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A Proof of Gromov's Algebraic Lemma

David Burguet

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Abstract : Following the analysis of differentiable mappings of Y. Yomdin, M. Gromov has stated a very elegant “Algebraic Lemma” which says that the “differentiable size” of an algebraic subset may be bounded in terms only of its dimension, degree and diameter - regardless of the size of the underlying coefficients! We give a complete and elementary proof of Gromov's result using the ideas presented in his Bourbaki talk as well as other necessary ingredients.

1 Introduction

Several problems in, e.g., Analysis and Dynamical Systems, requires estimating the differentiable size of semi-algebraic subsets. Y. Yomdin developed many tools to this end [7]. M. Gromov observed that one of this tools could be refined to give the following very elegant statement :

Theorem 1 *Let $r \geq 1$, $d \geq 0$, $\delta \geq 0$ be integers, there exists $M < \infty$ with the following properties. For all semi-algebraic compact $A \subset [0, 1]^d$ of maximum dimension l and of degree $\leq \delta$, there exist an integer N , integers $0 \leq l_1, \dots, l_N \leq l$ and maps $(\phi_i : [0, 1]^{l_i} \mapsto [0, 1]^d)_{i=1 \dots N}$ satisfying $\|\phi_i|_{[0, 1]^{l_i}}\|_r := \max_{\beta: |\beta| \leq r} \|\partial^\beta \phi_i\|_\infty \leq 1$, such that $\bigcup_{i=0}^N \phi_i([0, 1]^{l_i}) = A$, where $N \leq M$ and $\deg(\phi_i) \leq M$. Furthermore, $\phi_i|_{[0, 1]^{l_i}}$ is a diffeomorphism onto its image and the semi-algebraic sets $(\phi_i([0, 1]^{l_i}))_{1 \leq i \leq N}$ are disjoint.*

In his Seminaire Bourbaki [10], M. Gromov gives many ideas but stops short of a complete proof which has never been given. On the other hand, this result has been put to much use, especially in Dynamical System Theory. Y. Yomdin [13],[14] used it to compare the topological entropy and the “homological size” for C^r maps (in particular, Y. Yomdin shows in [13] Shub's conjecture in the case of C^∞ maps). S. Newhouse [11] then showed, using Pesin's theory, how this gives, for C^∞ smooth maps, upper-semicontinuity of the metric entropy and therefore the existence of invariant measures with maximum entropy. J. Buzzi [5] observed that in fact Y. Yomdin's estimates give an even more uniform result called asymptotic h-expansiveness, which was in turn used by M. Boyle, D. Fiebig and U. Fiebig [3] to prove existence of symbolic extensions. This theory, the dynamical consequence of the above theorem, is still developing in the works of M. Boyle, T. Downarowicz [2], S. Newhouse and others.

The proof of this theorem is trivial for the dimension 1 and easy for the dimension 2 (See part 6). To prove the theorem in higher dimensions, we introduce the notion of triangular (C^α, K) -Nash maps : it is the subject of the part 3. The fourth part is devoted to the structure

of semi-algebraic sets. In the fifth part, by taking the limit of “good” parametrizations, we reduce the main theorem to a proposition about the parametrization of semi-algebraic “smooth” maps in order to avoid the singularities. The other difficulties are dealt with as suggested by M. Gromov. The proof by induction of this proposition will be dealt with in the last section.

As I was completing the submission of this paper, I learnt that A. Wilkie had written a proof of the same theorem [12]. I am grateful to M. Coste for this reference.

2 Semi-algebraic sets and maps

First recall some basic results concerning semi-algebraic sets. We borrow from [8]. For completeness, other references are [1],[6],[7].

Definition 1 $A \subset \mathbb{R}^d$ is a semi-algebraic set, if it can be written as the union of sets of the form $\{x \in \mathbb{R}^d \mid P_1(x) > 0, \dots, P_r(x) > 0, P_{r+1}(x) = 0, \dots, P_{r+s}(x) = 0\}$, where $r, s \in \mathbb{N}$ and $P_1, \dots, P_{r+s} \in \mathbb{R}[X_1 \dots X_d]$. Such a formula is called a presentation of A .

The degree of a presentation is the sum of the total degrees of the polynomials involved with multiplicities. The degree of a semi-algebraic set is the minimum degree of its presentations.

Definition 2 $f : A \subset \mathbb{R}^d \rightarrow \mathbb{R}^n$ is a semi-algebraic map if the graph of f is a semi-algebraic set.

Definition 3 A Nash manifold is an analytic submanifold of \mathbb{R}^d , which is a semi-algebraic set.

A Nash map is a map defined on a Nash manifold, which is analytic and semi-algebraic.

We have the following description of a semi-algebraic set (See [8], Prop 3.5 p 124 and see [7] Prop 4.4 p 48) :

Theorem 2 Let $A \subset \mathbb{R}^n$ be a semi-algebraic set, there exist a minimal integer N (bounded by a function of $\deg(A)$) and connected Nash manifolds $(A_i)_{i=1 \dots N}$ such that $A = \coprod_{i=1}^N A_i$ and $\forall j \neq i (A_i \cap \text{adh}(A_j) \neq \emptyset) \Rightarrow (A_i \subset \text{adh}(A_j) \text{ et } \dim(A_i) < \dim(A_j))$. (\coprod : disjoint union).

Definition 4 (with the notations of the previous proposition) The maximum dimension of A is the maximum dimension of the Nash manifolds A_1, \dots, A_N .

3 (C^α, K) -Nash maps and triangular maps

Definition 5 \mathbb{N}^d is provided with a total order \preceq , defined as follows :
For $\alpha = (\alpha_1, \dots, \alpha_d), \beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$

$$\alpha \preceq \beta \text{ iff } (|\alpha| := \sum_i \alpha_i < |\beta|) \text{ or } (|\alpha| = |\beta| \text{ et } \alpha_k \leq \beta_k, \text{ where } k := \max\{l \leq n : \alpha_l \neq \beta_l\})$$

Notations 1 $\alpha + 1$ (resp $\alpha - 1$) represents the following element (resp. preceding) α for the order \preceq .

Definition 6 Let $K \in \mathbb{R}^+$, $d \in \mathbb{N}$, $\alpha \in \mathbb{N}^d - \{0\}$. Let $A \subset [0, 1]^d$ a semi-algebraic open set of \mathbb{R}^d . A map $f : A \rightarrow [0, 1]^d$ is a \mathcal{C}^0 -Nash map, if $f := (f_1, \dots, f_d)$ is a Nash-map, which can be continuously extending to $\text{adh}(A)$ (we call again f this single extension). A map $f : A \rightarrow [0, 1]^d$ is a (\mathcal{C}^α, K) -Nash map, if f is a \mathcal{C}^0 -Nash map and if $\|f\|_\alpha := \max_{\beta \leq \alpha, 1 \leq i \leq d} \|\partial^\beta f_i\|_\infty \leq K$. If $\alpha = (0, 0, \dots, r)$, we write (\mathcal{C}^r, K) instead of (\mathcal{C}^α, K) and we note $\|f\|_r := \|f\|_\alpha$.

