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# Error estimate for the Finite Volume Scheme applied to the advection equation

Benoît Merlet\* and Julien Vovelle†

## Abstract

We study the convergence of the Finite Volume Scheme for the advection equation with a divergence free  $C^1$  speed in a domain without boundary. We show that the rate of the  $L^\infty(0, T; L^1)$ -error estimate is  $h^{1/2}$  for  $BV$  data. This result was expected from numerical experiment and is optimal. The proof is based on Kuznetsov's method. This method has been introduced for non-linear hyperbolic equations but for the improvements presented in this paper, the linearity of the initial equation is crucial.

**Keywords:** scalar conservation laws, advection equation, Finite Volume method, error estimate

**MSC Number:** 35L65, 65M15

## 1 Introduction

The Finite Volume method is well adapted to the computation of the solution of pdes which are conservation (or balance) laws, for the reason that it respects the property of conservation (or balance) which is the root of the pde under study. The mathematical analysis of the application of the Finite Volume method to hyperbolic first-order conservation laws can be dated from the mid sixties (see [TS62] for example). Concerning the specific problem of the estimate of the rate of convergence of the method, the first result is due to Kuznetsov [Kuz76], who proves that this rate of convergence in  $L^\infty(0, T; L^1)$  is of order  $h^{1/2}$ , where  $h$  is the size of the mesh, provided that the initial data is in  $BV$  and the mesh is a structured cartesian grid. Ever since, several studies and results have come to supplement the error estimate of Kuznetsov. Before describing them, let us emphasize two points:

1. The analysis of the speed of convergence of the Finite Volume method is distinct from the analysis of the order of the method. In the analysis of the speed of convergence

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of the method, general data (e.g.  $BV$  data) are considered. Indeed, here, the problem is to show that the Finite Volume method behaves well regarding the approximation of the continuous evolution problem in all his features (in particular the creation and the transport of discontinuities). On the other hand, in the analysis of the order of the method, restrictions on the regularity of the data are of no importance.

2. If, numerically, the speed of convergence of the Finite Volume method applied to first-order conservation laws is observed to be (at least) of order  $h^{1/2}$  in the  $L^\infty(0, T; L^1)$ -norm, whether the mesh is structured or not, the theoretical and rigorous proof of this result appears to be strongly related to the structure of the mesh.

Indeed, in the case where the mesh is unstructured (see Section 1.2 – what we call a structured mesh is a cartesian mesh with identical cells but this can be slightly relaxed [CGY98]), the result of Kuznetsov has been extended, but to the price of a fall in the order of the error estimate:  $h^{1/4}$  error estimate in the  $L^\infty(0, T; L^1)$ -norm for the Finite Volume method applied to hyperbolic conservation laws on unstructured meshes has been proved by Cockburn, Coquel, Lefloch [CCL94], Vila [Vil94] and Eymard, Gallouët, Herbin [EGH00] for the Cauchy-Problem ( $h^{1/6}$  error estimate for the Cauchy-Dirichlet Problem [OV04]). We repeat that numerical tests gives an order  $h^{1/2}$  for structured as well as unstructured meshes; still, concerning these latter, numerical analysis did not manage to give the rigorous proof of the order  $h^{1/2}$ : there is an upper limit at the order  $h^{1/4}$ . In this paper we prove this expected  $h^{1/2}$ -error estimate in the case where the conservation law is *linear*, i.e. for linear advection equation with a free divergence speed: see Theorem 2.

This result is optimal [TT95, Şab97]. If one is interested in the order of the Finite Volume method applied to the approximation of linear advection equation, then things are different: the rate  $h^{1/2}$  is no more optimal, i.e: for quite regular initial data, the scheme can converge at speed  $h$ : see the recent work of Bouche, Ghidaglia, Pascal [BGP05] on that purpose. In the same direction (the study of the order of the Finite Volume method on unstructured meshes), we also make reference to the works of Després [Des04b, Des04a] on the one hand, who proves an  $h^{1/2}$  error estimate in the  $L_t^\infty L_x^2$ -norm for the upwind Finite Volume method applied to linear advection equations with constant speed and  $H^2$  initial data. On the other hand, we refer to the work of Vila and Villedieu [VV03], who prove an  $h^{1/2}$  error estimate in the  $L^2$  space-time norm for the approximation of Friedrichs hyperbolic systems with  $H^1$  data by energy estimates (they consider explicit in time Finite Volume schemes, for implicit schemes in the case of scalar advection equation, see, as they underline it, the result of Johnson and Pitkäranta [JP86] who show an  $h^{1/2}$  error estimate in the  $L^2$  space-time norm for  $H^1$  data).

In the three papers [Des04b, Des04a, BGP05], the interested reader will also find references to the finite difference approach for error estimates. However, this approach is mainly devoted to the study of Finite Volume schemes on structured meshes. More central in our context (Finite Volume schemes on unstructured meshes applied to hyperbolic conservation laws with  $BV$  initial data) is the question of a  $BV$ -estimate on the approximate numerical solution. Indeed, a uniform bound on the  $BV$  norm of the numerical

solution is known to be a key to the proof of an  $h^{1/2}$ -error estimate (whatever the equation and the scheme are linear or not) but such bounds are impossible to obtain (see the counter-example of Després in [Des04a]). Fortunately, uniform bounds on the total variation of the numerical flux and not on the numerical solution itself are sufficient to get an  $h^{1/2}$ -error estimate. We prove this bound as a by-product of our error estimate (see (13) in Theorem 2)

The paper is divided into three parts: first we continue this introduction by describing the linear advection problem, the Finite Volume scheme, our results and we give the main lines of the proof. In Section 2 we give some classical results on the Finite Volume scheme, in Section 3 we derive the approximate entropy inequality satisfied by the numerical solution and settle the technic of the doubling of variables for comparison of solutions. In Section 4 various and successive estimates are derived to complete the proofs of our results.

## Notations

If  $(X, \mu)$  is a measurable set with finite (positive) measure and  $\varphi \in L^1(X)$ , we denote the mean of  $\varphi$  over  $X$  by

$$\int_X \varphi d\mu := \frac{1}{\mu(X)} \int_X \varphi d\mu.$$

If  $X$  is a set,  $\mathbf{1}_X$  denotes the real function constant equal to 1 on  $X$ .

If  $U$  is an open subset of  $\mathbb{R}^q$ ,  $q \geq 1$ , we say that  $V \in \mathcal{C}_b^1(U, \mathbb{R}^d)$  if all the components of  $V$  are differentiable, are bounded and have continuous derivative on  $U$ .

If  $U$  is an open subset of  $\mathbb{R}^d$ ,  $d \geq 1$ , we let  $BV(U)$  denote the set of  $L^1$  function with bounded Radon measure as derivatives, which, by the Riesz representation theorem is known to be the set of  $L^1$  functions with bounded variation:

$$BV(U) = \left\{ u \in L^1(U); \sup_{\varphi \in \mathcal{C}_c^\infty(U)^d, \|\varphi\| \leq 1} \left| \int_U u(x) \operatorname{div} \varphi(x) dx \right| < +\infty \right\}$$

where  $\|\varphi\| := \|\sqrt{\varphi_1^2 + \dots + \varphi_d^2}\|_{L^\infty(U)}$  for  $\varphi \in \mathcal{C}_c^\infty(U)^d$  and  $\operatorname{div}$  is the divergence operator.

