \in R_+} -\nu(\alpha) \frac{\alpha_i}{\alpha \cdot x}$$

and

$$D_{t \wedge T_m} = \mathcal{E} \left( \sum_{i=1}^n \int_0^{t \wedge T_m} f_i(X_s) dW_s^i \right).$$

By Girsanov's theorem

$$Q|_{\mathcal{F}_t} = \mathcal{E} \left( \sum_{i=1}^n \int_0^t f_i(X_s) dW_s^i \right) P_x^{(\nu)}|_{\mathcal{F}_t} \quad (21)$$

and  $\hat{W}_t := (\hat{W}_t^1, \dots, \hat{W}_t^n)$ , where

$$\hat{W}_t^i := W_t^i - \int_0^t f_i(X_s) ds,$$

is a Brownian motion under  $Q$ . Now from (21)

$$Q|_{\mathcal{F}_{t \wedge T_m}} = D_{t \wedge T_m} P_x^{(\nu)}|_{\mathcal{F}_{t \wedge T_m}}.$$

But from (20)

$$X_t = X_0 + \hat{W}_t + \sum_{\beta \in R_+} \frac{1}{2} \int_0^t \frac{ds}{\beta \cdot X_s} \beta.$$

Hence by Corollary 14  $Q = P_x^{(0)}$ . In particular for any  $t_1 < \dots < t_n < t$  and any positive function  $F$

$$\mathbb{E}_x^{(0)} (F(X_{t_1}, \dots, X_{t_n}) \mathbb{I}_{\{t \leq T_m\}}) = \mathbb{E}_x^{(\nu)} (D_t F(X_{t_1}, \dots, X_{t_n}) \mathbb{I}_{\{t \leq T_m\}}),$$

where  $\mathbb{E}_x^{(0)}$  and  $\mathbb{E}_x^{(\nu)}$  are expectations under probabilities  $P_x^{(0)}$  and  $P_x^{(\nu)}$  respectively. Then by Beppo-Levi letting  $T_m \rightarrow T_0$ , one obtains

$$\mathbb{E}_x^{(0)} (F(X_{t_1}, \dots, X_{t_n}) \mathbb{I}_{\{t < T_0\}}) = \mathbb{E}_x^{(\nu)} (D_t F(X_{t_1}, \dots, X_{t_n}) \mathbb{I}_{\{t < T_0\}}),$$

which proves b). Using that  $P_x^{(\nu)}(T_0 = +\infty) = 1$  if  $\nu(\alpha) \geq 0$  for any  $\alpha \in R$  one obtains a). From a) and b) one easily gets c). ■

From Proposition 15 one can easily derive the following corollary.

**Corollary 16** *Let  $\mu, \nu : R \rightarrow \mathbb{R}_+ \cup \{0\}$  be two functions invariant by the action of  $W$ , then for any  $\mathcal{F}_t$ -measurable variable  $Y$*

$$\begin{aligned} & \mathbb{E}_x^{(\mu)} \left[ Y \exp \left( -\frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha) \nu(\beta)}{(\alpha \cdot X_s)(\beta \cdot X_s)} ds \right) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\mu(\alpha)} \right] \\ &= \mathbb{E}_x^{(\nu)} \left[ Y \exp \left( -\frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \mu(\alpha) \mu(\beta)}{(\alpha \cdot X_s)(\beta \cdot X_s)} ds \right) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\nu(\alpha)} \right]. \end{aligned}$$



The following Corollary is a generalization of Theorem 4.7 in [Yor80].

**Corollary 17** *Let  $\mu, \lambda : R \rightarrow \mathbb{R}_+ \cup \{0\}$  be two functions invariant by the action of  $W$  and for any  $\alpha, \beta \in R$  define  $\nu(\alpha, \beta) := \lambda(\alpha)\lambda(\beta) - \mu(\alpha)\mu(\beta)$ , then for any  $y \in C$*

$$\begin{aligned} & \mathbb{E}_x^{(\mu)} \left[ \exp \left( -\frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha, \beta)}{(\alpha \cdot X_s)(\beta \cdot X_s)} ds \right) \middle| X_t = y \right] \\ &= \frac{c(\mu)}{c(\lambda)} t^{\gamma(\mu) - \gamma(\lambda)} \frac{D_{(\lambda)}^W \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)}{D_{(\mu)}^W \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)} \prod_{\alpha \in R_+} ((\alpha \cdot y)(\alpha \cdot x))^{\lambda(\alpha) - \mu(\alpha)} \end{aligned} \quad (22)$$

**Proof.** From Corollary 16 one obtains for any Borel, positive  $f$

$$\begin{aligned} & \mathbb{E}_x^{(\mu)} \left[ f(X_t) \exp \left( -\frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha, \beta)}{(\alpha \cdot X_s)(\beta \cdot X_s)} ds \right) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\mu(\alpha)} \right] \\ &= \mathbb{E}_x^{(\lambda)} \left[ f(X_t) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\lambda(\alpha)} \right]. \end{aligned} \quad (23)$$

Denote by  $G$  a Borel version of the following conditional probability

$$G(\cdot) = \mathbb{E}_x^{(\mu)} \left[ \exp \left( -\frac{1}{2} \sum_{\alpha, \beta \in R_+} \int_0^t \frac{(\alpha \cdot \beta) \nu(\alpha, \beta)}{(\alpha \cdot X_s)(\beta \cdot X_s)} ds \right) \middle| X_t = \cdot \right],$$

then from (23)

$$\mathbb{E}_x^{(\mu)} \left[ f(X_t) G(X_t) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\mu(\alpha)} \right] = \mathbb{E}_x^{(\lambda)} \left[ f(X_t) \prod_{\alpha \in R_+} \left( \frac{X_t \cdot \alpha}{x \cdot \alpha} \right)^{-\lambda(\alpha)} \right]. \quad (24)$$

From (6) one obtains

$$\begin{aligned} & \prod_{\alpha \in R_+} \left( \frac{y \cdot \alpha}{x \cdot \alpha} \right)^{-\mu(\alpha)} p_t^{(\mu)}(x, y) \\ &= \frac{1}{c(\mu) t^{\gamma(\mu) + \frac{n}{2}}} \exp \left( -\frac{|x|^2 + |y|^2}{2t} \right) D_{(\mu)}^W \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \prod_{\alpha \in R_+} ((\alpha \cdot y)(\alpha \cdot x))^{\mu(\alpha)} \prod_{\alpha \in R_+} (\alpha \cdot y). \end{aligned}$$

The same expression can be written for  $p_t^{(\lambda)}(x, y)$ . Then from (24) one gets (22). ■

**Remark 18** *It is interesting to answer the following question. Given  $\nu, \mu : R \rightarrow \mathbb{R}_+ \cup \{0\}$  -two functions invariant by the action of  $W$  define  $\lambda$  by  $\lambda(\alpha) := \sqrt{\nu^2(\alpha) + \mu^2(\alpha)}$ . When is it true that for any  $\alpha, \beta \in R$*

$$\lambda(\alpha)\lambda(\beta) = \nu(\alpha)\nu(\beta) + \mu(\alpha)\mu(\beta)? \quad (25)$$

If  $\alpha$  and  $\beta$  is in the same orbit then  $\lambda(\alpha) = \lambda(\beta)$ ,  $\nu(\alpha) = \nu(\beta)$ ,  $\mu(\alpha) = \mu(\beta)$  and (25) is true. If  $\alpha$  and  $\beta$  are in different orbits then by Cauchy-Schwarz inequality, the equality (25)

$$\nu(\alpha)\nu(\beta) + \mu(\alpha)\mu(\beta) = \sqrt{\nu^2(\alpha) + \mu^2(\alpha)}\sqrt{\nu^2(\beta) + \mu^2(\beta)}$$

implies that  $\exists \bar{c} \geq 0$  such that  $\nu(\alpha) = \bar{c}\nu(\beta)$ ,  $\mu(\alpha) = \bar{c}\mu(\beta)$ . But it is easy to see that this implies that  $\exists c \geq 0$  such that for any  $\alpha \in R$   $\nu(\alpha) = c\mu(\alpha)$ .

**Remark 19** Denote

$$V(X_s) := \sum_{\alpha \in R_+} \frac{\alpha \cdot \lambda(\alpha)}{\alpha \cdot X_s},$$

then from (22) for any  $u > 0$

$$\begin{aligned} & \mathbb{E}_x^{(0)} \left[ \exp \left( -\frac{1}{2} u^2 \int_0^t \|V(X_s)\|^2 ds \right) \middle| X_t = y \right] \\ &= \frac{c_{(0)}}{c_{(u\lambda)}} t^{\gamma(0) - \gamma(u\lambda)} \frac{D_{(u\lambda)}^W \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)}{D_{(0)}^W \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)} \prod_{\alpha \in R_+} ((\alpha \cdot y)(\alpha \cdot x))^{u\lambda(\alpha)}. \end{aligned}$$

This is an analog of the fact that the Laplace transform of the Hartman-Watson distribution is given by a quotient of two Bessel functions (see [Yor80]).

## 5 Skew-product decomposition of jumps.

Let  $X$  be a Dunkl Markov process,  $\pi : \mathbb{R}^n \rightarrow \bar{C}$  be given by (4). Since  $X$  is càdlàg and  $\pi(X)$  is continuous it will be useful to introduce the space  $\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty)$  defined by

$$\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty) := \{ \omega \in \mathbf{D}_{\mathbb{R}^n} [0, \infty) \mid \pi \circ \omega \in \mathbf{C}_{\bar{C}} [0, \infty) \}. \quad (26)$$

Let  $E := \mathbb{R}^n \setminus \cup_{\alpha \in R} H_\alpha$ . Denote

$$\mathbf{D}_E^\pi [0, \infty) := \{ \omega \in \mathbf{D}_E [0, \infty) \mid \pi \circ \omega \in \mathbf{C}_C [0, \infty) \}.$$

We will say that  $X$  is a *solution of the  $\mathbf{D}_E^\pi [0, \infty)$  ( $\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty)$ ) martingale problem  $(\mathcal{A}, \nu)$*  if  $X$  is a process with sample paths in  $\mathbf{D}_E^\pi [0, \infty)$  ( $\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty)$ ) and  $X$  is a solution of the  $\mathbf{D}_E [0, \infty)$  ( $\mathbf{D}_{\mathbb{R}^n} [0, \infty)$ ) martingale problem  $(\mathcal{A}, \nu)$ .