The two following lemmas are dealing with the composition of $(\mathcal{C}^\alpha, 1)$ -Nash maps.

Lemma 1 For all $d, r \in \mathbb{N}^*$, there exists $K < +\infty$, such that if $\psi, \phi : [0, 1]^d \rightarrow [0, 1]^d$ are two $(\mathcal{C}^r, 1)$ -Nash maps, then $\psi \circ \phi$ is a (\mathcal{C}^r, K) -Nash map.

PROOF : Immediate. □

One of the key points of the proof of Yomdin-Gromov theorem is the idea of triangular maps, which allows us to control the derivatives one after one.

Definition 7 We say that a map $\psi : [0, 1]^l \rightarrow [0, 1]^d$ is triangular if $l \leq d$ and if there exists a family of maps $(\psi_i : [0, 1]^{\min(l, d+1-i)} \rightarrow [0, 1])_{i=1 \dots d}$, such that

$$\psi = (\psi_1(x_1 \dots x_l), \dots, \psi_{d-l+1}(x_1 \dots x_l), \psi_{d-l+2}(x_2 \dots x_l), \dots, \psi_{d-l+k}(x_k \dots x_l), \dots, \psi_d(x_l))$$

Remark 1 If $\psi : [0, 1]^n \rightarrow [0, 1]^m$ and $\phi : [0, 1]^m \rightarrow [0, 1]^p$ are two triangular maps, then $\phi \circ \psi : [0, 1]^n \rightarrow [0, 1]^p$ is triangular.

In the case of triangular maps, we give the following version of the lemma 1, allowing an induction on $\alpha \in \mathbb{N}^d$ rather than $r \in \mathbb{N}$, in the proof of the proposition 4.

Lemma 2 For all $d, r \in \mathbb{N}^*$, there exists $K < +\infty$ such that if $\psi, \phi : [0, 1]^d \rightarrow [0, 1]^d$ are two triangular $(\mathcal{C}^\alpha, 1)$ -Nash maps with $|\alpha| = r$, then $\psi \circ \phi$ is a (\mathcal{C}^α, K) -Nash map.

PROOF : Immediate. □

Definition 8 (resolution of a semi-algebraic set) Let $M : \mathbb{N}^3 \rightarrow \mathbb{R}^+$ and let $K \in \mathbb{R}^+$, $d \in \mathbb{N}^*$, $A \subset [0, 1]^d$ a Nash submanifold of dimension l and $\alpha \in \mathbb{N}^d - \{0\}$. The family of maps $(\phi_i : [0, 1]^l \rightarrow [0, 1]^d)_{i=1 \dots N}$ is a (\mathcal{C}^0, M) -resolution [resp. $(\mathcal{C}^\alpha, K, M)$ -resolution] of A if :

- each ϕ_i is triangular ;
- each ϕ_i is a \mathcal{C}^0 -Nash map [resp. a (\mathcal{C}^α, K) -Nash map] ;
- $\text{adh}(A) = \bigcup_{i=1}^N \phi_i([0, 1]^l)$ and each $\phi_i|_{[0, 1]^l}$ is a diffeomorphism onto its image ;
- $N, \text{deg}(\phi_i)$ are less than $M(0, d, \text{deg}(A))$ [resp. $M(|\alpha|, d, \text{deg}(A))$].

Definition 9 (resolution of a family of maps) Let $M : \mathbb{N}^4 \rightarrow \mathbb{R}^+$, $K \in \mathbb{R}^+$, $d \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^d - \{0\}$ and let $f_1, \dots, f_k : A \rightarrow [0, 1]$ be k \mathcal{C}^0 -Nash maps on $A \subset [0, 1]^d$, a semi-algebraic open set of \mathbb{R}^d . The family of maps $(\phi_i : [0, 1]^d \rightarrow [0, 1]^d)_{i=1 \dots N}$ is a (\mathcal{C}^0, M) -resolution [resp. $(\mathcal{C}^\alpha, K, M)$ -resolution] of $(f_i)_{i=1 \dots k}$ if :

- each ϕ_i is triangular ;

- each $\phi_i, f_j \circ \phi_i$ is a \mathcal{C}^0 -Nash map [resp. (\mathcal{C}^α, K) -Nash] ;
- $\text{adh}(A) = \bigcup_{i=1}^N \phi_i([0, 1]^d)$ and each $\phi_i|_{[0, 1]^d}$ is a diffeomorphism onto its image ;
- $N, \text{deg}(\phi_i), \text{deg}(f_j \circ \phi_i)$ are less than $M(0, d, k, \max_j(\text{deg}(f_j)))$ [resp. $M(|\alpha|, d, k, \max_j(\text{deg}(f_j)))$].

Eventually the function M can be chosen independently of the semi-algebraic datas (above A or f_1, \dots, f_k). We often imply such a function ; we note \mathcal{C}^0 [resp. (\mathcal{C}^α, K)] instead of (\mathcal{C}^0, M) [resp. $(\mathcal{C}^\alpha, K, M)$] and we will say " $N, \text{deg}(\phi_i) \dots$ are bounded by a function of $|\alpha|, d, \dots$ ".

The following remark is very useful later on :

Lemma 3 For all $M : \mathbb{N}^2 \rightarrow \mathbb{R}^+$, there exists $M' : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ such that we have the following property.

Let $d \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^d - \{0\}$, and $f : [0, 1]^d \rightarrow [0, 1]$ a $(\mathcal{C}^\alpha, M(|\alpha|, d))$ -Nash map, then f admits a $(\mathcal{C}^\alpha, 1, M')$ -resolution.

PROOF : Linear reparametrizations. □

4 Tarski's Principle and "saucissonnage"

Proposition 1 (Tarski's principle) Let A a semi-algebraic set of \mathbb{R}^{d+1} and $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ the projection defined by $\pi(x_1, \dots, x_{d+1}) = (x_1, \dots, x_d)$ then $\pi(A)$ is a semi-algebraic set and $\text{deg}(\pi(A))$ is bounded by a function of $\text{deg}(A)$ and d .

PROOF : See [6] Thm 2.2.1, p 26 and [7] Prop 4.3 p 48 □

Corollary 1 Any formula combining sign conditions on semi-algebraic functions by conjunction, disjunction, negation and universal and existential quantifiers defines a semi-algebraic set.

Corollary 2 Let $f : A \subset \mathbb{R}^k \rightarrow \mathbb{R}^l$ be a semi-algebraic fonction, then A is a semi-algebraic set and $\text{deg}(A)$ is bounded by a function of $\text{deg}(f), k$ and l .

Corollary 3 Any linear expression or composition of semi-algebraic maps is semi-algebraic. Furthermore, if ϕ and ψ are two semi-algebraic maps, such that the composition $\phi \circ \psi$ is well defined, then $\text{deg}(\phi \circ \psi)$ is bounded by a function of $\text{deg}(\phi)$ and $\text{deg}(\psi)$.