### 1.1 The linear advection equation

Let  $\Omega := \mathbb{T}^d$  or  $\Omega := \mathbb{R}^d$  and let  $T > 0$ . We consider the linear advection problem with periodic boundary conditions

$$\begin{cases} u_t + \operatorname{div}(Vu) = 0, & x \in \Omega, t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where we suppose that  $V \in C_b^1(\Omega \times [0, T], \mathbb{R}^d)$  satisfies

$$\operatorname{div} V(\cdot, t) = 0 \quad \forall t \in [0, T]. \quad (2)$$

The problem (1) has a solution for  $u_0 \in L^1(\Omega)$ ; for the purpose of the error estimate, we will consider initial data in  $BV(\Omega)$ .

**Theorem 1** *Assume  $V \in C_b^1(\Omega \times [0, T], \mathbb{R}^d)$ . Let  $u_0 \in L^1(\Omega)$ . The problem (1) admits a unique weak solution  $u$ , in the sense that  $u \in L^1(\Omega \times (0, T))$  and: for all  $\varphi \in C^\infty(\Omega \times [0, T])$ ,*

$$\int_{\Omega} \int_0^T u(\varphi_t + V \cdot \nabla \varphi) dx dt + \int_{\Omega} u_0 \varphi(x, 0) dx = 0.$$

Moreover,  $u \in \mathcal{C}([0, T], L^1(\Omega))$ ,  $u$  is the entropy solution of (1) and its  $L^2$ -norm is conserved:

$$\forall t \in (0, T), \quad \int_{\Omega} u^2(x, t) dx = \int_{\Omega} u_0^2(x) dx. \quad (3)$$

Besides, there exists a constant  $C \geq 0$  independent from  $u_0$  such that, if additionally  $u_0 \in BV(\Omega)$  then

$$\left. \begin{aligned} \|u(\cdot, t)\|_{BV(\Omega)} &\leq C \|u_0\|_{BV(\Omega)} \quad \text{for all } t \in [0, T], \\ \|u\|_{BV(\Omega \times [0, T])} &\leq C \|u_0\|_{BV(\Omega)}. \end{aligned} \right\} \quad (4)$$

**Proof of Theorem 1** All the results cited in the theorem follow from the characteristic formula:

$$u(x, t) = u_0(X(0; x, t)) \quad (5)$$

where  $X(\tau; x, t)$  denotes the solution of the Cauchy Problem

$$\left\{ \begin{aligned} \frac{dX}{d\tau}(\tau; x, t) &= V(X(\tau, x, t), \tau), \quad \tau \in [0, T], \\ X(t; x, t) &= x. \end{aligned} \right. \quad (6)$$

We only give the sketch of the proof: the Cauchy Problem (6) admits a global solution  $X$  for  $V$  is locally Lipschitz continuous (it is continuously differentiable) and bounded. By regularity of the flow,  $X \in \mathcal{C}^1([0, T] \times \Omega \times [0, T])$ . Besides, by (2), the flow preserves the Lebesgue measure on  $\mathbb{R}^d$ : for every  $\tau, t \in [0, T]$ ,

$$X(\tau, \cdot, t) \# \lambda = \lambda \quad (7)$$

where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}^d$ . Existence for (1) follows from the explicit formula (5) and, by an argument of duality, uniqueness also. Formula (5) also gives the stability of  $L^\infty(\Omega)$  and, by a change of variable, the fact that  $u \in \mathcal{C}(0, T; L^1(\Omega))$ , that  $u$  satisfies the entropy conditions (see (24)). The conservation (3) is a consequence of (7), as well as (4) which also relies on the fact that  $X$  has bounded derivatives.  $\blacksquare$

**Remark 1** *Notice that we consider domains without boundary. In particular, for a Cauchy-Dirichlet Problem, equality (3) (used in the proof of Theorem 2) would not hold. However, the Cauchy-Dirichlet Problem would deserve a specific analysis.*

## 1.2 Finite Volume scheme

The Finite Volume Scheme which approximates (1) is defined on a mesh  $\mathcal{T}$  which is a family of disjoint connected polygonal subsets of  $\Omega$  (called control volumes) such that  $\overline{\Omega}$  is the union of the closures of the elements of this family. We also suppose that the partition  $\mathcal{T}$  satisfies the following properties: the common interface of two control volumes is included in an hyperplane of  $\mathbb{R}^d$ ; there exists  $\alpha > 0$  such that

$$\begin{cases} \alpha h^d \leq |K|, \\ |\partial K| \leq \frac{1}{\alpha} h^{d-1}, \quad \forall K \in \mathcal{T}, \end{cases} \quad (8)$$

where  $h$  is the size of the mesh:  $h := \sup\{\text{diam}(K), K \in \mathcal{T}\}$ ,  $|K|$  is the  $d$ -dimensional Lebesgue measure of  $K$  and  $|\partial K|$  is the  $(d-1)$ -dimensional Lebesgue measure of  $\partial K$ . If  $K$  and  $L$  are two control volumes having an edge  $\sigma$  in common we say that  $L$  is a neighbour of  $K$  and denote (quite abusively)  $L \in \partial K$ . We also denote  $K|L$  the common edge and  $n_{KL}$  the unit normal to  $K|L$  pointing outward  $K$ . Set

$$V_{KL}^n := \int_{K|L} \int_{n\delta t}^{(n+1)\delta t} V \cdot n_{KL}$$

We denote by  $\partial K_n^+$  the set

$$\partial K_n^+ := \{L \in \partial K, V_{KL}^n > 0\}.$$

We also fix a time step  $\delta t > 0$ . We denote by  $\mathcal{M} := \mathcal{T} \times \mathbb{N}$  the space-time mesh and by  $K_n := K \times [n\delta t, (n+1)\delta t)$  a generic space-time cell. In the same way we set  $K|L_n := K|L \times [n\delta t, (n+1)\delta t)$ .

We will assume that the so called Courant-Friedrich-Levy condition is satisfied: there exists  $\xi \in (0, 1)$  such that

$$\sum_{L \in \partial K_n^+} \delta t V_{KL}^n \leq (1 - \xi)|K|, \quad \forall K_n \in \mathcal{M}. \quad (9)$$

**Remark 2** Under condition (8), the CFL condition (9) holds as soon as

$$\delta t \leq (1 - \xi) \frac{\|V\|_{L^\infty}}{\alpha^2} h.$$

The Finite Volume Scheme with explicit time-discretization is defined by the following set of equations:

$$u_K^0 = \frac{1}{m(K)} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}, \quad (10)$$

$$\frac{u_K^{n+1} - u_K^n}{\delta t} + \frac{1}{|K|} \sum_{L \in \partial K_n^+} V_{KL}^n (u_K^n - u_L^n) = 0, \quad \forall K \in \mathcal{T}, \forall n \in \mathbb{N}. \quad (11)$$

We then denote by  $u_h$  the approximate solution of (1) defined by the Finite Volume scheme:

$$u_h(x, t) = u_K^n, \quad \forall (x, t) \in K_n. \quad (12)$$

From now on, we suppose that  $0 < h \leq 1$  and that (9) is satisfied. We fix a time  $T > 0$  and we denote by  $N$  the integer such that  $(N + 1)\delta t \leq T < (N + 2)\delta t$  (so that  $u(\cdot, T) = \sum_{K \in \mathcal{T}} u_K^{N+1} \mathbf{1}_{|K}$ ).