Let  $\lambda > 0$  and  $\alpha \in R_+$ . Let  $\mathcal{D}(\mathcal{A}) \subset \mathbf{C}_K^\infty(E)$  and  $\mathcal{R}(\mathcal{A}) \subset \mathbf{C}_K^\infty(E)$ . Suppose that for any  $\nu \in \mathcal{P}(\mathbb{R}^n)$  there exists  $X$  - a solution of the  $\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty)$  martingale problem for  $(\mathcal{A}, \nu)$ . We will need the following construction from ([EK86], p.256). Let  $\Omega = \prod_{k=1}^\infty (\mathbf{D}_{\mathbb{R}^n} [0, \infty) \times [0, \infty))$  and  $(X_k, \Delta_k)$  denote the coordinate random variables. Define  $\mathcal{G}_k = \mathcal{F}(X_l, \Delta_l : l \leq k)$  and  $\mathcal{G}^k = \mathcal{F}(X_l, \Delta_l : l \geq k)$ . Then there is a probability distribution on  $\Omega$  such that for each  $k$   $X_k$  is a solution of the  $\mathbf{D}_{\mathbb{R}^n}^\pi [0, \infty)$  martingale problem for  $\mathcal{A}$ ,  $\Delta_k$  is independent of  $\mathcal{F}(X_1, \dots, X_k, \Delta_1, \dots, \Delta_{k-1})$  and exponentially distributed with parameter  $\lambda$ , and for  $A_1 \in \mathcal{G}_k$  and  $A_2 \in \mathcal{G}^{k+1}$ ,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{E}(\mathbb{I}_{A_1} \mathbb{P}[A_2 \mid X_{k+1}(0) = \sigma_\alpha(X_k(\Delta_k))]) \quad (27)$$

and  $\mathbb{P}(X_1(0) \in \cdot) = \nu(\cdot)$ . Define  $\tau_0 = 0$ ,  $\tau_k = \sum_{i=1}^k \Delta_i$ , and  $N_t = k$  for  $\tau_k \leq t < \tau_{k+1}$ . Note that  $N$  is a Poisson process with parameter  $\lambda$ . Define

$$Y(t) = X_{k+1}(t - \tau_k), \quad \tau_k \leq t < \tau_{k+1}, \quad (28)$$

and  $\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_t^N$ .

**Notation 20** We will denote  $Y$  in (28) by  $X *_{\alpha} N$ .

**Lemma 21** Let  $\mathcal{D}(\mathcal{A}) \subset \mathbf{C}_K^{\infty}(E)$  and  $\mathcal{R}(\mathcal{A}) \subset \mathbf{C}_K^{\infty}(E)$ . Suppose that for any  $\nu \in \mathcal{P}(E)$  there exists a solution  $X$  of the  $\mathbf{D}_E^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \nu)$ . Then for any  $\nu \in \mathcal{P}(E)$  there exists a Poisson process  $N$  with parameter  $\lambda$  such that  $Y = X *_{\alpha} N$ , where  $*_{\alpha}$  is defined by (28), is a solution of the  $\mathbf{D}_E^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}_{\lambda, \alpha}, \nu)$ , where for any  $u \in \mathcal{D}(\mathcal{A})$  and  $x \in E$

$$\mathcal{A}_{\lambda, \alpha} u(x) = \mathcal{A}u(x) + \lambda(u(\sigma_{\alpha}x) - u(x)).$$

Furthermore if for any  $u \in \mathcal{D}(\mathcal{A})$  and  $x \in E$

$$\mathcal{A}(u \circ \sigma_{\alpha})(x) = \mathcal{A}u(\sigma_{\alpha}(x)), \quad (29)$$

then there exists a solution of  $(\mathcal{A}_{\lambda, \alpha}, \nu)$  given by

$$(Y_t)_{t \geq 0} := (\sigma_{\alpha}^{N_t} X_t)_{t \geq 0},$$

where  $N$  is a Poisson process with parameter  $\lambda$  independent of  $X$ .

**Proof.** We follow the proof of Proposition 10.2 in ([EK86], p.256). Since  $E$  is not complete one shall consider the  $\mathbf{D}_{\mathbb{R}^n}^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \tilde{\nu})$ . For any  $\tilde{\nu} \in \mathcal{P}(\mathbb{R}^n)$  let  $\tilde{X}_0$  be such that  $\mathbb{P}(\tilde{X}_0 \in \cdot) = \tilde{\nu}(\cdot)$ . If  $\tilde{\nu}(E) = 0$ , then  $\tilde{X}_t := \tilde{X}_0$ , for any  $t \geq 0$ , is a solution of the  $\mathbf{D}_{\mathbb{R}^n}^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \tilde{\nu})$ . If  $\tilde{\nu}(E) > 0$  let  $(X_t)_{t \geq 0}$  be a solution of the  $\mathbf{D}_E^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \bar{\nu})$  with  $\bar{\nu}(\cdot) = \tilde{\nu}(\cdot \cap E) / \tilde{\nu}(E)$ ,  $\bar{\nu} \in \mathcal{P}(E)$ . Let  $\tilde{X}_0$  be independent from  $(X_t)_{t \geq 0}$ , then  $\tilde{X}_t := X_t \mathbb{I}_{\{\tilde{X}_0 \in E\}} + \tilde{X}_0 \mathbb{I}_{\{\tilde{X}_0 \notin E\}}$  is a solution of the  $\mathbf{D}_{\mathbb{R}^n}^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \tilde{\nu})$ . Taking  $\mu(x, \cdot) = \delta_{\sigma_{\alpha}(x)}(\cdot)$  and using Proposition 10.2 in ([EK86], p.256) one obtains that  $Y = \tilde{X} *_{\alpha} N$  is a solution of the  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{A}_{\lambda, \alpha}, \tilde{\nu})$ . Let  $\tilde{\nu} = \nu \in \mathcal{P}(E)$ , then  $\tilde{X} \equiv X$  is a solution of the  $\mathbf{D}_E^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}, \nu)$ . Furthermore, from (27)

$$\mathbb{P}(\cap_{k \geq 1} \{X_{k+1}(0) = \sigma_{\alpha}(X_k(\Delta_k))\}) = 1$$

and

$$\mathbb{P}(\cap_{k \geq 1} \{Y(\tau_k) = \sigma_{\alpha}(Y(\tau_k-))\}) = 1.$$

Hence

$$\mathbb{P}(\cap_{k \geq 1} \{\pi(Y(\tau_k)) = \pi(Y(\tau_k-))\}) = 1.$$

Since  $X_k$  has paths in  $\mathbf{D}_E^{\pi}[0, \infty)$  for any  $k \geq 1$   $\pi(Y)$  is a.s. continuous and  $Y$  is a process with sample paths in  $\mathbf{D}_E^{\pi}[0, \infty)$ . Hence  $Y$  is a solution of the  $\mathbf{D}_E^{\pi}[0, \infty)$  martingale problem  $(\mathcal{A}_{\lambda, \alpha}, \nu)$ .

Suppose now that (29) is true. Let  $N$  be a Poisson process independent of  $X$  and  $\tau_k := \inf \{s \geq 0 \mid N_s = k\}$ . Let  $Y_t = \sigma_\alpha^{N_t} X_t$ . Note that for any  $u \in \mathbf{C}_K^\infty(E)$

$$M_t^u := u(X_t) - u(X_0) - \int_0^t \mathcal{A}u(X_s) ds$$

is a  $(\mathcal{F}_t^X)$ -martingale.

Since  $(\tau_i)_{i \geq 0}$  are independent from  $M^u$ , for any  $k \geq 0$

$$u(X((t \vee \tau_{2k}) \wedge \tau_{2k+1})) - u(X(\tau_{2k})) - \int_{\tau_{2k}}^{(t \vee \tau_{2k}) \wedge \tau_{2k+1}} \mathcal{A}u(X_s) ds \quad (30)$$

is a  $(\mathcal{F}_t)$ -martingale. From (29)

$$u(\sigma_\alpha X((t \vee \tau_{2k+1}) \wedge \tau_{2k+2})) - u(\sigma_\alpha X(\tau_{2k+1})) - \int_{\tau_{2k+1}}^{(t \vee \tau_{2k+1}) \wedge \tau_{2k+2}} \mathcal{A}u(\sigma_\alpha X_s) ds \quad (31)$$

is a  $(\mathcal{F}_t)$ -martingale. Summing (30) and (31) over  $k$  one gets that

$$u(\sigma_\alpha^{N_t} X_t) - u(X_0) - \int_0^t \mathcal{A}u(\sigma_\alpha^{N_s} X_s) ds - \sum_{k=1}^{N_t} (u(\sigma_\alpha^{k+1} X_{\tau_k}) - u(\sigma_\alpha^k X_{\tau_k})) \quad (32)$$

is a  $(\mathcal{F}_t)$ -martingale. But

$$\int_0^t (u(\sigma_\alpha Y_{s-}) - u(Y_{s-})) d(N_s - \lambda s) \quad (33)$$

also is a  $(\mathcal{F}_t)$ -martingale. Note that since  $N$  is independent of  $X$

$$\int_0^t (u(\sigma_\alpha Y_{s-}) - u(Y_{s-})) dN_s = \sum_{k=1}^{N_t} (u(\sigma_\alpha Y_{\tau_k}) - u(Y_{\tau_k})).$$

Adding (32) and (33) and noting that:  $\sigma_\alpha^k X_{\tau_k} = Y_{\tau_k}$ , one gets that

$$u(Y_t) - u(X_0) - \left( \int_0^t \mathcal{A}u(Y_s) ds + \lambda \int_0^t (u(\sigma_\alpha Y_s) - u(Y_s)) ds \right)$$

is a  $(\mathcal{F}_t)$ -martingale. ■

**Lemma 22** *Suppose that  $k : R \rightarrow [\frac{1}{2}, +\infty)$  is a  $W$ -invariant multiplicity function,  $l : R \rightarrow [0, +\infty)$ , and for any  $u \in \mathbf{C}_K^\infty(E)$ ,  $x \in E$*

$$\mathcal{L}_{k,l}u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} + \sum_{\alpha \in R_+} l(\alpha) \frac{u(\sigma_\alpha x) - u(x)}{(x \cdot \alpha)^2}.$$

Let  $\pi : E \rightarrow \bar{C}$  be defined by (4). For any  $\nu \in \mathcal{P}(E)$ , if there exists a solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{L}_{k,l}, \nu)$ , then it is unique.