PROOF : See [6] Prop 2.2.6 p 28 □

Proposition 2 (Cohen) For all $P_1, \dots, P_s \in \mathbb{R}[X_1 \dots X_{d+1}]$, there exist a partition of $[0, 1]^d$ into Nash manifolds $\{A_1, \dots, A_m\}$, a finite family of \mathcal{C}^0 -Nash maps $\zeta_{i,1}^1, \zeta_{i,1}^2, \dots, \zeta_{i,q_i}^2 : A_i \rightarrow [0, 1]$, for all $1 \leq i \leq m$, such that :

- for each i and each k , the sign $P_k(x_1, y)$, with $x_1 \in [0, 1]$ et $y := (x_2, \dots, x_{d+1}) \in A_i$, only depends on the signs of $x_1 - \zeta_{i,j}(y)$, $j = 1, \dots, q_i$;

¹in [8], M.Coste prove the result for Nash maps, but not necessarily \mathcal{C}^0 -Nash, but the maps ζ_i are onto $[0, 1]$, and adapting easily the lemma 2.5.6 in [6], we can assume that ζ_i extends continuously on $\text{adh}(A_i)$

²we can have $\zeta_{i,k}(x) = \zeta_{i,k+1}(x)$ for $x \in \text{adh}(A_i)$

- the zeros' set of P_k coincide with the graphs of $\zeta_{i,j}$;
- $m, q_i, \deg(A_i), \deg(\zeta_{i,j})$ are bounded by a function of $\sum_k \deg(P_k)$ and d .

PROOF : [8] Thm 2.3 p 112. □

From the above we deduce easily the following proposition :

Proposition 3 *For all semi-algebraic $A \subset [0, 1]^{d+1}$, there exist integers m, q_1, \dots, q_m , a partition of $[0, 1]^d$ in Nash manifolds A_1, \dots, A_m and \mathcal{C}^0 -Nash maps, $\zeta_{i,1} < \dots < \zeta_{i,q_i} : A_i \rightarrow [0, 1]$, for all $1 \leq i \leq m$, such that :*

- $\text{adh}(A)$ coincide with an union of slices of the following form $\{(x_1, y) \in [0, 1] \times \text{adh}(A_i) : \zeta_{i,k}(y) \leq x_1 \leq \zeta_{i,k+1}(y)\}$;
- $m, q_i, \deg(A_i), \deg(\zeta_{i,j})$ are bounded by a function of $\deg(A)$ and d .

Corollary 4 *For all $A \subset [0, 1]^{d+1}$ semi-algebraic open set of \mathbb{R}^{d+1} , there exist integers m, q_1, \dots, q_m , disjoint semi-algebraic open sets A_1, \dots, A_m and \mathcal{C}^0 -Nash maps, $\zeta_{i,1} < \dots < \zeta_{i,q_i} : A_i \rightarrow [0, 1]$, for all $1 \leq i \leq m$, such that :*

- $\text{adh}(A)$ coincide with the union of slices of the following form $\{(x_1, y) \in [0, 1] \times \text{adh}(A_i) : \zeta_{i,k}(y) \leq x_1 \leq \zeta_{i,k+1}(y)\} = \text{adh}(\{(x_1, y) \in [0, 1] \times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\})$;
- $m, q_i, \deg(A_i), \deg(\zeta_{i,j})$ are bounded by a function of $\deg(A)$ and d .

PROOF : Name A'_i the Nash manifolds as in the proposition 3, tiing up A like a sausage in the direction of x_1 . Then $\pi(A)$ being open, we have according to the Thm 2, the following options for each i : either A'_i is open, or else $\exists j$ such that A'_j is open and $A'_i \subset \text{adh}(A'_j)$. In the last case, if A contains a slice $T_i = \{(x_1, y_1) \in [0, 1] \times A'_i : \zeta_{i,1}(y_1) < x_1 < \zeta_{i,2}(y_1)\}$, then A contains a slice of the following form $T_j = \{(x_1, y_1) \in [0, 1] \times A'_j : \zeta_{j,1}(y_1) < x_1 < \zeta_{j,2}(y_1)\}$ such that $T_i \subset \{(x_1, y_1) \in [0, 1] \times \text{adh}(A'_j) : \zeta_{j,1}(y_1) < x_1 < \zeta_{j,2}(y_1)\}$, because A is open. From this fact we deduce the corollary, where A_i are the semi-algebraic sets A'_i , which are open. □

In the following corollary, we reparametrize a semi-algebraic set with \mathcal{C}^0 -Nash maps of bounded degree. In this way a first step is achieved. But in order to show the Yomdin-Gromov theorem the main difficulty is the control of the derivatives of the reparametrizations.

Corollary 5 *There exists $M : \mathbb{N}^3 \rightarrow \mathbb{R}^+$, such that all Nash manifold $A \subset [0, 1]^d$ admits a (\mathcal{C}^0, M) -resolution.*

PROOF : We argue by induction on d . We note $P(d)$ the claim of the above corollary. $P(0)$ is trivial. Assume $P(d)$.

Let $A \subset [0, 1]^{d+1}$ be a Nash manifold of dimension l . The proposition 4 gives us integers m, q_1, \dots, q_m , Nash manifolds $A_1, \dots, A_m \subset [0, 1]^d$ and \mathcal{C}^0 -Nash map, $\zeta_{i,1} < \dots < \zeta_{i,q_i} : A_i \rightarrow [0, 1]$ such that :

- $\text{adh}(A)$ coincide with a union of slices of the following form $\{(x_1, y) \in [0, 1] \times \text{adh}(A_i) : \zeta_{i,k}(y) \leq x_1 \leq \zeta_{i,k+1}(y)\}$;
- $m, q_i, \deg(A_i)$ are bounded by a function of $\deg(A)$ and d .

Let $1 \leq i \leq m$. We apply the induction hypothesis to $A_i \subset [0, 1]^d$: there exists a \mathcal{C}^0 -resolution of A_i , ie an integer N_i (bounded by a function of $\deg(A_i)$ and d , therefore by a function of $\deg(A)$ and d) and \mathcal{C}^0 -Nash maps $\phi_{i,1}, \dots, \phi_{i,N_i} : [0, 1]^l \rightarrow [0, 1]^d$, such that $\text{adh}(A_i) = \bigcup_{p=1}^{N_i} \phi_{i,p}([0, 1]^l)$. Then we define $\psi_{i,k,p} : [0, 1]^l \rightarrow [0, 1]^{d+1}$ as follows : $\psi_{i,k,p}(x_1, y) := (x_1(\zeta_{i,k+1}(y) - \zeta_{i,k}(y)) \circ \phi_{i,p}(x_1, \dots, x_l) + \zeta_{i,k} \circ \phi_{i,p}(x_1, \dots, x_l), \phi_{i,p}(x_1, \dots, x_l))$.