Finally, let us introduce the relative (to  $V$ ) variation of  $u_h$

$$Q(u_0, T) := \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n| + \sum_{n=0}^N \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^+} \delta t V_{KL}^n |u_L^n - u_K^n|.$$

In the sequel,  $C$  denotes various constants which are non decreasing functions of  $\alpha$ ,  $T$ ,  $1/\xi$ ,  $\|V\|_\infty$ ,  $\|\partial_t V\|_\infty$  and  $\|\nabla V\|_\infty$  but independent from  $h$  or  $u_0$ . We now state the main result of the paper.

**Theorem 2** *Assume  $V \in C_b^1(\Omega \times [0, T], \mathbb{R}^d)$  satisfy (2). Let  $u_0$  in  $L^1 \cap BV(\Omega)$  and  $u \in L^1(\Omega \times (0, T))$  be the solution to (1). Let  $u_h$  be its numerical approximation given by the upwind Finite Volume method (10)-(11)-(12). Then, if the CFL condition (9) holds with  $\xi \in (0, 1)$ , we have the following error estimate:*

$$\begin{aligned} \int_{\Omega} |u(x, T) - u_h(x, T)| dx &\leq C \|u_0\|_{BV} h^{1/2}, \\ Q(u_0, T) &\leq C \|u_0\|_{BV}, \end{aligned} \quad (13)$$

where  $C$  is a function of  $T$ ,  $\|V\|_\infty$ ,  $\|\nabla V\|_\infty$ ,  $\|\partial_t V\|_\infty$  and  $\xi$ .

### 1.3 Sketch of the proof of Theorem 2

We emphasize the main ideas of the proof of Theorem 2. Let us first describe the classical method (see [EGH00]). Using a discrete entropy inequality and performing the doubling of variable technic of Kruzhkov, we have, for  $\varepsilon > 0$ ,

$$\int_{\Omega} |u_h(x, T) - u(x, T)| dx \leq C (\|u_0\|_{BV} \varepsilon + (I + II)), \quad (14)$$

(In this summary, we do not express  $II$ , estimates on  $II$  are similar to those on  $I$ ). We have

$$I = \sum_{n=0}^N \sum_{K \in \mathcal{T}} r_{K_n}, \quad (15)$$

with

$$\begin{aligned} r_{K_n} = \int_0^{+\infty} \int_{\Omega} &\left[ (|u_K^{n+1} - u(y, s)| - |u_K^n - u(y, s)|) \right. \\ &\left. \int_K \left( \int_{n\delta t}^{(n+1)\delta t} \rho_\varepsilon(x - y, t - s) dt - \rho_\varepsilon(x - y, (n+1)\delta t - s) \right) dt \right] dy ds. \end{aligned} \quad (16)$$

where the function  $\rho_\varepsilon$  is defined by  $\rho_\varepsilon(\cdot, \cdot) := (1/\varepsilon)^{d+1} \rho(\cdot/\varepsilon, \cdot/\varepsilon)$  where  $\rho$  is a smooth compactly supported non negative function satisfying  $\int \rho = 1$ .

In order to estimate  $r_{K_n}$  we bound  $|u_K^{n+1} - u(y, s)| - |u_K^n - u(y, s)|$  by  $|u_K^{n+1} - u_K^n|$  and we have

$$|r_{K_n}| \leq C|K||u_K^{n+1} - u_K^n|/\varepsilon. \quad (17)$$

Summing these estimates and using Cauchy-Schwartz inequality, we get

$$\begin{aligned} |I| &\leq (\delta t)^{1/2}/\varepsilon \left( \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K|\delta t \right)^{1/2} \left( \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K||u_K^{n+1} - u_K^n|^2 \right)^{1/2} \\ &\leq (\delta t)^{1/2}/\varepsilon (T|\Omega|)^{1/2} \left( \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K||u_K^{n+1} - u_K^n|^2 \right)^{1/2}. \end{aligned}$$

Then we use the  $L^2$ -estimate

$$\sum_{n=0}^N \sum_{K \in \mathcal{T}} |K||u_K^{n+1} - u_K^n|^2 \leq C\|u_0\|_{L^2}^2. \quad (18)$$

Plugging these estimates in (14) and optimizing in  $\varepsilon$ , we are led to

$$\int_{\Omega} |u_h(x, T) - u(x, T)| dx \leq C\|u_0\|_{L^2}^{1/2} \|u_0\|_{BV}^{1/2} h^{1/4}.$$

(thanks to the CFL condition, we have  $\delta t \leq Ch$ .)

The preceding method is classical and is also valid in the non-linear case. In this article, we remark that (in the linear case and) if  $u_0$  is a characteristic function, we have  $r_{K_n} = 0$  for most of the space-time cells  $K_n := K \times [n\delta t, (n+1)\delta t)$ . More precisely, if  $u_0$  is a characteristic function, then  $u$  is also the characteristic function of a subset  $B$  of  $\Omega \times [0, T]$ . Now, let us define

$$\mathcal{M}_{T,\varepsilon} := \{K_n : d(K_n, \partial B) \leq \varepsilon\}.$$

Roughly speaking, we will have

$$\sum_{K_n \in \mathcal{M}_{T,\varepsilon}} |K|\delta t \leq C\|u_0\|_{BV}\varepsilon.$$

If  $K_n \notin \mathcal{M}_{T,\varepsilon}$ , then  $(y, s) \mapsto u(y, s)$  is constant on the support of  $\rho_\varepsilon(x - \cdot, t - \cdot)$  for every  $(x, t)$  in  $K \times (n\delta t, (n+1)\delta t]$  and a direct computation yields to  $r_{K_n} = 0$ . For  $K_n \in \mathcal{M}_{T,\varepsilon}$ , we use inequality (17). Summing the inequalities and using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} |I| &\leq (\delta t)^{1/2}/\varepsilon \left( \sum_{K_n \in \mathcal{M}_{T,\varepsilon}} |K|\delta t \right)^{1/2} \left( \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K||u_K^{n+1} - u_K^n|^2 \right)^{1/2} \\ &\leq \|u_0\|_{BV}^{1/2} (\delta t/\varepsilon)^{1/2} \left( \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K||u_K^{n+1} - u_K^n|^2 \right)^{1/2}. \end{aligned} \quad (19)$$



Plugging (18) in (19) and optimizing in  $\varepsilon$  would lead to an error estimate of order  $h^{1/3}$ . To obtain  $h^{1/2}$ , we notice (see Lemma 4) that (18) may be improved in

$$\begin{aligned} E(T, u_0) &:= \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n|^2 & (20) \\ &\leq C (\|u_0\|_{L^2}^2 - \|u_h(\cdot, T)\|_{L^2}^2). & (21) \end{aligned}$$

Now, using the preceding inequality, (3) and the fact that  $\|u\|_\infty \leq 1$ , we compute

$$E(T, u_0) \leq C \int_{\Omega} |u_h(\cdot, T) - u(\cdot, T)|$$

Plugging this estimate in (19), using (14) and optimizing in  $\varepsilon$ , we obtain

$$\int_{\Omega} |u_h(\cdot, T) - u(\cdot, T)| \leq C \|u_0\|_{BV}^{2/3} \varepsilon^{1/3} h^{1/3} \left( \int_{\Omega} |u_h(\cdot, T) - u(\cdot, T)| \right)^{1/3}.$$

And the Theorem is proved when  $u_0$  is a characteristic function. The general result follows by linearity, using the decomposition of Lemma 5.