**Proof.** Let  $X$  be a solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{L}_{k,l}, \nu)$ . Let us fix a Weyl chamber  $C$  and a projection  $\pi : E \rightarrow \bar{C}$ , defined by (4). Denote  $Y_t := \pi(X_t)$ , then taking functions  $u \in \mathbf{C}_K^\infty(E)$  such that  $u(\sigma_\alpha x) = u(x)$  for any  $\alpha \in R$  and  $x \in E$  one sees that  $u(X_\cdot) = u(Y_\cdot)$  and

$$u(Y_t) - u(Y_0) - \int_0^t \mathcal{L}_k^W u(Y_s) ds$$

is a martingale. Hence from the results in the previous section  $Y$  is a unique solution of the  $\mathbf{C}_C[0, \infty)$  martingale problem  $(\mathcal{L}_k^W, \hat{\nu})$ , where  $\hat{\nu}(\cdot) = \nu(\pi^{-1}(\cdot))$ , i.e.  $Y$  is a radial Dunkl process. Define

$$B_m := \cup_{w \in W} w(C_m),$$

where  $C_m$  is defined by (14) and  $T_m := \inf \{s \geq 0 \mid X_s \notin B_m \text{ or } X_{s-} \notin B_m\}$ . Since  $X_s \notin B_m \iff Y_s \notin C_m$  and  $Y$  is a.s. continuous

$$T_m = \inf \{s \geq 0 \mid Y_s \notin C_m\} \rightarrow \infty \text{ as } m \rightarrow \infty \text{ a.s.} \quad (34)$$

and for any  $t \geq 0$  and  $m$  sufficiently large, such that  $X_0 \in \bar{C}_m$ ,

$$X_{t \wedge T_m} \in \bar{C}_m. \quad (35)$$

As in the proof of Proposition 6 there exist bounded functions  $g_m \in \mathbf{C}^\infty(\mathbb{R}^n)$ ,  $f_{\alpha,m} \in \mathbf{C}^\infty(\mathbb{R}^n)$ , such that  $g_m \equiv \log \bar{\omega}_k$  on  $B_{m+1}$  and  $g_m \equiv 0$  on  $\mathbb{R}^n \setminus B_{m+2}$ , where  $\bar{\omega}_k$  is defined by (8), and for any  $\alpha \in R$   $f_{\alpha,m}(x) = \frac{1}{(x \cdot \alpha)^2}$  for  $x \in B_{m+1}$  and  $f_{\alpha,m} \equiv 0$  on  $\mathbb{R}^n \setminus B_{m+2}$ . For any  $u \in \mathbf{C}_K^\infty(\mathbb{R}^n)$ , define  $\mathcal{A}_m$  by

$$\mathcal{A}_m u(x) = \frac{1}{2} \Delta u(x) + (\nabla u(x) \cdot \nabla g_m(x)) + \sum_{\alpha \in R_+} l(\alpha) f_{\alpha,m}(x) (u(\sigma_\alpha x) - u(x)).$$

By Theorem 3.3 in ([EK86], p.379) for any  $\nu \in \mathcal{P}(E)$  the  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{A}_m, \nu)$  is well-posed. By Theorem 6.1 in ([EK86], p.216) the stopped  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{A}_m, \nu, B_m)$  is well posed. Since for any  $u \in \mathbf{C}_K^\infty(\mathbb{R}^n)$ ,  $x \in B_{m+1}$   $\mathcal{A}_m u(x) = \mathcal{L}_{k,l} u(x)$  and using (35),  $X_{\cdot \wedge T_m}$  is a solution of the stopped  $\mathbf{D}_{\mathbb{R}^n}[0, \infty)$  martingale problem  $(\mathcal{A}_m, \nu, B_m)$ . Hence the distribution of  $(X_{t \wedge T_m})_{t \geq 0}$  is uniquely determined. Using (34) one obtains that the distribution of  $(X_t)_{t \geq 0}$  is uniquely determined. ■

Suppose that  $R = \{\pm \alpha_1, \pm \alpha_2, \dots, \pm \alpha_m\}$ ,  $R_+ = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ . For any  $i = 1, \dots, m$  define  $R^i = \{\pm \alpha_1, \pm \alpha_2, \dots, \pm \alpha_i\}$ ,  $R_+^i = \{\alpha_1, \alpha_2, \dots, \alpha_i\}$ ,  $R^0 = R_+^0 = \emptyset$ .

**Theorem 23** *Let  $X$  be the Dunkl Markov process, with extended generator given by (2), such that  $X_0 = x \in E$  a.s. and  $X^W$  -its radial part. Suppose that for any  $\alpha \in R_+$ ,  $k(\alpha) \geq \frac{1}{2}$ .*

*i) For  $i = 1, \dots, m$  there exist Poisson processes  $N^i$  with intensity  $k(\alpha_i)$  respectively and processes  $Y^i$ , defined recursively by*

$$Y_t^0 := X_t^W$$

and

$$Y_{\tau^i}^i := Y_{\bar{\tau}^{i-1}}^{i-1} *_{\alpha_i} N^i, \quad (36)$$

where

$$A_t^i := \int_0^t \frac{ds}{(Y_s^i \cdot \alpha_i)^2}, \quad \tau_t^i := \inf \{s \geq 0 \mid A_s^i > t\} \quad (37)$$

and

$$\tilde{A}_t^{i-1} := \int_0^t \frac{ds}{(Y_s^{i-1} \cdot \alpha_i)^2}, \quad \tilde{\tau}_t^{i-1} := \inf \left\{ s \geq 0 \mid \tilde{A}_s^{i-1} > t \right\}, \quad (38)$$

such that for any  $t > 0$   $A_t^i < +\infty$  a.s.,  $\tau_t^i < +\infty$  a.s.,  $\tilde{A}_t^{i-1} < +\infty$  a.s.,  $\tilde{\tau}_t^{i-1} < +\infty$  a.s., and  $(X_t)_{t \geq 0} \stackrel{(d)}{=} (Y_t^m)_{t \geq 0}$ , i.e.  $Y^m$  is a Dunkl Markov process with extended generator given by (2).

ii) For any  $i = 0, \dots, m$   $Y^i$  is a Markov process with extended generator  $\mathcal{G}^i$ , such that for any  $u \in \mathbf{C}_K^\infty(E)$

$$\mathcal{G}^i u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} + \sum_{\alpha \in R_+^i} k(\alpha) \frac{u(\sigma_\alpha x) - u(x)}{(x \cdot \alpha)^2}. \quad (39)$$

iii) If for some  $i = 1, \dots, m$

$$\sigma_{\alpha_i}(R^{i-1}) = R^{i-1}, \quad (40)$$

then  $Y^i$  can be given by

$$Y_t^i = \sigma_{\alpha_i}^{N_t^i} \int_0^t \frac{ds}{(Y_s^{i-1} \cdot \alpha_i)^2} Y_t^{i-1}, \quad (41)$$

where  $N^i$  is a Poisson process with intensity  $k(\alpha_i)$  independent from  $Y^{i-1}$ .

**Remark 24** Let  $k(\alpha) \geq \frac{1}{2}$  for any  $\alpha \in R_+$ . Fix a Weyl chamber  $C$  and  $x \in C$  and suppose that  $X_0^W = x$  a.s. From the previous section we now that  $X_t^W \in C$  for any  $t \geq 0$  a.s. Let  $Y^i$ ,  $i = 0, \dots, m$  be defined by Theorem 23. Then  $Y^i \in C_i$  for any  $t \geq 0$  a.s., where  $C_i$  are defined recursively as follows:

$$\begin{aligned} C_0 &= C \\ C_{i+1} &= C_i \cup \sigma_{\alpha_i}(C_i) \end{aligned} \quad (42)$$

and  $C_m = \mathbb{R}^n$ .

**Remark 25** The decomposition (36) depends on the way one enumerates the elements of  $R$ . Different enumerations lead to different skew-product decompositions of the Dunkl process.

**Remark 26** One has  $\sigma_{\alpha_m}(R^{m-1}) = R^{m-1}$  and  $\sigma_{\alpha_1}(R^0) = R^0$ . Therefore  $Y^1$  and  $Y^m$  can always be taken in the form (41).

**Remark 27** There is some analogy between this decomposition and the skew-product decomposition of Brownian motion on the  $n$ -dimensional sphere in terms of the Legendre processes and Brownian motion on  $n - 1$ -dimensional sphere in ([IM65], 7.15): one can iterate the skew-product decomposition of Brownian motion on  $n$ -dimensional sphere in order to get Brownian motion on  $n - 2$ -dimensional sphere, then Brownian motion on  $n - 3$ -dimensional sphere etc.

**Example 28** Take  $R = B_2$  and  $R_+ = \{\alpha_1, \dots, \alpha_4\}$ , where  $\alpha_1 := e_1 - e_2$ ,  $\alpha_2 := e_1 + e_2$ ,  $\alpha_3 := e_1$ ,  $\alpha_4 := e_2$ , and  $C = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \text{ and } x_1 > x_2\}$ . Then the condition (40) is true for  $i = 1, \dots, 4$  and we obtain the skew-product decomposition (41).