The $\psi_{i,k,p}$ are \mathcal{C}^0 -Nash triangular maps, such that

- the number of these parametrizations are bounded by $\sum_{i=1}^m q_i N_i$
- $\text{adh}(A) = \bigcup_{i,k,p} \psi_{i,k,p}([0, 1]^l)$
- $\deg(\psi_{i,k,p})$ is bounded by a function of $\deg(A)$ and d (See Corollaire 3)

Thus these maps form a \mathcal{C}^0 -resolution of A . □

The following lemma is another application of the Tarski's principle :

Lemma 4 *Let $A \subset \mathbb{R}^d$ be a semi-algebraic open set, $f : A \rightarrow \mathbb{R}^n$ a Nash map defined on A . The partial derivatives of f of all orders are also semi-algebraic maps of degree bounded by a function of $\deg(f)$, d and n .*

PROOF : Apply the corollary 1. □

5 Yomdin-Gromov Theorem

First we show the following technical proposition, in which we work with “smooth” functions. Finally we explain how we reduce the proof of the main theorem to this proposition.

Proposition 4 *Let $(f_i : A \rightarrow [0, 1])_{i=1\dots k}$ be \mathcal{C}^0 -Nash maps, where $A \subset [0, 1]^d$ is an open semi-algebraic set of \mathbb{R}^d and let $\alpha \in \mathbb{N}^d$. There exists a sequence $(A_n)_{n \in \mathbb{N}} \subset A^{\mathbb{N}}$ of semi-algebraic open sets of \mathbb{R}^d , such that*

- $a_n := \sup_{x \in A} d(x, A_n) \xrightarrow[n \rightarrow +\infty]{} 0$, where $d(x, A_n)$ is the distance between x and A_n ;
- $\deg(A_n)$ is bounded by a function of $\deg(A)$ and $|\alpha|$;
- f_i is analytic in a neighborhood of $\text{adh}(A_n)$;
- $(f_i/A_n)_{i=1\dots k}$ admits a $(\mathcal{C}^\alpha, 1)$ -resolution.

We will say that such a sequence $(A_n)_{n \in \mathbb{N}}$ is α -adapted to $(f_i)_{i=1\dots k}$.

The following corollary follows from the above proposition :

Corollary 6 *There exists $M : \mathbb{N}^4 \rightarrow \mathbb{R}^+$, such that all family $(f_i : A \rightarrow [0, 1])_{i=1\dots k}$ of \mathcal{C}^0 -Nash maps, where $A \subset [0, 1]^d$ is a semi-algebraic open set of \mathbb{R}^d , admits a $(\mathcal{C}^\alpha, 1, M)$ -resolution.*

Now we show how the proposition 4, the corollary 6 and the Yomdin-Gromov theorem deduce from the case $k = 1$ of the proposition 4. In fact we show stongers results, which are very useful for the proof by induction in the last section.

Notations 2 We consider the set E of pairs (α, d) , where $d \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}^d$. The set E is provided with the following order $\ll, :$

$$(\beta, e) \ll (\alpha, d) \text{ iff } (e < d) \text{ or } (e = d \text{ and } \beta \preceq \alpha)$$

We will write :

$P4(\alpha, d)$ the claim of the proposition 4 for all pair (β, e) with $(\beta, e) \ll (\alpha, d)$.

$C6(\alpha, d)$ the claim of the corollary 6 for all pair (β, e) with $(\beta, e) \ll (\alpha, d)$.

$YG(\alpha, d)$ the existence of a $(\mathcal{C}^\beta, 1)$ resolution for all Nash manifold $A \subset [0, 1]^e$, and for all pair $(\beta, e) \ll (\alpha, d)$, so that : theorem 1 $\iff YG(\alpha, d) \quad \forall (\alpha, d) \in E$.

Lemma 5 The claim $C6(\alpha, d)$ [resp. $P4(\alpha, d)$] for $k = 1$ implies the claim $C6(\alpha, d)$ [resp. $P4(\alpha, d)$] for all $k \in \mathbb{N}^*$.

PROOF :

We argue by induction on k .

Assume that for all k -family $g_1, \dots, g_k : B \rightarrow [0, 1]$ of \mathcal{C}^0 -Nash maps, with $B \subset [0, 1]^d$ a semi-algebraic open set of \mathbb{R}^d , admits a $(\mathcal{C}^\alpha, 1)$ -resolution.

Let $f_1, \dots, f_{k+1} : A \rightarrow [0, 1]$ $k + 1$ be \mathcal{C}^0 -Nash maps, with $A \subset [0, 1]^d$ a semi-algebraic open set of \mathbb{R}^d . According to the induction hypothesis for the maps (f_1, \dots, f_k) , there exists $(\phi_i)_{i=1 \dots N}$ a $(\mathcal{C}^\alpha, 1)$ -resolution of (f_1, \dots, f_k) .

Moreover, according to $C6(\alpha, d)$ for $k = 1$, there exists for each $i : (\psi_{i,j})_{j=1 \dots N_i}$ a $(\mathcal{C}^\alpha, 1)$ -resolution of $f_{k+1} \circ \phi_i$.

According to the lemma, the maps $\phi_i \circ \psi_{i,j}$, of wich the number is $\sum_{i=1}^N N_i$, are (\mathcal{C}^α, K) -Nash triangular maps, as well as the maps $f_p \circ \phi_i \circ \psi_{i,j}$ for all $1 \leq p \leq k$ (with $K = K(|\alpha|, d)$). Finally, for each i , $(\psi_{i,j})_{j=1 \dots N_i}$ being a $(\mathcal{C}^\alpha, 1)$ -resolution of $f_{k+1} \circ \phi_i$, the maps $f_{k+1} \circ \phi_i \circ \psi_{i,j}$ are $(\mathcal{C}^\alpha, 1)$ -Nash maps. Moreover, we have in a trivial way : $adh(A) = \bigcup_{i,j} \phi_i \circ \psi_j([0, 1]^d)$. We conclude the proof, thanks to the lemma 3, amounting to a $(\mathcal{C}^\alpha, 1)$ -resolution.

We adapt the above proof for $P4(\alpha, d)$ as follows. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence α -adapted to $(f_i)_{i=1 \dots k}$. For all $n \in \mathbb{N}$, we consider $(\phi_j^n)_{j=1 \dots N_n}$ a $(\mathcal{C}^\alpha, 1)$ resolution of $(f_i/A_n)_{i=1 \dots k}$ and $(A_p^{n,j})_{p \in \mathbb{N}}$ a sequence α -adpated to $f_{k+1} \circ \phi_j^n$. By uniform continuity of ϕ_j^n on $[0, 1]^d$, $\exists \delta_{j,n} > 0 \quad \forall x, y \in [0, 1]^d \quad d(x, y) < \delta_{j,n} \Rightarrow d(\phi_j^n(x), \phi_j^n(y)) < 1/n$. Choose $p_{j,n} \in \mathbb{N}$, such that $a_{p_{j,n}}^{n,j} < \delta_{j,n}^n$. Now, let show that $B_n := \bigcup_{j=1}^{N_n} \phi_j^n(A_{p_{j,n}}^{n,j})$ is a sequence α -adapted to $(f_i)_{i=1 \dots k+1}$.