## 2 Classical results

By continuity in  $BV$  of the  $L^2$ -projection on the functions piecewise constant with respect to the mesh  $\mathcal{T}$  (see [Coc91]), we have:

**Lemma 1** *There exists a constant  $C \geq 0$  which depends on  $\alpha, d$  only such that for every  $u_0 \in BV(\Omega)$ ,*

$$\begin{aligned} \|u_h(\cdot, 0)\|_{BV} &\leq C \|u_0\|_{BV}, \\ \|u_h(\cdot, 0) - u_0\|_{L^1} &\leq C \|u_0\|_{BV} h. \end{aligned}$$

The free divergence assumption (2) leads to the following identity

**Lemma 2** *Under the hypothesis of free divergence (2), one has*

$$\sum_{L \in \partial K_n^+} V_{KL}^n = \sum_{L: K \in \partial L_n^+} V_{LK}^n \quad \forall n \geq 0, \forall K \in \mathcal{T}.$$

### Monotony

Under a CFL condition the Finite Volume scheme is order-preserving (hence stable in  $L^\infty$ ):

**Proposition 1** *Under condition (9), the application  $T : (u_K^n) \mapsto (u_K^{n+1})$  defined by (11) is order-preserving.*

**Proof of Proposition 1:** Eq. (11) gives

$$u_K^{n+1} = \left( 1 - \sum_{L \in \partial K_n^+} \frac{V_{KL}^n \delta t}{|K|} \right) u_K^n + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n \delta t}{|K|} u_L^n \quad (22)$$

i.e., under (9),  $u_K^{n+1}$  is a convex combination of  $u_K^n, (u_L^n)_{L \in \partial K_n^+}$ . ■

**Corollary 1** *Assume (9), then, for all  $\xi \in \mathbb{R}$ ,*

$$\frac{|u_K^{n+1} - \xi| - |u_K^n - \xi|}{\delta t} + \frac{1}{|K|} \sum_{L \in \partial K_n^+} V_{KL}^n (|u_K^n - \xi| - |u_L^n - \xi|) \leq 0, \forall K \in \mathcal{T}, \forall n \in \mathbf{N}. \quad (23)$$

**Proof of Corollary 1:** Write  $|u - \xi| = u \top \xi - u \perp \xi$ , where  $a \top b := \max(a, b)$  and  $a \perp b := \min(a, b)$ . Since  $T : (u_K^n) \mapsto (u_K^{n+1})$  is non-decreasing,  $(u_K^{n+1}) \leq T(u_K^n \top \xi)$  and also  $(\xi) = T(\xi) \leq T(u_K^n \top \xi)$ . Therefore  $(u_K^{n+1} \top \xi) \leq T(u_K^n \top \xi)$ . Similarly,  $(u_K^{n+1} \perp \xi) \geq T(u_K^n \perp \xi)$  and by subtracting these two results we get (23).

## $L^1$ -stability

From Lemma 2 it is not difficult to see that the quantity  $\sum_{K \in \mathcal{T}} |K| u_K^n$  is conserved. This fact and Proposition 1 implies:

**Proposition 2** *Under condition (9), the scheme  $T : (u_K^n) \mapsto (u_K^{n+1})$  is stable for the  $L^1$ -norm:*

$$\sum_{K \in \mathcal{T}} |K| |u_K^{n+1}| \leq \sum_{K \in \mathcal{T}} |K| |u_K^n|, \quad \forall n \geq 0.$$

## 3 Entropy inequalities

The solution  $u$  of the linear problem (1) satisfies the following entropy equality (see Theorem 1):

**Proposition 3** *for all  $\xi \in [0, 1]$ , for all  $\varphi \in \mathcal{C}_c^\infty(\Omega \times [0, +\infty))$ ,  $\varphi \geq 0$ ,*

$$\int_0^\infty \int_\Omega |u - \xi| (\varphi_t + V \cdot \nabla \varphi) dx dt + \int_\Omega |u_0 - \xi| \varphi(\cdot, 0) = 0, \quad (24)$$

This is an entropy equality and not just entropy inequality since the problem is linear; somehow an entropy inequality would be sufficient. We intend to prove that  $u_h$  satisfies an entropy inequality up to an error term:

**Proposition 4** for all  $\xi \in [0, 1]$ , for all  $\varphi \in \mathcal{C}_c^\infty(\Omega \times [0, +\infty))$ ,  $\varphi \geq 0$ ,

$$\int_0^\infty \int_\Omega |u_h - \xi|(\varphi_t + V \cdot \nabla \varphi) dx dt + \int_\Omega |u_h(0) - \xi| \varphi(\cdot, 0) \geq -\eta_h(\xi, \varphi), \quad (25)$$

$$\begin{aligned} \eta_h(\xi, \varphi) := & \sum_{K_n \in \mathcal{M}} |K| \delta t \left( \frac{|u_K^{n+1} - \xi| - |u_K^n - \xi|}{\delta t} (\varphi_K^n - \varphi_K((n+1)\delta t)) \right) \\ & + \sum_{L \in \partial K_n^+} \frac{1}{|K|} (|u_K^n - \xi| - |u_L^n - \xi|) (V_{KL}^n \varphi_K^n - (V\varphi)_{KL}^n) \end{aligned} \quad (26)$$

where

$$\begin{aligned} \varphi_K^n &:= \int_{K_n} \varphi(x, t) dx dt, & \varphi_K(t) &:= \int_K \varphi(x, t) dx, \\ (V\varphi)_{KL}^n &:= \int_{K|L} \int_{n\delta t}^{(n+1)\delta t} \varphi(x, t) V(x, t) \cdot n_{KL}(x) dt dx. \end{aligned}$$

**Proof of Proposition 4:** We develop the first term of the inequality:

$$\begin{aligned} & \int_0^\infty \int_\Omega |u_h - \xi|(\varphi_t + V \cdot \nabla \varphi) dx dt \\ &= \sum_{K_n \in \mathcal{M}} |K| \delta t |u_K^n - \xi| \frac{1}{\delta t |K|} \int_{n\delta t}^{(n+1)\delta t} \int_K (\varphi_t + V \cdot \nabla \varphi) \\ &= \sum_{K_n \in \mathcal{M}} |K| \delta t |u_K^n - \xi| \left( \frac{1}{|K|} \int_K \varphi((n+1)\delta t) - \frac{1}{|K|} \int_K \varphi(n\delta t) \right) \\ & \quad + \sum_{L \in \partial K} \frac{1}{\delta t |K|} \int_{n\delta t}^{(n+1)\delta t} \int_{K|L} \varphi V \cdot n_{KL} \\ &= \sum_{K_n \in \mathcal{M}} |K| \delta t |u_K^n - \xi| \left( \varphi_K((n+1)\delta t) - \varphi_K(n\delta t) \right) \\ & \quad + \sum_{L \in \partial K} \frac{(V\varphi)_{KL}^n - (V\varphi)_{LK}^n}{|K|} \end{aligned}$$

and by rearranging the terms in the first and second sum we get

$$\begin{aligned} & \int_0^\infty \int_\Omega |u_h - \xi|(\varphi_t + V \cdot \nabla \varphi) dx dt + \int_\Omega |u_h(\cdot, 0) - \xi| \varphi(\cdot, 0) \\ &= \sum_{K_n \in \mathcal{M}} |K| \delta t \left( (|u_K^n - \xi| - |u_K^{n+1} - \xi|) \varphi_K((n+1)\delta t) \right) \\ & \quad + \sum_{L \in \partial K_n^+} \frac{1}{|K|} (|u_L^n - \xi| - |u_K^n - \xi|) (V\varphi)_{KL}^n \\ &= \sum_{K_n \in \mathcal{M}} |K| \delta t \left( (|u_K^n - \xi| - |u_K^{n+1} - \xi|) \right) \\ & \quad + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n}{|K|} (|u_L^n - \xi| - |u_K^n - \xi|) \varphi_K^n - \eta_h(\xi, \varphi). \end{aligned}$$