**Proof of Theorem 23.**

The proof is done by induction. Let us fix a Weyl chamber  $C$  of  $W$  and consider  $X^W$  -the radial part of the Dunkl Markov process. For any  $x \in E$  there is one and only one  $w_x \in W$  such that  $w_x(x) \in C$ . Define  $\tilde{X}^W$  started at  $x \in E$  by  $w_x^{-1}(X^W)$ , where  $X^W$  is the radial part of Dunkl Markov process started at  $w_x(x)$ . One can see that the extended radial part  $\tilde{X}^W$  is a Feller process on  $E$  which is a.s. continuous with extended generator given by,

$$\mathcal{L}_k^W u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha}, \quad (43)$$

( $u \in \mathbf{C}_K^\infty(E)$ ). Note also that  $\pi(\tilde{X}^W)$  is a.s. continuous. Let  $P_x$  be the law of  $X^W$  on  $\mathbf{C}_E[0, \infty)$ . For any  $\nu \in \mathcal{P}(E)$ , let  $P_\nu$  be the law on  $\mathbf{C}_E[0, \infty)$  given by

$$P_\nu(\cdot) = \int_E \nu(dx) P_x(\cdot).$$

Then the process  $\bar{X}^W$  with the law  $P_\nu$  is a solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^0, \nu)$  (in order to recover the radial Dunkl process  $X^W$  started at  $x \in C$ , one poses  $\nu = \delta_x$ ). By Lemma 22 this solution is unique. Furthermore  $\bar{X}^W$  is a Markov process.

For any  $\nu \in \mathcal{P}(E)$  let  $Y^{j-1}$  be the unique solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^{j-1}, \nu)$ . Since the paths of  $Y^{j-1}$  are in  $\mathbf{D}_E^\pi[0, \infty)$  the process  $\log(\alpha_j \cdot Y^{j-1})^2$  is càdlàg in  $\mathbb{R}$ . Therefore for any  $t > 0$   $\log(\alpha_j \cdot Y^{j-1})^2$  is uniformly bounded on  $[0, t]$  and  $(\alpha_j \cdot Y^{j-1})^2$  is uniformly bounded from zero on  $[0, t]$ . Hence

$$\tilde{A}_t^{j-1} = \int_0^t \frac{ds}{(Y_s^{j-1} \cdot \alpha_j)^2} < +\infty \text{ a.s.}$$

Next

$$\tilde{A}_t^{j-1} = \int_0^t \frac{ds}{(Y_s^{j-1} \cdot \alpha_j)^2} \geq \frac{1}{2} \int_0^t \frac{ds}{\|Y_s^{j-1}\|^2}.$$

But from (39) for any  $f \in \mathbf{C}_K^\infty((0, +\infty))$  denoting  $\eta_t := \|Y_t^{j-1}\|^2$

$$f(\eta_t) - f(\eta_0) - \int_0^t ds [2\eta_s f''(\eta_s) + (n + 2\gamma) f'(\eta_s)]$$

is a martingale. Since  $\eta$  is the square of an  $(n + 2\gamma)$ -dimensional Bessel process (started at  $\eta_0$  with the law  $\nu$ )

$$\int_0^t \frac{1}{\eta_s} ds \rightarrow \infty, \text{ as } t \rightarrow \infty \text{ a.s.}$$

Hence  $\tilde{A}_t^{j-1} \rightarrow \infty$ , as  $t \rightarrow \infty$ , and  $\tilde{\tau}_t^{j-1} < +\infty$  a.s. Finally  $\tilde{\tau}^{j-1}$  is a.s. strictly increasing continuous time-change with inverse  $\tilde{A}^{j-1}$ . As in ([Yor01], p.168) for any  $u \in \mathbf{C}_K^\infty(E)$

$$M_t^u := u\left(Y_{\tilde{\tau}_t^{j-1}}^{j-1}\right) - u\left(Y_0^{j-1}\right) - \int_0^t \left(Y_{\tilde{\tau}_s^{j-1}}^{j-1} \cdot \alpha_j\right)^2 \mathcal{G}^{j-1} u\left(Y_{\tilde{\tau}_s^{j-1}}^{j-1}\right) ds \quad (44)$$

is a local martingale. Since  $(M_t^u)_{t \geq 0}$  is a.s. bounded on bounded time intervals it is a martingale. Hence  $Y_{\tilde{\tau}^{j-1}}^{j-1}$  is a solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $((\cdot \cdot \alpha_j)^2 \mathcal{G}^{j-1}, \nu)$ . Next by Lemma 21 there exists a Poisson process  $N^j$  with intensity  $k(\alpha_j)$  such that the process  $Z^j = Y_{\tilde{\tau}^{j-1}}^{j-1} *_{\alpha_j} N^j$  is a solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{A}^j, \nu)$ , where for any  $u \in \mathbf{C}_K^\infty(E)$

$$\mathcal{A}^j u(x) = (x \cdot \alpha_j)^2 \mathcal{G}^{j-1} u(x) + k(\alpha_j) (u(\sigma_{\alpha_j} x) - u(x)).$$

Note that  $\left(\int_0^t (\alpha_j \cdot Z_s^j)^2 ds\right)_{t \geq 0}$  is an a.s. strictly increasing, finite, continuous time-change. Let  $\bar{\tau} := \lim_{t \rightarrow +\infty} \int_0^t (\alpha_j \cdot Z_s^j)^2 ds$ . For any  $t < \bar{\tau}$ , define  $\tau(t)$  as a unique solution of

$$t = \int_0^{\tau(t)} (\alpha_j \cdot Z_s^j)^2 ds.$$

Then

$$\frac{d}{dt} \tau(t) = \frac{1}{(\alpha_j \cdot Z_{\tau(t)}^j)^2}.$$

Define now  $Y_t^j := Z_{\tau(t)}^j$ , for  $t < \bar{\tau}$ . Then

$$\tau(t) = \int_0^t \frac{ds}{(\alpha_j \cdot Y_s^j)^2},$$

$\tau(t) < +\infty$  for any  $t < \bar{\tau}$  and  $\tau(t) \rightarrow +\infty$ , as  $t \rightarrow \bar{\tau}$ . Hence there exists a sequence  $\{\tau_n\} \subset (0, \bar{\tau})$ , such that  $\tau_n \rightarrow \bar{\tau}$  and  $(\alpha_j \cdot Y_{\tau_n}^j)^2 \rightarrow 0$ , as  $n \rightarrow +\infty$ . Since for any  $u \in \mathbf{C}_K^\infty(E)$

$$u(Z_t^j) - u(Z_0^j) - \int_0^t \left( (Z_s^j \cdot \alpha_j)^2 \mathcal{G}^{j-1} u(Z_s^j) + k(\alpha_j) (u(\sigma_{\alpha_j} Z_s^j) - u(Z_s^j)) \right) ds$$

is a martingale, as for (44),

$$u(Y_t^j) - u(Y_0^j) - \int_0^t \left( \mathcal{G}^{j-1} u(Y_s^j) + k(\alpha_j) (Y_s^j \cdot \alpha_j)^{-2} (u(\sigma_{\alpha_j} Y_s^j) - u(Y_s^j)) \right) ds, \quad t < \bar{\tau}$$

is a martingale. Denote  $Y^W := \pi(Y^j)$ , then for any  $u \in \mathbf{C}_K^\infty(C)$

$$u(Y_t^W) - u(Y_0^W) - \int_0^t \mathcal{L}_k^W u(Y_s^W) ds, \quad (t < \bar{\tau})$$

is a martingale. This shows that  $Y^W$  is a radial Dunkl Markov process up to time  $\bar{\tau}$ . Since  $(\alpha_j \cdot Y_{\tau_n}^j)^2 \rightarrow 0$ , as  $n \rightarrow +\infty$ ,  $\lim_{n \rightarrow +\infty} Y_{\tau_n}^W \notin C$  and for any  $\alpha \in R_+$   $k(\alpha) \geq \frac{1}{2}$ , by the results of the previous section  $\tau_n \rightarrow \infty$  a.s. and  $\bar{\tau} = +\infty$  a.s. Hence

$$\int_0^t \frac{ds}{(\alpha_j \cdot Y_s^j)^2} \rightarrow +\infty, \quad \text{as } t \rightarrow +\infty$$



and for any  $t > 0$   $A_t^j < +\infty$ ,  $\tau_t^j < +\infty$  a.s. Furthermore  $Y^j$  is a solution of  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^j, \nu)$ . Now by Lemma 22 the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^j, \nu)$  is well-posed. Since the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^j, \nu)$  is well-posed for any  $\nu \in \mathcal{P}(E)$ , by the same argument as in Theorem 4.2(a) in ([EK86], p.184),  $Y^j$  is a Markov process.

Finally when  $j = m$  one obtains that  $Y^m$  is the unique solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{G}^m, \nu)$ , but for any  $u \in \mathbf{C}_K^\infty(E)$

$$\mathcal{G}^m u(x) = \mathcal{L}_k u(x).$$

Therefore  $Y^m$  is a Dunkl Markov process.

In order to get *iii*) note that from (40) for any  $u \in \mathbf{C}_K^\infty(E)$  and  $x \in E$

$$\mathcal{G}^{i-1}(u \circ \sigma_{\alpha_i})(x) = \mathcal{G}^{i-1}u(\sigma_{\alpha_i}(x)),$$

and by Lemma 21 one can take  $Y_{\tau_t^i}^i = Z^i$ , where  $\tau_t^i = \int_0^t (\alpha_i \cdot Z_s^i)^2 ds$  and

$$Z^i = \sigma_{\alpha_i}^{N_{\tau_t^i}^i} Y_{\tau_t^i}^{i-1},$$

where  $N^i$  is a Poisson process with intensity  $k(\alpha_i)$  independent from  $Y_{\tilde{\tau}_t^{i-1}}^{i-1}$  (hence, independent from  $Y^{i-1}$ ). But

$$(\sigma_{\alpha_i} x \cdot \alpha_i)^2 = ((\alpha_i \cdot x) - (\alpha_i \cdot x)(\alpha_i \cdot \alpha_i))^2 = (x \cdot \alpha_i)^2$$

and

$$\tau_t^i = \int_0^t (\alpha_i \cdot Z_s^i)^2 ds = \int_0^t (\alpha_i \cdot Y_{\tilde{\tau}_s^{i-1}}^{i-1})^2 ds. \quad (45)$$

Finally differentiating the equality

$$t = \tilde{A}_{\tilde{\tau}_t^{i-1}}^{i-1}$$

one gets that

$$\frac{d}{dt} \tilde{\tau}_t^{i-1} = (\alpha_i \cdot Y_{\tilde{\tau}_t^{i-1}}^{i-1})^2$$

and from (45)  $\tau_t^i = \tilde{\tau}_t^{i-1}$ . Hence,

$$Y_t^i = \sigma_{\alpha_i}^{N_{\tilde{A}_t^{i-1}}^i} Y_t^{i-1}.$$

■

From Theorem 23 if (40) and (46) hold for a certain  $j$  we can deduce the following relationship between the semigroups of  $Y^j$  and  $Y^{j-1}$ .