B_n is open, because, for $p \in \mathbb{N}$, $A_p^{n,j} \subset [0, 1]^d$ is an open subset of \mathbb{R}^d and $\phi_j^n|_{[0, 1]^d}$ is a diffeomorphism. Furthermore, B_n is a semi-algebraic set because ϕ_j^n is a semi-algebraic map and $A_p^{n,j}$ are semi-algebraic sets and $deg(B_n)$ is bounded by a function of $deg(A)$ and $|\alpha|$, as it is so for N_n , $deg(\phi_j^n)$ and $deg(A_{p_{j,n}}^{n,j})$. Check the "density" condition. Let $x \in A$, $\exists x_n \in A_n$ such that $d(x, x_n) < \max(2a_n, 1/n)$. $\exists 1 \leq i \leq N_n$ and $\exists y_n \in [0, 1]^d$ such that $\phi_i^n(y_n) = x_n$. But $(A_p^{n,i})_{p \in \mathbb{N}}$ is adapted to $f_{k+1} \circ \phi_i^n$, thus there exists a sequence $z_n \in A_{p_{i,n}}^{n,i}$ such that $d(z_n, y_n) < \delta_{i,n}$. Therefore $d(\phi_i^n(z_n), x) < \max(2a_n, 1/n) + 1/n$ and $\sup_{y \in A} d(y, B_n) < \max(2a_n, 1/n) + 1/n \xrightarrow{n \rightarrow +\infty} 0$. \square

PROOF OF COROLLARY 6 (FROM $P4((0, \dots, 0, r+1), d)$ WE DEDUCE $C6((0, \dots, 0, r), d)$.) :

Let $(\phi_i^k)_{i \leq N_k}$ be a $(\mathcal{C}^{r+1}, 1)$ -resolution of $f|_{A_k}$. By hypothesis, N_k is bounded by a function of $\deg(A_k)$ and r and thus by a function of $\deg(A)$ and r ; consequently $(N_k)_{k \in \mathbb{N}}$ is a bounded sequence. By extracting a subsequence, we can assume $N_k = N$, for all $k \in \mathbb{N}$. According to the Ascoli theorem, $\prod_{i=1 \dots 2N} B(r+1)$ is a compact set in $\prod_{i=1 \dots 2N} B(r)$, where $B(r)$ is the closed unit ball of $\mathcal{C}^r([0, 1]^d)$ (set of \mathcal{C}^r maps on $]0, 1[^d$ onto \mathbb{R}). By extracting again a subsequence, we can assume that $(\phi_i^n)_{n \in \mathbb{N}}$ converge in norm $\|\cdot\|_r$ on $]0, 1[^d$ to a $(\mathcal{C}^r, 1)$ -Nash map, ψ_i , such that $f \circ \psi_i$ is a $(\mathcal{C}^r, 1)$ -Nash map, for each $i = 1 \dots N$. One only needs to see $\bigcup_{i=1 \dots N} \psi_i([0, 1]^d) = \text{adh}(A)$. It's enough to show that $A \subset \bigcup_{i=1 \dots N} \psi_i([0, 1]^d)$. Actually, by convergence of ϕ_i^n to ψ_i , we have $\psi_i([0, 1]^d) \subset \text{adh}(A)$, for all i . Let $y \in A$, recording to the density condition, there exists a sequence $x_n \in A_n$, such that $x_n \rightarrow y$. By extracting a subsequence, we can assume that there exist $1 \leq i \leq N$ and a sequence $(y_n \in [0, 1]^d)_{n \in \mathbb{N}}$ such that $x_n = \phi_i^n(y_n)$. By uniform convergence of ϕ_i^n to ψ_i , we have $\psi_i(y_n) \rightarrow y$. Now extracting a convergent subsequence we easily conclude that $\bigcup_{i=1 \dots N} \psi_i([0, 1]^d) = \text{adh}(A)$. Finally $(\psi_i)_{i \leq N}$ is a $(\mathcal{C}^r, 1)$ -resolution of f . \square

Now we show that the corollary 6 for the dimension d implies the Yomdin-Gromov theorem for the dimension $d+1$. More precisely, we will show $C6(\alpha, d) \Rightarrow YG(\alpha, d+1)$.

PROOF OF THEOREM 1 (WE ASSUME $C6(\alpha, d)$ AND FROM IT WE DEDUCE $YG(\alpha, d+1)$) :

Consider first the case, where $A \subset [0, 1]^{d+1}$ is a semi-algebraic open set of \mathbb{R}^{d+1} . According to the corollary 4, we can assume that $\text{adh}(A)$ is a slice of the form $\text{adh}(A) = \{(x, y) \in [0, 1] \times A' \mid \eta(y) \leq x \leq \zeta(y)\}$, where $A' \subset [0, 1]^d$ is a semi-algebraic open set of \mathbb{R}^d and $\eta, \zeta : A' \rightarrow [0, 1]$ \mathcal{C}^0 -Nash maps, where $\deg(\eta), \deg(\zeta), \deg(A')$ depend only on $\deg(A)$ and on d . Applying $C6(\alpha, d)$ to (ζ, η) , there exists $(\phi_i)_{i=1 \dots N}$ a $(\mathcal{C}^\alpha, 1)$ -resolution of (ζ, η) .

For each i , we define $\psi_i : [0, 1] \times [0, 1]^d \rightarrow [0, 1]^{d+1}$ par $\psi_i(x, y) = (x(\zeta \circ \phi_i - \eta \circ \phi_i)(y) + \eta \circ \phi_i(y), \phi_i(y))$. Then $(\psi_i)_{i=1 \dots N}$ is a $(\mathcal{C}^\alpha, 2)$ -resolution of A . We conclude the proof thanks to the lemma 3.

If A is a Nash manifold of dimension $l < d+1$, A admits a \mathcal{C}^0 -résolution $(\phi_i : [0, 1]^l \rightarrow [0, 1]^{d+1})_{i=1 \dots N}$, according to the corollary 5. We conclude the proof, using for each i , $C6(\alpha, d)$ on the coordinates of $\phi_i|_{]0, 1[^l}$. \square

6 Proof of Corollary 6 in dimension 1

First we study the case of dimension 1, where we can prove right away Corollary 6. The case of dimension 1 allows us to introduce simple ideas of parametrizations, which will be adapted in higher dimensions.

The semi-algebraic sets of $[0, 1]$ are the finite union of open intervals and points. So it's enough to prove the Corollary 6 for A of the form $]a, b[\subset [0, 1]$.

PROOF OF $C6(1, 1)$ (CASE OF THE FIRST DERIVATIVE) : : Let $f :]a, b[\rightarrow [0, 1]$ be a \mathcal{C}^0 -Nash map. We cut up the interval $]a, b[$ into a minimal number N of subintervals $(J_k)_{k=1 \dots N}$, such that $\forall x \in J_k, |f'(x)| \geq 1$ or $\forall x \in J_k, |f'(x)| \leq 1$.

The assertion on N results from Tarski principle.