This is the discrete entropy inequality (23) satisfied by  $u_h$  which shows that the first term of the last equality is non-negative (multiply Eq. (23) by  $\varphi_K^n$  and sum over  $n \geq 0$  and  $K \in \mathcal{T}$ ), from which follows the approximate continuous entropy inequality (25)-(26). ■

### 3.1 Doubling variables

To perform the technic of doubling of variables [Kru70, Kuz76], we introduce an approximation of the unit ( $\rho_\varepsilon$ ) and a test function  $\psi_T$ .

**Definition 1** We set  $\rho_\varepsilon^1(t) = \varepsilon^{-1}\rho^1(t/\varepsilon)$ ,  $\rho_\varepsilon^d(x) = \varepsilon^{-d}\rho^d(x/\varepsilon)$  and

$$\rho_\varepsilon(x, t) := \rho_\varepsilon^d(x)\rho_\varepsilon^1(t),$$

where  $\rho^1 \in C_c^1(\mathbb{R})$ ,  $\text{supp } \rho^1 \subset [-1, 0]$ ,  $\int \rho^1 = 1$ ,  $\rho^1 \geq 0$  and  $\rho^d \in C_c^1(\mathbb{R}^d)$ ,  $\text{supp } \rho^d \subset \bar{B}_{\mathbb{R}^d}(0, 1)$ ,  $\int \rho^d = 1$ ,  $\rho^d \geq 0$ .

Let us also introduce the characteristic function  $\psi_T := \mathbf{1}_{[0, (N+1)\delta t)}$ . Eventually, we set  $\varphi_\varepsilon : \Omega^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y, t, s) \mapsto \psi_T(t)\rho_\varepsilon(x - y, t - s)$ .

We will need a sequence  $(\psi_T^k)_k \geq 1$  approximating  $\psi_T$ , satisfying  $\psi_T^k \in C^1(\mathbb{R}_+, \mathbb{R})$ ,  $0 \leq \psi_T^k \leq 1$  and

$$\begin{aligned} \psi_T^k(t) &= 1 & \text{for } t &\leq (N+1)\delta t, \\ \psi_T^k(t) &= 0 & \text{for } t &\geq (N+1+1/k)\delta t. \end{aligned}$$

We then set  $\varphi_\varepsilon^k : \Omega^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y, t, s) \mapsto \psi_T^k(t)\rho_\varepsilon(x - y, t - s)$ .

**Proposition 5** Assume the CFL condition (9). Suppose  $u_0 \in L^1 \cap BV(\Omega)$ . Let  $u$  be the solution of (1) and  $u_h$  the numerical approximation given by the Finite Volume method (10)-(11)-(12). Then, for  $T \geq 0$ ,

$$\begin{aligned} &\int_{\Omega} |u(x, T) - u_h(x, T)| dx \\ &\leq C \|u_0\|_{BV}(\varepsilon + h + \delta t) + \int_{\mathbb{R}} \int_{\Omega} \eta_h(u(y, s), \varphi_\varepsilon(\cdot, y, \cdot, s)) dy ds. \end{aligned} \quad (27)$$

#### Proof of Proposition 5:

Let  $\varepsilon > 0$ . For  $k \geq 1$ , let us write (25) with  $\xi = u(y, s)$ , and

$$(x, t) \mapsto \varphi_\varepsilon^k(x, y, t, s)$$

as a test function and sum the result with respect to  $(y, s)$  to get

$$\begin{aligned} &\int_0^\infty \int_{\Omega} \int_0^\infty \int_{\Omega} |u_h(x, t) - u(y, s)| (\partial_t + V \cdot \nabla_x)(\psi_T^k \rho_\varepsilon) dx dt dy ds \\ &+ \int_0^\infty \int_{\Omega} \int_{\Omega} |u_h(x, 0) - u(y, s)| \psi_T(0) \rho_\varepsilon(x - y, -s) dx dy ds \geq - \int_0^\infty \int_{\Omega} \eta_h(u(y, s), \varphi_\varepsilon^k) dy ds. \end{aligned} \quad (28)$$

Similarly, write (24) in the  $(y, s)$ -variables, choose  $\xi = u_h(x, t)$ ,  $(y, s) \mapsto \varphi_\varepsilon^k(x, y, t, s)$  as a test function and sum the result with respect to  $(x, t)$  to get

$$\int_0^\infty \int_\Omega \int_0^\infty \int_\Omega |u_h(x, t) - u(y, s)| (\partial_s + V \cdot \nabla_y) (\psi_T^k \rho_\varepsilon) dx dt dy ds = 0 \quad (29)$$

(The term involving the initial condition vanishes because  $\rho_\varepsilon(x - y, t) = 0$  for non negative times  $t$ .) Use the identities  $(\partial_t + \partial_s) \rho_\varepsilon(x - y, t - s) = 0$  and  $(\nabla_x + \nabla_y) \rho_\varepsilon(x - y, t - s) = 0$ , sum (28) and (29) and let  $k$  tend to  $+\infty$  to obtain the following inequality:

$$\begin{aligned} \int_0^\infty \int_\Omega \int_\Omega |u_h(x, T) - u(y, s)| \rho_\varepsilon(x - y, (N + 1)\delta t - s) dx dy ds \\ + \int_0^\infty \int_\Omega \int_\Omega |u_h(x, 0) - u(y, s)| \rho_\varepsilon(x - y, -s) dx dy ds \\ \geq - \int_0^\infty \int_\Omega \eta_h(u(y, s), \varphi_\varepsilon) dy ds. \end{aligned}$$

And since  $\int_0^\infty \int_\Omega \int_\Omega \rho_\varepsilon(x - y, -s) dx dy ds = 1$ , we have

$$\begin{aligned} \int_\Omega |u_h(x, T) - u(x, T)| dx + \int_0^\infty \int_\Omega \eta_h(u(y, s), \varphi_\varepsilon) dy ds \\ \geq - \int_0^\infty \int_\Omega \int_\Omega |u(x, T) - u(y, s)| \rho_\varepsilon(x - y, (N + 1)\delta t - s) dx dy ds \\ - \int_0^\infty \int_\Omega \int_\Omega |u_0(x) - u(y, s)| \rho_\varepsilon(x - y, -s) dx dy ds - \int_\Omega |u_0(x) - u_h(x, 0)| dx. \quad (30) \end{aligned}$$

The speed of convergence of the remaining terms of (30) is related to the speed of convergence of the translations of  $u_0$  and  $u_h(\cdot, 0)$  in  $L^1$ , which is known since they are  $BV$  functions (see e.g. [Kru70] for the proof of the following lemma)