**Proposition 29** *Under the conditions of Theorem 23 suppose that for a certain  $j$  (40) holds. Fix a Weyl chamber  $C$ . Let  $Y_0^0 \in C$  a.s., then  $Y^{j-1} \in C_{j-1}$  and  $Y^j \in C_j$  a.s., where  $(C_i)$  are defined by (42). Suppose that  $P_t^{j-1}(x, dy)$  and  $P_t^j(x, dy)$  are the semi-groups of  $Y^{j-1}$  and  $Y^j$  respectively, and*

$$C_{j-1} \cap \sigma_{\alpha_j}(C_{j-1}) = \emptyset, \quad (46)$$

then for any  $x, y \in C_{j-1}$

$$P_t^{j-1}(x, dy) = P_t^j(x, dy) + P_t^j(x, \sigma_{\alpha_j}(dy)). \quad (47)$$

**Proof.** By Remark 24

$$C_j = C_{j-1} \cup \sigma_{\alpha_j}(C_{j-1}).$$

For any bounded measurable  $f : C_{j-1} \rightarrow \mathbb{R}$  define  $g : C_j \rightarrow \mathbb{R}$  such that for any  $x \in C_j$

$$g(x) := f(x) \mathbb{I}_{\{x \in C_{j-1}\}} + f(\sigma_{\alpha_j}(x)) \mathbb{I}_{\{x \in \sigma_{\alpha_j}(C_{j-1})\}},$$

then for any  $x \in C_{j-1}$

$$\begin{aligned} \mathbb{E}_x g(Y_t^j) &= \int_{C_{j-1}} f(y) P_t^j(x, dy) + \int_{\sigma_{\alpha_j}(C_{j-1})} f(\sigma_{\alpha_j}(y)) P_t^j(x, dy) \\ &= \int_{C_{j-1}} f(y) (P_t^j(x, dy) + P_t^j(x, \sigma_{\alpha_j}(dy))). \end{aligned} \quad (48)$$

On the other hand

$$\mathbb{E}_x g(Y_t^j) = \mathbb{E}_x f(Y_t^{j-1}) = \int_{C_{j-1}} f(y) P_t^{j-1}(x, dy). \quad (49)$$

Comparing (48) and (49) one obtains (47). ■

Using Lemma 22 and the analog of Theorem 23 (one should change  $k$  to  $k'$  in the proper places) one can introduce a slightly more general class of Markov processes than Dunkl processes - the  $(k, k')$ -Dunkl Markov processes. These processes are introduced in dimension 1 in [GY05d] and in general dimensions in [GY05a] (see also [Law05]) and are two parameter analogs of Dunkl Markov processes. They are characterized by their extended generator, for any  $u \in \mathbf{C}_K^\infty(E)$ ,

$$\mathcal{L}_{k,k'} u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} - \sum_{\alpha \in R_+} k'(\alpha) \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2}, \quad (50)$$

where  $k : R \rightarrow [\frac{1}{2}, \infty)$  and  $k' : R \rightarrow [0, \infty)$  are two multiplicity functions invariant by the finite reflection group  $W$  associated with  $R$ . The rest of the notations is the same as in (2). Denote such processes by  $X^{(k,k')}$ . It is simple to see that the radial part of  $X^W = \pi(X^{(k,k')})$  is the same as for the Dunkl Markov process  $X^{(k)}$ . Note that by Lemma 22  $X^{(k,k')}$ , started at  $x \in E$ , is a unique solution of the  $\mathbf{D}_E^\pi[0, \infty)$  martingale problem  $(\mathcal{L}_{k,k'}, \delta_x)$ .

It is proven in [GY05a] that one can pass between  $X^{(k)}$  and  $X^{(k,k')}$  by changing the probability. Here, we will give a proof based on a slight modification of Theorem 23.

**Proposition 30** *Let  $k : R \rightarrow [\frac{1}{2}, \infty)$ ,  $k' : R \rightarrow (0, \infty)$ , and  $k'' : R \rightarrow (0, \infty)$  be multiplicity functions invariant by the finite reflection group  $W$  associated with  $R$ . Suppose that under  $\mathbb{P}$   $X$  is  $(k, k')$ -Dunkl Markov process, started at  $x \in E$ , with extended generator given by (50). Define the change of probability*

$$\mathbb{Q}|_{\mathcal{F}_t^X} = \prod_{\alpha \in R_+} \exp\left(- (k''(\alpha) - k'(\alpha)) \int_0^t \frac{ds}{(\alpha \cdot X_s)^2}\right) \left(\frac{k''(\alpha)}{k'(\alpha)}\right)^{n_t^{(\alpha)}} \mathbb{P}|_{\mathcal{F}_t^X}, \quad (51)$$

where  $n_t^{(\alpha)}$  is the number of jumps of  $X$  in direction  $\alpha$  (i.e.  $X_s = \sigma_\alpha(X_{s-})$ ) on  $[0, t]$ , then under  $\mathbb{Q}$   $X$  is a  $(k, k'')$ -Dunkl Markov process.

**Proof.** Suppose that under  $\mathbb{P}$   $X$  is  $(k, k')$ -Dunkl Markov process. Let us enumerate the elements of  $R_+$  as  $\alpha_1, \dots, \alpha_m$ . One can see that the analog of Theorem 23 is still true for  $(k, k')$ -Dunkl Markov processes (one should change  $k$  to  $k'$  in the proper places), in particular since  $\sigma_{\alpha_m}(R^{m-1}) = R^{m-1}$ , where  $R^{m-1} = \{\pm\alpha_1, \dots, \pm\alpha_{m-1}\}$ , as in (41) one has

$$X_t = \sigma_{\alpha_m}^{N_{A_t}^m} Y_t^{m-1},$$

where  $N^m$  is a Poisson process with the intensity  $k'(\alpha_m)$  independent from  $Y^{m-1}$  and

$$A_t := \int_0^t \frac{ds}{(Y_s^{m-1} \cdot \alpha_m)^2}.$$

Note also that  $n_t^{(\alpha_m)} := N_{A_t}^m$  is the number of jumps of  $X$  in direction  $\alpha_m$  (i.e.  $X_s = \sigma_{\alpha_m}(X_{s-})$ ) on  $[0, t]$ . In particular  $(N_{A_t}^m)$  is  $(\mathcal{F}_t^X)$ -adapted and  $(Y_t^{m-1})$  is  $(\mathcal{F}_t^X)$ -adapted.

Define a new probability  $\mathbb{Q}$  by

$$\mathbb{Q}|_{\mathcal{F}_t^{N^m}} = \exp(-(k''(\alpha_m) - k'(\alpha_m))t) \left( \frac{k''(\alpha_m)}{k'(\alpha_m)} \right)^{N_t^m} \mathbb{P}|_{\mathcal{F}_t^{N^m}}, \quad (52)$$

then under  $\mathbb{Q}$   $N^m$  is a Poisson process with the intensity  $k''(\alpha_m)$ . Note that  $N^m$  is independent from  $Y^{m-1}$  under  $\mathbb{Q}$ . Let  $u \in \mathbf{C}_K^\infty(E)$ . Under  $\mathbb{P}$   $X$  is  $(k, k')$ -Dunkl Markov process with extended generator

$$\mathcal{L}_{k,k'} u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{(\nabla u(x) \cdot \alpha)}{x \cdot \alpha} - \sum_{\alpha \in R_+} k'(\alpha) \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2}.$$

Using (52) one obtains that for any bounded, measurable function  $F$

$$\mathbb{E}_{\mathbb{Q}} [F(Y_u^{m-1}, N_{A_u}^m; u \leq t)] = \mathbb{E}_{\mathbb{P}} [F(Y_u^{m-1}, N_{A_u}^m; u \leq t) D_{A_t}],$$

where

$$D_s := \exp(-(k''(\alpha_m) - k'(\alpha_m))s) \left( \frac{k''(\alpha_m)}{k'(\alpha_m)} \right)^{N_s^m}.$$

Then

$$\mathbb{Q}|_{\mathcal{F}_t^X} = \exp(-(k''(\alpha_m) - k'(\alpha_m))A_t) \left( \frac{k''(\alpha_m)}{k'(\alpha_m)} \right)^{N_{A_t}^m} \mathbb{P}|_{\mathcal{F}_t^X} \quad (53)$$

and under  $\mathbb{Q}$   $X$  is  $(k, \hat{k}')$ -Dunkl Markov process with extended generator

$$\begin{aligned} \mathcal{L}_{k,k'} u(x) &= \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{(\nabla u(x) \cdot \alpha)}{x \cdot \alpha} \\ &\quad - \sum_{\alpha \in R_+ \setminus \{\alpha_m\}} k'(\alpha) \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2} - k''(\alpha_m) \frac{u(x) - u(\sigma_{\alpha_m} x)}{(x \cdot \alpha_m)^2} \end{aligned}$$

(i.e. we have defined  $\hat{k}'$  as follows:  $\hat{k}'(\alpha) := k'(\alpha)$ , for  $\alpha \in R_+ \setminus \{\alpha_m\}$ , and  $\hat{k}'(\alpha_m) := k''(\alpha_m)$ ). Note that from the proof of Theorem 23 *iii*)

$$\int_0^t \frac{ds}{(Y_s^{m-1} \cdot \alpha_m)^2} = \int_0^t \frac{ds}{(X_s \cdot \alpha_m)^2}.$$

Hence we can rewrite (53) as

$$\mathbb{Q}|_{\mathcal{F}_t^X} = \exp\left(- (k''(\alpha_m) - k'(\alpha_m)) \int_0^t \frac{ds}{(X_s \cdot \alpha_m)^2}\right) \left(\frac{k''(\alpha_m)}{k'(\alpha_m)}\right)^{N_{\int_0^t \frac{ds}{(X_s \cdot \alpha_m)^2}}^{(m)}} \mathbb{P}|_{\mathcal{F}_t^X}.$$