On each interval J_k , we consider the following parametrization of $\text{adh}(J_k) = [c, d] \subset [0, 1]$, $\phi :$

- $\phi(t) = c + t(d - c)$ if $|f'| \leq 1$ and then we have $\deg(\phi) = 1$, $\deg(f \circ \phi) = \deg(f)$
- $\phi(t) = f_{|[c,d]}^{-1}(f(c) + t(f(d) - f(c)))$ if $|f'| \geq 1$ and then we have $\deg(\phi) = \deg(f)$ (indeed $\deg(f^{-1}) = \deg(f)$) and $\deg(f \circ \phi) = 1$

□

PROOF OF $C6(r, 1)$ (CASE OF HIGHER DERIVATIVES) : : we argue by induction on r : assume $C6(r, 1)$, with $r \geq 1$ and prove $C6(r + 1, d)$.

Let $f :]a, b[\subset [0, 1] \rightarrow [0, 1]$ be a C^0 -Nash map. By considering $(f \circ \phi_i)_{i=1\dots N}$, where $(\phi_i)_{i=1\dots N}$ is a $(C^r, 1)$ -resolution of f , we can assume that f is a $(C^r, 1)$ -Nash map.

We divide the interval $]a, b[$ into a minimal number n_i of subintervals on which $|f^{(r+1)}|$ is either increasing or decreasing, ie the sign of $f^{(r+1)}f^{(r+2)}$ is constant. Consider the case where $|f^{(r+1)}|$ is decreasing, the creasing case is similar : we have just to do the following change of variable $x \mapsto 1 - x$. We reparametrize those intervals from $[0, 1]$ with linear maps $\tilde{\phi}_i$. We define $f_i = f \circ \tilde{\phi}_i$, noted g in the following computations.

Define $h(x) = x^2$; we compute :

$$(g \circ h)^{(r+1)}(x) = (2x)^{r+1}g^{(r+1)}(x^2) + R(x, f(x), \dots, f^{(r)}(x))$$

where R is a polynomial depending only on r . Therefore

$$\forall x \in [0, 1] \quad |(g \circ h)^{(r+1)}(x)| \leq |(2x)^{r+1}g^{(r+1)}(x^2)| + C(r),$$

where $C(r)$ is a function of r .

Furthermore, we have

$$x|g^{(r+1)}(x)| = \int_0^x |g^{(r+1)}(t)|dt \leq \left| \int_0^x g^{(r+1)}(t)dt \right| = |g^{(r)}(x) - g^{(r)}(0)| \leq 2 \quad (1)$$

thus

$$|(g \circ h)^{(r+1)}(x)| \leq C(r) + 2\frac{(2x)^{r+1}}{x^2} \leq C(r) + 2^{r+2}$$

Enfin $\deg(\tilde{\phi}_i \circ h) = 2$ and $\deg(g \circ h) = 2\deg(f)$. The claim concerning the integers n_i results from the Tarski's principle. We conclude the proof of $C6(r + 1, d)$ thanks to the lemma 3.

□

7 Proof of Proposition 4

Let us fix two integers $r \geq 2, c \geq 1$. In this section we show $P4((0, \dots, 0, r - 1), c)$ for $k = 1$, the general case deducing from it (See section 4).

We argue by induction on the set E_{rc} of pairs (α, d) , where $d \in \mathbb{N}^*, d \leq c$ and $\alpha \in \mathbb{N}^d, |\alpha| \leq r + c - d$. E_{rc} is provided with the order \ll .

We assume now that $P4(\alpha, d)$ is checked and we distinguish three cases depending on the values of the pair (α, d) :

$$\boxed{\text{Increase of the dimension : } P4((0\dots 0, r+c-d), d) \Rightarrow P4((1, 0\dots, 0), d+1)}$$

PROOF : Let $f : A \subset [0, 1]^{d+1} \rightarrow [0, 1]$ a \mathcal{C}^0 -Nash map, defined on a semi-algebraic open set of \mathbb{R}^{d+1} . We consider the following semi-algebraic open sets : $A_+ = \text{int}(\{x \in A, |\partial_{x_1} f(x)| > 1\})$ et $A_- = \text{int}(\{x \in A, |\partial_{x_1} f(x)| \leq 1\})$. We have $\text{adh}(A) = \text{adh}(A_+) \cup \text{adh}(A_-)$. Actually, we see easily that $\text{adh}(A_+) \cup \text{adh}(A_-) \subset \text{adh}(A)$. Let show $A \subset \text{adh}(A_+) \cup \text{adh}(A_-)$. Let $x \in A$. If $d(x, A_+) = 0$, then $x \in \text{adh}(A_+)$; if not, as A is an open set, there exists $r > 0$, such that the ball $B(x, r) \subset A \cap A_+^c \subset \{x \in A, |\partial_{x_1} f(x)| \leq 1\}$ and thus $x \in A_-$.

According to $P4((0, \dots, 0, r+c-d), d) \Rightarrow P4((0, \dots, 0, 2), d) \Rightarrow C6((0, \dots, 0, 1, d)) \Rightarrow YG((1, \dots, 0), d+1)$, there exist $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -Nash triangular maps $(\phi_j)_{1 \leq j \leq N_-}$ such that $\text{adh}(A_-) = \bigcup_{1 \leq j \leq N_-} \phi_j([0, 1]^d)$ and such that $N_-, \text{deg}(\phi_j)$ are bounded by a function of $\text{deg}(A_-)$, and thus by a function of $\text{deg}(f)$ (according to the lemma 4 and the corollary 6). We have $|\partial_{x_1}(f \circ \phi_j)| \leq 1$. The maps ϕ_i can be used to build a resolution of f .

For A_+ , we consider the inverse of f . Observe first, that according to the corollary 4, we can assume that A_+ is a slice of the following form $\{(x_1, y) \in [0, 1] \times A'_+ : \zeta(y) < x_1 < \eta(y)\}$, où $A'_+ \subset [0, 1]^d$ is a semi-algebraic open set of \mathbb{R}^d and $\zeta, \eta : A'_+ \rightarrow [0, 1]$ are \mathcal{C}^0 -Nash maps.

Define $D_+ = \{(f(x_1, y), y) : (x_1, y) \in A_+\}$. We define $g : A_+ \rightarrow D_+$, $g(x_1, y) = (f(x_1, y), y)$. This map g is a local diffeomorphism, by the local inversion theorem. Moreover, g is one to one, because $g(x_1, y) = g(x'_1, y')$ implies $y = y'$, and $f(x_1, y) = f(x'_1, y)$ implies $x_1 = x'_1$, because $|\partial_{x_1} f(x)| \geq 1$ for $x \in A_+$. Therefore g extends to $g : \text{adh}(A_+) \rightarrow \text{adh}(D_+)$, a one to one bicontinuous map.