**Lemma 3** *Let  $q \in \mathbb{N}^*$ , let  $U$  be an open convex bounded subset of  $\mathbb{R}^q$ , let  $(\theta_\varepsilon)$  be a sequence of non-negative functions on  $\mathbb{R}^q \times \mathbb{R}^q$  such that  $\text{supp}(\theta_\varepsilon)$  is a subset of the  $\varepsilon$ -neighbourhood of the diagonal  $\{(x, y) \in \mathbb{R}^2; x = y\}$  and such that for all  $x, y \in \mathbb{R}^q$ ,*

$$\int_U \theta_\varepsilon(x, \cdot) = \int_U \theta_\varepsilon(\cdot, y) = M.$$

*Then, for  $f \in L^1 \cap BV(U)$ ,*

$$\int_U \int_U |f(x) - f(y)| \theta_\varepsilon(x, y) dx dy \leq M |U| \|f\|_{BV} \varepsilon.$$

From Lemma 3, inequality (4) of Theorem 1 and the second inequality of Lemma 1, we have

$$\begin{aligned} \int_0^\infty \int_\Omega \int_\Omega |u(x, T) - u(y, s)| \rho_\varepsilon(x - y, (N + 1)\delta t - s) dx dy ds &\leq C \|u_0\|_{BV} (\varepsilon + \delta t), \\ \int_0^\infty \int_\Omega \int_\Omega |u(x, 0) - u(y, s)| \rho_\varepsilon(x - y, -s) dx dy ds &\leq C \|u_0\|_{BV} \varepsilon, \\ \int_0^\infty \int_\Omega \int_\Omega |u(x, 0) - u_0(x)| \rho_\varepsilon(x - y, -s) dx dy ds &\leq C \|u_0\|_{BV} h. \end{aligned}$$

The Proposition follows from these estimates and (30).  $\blacksquare$

## 4 Proof of Theorem 2

We begin with three lemmas

**Lemma 4 (Energy estimate)** *Suppose  $u_0 \in L^2(\Omega)$ . Let  $u_h$  be the numerical solution defined by (10)-(11)-(12). Assume (9), then, for all  $t > \delta t$ , we have the following bound*

$$\begin{aligned} E(t) := &\sum_{n < t/\delta t - \delta t} \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n|^2 \\ &+ \sum_{n < t/\delta t - \delta t} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \partial K} V_{KL} |u_K^n - u_L^n|^2 \leq C \left( \|u_h(0)\|_{L^2(0,1)}^2 - \|u_h(t)\|_{L^2(0,1)}^2 \right) \quad (31) \end{aligned}$$

where  $C = (2 - \xi)/\xi$  is a positive constant.

**Proof of Lemma 4:** Multiply Eq. (11) by  $u_K^n$ , use the identity

$$a(b - a) = \frac{1}{2}(b^2 - a^2) - \frac{1}{2}(b - a)^2$$

with  $(a, b) = (u_K^n, u_K^{n+1})$  and  $(a, b) = (u_K^n, u_L^n)$  and sum the result over  $K \in \mathcal{T}$ ,  $0 \leq n \leq t/\delta t - 1$  to get

$$\begin{aligned} - &\sum_{0 \leq n \leq t/\delta t - 1} \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n|^2 \\ &+ \sum_{0 \leq n \leq t/\delta t - 1} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \partial K} V_{KL} |u_K^n - u_L^n|^2 = \|u_h(\cdot, 0)\|_{L^2(\Omega)}^2 - \|u_h(\cdot, t)\|_{L^2(\Omega)}^2. \quad (32) \end{aligned}$$

By (11), we have

$$\begin{aligned}
& \sum_{0 \leq n \leq t/\delta t - 1} \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n|^2 = \sum_{0 \leq n \leq t/\delta t - 1} \sum_{K \in \mathcal{T}} \delta t^2 / |K| \left| \sum_{L \in \partial K_n^+} V_{KL}^n (u_K^n - u_L^n) \right|^2 \\
& \leq \sum_{0 \leq n \leq t/\delta t - 1} \sum_{K \in \mathcal{T}} \delta t \left( \sum_{L \in \partial K_n^+} \delta t V_{KL}^n / |K| \right) \left( \sum_{L \in \partial K_n^+} V_{KL}^n |u_K^n - u_L^n|^2 \right) \\
& \leq (1 - \xi) \sum_{0 \leq n \leq t/\delta t - 1} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^+} V_{KL}^n |u_K^n - u_L^n|^2 \tag{33}
\end{aligned}$$

by (9). Using this estimate in (32) gives

$$\sum_{0 \leq n \leq t/\delta t - 1} \delta t \sum_{K \in \mathcal{T}} \sum_{L \in \partial K} V_{KL}^n |u_K^n - u_L^n|^2 \leq \frac{1}{\xi} \left( \|u_h(0)\|_{L^2(0,1)}^2 - \|u_h(t)\|_{L^2(0,1)}^2 \right)$$

Together with (33), this yields (31). ■

**Lemma 5** [Fed69, Bre84] *Let  $u_0 \in BV(\Omega)$ . For every  $\zeta \in \mathbb{R}$ , define*

$$\chi_{u_0}(x, \zeta) := \begin{cases} 1 & \text{if } 0 < u_0(x) < \zeta, \\ -1 & \text{if } \zeta < u_0(x) < 0, \\ 0 & \text{in the other cases.} \end{cases}$$

We have  $u_0 = \int_{\mathbb{R}} \chi_{u_0}(\cdot, \zeta) d\zeta$ , almost every where. Moreover, by the co-area formula, we have

$$\|u_0\|_{BV} = \int_{\mathbb{R}} \|\chi_{u_0}(\cdot, \zeta)\|_{BV} d\zeta.$$

**Lemma 6** [Translations of  $\varphi_\varepsilon$ ]

Recall that the test-function  $\varphi_\varepsilon$  is defined in Definition 1. For all  $K_n \in \mathcal{M}$ , for all  $L \in \partial K$ , we have

$$\int_{\Omega \times \mathbb{R}_+} \int_K \left| \int_{n\delta t}^{(n+1)\delta t} \varphi_\varepsilon(x', y, t', s) dt' - \varphi_\varepsilon(x', y, (n+1)\delta t, s) \right| dx' dy ds \leq C|K|(\delta t/\varepsilon) \tag{34}$$

$$\int_{\Omega \times \mathbb{R}_+} \int_{K|L_n} \left| \int_{K_n} \varphi(x', y, s, t') dx' dt' - \varphi_\varepsilon(x, y, s, t) \right| dx dt dy ds \leq C\delta t|K|L/h/\varepsilon. \tag{35}$$

**Proof of Lemma 6:** We prove (34), the proof of (35) is similar. Set  $t_n := (n + 1)\delta t$ . Since  $\psi_T$  is constant on  $(n\delta t, (n + 1)\delta t]$ , we have