Of course the proof does not depend on the way we choose to enumerate  $R_+$ . Hence, for any  $\beta \in R_+$ , define the change of measure

$$\mathbb{Q}|_{\mathcal{F}_t^X} = \exp\left(- (k''(\beta) - k'(\beta)) \int_0^t \frac{ds}{(X_s \cdot \beta)^2}\right) \left(\frac{k''(\beta)}{k'(\beta)}\right)^{N_{\int_0^t \frac{ds}{(X_s \cdot \beta)^2}}^{(\beta)}} \mathbb{P}|_{\mathcal{F}_t^X},$$

where  $N^{(\beta)}$  is a Poisson process with the intensity  $k'(\beta)$ , then if under  $\mathbb{P}$   $X$  is  $(k, k')$ -Dunkl Markov process with extended generator

$$\mathcal{L}_{k,k'} u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{(\nabla u(x) \cdot \alpha)}{x \cdot \alpha} - \sum_{\alpha \in R_+} k'(\alpha) \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2},$$

then under  $\mathbb{Q}$   $X$  is  $(k, \tilde{k}')$ -Dunkl Markov process with extended generator

$$\begin{aligned} \mathcal{L}_{k,\tilde{k}'} u(x) &= \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla u(x) \cdot \alpha}{x \cdot \alpha} \\ &\quad - \sum_{\alpha \in R_+ \setminus \{\beta\}} k'(\alpha) \frac{u(x) - u(\sigma_\alpha x)}{(x \cdot \alpha)^2} - k''(\beta) \frac{u(x) - u(\sigma_\beta x)}{(x \cdot \beta)^2} \end{aligned}$$

( $\tilde{k}'$  is defined as follows:  $\tilde{k}'(\alpha) = k'(\alpha)$ , for  $\alpha \in R_+ \setminus \{\beta\}$ , and  $\tilde{k}'(\beta) = k''(\beta)$ ). Now one can use the fact that if  $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$  are probability measures such that

$$\mathbb{P}_1|_{\mathcal{F}_t} = M_t^1 \mathbb{P}_2|_{\mathcal{F}_t}, \quad \mathbb{P}_2|_{\mathcal{F}_t} = M_t^2 \mathbb{P}_3|_{\mathcal{F}_t},$$

then

$$\mathbb{P}_1|_{\mathcal{F}_t} = M_t^1 M_t^2 \mathbb{P}_3|_{\mathcal{F}_t}.$$

Hence iterating changes of measure over  $\beta \in R_+$  one obtains (51). ■

## 6 Skew-product decomposition of the radial process and applications.

Let  $n \geq 2$ . It is interesting to obtain the skew-product decomposition of the radial part of the Dunkl process. For this we will pass to the polar coordinates in the extended generator

$$\mathcal{L}_k^W u(x) = \frac{1}{2} \Delta u(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{(\nabla u(x) \cdot \alpha)}{x \cdot \alpha},$$

( $u \in \mathbf{C}_0^2(\bar{C})$ ), such that  $\nabla u(x) \cdot \alpha = 0$  for  $x \in H_\alpha$ ,  $\alpha \in R_+$ ). First of all it is simple to see that

$$2\mathcal{L}_k^W u(x) = \Delta u(x) + \frac{1}{\omega_k(x)} (\nabla u(x) \cdot \nabla \omega_k(x)),$$

where  $\omega_k(x) = \prod_{\alpha \in R_+} (\alpha \cdot x)^{2k(\alpha)}$ . Denote  $r = \|x\|$ ,  $\theta = (\theta_1, \dots, \theta_n)$ , where

$$\theta_i := \frac{x_i}{r}$$

and  $x = r\theta$ . One has

$$\Delta = \left( \frac{\partial}{\partial r} \right)^2 + (n-1) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}},$$

where  $\Delta_{S^{n-1}}$  is the Laplacian on the unit sphere. One can check that

$$\nabla u(x) = \frac{\partial u(x)}{\partial r} \nabla(\|x\|) + J(\|x\|) \nabla_\theta u(x),$$

where  $(\nabla_\theta u)^t(x) = \left( \frac{\partial}{\partial \theta_1} u(x), \dots, \frac{\partial}{\partial \theta_n} u(x) \right)$  and

$$J(\|x\|) = \left( \frac{\partial^2(\|x\|)}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$$

is the Hessian of  $\|x\|$ . One obtains also that for  $1 \leq i, j \leq n$

$$\|x\| \frac{\partial^2(\|x\|)}{\partial x_i \partial x_j} = \delta_{ij} - \theta_i \theta_j = P_{ij}(\theta),$$

where  $P = (P_{ij}(\theta))_{1 \leq i, j \leq n}$  is the matrix of the orthogonal projection on the tangent space to the unit sphere at the point  $\theta$ . Finally,

$$\frac{1}{r} \frac{\nabla_\theta \omega_k(\theta)}{\omega_k(\theta)} = \frac{\nabla \omega_k(x)}{\omega_k(x)},$$

so that

$$\begin{aligned} \frac{1}{\omega_k(x)} (\nabla u(x) \cdot \nabla \omega_k(x)) &= \left( \frac{\partial u(x)}{\partial r} \nabla(\|x\|) + J(\|x\|) \nabla_\theta u(x) \right) \cdot \left( \frac{1}{r} \frac{\nabla_\theta \omega_k(\theta)}{\omega_k(\theta)} \right) \\ &= \frac{1}{r} \frac{\partial u(x)}{\partial r} \left( \theta \cdot \left( \frac{\nabla_\theta \omega_k(\theta)}{\omega_k(\theta)} \right) \right) + \frac{1}{r} (J(\|x\|) \nabla_\theta u(x)) \cdot \left( \frac{\nabla_\theta \omega_k(\theta)}{\omega_k(\theta)} \right). \end{aligned}$$

Since  $\left( \theta \cdot \frac{\nabla_\theta \omega_k(\theta)}{\omega_k(\theta)} \right) = \sum_{\alpha \in R_+} 2k(\alpha) = 2\gamma$  one obtains

$$\frac{1}{\omega_k(x)} (\nabla u(x) \cdot \nabla \omega_k(x)) = \frac{2\gamma}{r} \frac{\partial u(x)}{\partial r} + \frac{1}{r} \frac{1}{\omega_k(\theta)} ((J(\|x\|) \nabla_\theta u(x)) \cdot \nabla_\theta \omega_k(\theta))$$

and finally

$$\begin{aligned} \mathcal{L}_k^W u(x) &= \frac{1}{2} \frac{\partial^2 u(x)}{\partial r^2} + \frac{n-1+2\gamma}{2r} \frac{\partial u(x)}{\partial r} \\ &\quad + \frac{1}{r^2} \left[ \frac{1}{2} \Delta_{S^{n-1}} u(x) + \frac{1}{2\omega_k(\theta)} ((P(\theta) \nabla_\theta u(x)) \cdot \nabla_\theta \omega_k(\theta)) \right]. \end{aligned}$$

For  $\mathcal{L}_k$  the polar decomposition is given by

$$\begin{aligned} \mathcal{L}_k u(x) &= \frac{1}{2} \frac{\partial^2 u(x)}{\partial r^2} + \frac{n-1+2\gamma}{2r} \frac{\partial u(x)}{\partial r} \\ &+ \frac{1}{r^2} \left[ \frac{1}{2} \Delta_{S^{n-1}} u(x) + \frac{1}{2\omega_k(\theta)} ((P(\theta) \nabla_{\theta} u(x)) \cdot \nabla_{\theta} \omega_k(\theta)) + \sum_{\alpha \in R_+} k(\alpha) \frac{u(\sigma_{\alpha} x) - u(x)}{(\theta \cdot \alpha)^2} \right], \end{aligned}$$

for any  $u \in \mathbf{C}_K^2(\mathbb{R}^n)$ .

Let us prove that this polar decomposition leads to some skew-product decomposition of the Dunkl Markov process. First of all we give a definition of spherical Dunkl process. The existence of such a process will be proved in the next theorem.

**Definition 31** Let  $(\Theta_t^W)_{t \geq 0}$  be a Markov process on the unit sphere  $S^{n-1} \subset \mathbb{R}^n$  with extended generator given by

$$\mathcal{L}_k^{\Theta} u(\theta) = \frac{1}{2} \Delta_{S^{n-1}} u(\theta) + \frac{1}{2\omega_k(\theta)} ((P(\theta) \nabla_{\theta} u(\theta)) \cdot \nabla_{\theta} \omega_k(\theta)) + \sum_{\alpha \in R_+} k(\alpha) \frac{u(\sigma_{\alpha} \theta) - u(\theta)}{(\theta \cdot \alpha)^2}, \quad (54)$$

( $u \in \mathbf{C}_K^2(S^{n-1})$ ). Then  $(\Theta_t)_{t \geq 0}$  is called spherical Dunkl Markov process.

**Theorem 32** Let  $(X_t)_{t \geq 0}$  be a multidimensional Dunkl Markov process with extended generator given by (2), then it can be decomposed as a skew-product

$$X_t = r_t \Theta_{\int_0^t \frac{ds}{r_s^2}},$$

where  $(r_t)_{t \geq 0}$  is a Bessel process of dimension  $2\gamma + n$  and  $(\Theta_t)_{t \geq 0}$  is spherical Dunkl Markov process, with extended generator given by (54), independent from  $(r_t)_{t \geq 0}$ .

**Notation 33** For a process  $X$ , if necessary, we will denote by  $X^{(x)}$  the process started at  $x$ .

**Proof.** Let  $X$  be a Dunkl Markov process with extended generator given by (2). Since  $r := \|X\|$  is a Bessel process of index  $\gamma + n/2 - 1 > 0$

$$A_t := \int_0^t r_u^{-2} du$$

is a continuous strictly increasing time-change. Since

$$\int_0^t r_u^{-2} du \rightarrow +\infty, \text{ as } t \rightarrow +\infty \text{ a.s.}$$

one has:

$$\tau_t := \inf \left\{ s \geq 0 \mid \int_0^s r_u^{-2} du = t \right\} < +\infty$$

for any  $t \geq 0$ . Denote  $Y_t := X_{\tau_t}$ ,  $\xi_t := r_{\tau_t}$ . Then  $\xi$  is a Markov process. Note that differentiating the equality

$$t = \int_0^{\tau_t} r_u^{-2} du,$$

which is true for any  $t \geq 0$  one obtains that

$$\tau_t = \int_0^t \xi_u^2 du. \quad (55)$$

$Y$  is a strong Markov process with respect to  $(\mathcal{G}_t) := (\mathcal{F}_{\tau_t})$ .