We see that D_+ is a semi-algebraic open set of \mathbb{R}^{d+1} . On D_+ we define $\phi : \phi(t, u) := g^{-1}(t, u) = (f(\cdot, u)^{-1}(t), u)$. Observe that ϕ is a \mathcal{C}^0 -Nash triangular map and $\text{deg}(\phi) = \text{deg}(f)$. Define $\phi(t, u) = (x_1, y)$. We compute :

$$D\phi(t, u) = \begin{pmatrix} \frac{1}{\partial_{x_1} f(x_1, y)} & -\frac{1}{\partial_{x_1} f} \nabla_y f(x_1, y) \\ 0 & Id \end{pmatrix}$$

As $(x_1, y) \in A_+$, we have $|\partial_{x_1} \phi| \leq 1$. Furthermore, we check

$$f \circ \phi(t, u) = t.$$

Therefore, ϕ and $f \circ \phi$ are $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -Nash triangular maps. In order to obtain a resolution, we apply again $YG((1, 0\dots, 0), d+1)$ to D_+ . That gives $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -Nash triangular parametrization $\psi_j : [0, 1]^{d+1} \rightarrow \text{adh}(D_+)$, $j \leq N_+$, such that $N_+, \text{deg}(\psi_j)$ are bounded by a function of $\text{deg}(D_+)$, thus by a function of $\text{deg}(f)$. Moreover

$$|\partial_{x_1}(\phi \circ \psi_j)| = |\partial_{x_1}(\phi)| \cdot |\partial_{x_1}(\psi_j^1)| \leq 1$$

because ψ_j is triangular and

$$|\partial_{x_1}(f \circ \phi \circ \psi_j)| = |\partial_{x_1} \psi_j^1| \leq 1,$$

where $\psi_j := (\psi_j^1, \dots, \psi_j^{d+1})$. The following parametrizations $\phi \circ \psi_j : [0, 1]^{d+1} \mapsto [0, 1]^{d+1}$ are therefore $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -Nash triangular maps such that :

- $adh(A_+) = \bigcup_{j=1}^{N_+} \phi \circ \psi_j([0, 1]^{d+1})$;
- each $f \circ \phi \circ \psi_j$ is a $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -Nash map ;
- $deg(\phi \circ \psi_j)$, $deg(f \circ \phi \circ \psi_j)$ are bounded by a function of $|\alpha|$, d , and $deg(f)$ (See Corollary 3) ;
- $\phi \circ \psi_j /]0, 1[^{d+1}$ is a diffeomorphism, because ϕ and $\psi_j /]0, 1[^{d+1}$ are also diffeomorphism.

Finally, we combine the maps $\phi_1, \dots, \phi_{N_-}$ with the maps $\phi \circ \psi_1, \dots, \phi \circ \psi_{N_+}$, so that we obtain a $(\mathcal{C}^{(1,0,\dots,0)}, 1)$ -resolution of f . The bound on the number of parametrizations is the result of the bounds on N_- and N_+ from the Yomdin-Gromov theorem and of the bounds from the proposition 4. □

Increase of the derivation order : $P4((0\dots 0, s), d) \Rightarrow P4((s+1, 0\dots, 0), d)$ pour $s < r + c - d$

(untill the end, $C(|\alpha|, d)$ are functions of $|\alpha|$ and d)

PROOF :

Remark 2 *By applying induction hypothesis to f , we obtain a $(\mathcal{C}^s, 1)$ -resolution $(\phi_{i,n})_{i=1\dots N_n}$ of f / A_n , with A_n an adapted sequence. Replacing f with the family of maps $(f \circ \phi_{i,n}, \phi_{i,n})$, and by applying lemma 5, we observe that it is enough to show the result for $(\mathcal{C}^s, 1)$ -Nash maps.*

Let $A \subset [0, 1]^d$ be a semi-algebraic open set of \mathbb{R}^d and let $f : A \rightarrow [0, 1]$ be a $(\mathcal{C}^s, 1)$ -Nash map. In this case, we adapt the proof in dimension 1.

We cut up A according to the sign $\frac{\partial^{s+1} f}{\partial x_1^{s+1}} \frac{\partial^{s+2} f}{\partial x_1^{s+2}}$, and we can assume (See corollary 4) that A is a slice of the following form $\{(x_1, y) \in [0, 1] \times A' : \zeta(y) < x_1 < \eta(y)\}$, where $A' \subset [0, 1]^{d-1}$ is a semi-algebraic open set of \mathbb{R}^d and $\zeta, \eta : A' \rightarrow [0, 1]$ are \mathcal{C}^0 -Nash maps.

Applying the estimate (1) obtained in section 6 to the following function $x_1 \mapsto \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1, y)$ (we fix y), we get

$$\left| \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1, y) \right| \leq \frac{2}{|x_1 - \zeta(y)|} \quad (2)$$

ou

$$\left| \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1, y) \right| \leq \frac{2}{|x_1 - \eta(y)|}, \quad (3)$$

suisant le signe de $\frac{\partial^{s+1} f}{\partial x_1^{s+1}} \frac{\partial^{s+2} f}{\partial x_1^{s+2}}$.

The induction hypothesis $P4((0\dots 0, s), d)$ implies the claim $P4((0, \dots, 0, s+2), d-1)$, because $(0\dots 0, s+2), d-1 \ll ((0\dots 0, s), d)$ and the claim $P4((0, \dots, s+2), d-1)$ implies $C6((0, \dots, 0, s+1), d-1)$. Apply $C6((0, \dots, 0, s+1), d-1)$ to (ζ, η) : there exist $(\mathcal{C}^{s+1}, d-1)$ -Nash triangular maps $h : [0, 1]^{d-1} \rightarrow [0, 1]^{d-1}$, of which the images cover $adh(A')$, such that $\zeta \circ h$ and $\eta \circ h$ are $(\mathcal{C}^{s+1}, d-1)$ -Nash maps. Define $\psi : [0, 1] \times [0, 1]^{d-1} \rightarrow adh(A)$,

$$\psi(v_1, w) = (\zeta \circ h(w) \cdot (1 - v_1^2) + \eta \circ h(w) \cdot v_1^2, h(w))$$

ψ is triangular, $\psi /]0, 1[^d$ is a diffeomorphisme and $\|\psi\|_{s+1} \leq 2$.

In the new coordinates $(v_1, v_2 \dots v_d)$, the previous estimates (2) and (3) become, with $w = (v_2, \dots, v_d)$:

$$\left| \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(\psi(v_1, w)) \right| \leq \frac{2}{v_1^2 |\eta \circ h(w) - \zeta \circ h(w)|}$$

Moreover, $\frac{\partial^{s+1}(f \circ \psi)}{\partial v_1^{s+1}}(v_1, w) = (2v_1)^{s+1}(\eta \circ h(w) - \zeta \circ h(w)) \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(\psi(v_1, w)) + R(\eta \circ h(w) - \zeta \circ h(w), v_1, (\frac{\partial^k f}{\partial x_1^k}(\psi(v_1, w)))_{k \leq s})$, where R is a polynomial, which depends only on s . The first part is less than 2^{s-1} . Consider the second part. The map f is a $(\mathcal{C}^s, 1)$ -Nash map, thus $|\frac{\partial^k f}{\partial x_1^k}| \leq 1$, for $k \leq s$; thus $|R(\eta \circ h(w) - \zeta \circ h(w), v_1, (\frac{\partial^k f}{\partial x_1^k}(\psi(v_1, w)))_{k \leq s})|$ is bounded by a function of s , and therefore $|\frac{\partial^{s+1}(f \circ \psi)}{\partial v_1^{s+1}}| \leq C(s, d)$. The derivatives of lower order than s of $f \circ \psi$ are also bounded by a function of s . Thanks to the lemma 3, we can assume that ψ is a $(\mathcal{C}^{s+1}, 1)$ -Nash map and $f \circ \psi$ is a $(\mathcal{C}^{(s+1, 0 \dots 0)}, 1)$ -Nash map.

We check with the lemma 4 and the proposition 2 that the number of cuttings is bounded by a function of d and $deg(f)$. Then by $C6((0, \dots, 0, s+1), d-1)$ the number of parametrizations h and their degree are also bounded by such a function. It follows that the total number of parametrizations ψ is bounded by a function of d and of $deg(f)$. At last, we conclude thanks to the corollary 3, that the same holds for the degree of the parametrizations ψ . □

Control of the following derivative : $P4(\alpha, d) \Rightarrow P4(\alpha + 1, d)$ with $\alpha \neq (0, \dots, 0, s + 1)$

PROOF : According to the remark 2, we can assume that $f :]0, 1[^d \rightarrow [0, 1]$ is a $(\mathcal{C}^\alpha, 1)$ -Nash map.

Define $A_n =]1/n, 1 - 1/n[^d$ and $b_n = 1 - 2/n$.

According to the Tarski's principle (See Corollary 1), $B = \{(x_1, y) \in adh(A_n) : |\frac{\partial^{\alpha+1} f}{\partial x_1^{\alpha+1}}(x_1, y)| = \sup_{t \in [0, 1]} (|\frac{\partial^{\alpha+1} f}{\partial x_1^{\alpha+1}}(t, y)|)\}$ is a emi-algebraic set. We have introduce the concept of adapted sequence, so that the sup above is bounded and attained (recall that f is not supposed analytic near the boundary of A). According to the corollary 4), B is covered by sets $(B_i)_{i=1 \dots N}$, $B_i = \{(x_1, y) \in [0, 1] \times adh(B'_i) : \gamma_i(y) \leq x_1 \leq \sigma_i(y)\}$, where $B'_i \subset [0, 1]^{d-1}$ are semi-algebraic open sets of \mathbb{R}^{d-1} , such that $\bigcup_{i=1}^N adh(B'_i) = [1/n, 1 - 1/n]^{d-1}$ and where $\sigma_i, \gamma_i : B'_i \rightarrow [0, 1]$ are \mathcal{C}^0 -Nash maps. Moreover, we check thanks to the Tarski's principle and the proposition 4 that N and the degree of σ_i are bounded by a function of $deg(f)$ and s . Define $g_i(y) = \frac{\partial^{(\alpha+1)_1} f}{\partial x_1^{(\alpha+1)_1}}(\sigma_i(y), y)$ with $y_1 \in adh(B'_i)$, where $(\alpha + 1)_i$ represent the i^{eme} coordinate of $\alpha + 1$. The induction hypothesis $P4(\alpha, d)$ implies $P4((0, \dots, 0, |\alpha| + 1), d - 1)$ and thus $C6((0 \dots 0, |\alpha|), d - 1)$, which applied to σ_i et g_i gives $(\mathcal{C}^{|\alpha|}, 1)$ -Nash triangular maps $h_{i,k} : [0, 1]^{d-1} \rightarrow [0, 1]^{d-1}$, such that $g_i \circ h_{i,k}$ and $\sigma_i \circ h_{i,k}$ are $(\mathcal{C}^{|\alpha|}, 1)$ -Nash and such that $\bigcup_k h_{i,k}([0, 1]^{d-1}) = adh(B'_i)$.

Then,

$$\frac{\partial^{(\alpha+1)_2, \dots, (\alpha+1)_d}(g_i \circ h_{i,k})}{\partial x^{(\alpha+1)_2, \dots, (\alpha+1)_d}}(y) = \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times \left(\frac{\partial h_{i,k}}{\partial x_2}\right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d}\right)^{(\alpha+1)_d} + R$$

where R is a polynomial of derivatives of order $\preceq \alpha$, and of the derivatives of $h_{i,k}$ and $\sigma_i \circ h_{i,k}$ of order less than $|\alpha|$, R depending only on α . The map $h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$ -Nash map

and by hypothesis f is a $(\mathcal{C}^\alpha, 1)$ -Nash map, so that we have $|R| < C(|\alpha|, d)$.
After all $g_i \circ h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$ -Nash map. Hence we have

$$\left| \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \left(\frac{\partial h_{i,k}}{\partial x_2} \right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d} \right)^{(\alpha+1)_d} \right| \leq \left| \frac{\partial^{((\alpha+1)_2, \dots, (\alpha+1)_d)}(g_i \circ h_{i,k})}{\partial x^{((\alpha+1)_2, \dots, (\alpha+1)_d)}} \right| + |R| < C(|\alpha|, d)$$

Define $\phi_{i,k} : [0, 1]^d \rightarrow [0, 1]^d$,

$$\phi_{i,k}(x_1, y) = (1/n + b_n x_1, h_{i,k}(y))$$

$\phi_{i,k}$ is a $(\mathcal{C}^{\alpha+1}, 1)$ -Nash triangular map and $\phi_{i,k}/]0, 1[^d$ is a diffeomorphism onto its image.
We check the two following points :

- $\frac{\partial^{\alpha+1}(f \circ \phi_{i,k})}{\partial x^{\alpha+1}} = \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times (b_n)^{(\alpha+1)_1} \left(\frac{\partial h_{i,k}}{\partial x_2} \right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d} \right)^{(\alpha+1)_d} + S$ where S is a polynomial of the derivatives of f of order $\beta \preceq \alpha$ (because $h_{i,k}$ is triangular) and of the derivatives of $h_{i,k}$ of order less than $|\alpha|$, S depending only on α . From above we deduce that $|S| < C(|\alpha|, d)$.

Moreover by definition of σ_i , $\left| \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times \left(\frac{\partial h_{i,k}}{\partial x_2} \right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d} \right)^{(\alpha+1)_d} \right| \leq \left| \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times \left(\frac{\partial h_{i,k}}{\partial x_2} \right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d} \right)^{(\alpha+1)_d} \right| < C(|\alpha|, d)$,
thus $\left| \frac{\partial^{\alpha+1}(f \circ \phi_{i,k})}{\partial x^{\alpha+1}} \right| \leq \left| \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times \left(\frac{\partial h_{i,k}}{\partial x_2} \right)^{(\alpha+1)_2} \dots \left(\frac{\partial h_{i,k}}{\partial x_d} \right)^{(\alpha+1)_d} \right| + |S| < C(|\alpha|, d)$

- finally for $\beta \preceq \alpha$, in the expression $\frac{\partial^\beta(f \circ \phi_{i,k})}{\partial x^\beta}$ take part only the derivatives of f of order $\preceq \alpha$, still because of the triangularity of $h_{i,k}$. Hence $\left| \frac{\partial^\beta(f \circ \phi_{i,k})}{\partial x^\beta} \right| < C(|\alpha|, d)$.

The lemma 3 gives us a $(\mathcal{C}^\alpha, 1)$ -resolution of f/A_n .

□

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