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}_+} \int_K \left| \int_{n\delta t}^{(n+1)\delta t} \varphi_\varepsilon(x', y, t', s) dt' - \varphi_\varepsilon(x', y, (n + 1)\delta t, s) \right| dx' dy ds \\
& \leq \|\psi_T\|_\infty |K| \int_{\mathbb{R}_+} \left| \int_{n\delta t}^{(n+1)\delta t} \rho_\varepsilon^1(t' - s) dt' - \rho_\varepsilon^1(t_n - s) \right| ds \\
& = \|\psi_T\|_\infty |K| \int_{\mathbb{R}_+} \left| \int_{n\delta t}^{(n+1)\delta t} (\rho_\varepsilon^1(t' - s) - \rho_\varepsilon^1(t_n - s)) dt' \right| ds \\
& \leq |K| \int_{n\delta t}^{(n+1)\delta t} \int_{\mathbb{R}_+} |\rho_\varepsilon^1(t' - s) - \rho^1(t_n - s)| ds dt \\
& \leq |K| \|\rho_\varepsilon^1\|_{BV} \int_{n\delta t}^{(n+1)\delta t} |t' - t_n| dt' \\
& \leq |K| \|\rho_\varepsilon^1\|_{BV} \delta t \leq C|K|\delta t/\varepsilon.
\end{aligned}$$

■

The main step in the proof of Theorem 2 is the following

**Proposition 6** *Suppose that  $V \in C^1(\Omega \times [0, T], \mathbb{R}^d)$  satisfies (2). Let  $u_0$  be the characteristic function of a connected set  $A \subset \Omega$ . We suppose that  $u_0 \in BV(\Omega)$ . Let  $u$  be the solution of (1) with initial data  $u_0$  and  $u_h$  be the approximate solution defined by (10)-(11)-(12). Then*

$$\int_{\Omega} |u_h(x, T) - u(x, T)| \leq C \|u_0\|_{BV} h^{1/2}.$$

Moreover  $Q(u_0, T) \leq C \|u_0\|_{BV}$ .

**Proof of Proposition 6** Notice that, insofar as  $u_0$  is the characteristic function of the set  $A$ , the fact that  $u_0 \in BV(\Omega)$  precisely means that  $A$  has finite perimeter. Indeed,  $\|u_0\|_{BV} = |\partial A|$ . Similarly, the function  $u$  is the characteristic function of a set  $B$  and we have  $|\partial(B \cap (\Omega \times [0, T]))| = \|u\|_{BV(\Omega \times [0, T])}$ .

**Remark 3** *Along the proof, we will use a small parameter  $\varepsilon$ . At the end of the proof this parameter will be set to  $h^{1/2}$ . In particular, since  $0 < h \leq 1$ , we have  $h \leq \varepsilon$ .*

From the isoperimetric inequality, we have  $\int_{\Omega} |u_0(x)| dx \leq C \|u_0\|_{BV}^{d/(d-1)}$ . Thus if  $\|u_0\|_{BV} \leq \varepsilon^{d-1}$ ,

$$\int_{\Omega} |u_0(x)| dx \leq C \|u_0\|_{BV} \varepsilon,$$

and, in this case, the Proposition is a consequence of Remark 3 and of the  $L^1$ -stability of the scheme.



We now suppose that

$$\|u_0\|_{BV} > \varepsilon^{d-1}. \quad (36)$$

Using Proposition 5, we have to bound the quantity

$$\begin{aligned} & \int_0^{+\infty} \int_{\Omega} \eta_h(u(y, s), \varphi_\varepsilon(\cdot, y, \cdot, s)) dy ds \\ &= \int_0^{+\infty} \int_{\Omega} \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^+} (|u_K^{n+1} - u(y, s)| - |u_K^n - u(y, s)|) \\ & \times \left( \int_K \left[ \int_{n\delta t}^{(n+1)\delta t} \varphi_\varepsilon(x', y, t', s) dt' - \varphi_\varepsilon(x', y, (n+1)\delta t, s) \right] dx' \right) dy ds \\ &+ \int_0^{+\infty} \int_{\Omega} \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^+} (|u_K^n - u(y, s)| - |u_L^n - u(y, s)|) \\ & \times \left( \int_{K|L_n} \left[ \int_{K_n} \varphi_\varepsilon(x', y, t', s) dx' dt' - \varphi_\varepsilon(x, y, t, s) \right] V(x, t) \cdot n_{KL} dx \right) dy ds. \quad (37) \end{aligned}$$

**Remark 4** Since  $\psi_T = \mathbf{1}_{[0, \mathbf{N}+1)}$ , the sums on  $\mathcal{M}$  have been restricted to

$$\mathcal{M}_T := \{K_n \in \mathcal{M} : 0 \leq n \leq N\}.$$

Let us note  $I$  and  $II$  the two terms of the right hand side of (37).

**Step 1.** Estimate of  $II$ :

In order to bound  $II$ , we introduce the set

$$\mathcal{M}_{T, \varepsilon} := \{K_n \in \mathcal{M}_T : d(K_n, \partial B) \leq \varepsilon\}.$$

An important fact is that, up to terms of order  $h + \delta t$ , the volume of  $\bigcup_{K_n \in \mathcal{M}_{T, \varepsilon}} K_n$  is bounded by  $C\varepsilon$ . Indeed, the Hausdorff measure of dimension  $d$  of  $\partial B$  is  $\|u\|_{BV(\Omega \times [0, T])}$ , thus

$$\begin{aligned} \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \delta t |K| &\leq \|u\|_{BV(\Omega \times [0, T])} (\varepsilon + h + \delta t) + C(\varepsilon + h + \delta t)^d \\ &\leq C \|u_0\|_{BV} \varepsilon, \end{aligned} \quad (38)$$

where the second inequality is a consequence of (4), (9), Remark 3 and (36).

Using Fubini's Theorem, we write:

$$II = \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^+} p_{K_n, L},$$

with

$$\begin{aligned}
p_{K_n, L} &:= \int_{K|L_n} \left[ \int_{K_n} q_{K_n, L}(x', t') dx' dt' - q_{K_n, L}(x, t) \right] V(x, t) \cdot n_{KL} dx dt, \\
q_{K_n, L}(x, t) &:= \int_0^{+\infty} \int_{\Omega} (|u_K^n - u(y, s)| - |u_L^n - u(y, s)|) \varphi_{\varepsilon}(x, y, t, s) dy ds.
\end{aligned}$$

If  $K_n \notin \mathcal{M}_{T, \varepsilon}$  and  $(x'', t'') \in K_n$ , then the function  $(y, s) \mapsto u(y, s)$  is constant on the set  $\text{supp } \varphi_{\varepsilon}(x'', \cdot, t'', \cdot)$  (in the sequel, the corresponding constant is denoted by  $\zeta$ ). In this case we have

$$\begin{aligned}
q_{K_n, L}^n(x'', t'') &= (|u_K^n - \zeta| - |u_L^n - \zeta|) \int_0^{+\infty} \int_{\Omega} \varphi_{\varepsilon}(x'', y, t'', s) dy ds \\
&= (|u_K^n - \zeta| - |u_L^n - \zeta|).
\end{aligned}$$

Thus, for  $K_n \notin \mathcal{M}_{T, \varepsilon}$ , we have  $p_{K_n, L} = 0$

If  $K_n \in \mathcal{M}_{T, \varepsilon}$ , we bound  $\left| |u_K^n - u(y, s)| - |u_L^n - u(y, s)| \right|$  by  $|u_K^n - u_L^n|$ . We have

$$\begin{aligned}
|p_{K_n, L}^n| &\leq |u_K^n - u_L^n| \\
&\quad \times \int_{\Omega \times \mathbb{R}_+} \int_{K|L_n} \left| \int_{K_n} \varphi(x', y, s, t') dx' dt' - \varphi_{\varepsilon}(x, y, s, t) \right| |V(x, t) \cdot n_{KL}| dx dt dy ds.
\end{aligned}$$

Since  $V \in \mathcal{C}^1(\Omega \times [0, T])$ , we have  $|V(x, t) \cdot n_{KL}| \leq \frac{1}{|K|L|} V_{KL}^n + C(h + \delta t)$  for every  $(x, t) \in K|L_n$ , and therefore, by (35):

$$\begin{aligned}
|p_{K_n, L}^n| &\leq C(V_{KL}^n \delta t |u_K^n - u_L^n| + |K|L| \delta t (h + \delta t) |u_K^n - u_L^n|) h / \varepsilon \\
&\leq C(V_{KL}^n \delta t |u_K^n - u_L^n| + \delta t |K|) h / \varepsilon.
\end{aligned}$$

We used (8), (9) and the  $L^\infty$  bound on  $u_h$  to derive the last estimate.