Let us prove that  $\Theta := Y/\|Y\|$  is a Markov process. One has for any  $c > 0$

$$\frac{\tau_t}{c} = \inf \left\{ s \geq 0 \left| \int_0^s \left\| \frac{1}{\sqrt{c}} X_{cu} \right\|^{-2} du > t \right. \right\}.$$

Since

$$\left( \frac{1}{\sqrt{c}} X_{ct}^{(x)} \right)_{t \geq 0} \stackrel{(d)}{=} \left( X_t^{(\frac{x}{\sqrt{c}})} \right)_{t \geq 0}$$

one obtains that

$$\left( \frac{1}{\sqrt{c}} Y_t^{(x)} \right)_{t \geq 0} = \left( \frac{1}{\sqrt{c}} X_{c\tau_t}^{(x)} \right)_{t \geq 0} \stackrel{(d)}{=} \left( Y_t^{(\frac{x}{\sqrt{c}})} \right)_{t \geq 0}. \quad (56)$$

Since  $Y_t$  is Markov, for any bounded, measurable function  $f$  and  $s < t$

$$\mathbb{E}_x \left[ f \left( \frac{Y_t}{\|Y_t\|} \right) \middle| \mathcal{F}_s^Y \right] = g_{t-s}(Y_s)$$

for a certain measurable function  $g_{t-s}$ . But from (56) for any  $y \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned} g_{t-s}(y) &= \mathbb{E}_y \left[ f \left( \frac{Y_{t-s}}{\|Y_{t-s}\|} \right) \right] = \mathbb{E}_y \left[ f \left( \frac{\frac{1}{\|y\|} Y_{t-s}}{\frac{1}{\|y\|} \|Y_{t-s}\|} \right) \right] \\ &= \mathbb{E}_{\frac{y}{\|y\|}} \left[ f \left( \frac{Y_{t-s}}{\|Y_{t-s}\|} \right) \right] = g_{t-s} \left( \frac{y}{\|y\|} \right). \end{aligned}$$

Hence  $\Theta$  is a Markov process. Note that  $\xi$  and  $\Theta$  are both Markov processes with respect to  $(\mathcal{G}_t)$ .

Note that for any  $u \in \mathbf{C}_K^2(\mathbb{R}^n) \subset \mathbb{D}_X$  ( $\mathbb{D}_X$  is the domain of the extended infinitesimal generator of  $X$ )

$$u(X_t) - u(X_0) - \int_0^t \mathcal{L}_k u(X_s) ds$$

is a martingale. By Volkonsky's formula (see [Yor01], p.168)  $\mathbf{C}_K^2(\mathbb{R}^n) \subset \mathbb{D}_Y$  and

$$u(Y_t) - u(Y_0) - \int_0^t \bar{\mathcal{L}}u(Y_s) ds$$

is a martingale, where  $\bar{\mathcal{L}}u(x) := \|x\|^2 \mathcal{L}_k u(x)$ .

Let  $\mathcal{L}^\Theta$  be the restriction of  $\bar{\mathcal{L}}$  to the functions  $f \in \mathbf{C}_K^2(\mathbb{R}^n)$ , such that for any  $x \in \mathbb{R}^n$   $f(x) = f(x/\|x\|)$ . Then for any  $\theta \in S^{n-1}$

$$\mathcal{L}^\Theta f(\theta) = \frac{1}{2} \Delta_{S^{n-1}} f(\theta) + \frac{1}{2\omega_k(\theta)} ((P(\theta) \nabla_\theta f(\theta)) \cdot \nabla_\theta \omega_k(\theta)) + \sum_{\alpha \in R_+} k(\alpha) \frac{f(\sigma_\alpha \theta) - f(\theta)}{(\theta \cdot \alpha)^2}.$$

and

$$D_t^f := f(\Theta_t) - f(\Theta_0) - \int_0^t \mathcal{L}^\Theta f(\Theta_s) ds$$

is a martingale. For any  $g \in \mathbf{C}_K^2([0, \infty))$  and  $r > 0$  define  $\mathcal{L}^\xi$  by

$$\mathcal{L}^\xi g(r) = \frac{1}{2} r^2 \frac{\partial^2 g(r)}{\partial r^2} + r \frac{n-1+2\gamma}{2} \frac{\partial g(r)}{\partial r}.$$

Since  $g(\|\cdot\|) \in \mathbb{D}_Y$

$$C_t^g := g(\xi_t) - g(\xi_0) - \int_0^t \mathcal{L}^\xi g(\xi_s) ds$$

is a martingale.

Take now a copy of  $\hat{\Theta}^{(x/\|x\|)}$  of  $\Theta^{(x/\|x\|)}$  such that  $\xi^{(\|x\|)}$  and  $\hat{\Theta}^{(x/\|x\|)}$  are independent processes. We will show that the process  $Z$ , defined by

$$Z_{\int_0^t \xi_s^2 ds} := \xi_t^{(\|x\|)} \hat{\Theta}_t^{(x/\|x\|)}$$

is a Dunkl Markov process. Note that since  $\hat{\Theta}^{(x/\|x\|)}$  is a copy of  $\Theta^{(x/\|x\|)}$

$$\hat{D}_t^f := f(\hat{\Theta}_t) - f(\hat{\Theta}_0) - \int_0^t \mathcal{L}^\Theta f(\hat{\Theta}_s) ds$$

is a martingale. Then by Ito's formula

$$g(\xi_t) f(\hat{\Theta}_t) \sim \int_0^t g(\xi_s) \mathcal{L}^\Theta f(\hat{\Theta}_s) ds + \int_0^t f(\hat{\Theta}_s) \mathcal{L}^\xi g(\xi_s) ds + [g(\xi), f(\hat{\Theta})]_t, \quad (57)$$

where  $M_t \sim N_t$  means that  $M_t - N_t$  is a martingale.

Since  $\xi$  is continuous

$$[g(\xi), f(\hat{\Theta})]_t = [g(\xi), f(\hat{\Theta})]_t^c + g(\xi_0) f(\hat{\Theta}_0).$$

Since  $\xi^{(\|x\|)}$  and  $\hat{\Theta}^{(x/\|x\|)}$  are independent processes  $\langle C^g, (\hat{D}^f)^c \rangle_t = [g(\xi), f(\hat{\Theta})]_t^c = 0$ . Hence

$$g(\xi_t) f(\hat{\Theta}_t) - \int_0^t g(\xi_s) \mathcal{L}^\Theta f(\hat{\Theta}_s) ds + \int_0^t f(\hat{\Theta}_s) \mathcal{L}^\xi g(\xi_s) ds$$

is a martingale. Let  $h(x) := g(\|x\|) f(x/\|x\|)$ , then

$$g(\xi_s) \mathcal{L}^\Theta f(\hat{\Theta}_s) + f(\hat{\Theta}_s) \mathcal{L}^\xi g(\xi_s) = \bar{\mathcal{L}} h(\xi_s \hat{\Theta}_s)$$



and

$$h\left(\xi_t \hat{\Theta}_t\right) - h\left(\xi_0 \hat{\Theta}_0\right) - \int_0^t \bar{\mathcal{L}}h\left(\xi_s \hat{\Theta}_s\right) ds$$

is a martingale. For any  $R > 0$  let  $\lambda_R$  be a function such that  $\lambda_R \in \mathbf{C}_K^\infty(\mathbb{R})$ ,  $\lambda_R \geq 0$ , and  $\lambda_R \equiv 1$  on  $[0, R]$ . For any polynomial  $p$  define  $p_R(\cdot) := p(\cdot) \lambda_R(\|\cdot\|)$ , then

$$p_R\left(\xi_t \hat{\Theta}_t\right) - p_R\left(\xi_0 \hat{\Theta}_0\right) - \int_0^t \bar{\mathcal{L}}p_R\left(\xi_s \hat{\Theta}_s\right) ds \quad (58)$$

is a martingale. Now for  $1 \leq i \leq n$  and  $u \in \mathbf{C}^1(\mathbb{R}^n)$  define  $T_i$  by

$$T_i u(x) = \frac{\partial u(x)}{\partial x_i} + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\alpha \cdot x},$$

then from ([Rös03], 2.2)

$$\mathcal{L}_k = \frac{1}{2} \sum_i T_i^2 \quad (59)$$

and

$$\frac{u(x) - u(\sigma_\alpha x)}{\alpha \cdot x} = \int_0^1 \partial_\alpha u(x - t(\alpha \cdot x)\alpha) dt \quad (60)$$

for any  $u \in \mathbf{C}^1(\mathbb{R}^n)$  and  $\alpha \in R$ . Note that polynomials are dense in  $\mathbf{C}_K^2(\mathbb{R}^n)$  for the topology of uniform convergence on compact sets of the functions and their two first order derivatives. Hence using (58), (59), and (60) one obtains that for any  $u \in \mathbf{C}_K^2(\mathbb{R}^n)$

$$u\left(\xi_t \hat{\Theta}_t\right) - u\left(\xi_0 \hat{\Theta}_0\right) - \int_0^t \bar{\mathcal{L}}u\left(\xi_s \hat{\Theta}_s\right) ds$$

is a martingale. Denote  $\bar{r}_t := \inf\{s \geq 0 \mid \int_0^s \xi_u^2 du = t\} = \int_0^t r_u^{-2} du$ , then from (55)  $r_t = \xi_{\bar{r}_t}$  and again by Volkonsky's formula (see [Yor01], p.168) for any  $u \in \mathbf{C}_K^2(\mathbb{R}^n)$

$$u\left(r_t \hat{\Theta}_{\bar{r}_t}\right) - u\left(r_t \hat{\Theta}_{\bar{r}_0}\right) - \int_0^t \mathcal{L}_k u\left(r_s \hat{\Theta}_{\bar{r}_s}\right) ds$$

is a martingale. Then by Theorem 4  $r_t \hat{\Theta}_{\bar{r}_t}$  is a Dunkl Markov process.