Summing these estimates, we get

$$|II| \leq C \left( \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \sum_{L \in \partial K_n^+} \delta t V_{KL}^n |u_K^n - u_L^n| + \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \delta t |K| \right) h / \varepsilon.$$

By Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|II| &\leq C \left( \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \sum_{L \in \partial K_n^+} \delta t V_{KL}^n |u_K^n - u_L^n|^2 \right)^{1/2} \left( \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \sum_{L \in \partial K_n^+} \delta t V_{KL}^n \right)^{1/2} h / \varepsilon \\
&\quad + C \sum_{K_n \in \mathcal{M}_{T, \varepsilon}} \delta t |K| h / \varepsilon.
\end{aligned}$$

and from inequality (38) and assumption (8), we have

$$\sum_{K_n \in \mathcal{M}_{T,\varepsilon}} \delta t |K| \leq C \|u_0\|_{BV} \varepsilon, \quad \sum_{K_n \in \mathcal{M}_{T,\varepsilon}} \sum_{L \in \partial K_n^+} \delta t h V_{KL}^n \leq C \|u_0\|_{BV} \varepsilon.$$

Thus

$$|II| \leq CE(u_0, T)^{1/2} \|u_0\|_{BV}^{1/2} h^{1/2} / \varepsilon^{1/2} + C \|u_0\|_{BV} \varepsilon. \quad (39)$$

**Step 2.** Estimate of  $I$ :

We rewrite

$$I = \sum_{K_n \in \mathcal{M}_T} r_{K_n},$$

with

$$\begin{aligned} r_{K_n} &:= \int_K \left[ \int_{n\delta t}^{(n+1)\delta t} s_{K_n}(x, t) dt - s_{K_n}(x, (n+1)\delta t) \right] dx, \\ s_{K_n}(x, t) &:= \int_0^{+\infty} \int_{\Omega} (|u_K^{n+1} - u(y, s)| - |u_K^n - u(y, s)|) \varphi_\varepsilon(x, y, t, s) dy ds. \end{aligned}$$

As previously, we split the sum over  $\mathcal{M}_T$  in the definition of  $I$  in two blocks. Let us first consider that  $K_n \notin \mathcal{M}_{T,\varepsilon}$ . For  $(x, t) \in K_n$ , the function  $(y, s) \mapsto u(y, s)$  is constant on the support of  $\varphi_\varepsilon(x, \cdot, t, \cdot)$  and we have

$$\begin{aligned} s_{K_n}(x, t) &= (|u_K^{n+1} - \zeta| - |u_K^n - \zeta|) \int_{\mathbb{R}_+} \int_{\Omega} \varphi_\varepsilon(x, y, t, s) dy ds \\ &= (|u_K^{n+1} - \zeta| - |u_K^n - \zeta|) \end{aligned}$$

and a direct computation yields  $r_{K_n} = 0$ .

If  $K_n \in \mathcal{M}_{T,\varepsilon}$ , similar estimates as those used to bound the term  $II$  (in particular we use (34)) gives

$$|r_{K_n}| \leq C |K| |u_K^{n+1} - u_K^n| (\delta t / \varepsilon).$$

By summing the result over  $K_n \in \mathcal{M}_T$ , we get

$$\begin{aligned} |I| &\leq C \delta t / \varepsilon \sum_{K_n \in \mathcal{M}_{T,\varepsilon}} |K| |u_K^{n+1} - u_K^n| \\ &\leq C (\delta t)^{1/2} / \varepsilon E(u_0, T)^{1/2} \left( \sum_{K_n \in \mathcal{M}_{T,\varepsilon}} |K| \delta t \right)^{1/2} \\ &\leq C \|u_0\|_{BV}^{1/2} E(u_0, T)^{1/2} (\varepsilon^{1/2} + (\delta t / \varepsilon)^{1/2}). \end{aligned} \quad (40)$$

**Step 3: End of the Proof**

Collecting the estimates (39) and (40) and using Proposition 5, we get:

$$\int_{\Omega} |u(x, T) - u_h(x, T)| dx \leq C \left( \|u_0\|_{BV} (h/\varepsilon + \varepsilon) + E(u_0, T)^{1/2} \|u_0\|_{BV}^{1/2} (h/\varepsilon)^{1/2} \right),$$

Setting  $\varepsilon := h^{1/2}$ , we get

$$\int_{\Omega} |u(x, T) - u_h(x, T)| dx \leq C \left( \|u_0\|_{BV} h^{1/2} + E(u_0, T)^{1/2} \|u_0\|_{BV}^{1/2} h^{1/4} \right). \quad (41)$$

Now, we may use the last inequality to estimate  $E(u_0, T)$ . Indeed, from (3) and Lemma 4, we have

$$\begin{aligned} E(u_0, T) &\leq C \int_{\Omega} (u_h(x, 0)^2 - u_h(x, T)^2) dx \\ &\leq C \int_{\Omega} (u(x, 0)^2 - u_h(x, T)^2) dx \\ &= C \int_{\Omega} (u(x, T)^2 - u_h(x, T)^2) dx \\ &\leq 2C \|u_0\|_{\infty} \int_{\Omega} |u(x, T) - u_h(x, T)| dx \\ &\leq 2C \left( \|u_0\|_{BV} h^{1/2} + E(u_0, T)^{1/2} \|u_0\|_{BV}^{1/2} h^{1/4} \right). \end{aligned}$$

Solving this inequality in  $E(u_0, T)$ , we get

$$E(u_0, T) \leq C \|u_0\|_{BV} h^{1/2}. \quad (42)$$

Plugging this estimate in (41), the proof of the first part of Proposition 6 is achieved.

Since  $E(u_0, T)$  is bounded by  $\int_{\Omega} (u(x, T)^2 - u_h(x, T)^2) dx$ , the estimate (42) is also valid when (36) is not satisfied. Thus from (42) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} Q(u_0, T) &\leq CE(u_0, T)^{1/2} \left( \sum_{K_n \in \mathcal{M}_T} |K| + \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^+} V_{KL}^n \delta t \right)^{1/2} \\ &\leq C \|u_0\|_{BV}, \end{aligned}$$

which ends the proof of Proposition 6. ■

**Proof of Theorem 2.** Taking into account Lemma 5 and the fact that the scheme and the equation are linear, it is sufficient to prove Theorem 2 when  $u_0$  is a step function:  $u_0 = \mathbf{1}_A$ . This is Proposition 6. ■

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