Since  $\sigma(\xi_t, t \geq 0) = \sigma(r_t, t \geq 0)$ ,  $r$  and  $\hat{\Theta}$  are independent processes. ■

Using that  $X_t^W = \pi(X_t)$  one can introduce the following

**Definition 34** Let  $(\Theta_t^W)_{t \geq 0}$  be a Markov process on  $\bar{C} \cap S^{n-1} \subset \mathbb{R}^n$  with extended generator given by

$$\mathcal{L}_k^{W, \Theta} u(\theta) = \frac{1}{2} \Delta_{S^{n-1}} u(\theta) + \frac{1}{2\omega_k(\theta)} ((P(\theta) \nabla_\theta u(\theta)) \cdot \nabla_\theta \omega_k(\theta)), \quad (61)$$

( $u \in \mathbf{C}_0^2(\bar{C} \cap S^{n-1})$ ), such that  $\nabla u(x) \cdot \alpha = 0$  for  $x \in H_\alpha \cap \bar{C} \cap S^{n-1}$ ,  $\alpha \in R_+$ ). Then we shall call  $(\Theta_t)_{t \geq 0}$  the spherical part of the radial Dunkl Markov process.

One can also easily get that

$$X_t = r_t \Theta_{\int_0^t \frac{ds}{r_s^2}}^W.$$

**Remark 35** In the case  $\mathbb{R}^2$  one can further compute explicitly the extended generator (5). Passing to polar coordinates  $\theta = (\cos \phi, \sin \phi)^t$  one obtains

$$\mathcal{L}_k^{W, \Theta} u(\cos \phi, \sin \phi) = \frac{1}{2} \frac{d^2 u}{d\phi^2}(\cos \phi, \sin \phi) + \frac{\frac{d}{d\phi} u(\cos \phi, \sin \phi) \frac{d}{d\phi} \omega_k(\cos \phi, \sin \phi)}{2\omega_k(\cos \phi, \sin \phi)}.$$

Consider now a Dunkl Markov process  $X^{(k),x}$  with drift  $x$  started at 0. From [GY05c] this is a Markov process with infinitesimal generator given by

$$\begin{aligned} \mathcal{L}^{(k),x} u(y) &= \mathcal{L}_k(u)(y) + \frac{1}{D_k(x,y)} (\nabla_y (D_k(x,y)) \cdot \nabla u(y)) \\ &+ \frac{1}{D_k(x,y)} \sum_{\alpha \in R_+} \frac{k(\alpha)}{(y \cdot \alpha)^2} (D_k(x, \sigma_\alpha y) - D_k(x,y)) (u(\sigma_\alpha y) - u(y)). \end{aligned} \quad (62)$$

Note that if  $k(\alpha) \equiv 0$ , this process is the usual Brownian motion with drift  $x$ . Moreover

$$X_t^{(k),x} = t X_{\frac{1}{t}}^{(k,x)}, \quad (63)$$

where  $(X_t^{(k,x)})_{t \geq 0}$  is a Dunkl Markov process started at  $x$ .

From (63)

$$\|X_t^{(k),x}\| = t \|X_{\frac{1}{t}}^{(k,x)}\|.$$

Since  $\|X^{(k,x)}\|$  is a Bessel process with index  $\nu := \gamma + n/2 - 1$ ,  $\|X_t^{(k),x}\|$  is a Bessel process "with drift" (see [Wat75], [PY81], [GY05c]), i.e. the process with extended generator given by  $\mathcal{L}^{(x)}$  on p.158 in [GY05c]:

$$\mathcal{L}^{(x)} = \frac{1}{2} \frac{d^2}{dy^2} + \left( \frac{2\nu + 1}{2y} + x \frac{I_{\nu+1}}{I_\nu}(xy) \right) \frac{d}{dy}. \quad (64)$$

Extending the results of [PY81] which are given for Brownian motion with drift one obtains

**Proposition 36** Let  $(X_t)$  be a Dunkl Markov process  $X^{(k),x}$  with drift  $x \neq 0$  started at 0 and let  $R_t = \|X_t^{(k),x}\|$ . Then

$$X_t^{(k),x} = R_t \Theta \left( \int_t^\infty \frac{ds}{R_s^2} \right), \quad (65)$$

where  $(\Theta_t)_{t \geq 0}$  is a Dunkl Markov process on the unit sphere  $S^{n-1}$ , starting at  $\Theta_0 = \frac{x}{\|x\|}$ , and independent of a Bessel process "with drift"  $(R_t)_{t \geq 0}$ .

**Proof.** Take  $X_t$  a Dunkl Markov process starting at  $x \neq 0$ . Then

$$X_s = \|X_s\| \Theta_{\int_0^s \frac{du}{\|X_u\|^2}},$$

where  $(\Theta_t)_{t \geq 0}$  is a Dunkl Markov process on the unit sphere independent from  $(\|X_t\|)_{t \geq 0}$ . Now making substitutions  $s \rightarrow \frac{1}{s}$  and  $u \rightarrow \frac{1}{u}$  one obtains

$$sX_{\frac{1}{s}} = \left\| sX_{\frac{1}{s}} \right\| \Theta_{\int_s^\infty \frac{du}{\left\| uX_{\frac{1}{u}} \right\|^2}},$$

which leads (65). ■

Now it is not difficult to prove the following proposition which extends a similar result in [PY81].

**Proposition 37** *Let  $(X_t)$  be a Dunkl Markov process  $X^{(k),x}$  with drift  $x \neq 0$  started at 0 and let  $R_t = \|X^{(k),x}\|$ . Define  $T_a := \inf \{s \geq 0 \mid R_s = a\}$  and  $\vartheta_t := \frac{X_t^{(k),x}}{R_t}$ . Then the hitting angle  $\vartheta(T_a)$  is independent from  $T_a$ .*

**Proof.** It is plain from (65) that  $\vartheta(T_a) = \Theta\left(\int_{T_a}^\infty \frac{ds}{R_s^2}\right)$  is independent from the whole radial motion prior to  $T_a$  and the result follows. ■

**Remark 38** *When  $(X_t)$  is transient one obtains the same result for a last exit time  $L_a := \sup \{s \geq 0 \mid R_s = a\}$ , i.e.  $\vartheta(L_a)$  is independent from  $L_a$ .*

## References

- [ABJ02] J.-P. Anker, P. Bougerol, and T. Jeulin, *The infinite Brownian loop on a symmetric space*, Rev. Mat. Iberoamericana **18** (2002), no. 1, 41–97.
- [BBO05] P. Biane, P. Bougerol, and N. O’Connell, *Littlemann paths and brownian paths*, To appear in Duke Mathematical Journal (2005).
- [DX01] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001.
- [EK86] S. N. Ethier and T. G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence.
- [Gal63] A. R. Galmarino, *Representation of an isotropic diffusion as a skew product*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/1963), 359–378.
- [GY05a] L. Gallardo and M. Yor, *An absolute continuity relationship between two multidimensional Dunkl processes*, Working paper (2005).

- [GY05b] L. Gallardo and M. Yor, *A chaotic representation property of the multidimensional Dunkl processes*, To appear in *Annals of Probability* (2005).
- [GY05c] L. Gallardo and M. Yor, *Some new examples of Markov processes which enjoy the time-inversion property*, *Probab. Theory Related Fields* **132** (2005), no. 1, 150–162.
- [GY05d] L. Gallardo and M. Yor, *Some remarkable properties of the Dunkl martingales*, To appear in *Séminaire de Probabilités XXXIX. Dedicated to Paul-André Meyer*, *Lecture Notes in Math.*, Springer, Berlin, 2005.
- [IM65] K. Itô and H. P. McKean, Jr., *Diffusion processes and their sample paths*, *Die Grundlehren der Mathematischen Wissenschaften, Band 125*, Academic Press Inc., Publishers, New York, 1965.
- [IW81] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, *North-Holland Mathematical Library*, vol. 24, North-Holland Publishing Co., Amsterdam, 1981.
- [Kun69] H. Kunita, *Absolute continuity of Markov processes and generators*, *Nagoya Math. J.* **36** (1969), 1–26.
- [Law05] S. Lawi, *A characterization of Markov processes enjoying the time-inversion property*, Working paper (2005).
- [McK69] H. P. McKean, Jr., *Stochastic integrals*, *Probability and Mathematical Statistics*, No. 5, Academic Press, New York, 1969.
- [PR88] E. J. Pauwels and L. C. G. Rogers, *Skew-product decompositions of Brownian motions*, *Geometry of random motion (Ithaca, N.Y., 1987)*, *Contemp. Math.*, vol. 73, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.
- [PY81] J. Pitman and M. Yor, *Bessel processes and infinitely divisible laws*, *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, *Lecture Notes in Math.*, vol. 851, Springer, Berlin, 1981, pp. 285–370.
- [Rös98] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, *Comm. Math. Phys.* **192** (1998), no. 3, 519–542.
- [Rös03] M. Rösler, *Dunkl operators: theory and applications*, *Orthogonal polynomials and special functions (Leuven, 2002)*, *Lecture Notes in Math.*, vol. 1817, Springer, Berlin, 2003, pp. 93–135.
- [RV98] M. Rösler and M. Voit, *Markov processes related with Dunkl operators*, *Adv. in Appl. Math.* **21** (1998), no. 4, 575–643.
- [RY99] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, third ed., *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 293, Springer-Verlag, Berlin, 1999.

- [Wat75] S. Watanabe, *On time inversion of one-dimensional diffusion processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **31** (1974/75), 115–124.
- [Yor80] M. Yor, *Loi de l'indice du lacet brownien, et distribution de Hartman-Watson*, Z. Wahrsch. Verw. Gebiete **53** (1980), no. 1, 71–95.
- [Yor01] M. Yor, *Exponential functionals of Brownian motion and related processes*, Springer Finance, Springer-Verlag, Berlin, 2001, With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson.