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To cite this version:

Raphael Rouquier. q-Schur algebras and complex reflection groups, I. 2007. hal-00008615v2

HAL Id: hal-00008615

https://hal.science/hal-00008615v2

Preprint submitted on 3 Dec 2007

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hal-00008615, version 2 - 3 Dec 2007

q-SCHUR ALGEBRAS AND COMPLEX REFLECTION GROUPS

RAPHAËL ROUQUIER

To Victor Ginzburg, on his fiftieth birthday

Abstract. We show that the category $\mathcal{O}$ for a rational Cherednik algebra of type A is equivalent to modules over a q-Schur algebra (parameter $\notin \frac{1}{2}+\mathbf{Z}$), providing thus character formulas for simple modules. We give some generalization to $B_n(d)$. We prove an "abstract" translation principle. These results follow from the unicity of certain highest weight categories covering Hecke algebras. We also provide a semi-simplicity criterion for Hecke algebras of complex reflection groups and show the isomorphism type of Hecke algebras is invariant under field automorphisms acting on parameters.

1. Introduction

This paper (and its sequel) develops a new aspect of the representation theory of Hecke algebras of complex reflection groups, namely the study of quasi-hereditary covers, analogous to q-Schur algebras in the symmetric groups case. An important point is the existence of a family of such covers: it depends on the choice of "logarithms" of the parameters.

The theory we develop is particularly interesting when the ring of coefficients is not specialized: it blends features of representation theory over $\mathbf{C}$ at roots of unity and features away from roots of unity, where Lusztig's families of characters show up (in that respect, it is a continuation of [Rou1], where combinatorial objects are given homological definitions, which led to generalizations from real to complex reflection groups).

The main idea of this first paper is the unicity of certain types of quasi-hereditary covers. This applies in particular to the category $\mathcal{O}$ of rational Cherednik algebras : we show that, in type A, when the parameter is not in $\frac{1}{2}+\mathbf{Z}$, the category $\mathcal{O}$ is equivalent to the module category of a q-Schur algebra, solving a conjecture of [GGOR]. As a consequence, we obtain character formulas for simple objects of $\mathcal{O}$ in this case. We also obtain a general translation principle for category $\mathcal{O}$ of a Cherednik algebra.

In §3, we introduce a function "c" on the set of irreducible characters of W, with values linear functions of the logarithms of the parameters and we construct an order on the set of irreducible characters of W. This is suggested by [DunOp, Lemma 2.5] ("roots of unity" case) as well as by [Lu1] ("away from roots of unity").

In §4, we develop a general theory of (split) highest weight categories over a commutative ring. This is a categorical version of Cline-Parshall-Scott's integral quasi-hereditary algebras. We study covers of finite dimensional algebras by highest weight categories and consider different levels of "faithfulness". The simplest situation is that of a "double centralizer Theorem". The key results are Proposition 4.42 (deformation principle) and Theorem 4.49 (unicity).

§5 shows that category $\mathcal{O}$ for Cherednik algebras gives a cover of Hecke algebras of complex reflection groups, and that it has the faithfulness property when the rank 1 parabolic Hecke subalgebras are semi-simple. This provides a translation principle for category $\mathcal{O}$. We also give

a simple criterion for semi-simplicity of Hecke algebras in characteristic 0, generalizing the usual property for Coxeter groups and the "one-parameter case", that the algebra is semi-simple if that parameter is not a root of unity (Theorem 3.5). We prove that rescaling the parameters by a positive integer (without affecting "denominators") doesn't change category $\mathcal{O}$, up to equivalence, and that the Hecke algebra is unchanged, up to isomorphism of $\mathbf{C}$-algebras, by field automorphisms acting on parameters. In the last part, we describe blocks with "defect 1".

Finally, in §6, we consider the case $W = B_n(d)$. We show that, for a suitable choice of "logarithms of parameters", the category $\mathcal{O}$ is equivalent to modules over Dipper-James-Mathas q-Schur algebra (Theorem 6.8). Otherwise, we obtain new q-Schur algebras. Putting our work together with Yvonne's [Yv] suggests that the decomposition matrices should be given by Uglov's canonical bases of the level d Fock spaces.

Relations between Kazhdan-Lusztig theory and modular representations at roots of unity have been investigated by various authors [Ge2, GeRou, Ge3, Jac1, Jac2], and [DuPaSc1, DuPaSc2, DuPaSc3], whose "integral" approach influenced our §4. We hope our approach provides some new insight.

The second part will deal with integral aspects, bad primes and dualities and will address the relations between the representation theory "at $t = 0$" of the rational Cherednik algebra and Lusztig's asymptotic Hecke algebra. We will discuss more thoroughly the case of finite Coxeter groups and present a number of conjectures.

I thank Susumu Ariki, Steve Donkin, Karin Erdmann, Pavel Etingof, Victor Ginzburg and Bernard Leclerc for useful discussions.

2. Notations

Let k be a commutative ring and A a k-algebra. We denote by A-mod the category of finitely generated A-modules and by A-proj the category of finitely generated projective A-modules. We write $\otimes$ for $\otimes_k$. Let M be a k-module. We put $M^* = \mathrm{Hom}_k(M, k)$ and given n a non-negative integer, we write $M^{\oplus n}$ for M^n, the direct sum of n copies of M, when there is a risk of confusion.

Let k' be a commutative k-algebra. We put $k'M = k' \otimes M$. We put $k'(A\text{-mod}) = (k'A)\text{-mod}$. We denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of simple A-modules. If $\mathfrak{m}$ is a maximal ideal of k, then we put $M(\mathfrak{m}) = (k/\mathfrak{m})M$, etc... Given B another k-algebra, we write $(A\text{-mod}) \otimes (B\text{-mod})$ for $(A \otimes B)$-mod.

Let $\mathcal{A}$ be an abelian category. We denote by $D^b(\mathcal{A})$ the derived category of bounded complexes of objects of $\mathcal{A}$. We denote by $\mathcal{A}$-proj the full subcategory of $\mathcal{A}$ of projective objects. Given I a set of objects of $\mathcal{A}$, we denote by $\mathcal{A}^I$ the full exact subcategory of $\mathcal{A}$ of I-filtered objects, *i.e.*, objects that have a finite filtration whose successive quotients are isomorphic to objects of I.

Given G a finite group, we denote by $\mathrm{Irr}(G)$ the set of irreducible (complex) characters of G.

Let Λ be a set. Given $\leq_1$ and $\leq_2$ two orders on Λ, we say that $\leq_1$ refines $\leq_2$ if $\lambda \leq_2 \lambda'$ implies $\lambda \leq_1 \lambda'$. Fix an order on Λ. A subset I of Λ is an ideal (resp. a coideal) if $\lambda' \leq \lambda$ (resp. $\lambda \leq \lambda'$) and $\lambda \in I$ imply $\lambda' \in I$. Given $\lambda \in \Lambda$, we put $\Lambda_{<\lambda} = \{\lambda' \in \Lambda | \lambda' < \lambda\}$, etc...

3. Parameters for Hecke algebras

3.1. **Definitions.**

3.1.1. *Hecke algebra.* Let W be a finite reflection group on a complex vector space V. Let $\mathcal{A}$ be the set of reflecting hyperplanes of W and for $H \in \mathcal{A}$, let W_H be the pointwise stabilizer of H in W, let $e_H = |W_H|$, and let o_H be the cardinality of $W(H)$ (=orbit of H under W).

Let $U = \coprod_{H\in\mathcal{A}/W} \mathrm{Irr}(W_H)$. We have a bijection $\mathbf{Z}/e_H \xrightarrow{\sim} \mathrm{Irr}(W_H),\ j \mapsto \det^j_{|W_H}$, and we denote by (H, j) the corresponding element of U. Let $\mathbf{G}_m$ be the multiplicative group over $\mathbf{Z}$. Let $\mathbf{T} = (\mathbf{G}_m)^U$ and $\mathbf{k} = \mathbf{Z}[\mathbf{T}] = \mathbf{Z}[\{\mathbf{x}_u^{\pm 1}\}_{u\in U}]$.

Let $V_{\mathrm{reg}} = V - \bigcup_{H\in\mathcal{A}} H$, let $x_0 \in V_{\mathrm{reg}}$, and let $B_W = \pi_1(V_{\mathrm{reg}}/W, x_0)$ be the braid group of W. Let $\mathbf{H}$ be the Hecke algebra of W over $\mathbf{k}$ [BrMaRou, §4.C], quotient of $\mathbf{k}[B_W]$ by the relations

$$\prod_{0\le j<e_H} (\sigma_H - \mathbf{x}_{H,j}) = 0.$$

There is one such relation for each $H \in \mathcal{A}$. Here, σ_H is an s_H-generator of the monodromy around H, where s_H is the reflection around H with determinant $e^{2i\pi/e_H}$.

In the rest of the paper, we make the following assumption, which is known to hold for all but finitely many irreducible complex reflection groups, for which it is conjectured to be true [BrMaRou, §4.C] (cf [EtRa, §6] for a proof of a weak version of the conjecture, when $\dim V = 2$).

Hypothesis 1. The algebra $\mathbf{H}$ is free over $\mathbf{k}$, of rank $|W|$.

3.1.2. *Specialization.* Let k be a commutative ring. A *parameter* for W is an element $x_. = \{x_u\}$ of $\mathbf{T}(k)$. This is the same data as a morphism of groups $X(\mathbf{T}) \to k^\times$ or a morphism of rings $\mathbf{k} \to k,\ \mathbf{x}_u \mapsto x_u$.

Let $m = \mathrm{lcm}(\{e_H\}_{H\in\mathcal{A}})$ and $\Phi_m(t) \in \mathbf{Z}[t]$ be the m-th cyclotomic polynomial. Let $k_m = \mathbf{Z}[t]/(\Phi_m(t))$. We will identify k_m with its image through the embedding $k_m \to \mathbf{C},\ t \mapsto e^{2i\pi/m}$.

The canonical morphism $\mathbf{k}[B_W] \to \mathbf{k}[W],\ \sigma_H \mapsto s_H$, induces an isomorphism $k_m \otimes_{\mathbf{k}} \mathbf{H} \xrightarrow{\sim} k_m[W]$ where the specialization $\mathbf{k} \to k_m$ is given by $\rho = \{t^{jm/e_H}\}_{(H,j)\in U} \in \mathbf{T}(k_m)$. It is convenient to shift the elements of $\mathbf{T}$ by ρ. We put $\mathbf{q}_{H,j} = \mathbf{x}_{H,j} t^{-jm/e_H} \in k_m[\mathbf{T}]$. Given a specialization $k_m[\mathbf{T}] \to k$, we denote by $q_.$ the image of $\mathbf{q}_.$. We put $\mathbf{H}(q_.) = k\mathbf{H}$.

The algebra $k_m\mathbf{H}$ is a deformation of $k_m[W]$. It follows that $\mathbf{C}(\mathbf{T})\mathbf{H}$ is semi-simple. Let K be a field extension of $\mathbf{C}(\mathbf{T})$ such that $K\mathbf{H}$ is split semi-simple. Let S be a local $\mathbf{C}[\mathbf{T}]$-subalgebra of K, integrally closed in K, and whose maximal ideal contains $\{\mathbf{q}_u - 1\}_{u\in U}$. Then we have a canonical isomorphism $\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(K\mathbf{H})$ ("Tits deformation Theorem"). More generally, let k be a field such that $k\mathbf{H}$ is split semi-simple, together with an integrally closed local $k_m[\frac{1}{|W|}][\mathbf{T}]$-subalgebra S of k whose maximal ideal contains $\{\mathbf{q}_u - 1\}_{u\in U}$. Then we have a canonical isomorphism $\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(k\mathbf{H}), \chi \mapsto \chi_k$.

By [Mal, Corollary 4.8], if the representation V of W is defined over a subfield K_0 of $\mathbf{C}$ and the group of roots of unity in K_0 is finite of order l, then $K_0(\{\mathbf{q}_u^{1/l}\}_{u\in U})$ is a splitting field for $\mathbf{H}$. We choose $S = K_0[\{\mathbf{q}_u^{1/l}\}]_{(\mathbf{q}_u^{1/l}-1)_u}$ to define the bijection $\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(K_0(\{\mathbf{q}_u^{1/l}\}_{u\in U})\mathbf{H})$.

Remark 3.1. One could also work with the smaller coefficient ring $\mathbf{Z}[\{\mathbf{a}_u\}][\{\mathbf{a}_{H,0}^{\pm 1}\}]$ instead of $\mathbf{k}$ and define $\mathbf{H}$ with the relations $\sigma_H^{e_H} + \mathbf{a}_{H,e_H-1}\sigma_H^{e_H-1} + \cdots + \mathbf{a}_{H,0} = 0$.

3.2. **Logarithms of the parameters.**

3.2.1. *Function c.* Let $\mathfrak{t}$ be the Lie algebra of $\mathbf{T}$ over k_m. Let $\{\mathbf{h}_u\}_{u\in U}$ be the basis of $X(\mathfrak{t})$ giving the isomorphism $\sum_u \mathbf{h}_u : \mathfrak{t} \xrightarrow{\sim} k_m^U$ corresponding to the isomorphism $\sum_u \mathbf{q}_u : \mathbf{T}(k_m) \xrightarrow{\sim} \mathbf{G}_m(k_m)^U$. We denote by $\mathfrak{t}_{\mathbf{Z}} = \bigoplus_u \mathbf{h}_u^{-1}(\mathbf{Z})$ the corresponding $\mathbf{Z}$-Lie subalgebra of $\mathfrak{t}$.

Let $\chi \in \mathrm{Irr}(W)$. We put

$$n_{H,j}(\chi) = \frac{o_H e_H}{\chi(1)} \langle \chi_{|W_H}, \det{}^j_{|W_H} \rangle.$$

This is the scalar by which $\sum_{H'\in W(H), w\in W_{H'}} \det(w)^{-j} w$ acts on an irreducible representation of W with character χ. In particular, this is a non-negative integer.

We define a map $\mathbf{c} : \mathrm{Irr}(W) \to X(\mathfrak{t})$ by

$$\chi \mapsto \mathbf{c}_\chi = \sum_{(H,j)\in U} n_{H,j}(\chi) \mathbf{h}_{H,j}.$$

We also put

$$\mathbf{c}'_\chi = \sum_{(H,j)\in U} n_{H,j}(\chi)(\mathbf{h}_{H,j} - \mathbf{h}_{H,0}) = \mathbf{c}_\chi - \sum_{H\in\mathcal{A}/W} o_H e_H \mathbf{h}_{H,0}.$$

So, $\mathbf{c}'_\chi = 0$ if and only if χ is the trivial character.

3.2.2. *Lift.* Let k be a commutative ring and $q_. \in (k^\times)^U$. Let Γ be the subgroup of $k^\times$ generated by $\{q_u\}_{u\in U}$. We denote by Γ_{tor} its subgroup of elements of finite order.

Let $\tilde\Gamma$ be a free abelian group together with a surjective morphism $\exp : \tilde\Gamma \to \Gamma$ and an isomorphism $\mathbf{Z} \xrightarrow{\sim} \ker(\exp)$:

$$0 \to \mathbf{Z} \to \tilde\Gamma \xrightarrow{\exp} \Gamma \to 0.$$

Let us fix an order on $\tilde\Gamma$ with the following properties:

- it extends the natural order on $\mathbf{Z}$
- it is compatible with the group law
- if $x \notin \mathbf{Z}$ and $x > 0$, then $x + n > 0$ for all $n \in \mathbf{Z}$, *i.e.*, the order on $\tilde\Gamma$ induces an order on Γ.

We define the *coarsest* order to be the one given by $x > 0$ if and only if $x \in \mathbf{Z}_{>0}$.

Let $h_. \in \tilde\Gamma \otimes_{\mathbf{Z}} \mathfrak{t}_{\mathbf{Z}}$ with $q_. = \exp(h_.)$: this is the data of $\{h_u\} \in \tilde\Gamma^U$ with $q_u = \exp(h_u)$. To $h_.$ corresponds a morphism $X(\mathfrak{t}) \to \tilde\Gamma$. We denote by $c : \mathrm{Irr}(W) \to \tilde\Gamma$ the map deduced from $\mathbf{c}$.

Let $\pi \in B_W$ be the element given by the loop $t \in [0,1] \mapsto e^{2i\pi t}$. This is a central element of B_W and we denote by T_π its image in $\mathbf{H}$.

We have $\chi_k(T_\pi) = \exp(c_\chi)$ [BrMi, Proposition 4.16]. Cf also [BrMaMi, §1] for a more detailed discussion.

Remark 3.2. Note that given Γ, there exists always $\tilde\Gamma$ as above, when k is a domain: take $\Gamma = \Gamma_{tor} \times L$ with L free and g a generator of Γ_{tor}. Let $\tilde\Gamma = \langle \tilde g \rangle \times L$. Define exp by $\tilde g \mapsto g$ and as the identity on L. The coarsest order on $\tilde\Gamma$ satisfies the conditions above.

Example 3.3. Assume the q_u's are roots of unity and k is a domain. Then, Γ is a finite cyclic group and $\tilde\Gamma$ is free of rank 1. The order on $\tilde\Gamma$ is the coarsest order.

3.2.3. *Order on* $\mathrm{Irr}(W)$. We define now an order on $\mathrm{Irr}(W)$. Let $\chi, \chi' \in \mathrm{Irr}(W)$. We put $\chi > \chi'$ if $c_\chi < c_{\chi'}$ (equivalently, $c'_\chi < c'_{\chi'}$).

3.3. **Change of parameters.**

3.3.1. *Twist.* Let $W^\wedge = \mathrm{Hom}(W, \mathbf{C}^\times)$ be the group of one-dimensional characters of W. We have an isomorphism given by restriction $W^\wedge \xrightarrow{\sim} \left(\prod_{H\in\mathcal{A}} \mathrm{Irr}(W_H)\right)/W$. The group $W^\wedge$ acts by multiplication on U, and this gives an action on $\mathbf{k}_m = k_m \otimes_{\mathbf{Z}} \mathbf{k}$. Let $\xi \in W^\wedge$. The action of ξ on $\mathbf{k}_m$ is given by $\mathbf{q}_{H,j} \mapsto \mathbf{q}_{H,j+r_H}$, where $\xi_{|W_H} = \det_{|W_H}^{r_H}$. It extends to an automorphism of k_m-algebras $\mathbf{k}_m[B_W] \xrightarrow{\sim} \mathbf{k}_m[B_W]$, $\sigma_H \mapsto \xi(s_H)^{-1}\sigma_H$. It induces automorphisms of k_m-algebras $k_m\mathbf{H} \xrightarrow{\sim} k_m\mathbf{H}$ and $k_m[W] \xrightarrow{\sim} k_m[W]$, $w \mapsto \xi(w)^{-1}w$ for $w \in W$.

There is a similar action of ξ on $X(\mathbf{t})$ given by $\mathbf{h}_{H,j} \mapsto \mathbf{h}_{H,j+r_H}$. We denote by θ_ξ these automorphisms induced by ξ. We have $\theta_\xi(\mathbf{c}_\chi) = \mathbf{c}_{\chi\otimes\xi}$.

3.3.2. *Permutation of the parameters.* Consider $G = \prod_{H\in\mathcal{A}/W} \mathfrak{S}(\mathrm{Irr}(W_H)) \subset \mathfrak{S}(U)$. It acts on $\mathbf{T}$, hence on $\mathbf{k}$. Let $g \in G$. We denote by $\mathbf{k}_g$ the ring $\mathbf{k}$ viewed as a $\mathbf{k}$-module by letting $a \in \mathbf{k}$ act by multiplication by $g(a)$. There is an isomorphism of $\mathbf{k}$-algebras $\mathbf{H} \xrightarrow{\sim} \mathbf{k}_g\mathbf{H}$, $\sigma_H \mapsto \sigma_H, a \mapsto g(a)$ for $a \in \mathbf{k}$. The action on $\mathbf{t}$ is given by $h_{H,j} \mapsto h_{H,g(j)} + \frac{g(j)-j}{e_H}$ (we view g as an automorphism of $\{0, \dots, e_H - 1\}$).

Let $K = \mathbf{C}(\{\mathbf{q}_u^{1/l}\}_u)$. We extend the action of G to an action by $\mathbf{C}$-algebra automorphisms on K : the element g sends $\mathbf{q}_{H,j}^{1/l}$ to $\mathbf{q}_{H,g(j)}^{1/l} e^{2i\pi(g(j)-j)/(le_H)}$. We deduce an action (by ring automorphisms) on $K\mathbf{H}$ fixing the image of B_W. The action of G on $\mathrm{Irr}(K\mathbf{H})$ induces an action on $\mathrm{Irr}(W)$.

3.3.3. *Normalization.* Consider a map $f : \mathcal{A}/W \to k^\times$. Let $q'_.$ be given by $q'_{H,j} = f(H)q_{H,j}$. Let k' be k as a ring, but viewed as a $\mathbf{k}$-algebra through $q'_.$. Then, we have an isomorphism of k-algebras $k\mathbf{H} \xrightarrow{\sim} k'\mathbf{H}$, $T_H \mapsto f(H)^{-1}T_H$ (here, T_H is the image of σ_H).

So, up to isomorphism, $k\mathbf{H}$ depends only on $q_.$ modulo the "diagonal" subgroup $(\mathbf{G}_m)^{\mathcal{A}/W}$ of $\mathbf{T}$. In particular every Hecke algebra over k is isomorphic to one where $q_{H,0} = 1$ for all $H \in \mathcal{A}$.

Similarly, consider a map $\tilde{f} : \mathcal{A}/W \to \tilde{\Gamma}$. Put $h'_{H,j} = \tilde{f}(H) + h_{H,j}$. Then, $\mathbf{c}'_{\chi|\mathbf{h}_.=h'_.} = \mathbf{c}'_{\chi|\mathbf{h}_.=h_.}$: we can reduce to the study of the order on $\mathrm{Irr}(W)$ to the case where $h_{H,0} = 0$ for all H.

Remark 3.4. Assume there is $\kappa \in \tilde{\Gamma}$ with $h_{H,j} = 0$ for all H and $j \neq 0$ and $h_{H,0} = \kappa$ ("spetsial case"). Then, $c'_\chi = -\frac{N(\chi)+N(\chi^*)}{\chi(1)}\kappa$, where $N(\chi)$ is the derivative at 1 of the fake degree of χ (cf [BrMi, §4.B]).

Assume furthermore that W is a Coxeter group. Let a_χ (resp. A_χ) be the valuation (resp. the degree) of the generic degree of χ. Then, $\frac{N(\chi)+N(\chi^*)}{\chi(1)} = a_\chi + A_\chi$ [BrMi, 4.21], hence $c'_\chi = -(a_\chi + A_\chi)\kappa$.

3.4. **Semi-simplicity.** Let us close this part with a semi-simplicity criterion for Hecke algebras of complex reflection groups over a field of characteristic 0. It generalizes the classical idea for Coxeter groups, that, in the equal parameters case ($(q_{H,0}, q_{H,1}) = (q, -1)$), the Hecke algebra is semi-simple if q is not a non-trivial root of unity.

Theorem 3.5. *Let $\mathbf{k} \to k$ be a specialization with k a characteristic 0 field. Assume that $\Gamma_{tor} = 1$. Then, $k\mathbf{H}$ is semi-simple.*

The proof uses rational Cherednik algebras and will be given in §5.2.1.

In general, using the action of T_π (cf §3.2.2), we have the following weaker statement:

Proposition 3.6. *Let R be a local commutative noetherian $\mathbf{k}$-algebra with field of fractions K and residue field k. Assume $K\mathbf{H}$ is split semi-simple.*

If χ_K and χ'_K are in the same block of $R\mathbf{H}$, then $c_\chi - c_{\chi'} \in \mathbf{Z}$.

4. Quasi-hereditary covers

4.1. **Integral highest weight categories.** In this part, we define and study highest weight categories over a commutative noetherian ring (extending the classical notion for a field). This matches the definition of split quasi-hereditary algebras [CPS2]. In the case of a local base ring, a different but equivalent approach is given in [DuSc1, §2] (cf also *loc. cit.* for comments on the general case).

4.1.1. *Reminders.* Let k be a commutative noetherian ring. Let A be a finite projective k-algebra (*i.e.*, a k-algebra, finitely generated and projective as a k-module). Let $\mathcal{C} = A$-mod.

Let us recall some basics facts about projectivity.

Let M be a finitely generated k-module. The following assertions are equivalent:

- M is a projective k-module.
- $k_{\mathfrak{m}}M$ is a projective $k_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of k.
- $\mathrm{Tor}_1^k(k/\mathfrak{m}, M) = 0$ for every maximal ideal $\mathfrak{m}$ of k.

Let M be a finitely generated A-module. The following assertions are equivalent:

- M is a projective A-module.
- $k_{\mathfrak{m}}M$ is a projective $k_{\mathfrak{m}}A$-module for every maximal ideal $\mathfrak{m}$ of k.
- M is a projective k-module and $M(\mathfrak{m})$ is a projective $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ of k.
- M is a projective k-module and $\mathrm{Ext}^1_A(M, N) = 0$ for all $N \in \mathcal{C} \cap k$-proj.

We say that a finitely generated A-module M is *relatively k-injective* if it is a projective k-module and $\mathrm{Ext}^1_{\mathcal{C}}(N, M) = 0$ for all $N \in \mathcal{C} \cap k$-proj. So, M is relatively k-injective if and only if M is a projective k-module and M^* is a projective right A-module.

4.1.2. *Heredity ideals and associated modules.*

Definition 4.1. *An ideal J of A is an* indecomposable split heredity ideal [CPS2, Definition 3.1] *if the following conditions hold*

(i) *A/J is projective as a k-module*
(ii) *J is projective as a left A-module*
(iii) *$J^2 = J$*
(iv) *$\mathrm{End}_A(J)$ is Morita equivalent to k.*

Remark 4.2. Note that a split heredity ideal, as defined in [CPS2, Definition 3.1], is a direct sum of indecomposable split heredity ideals, corresponding to the decomposition of the endomorphism ring into a product of indecomposable algebras. Note further that J is a split heredity ideal for A if and only if it is a split heredity ideal for the opposite algebra A^{opp} [CPS2, Corollary 3.4].

Given L an A-module, we denote by

$$\tau_L : L \otimes_{\mathrm{End}_A(L)} \mathrm{Hom}_A(L, A) \to A, l \otimes f \mapsto f(l)$$

the canonical morphism of (A, A)-bimodules.

Given P an A-module, we define similarly $\tau'_{L,P} : L \otimes \mathrm{Hom}_A(L, P) \to P$.

Lemma 4.3. *Let L be a projective A-module. Then, $J = \mathrm{im}\, \tau_L$ is an ideal of A and $J^2 = J$.*

Proof. Since τ_L is a morphism of (A, A)-bimodules, it is clear that J is an ideal of A. Let $E = \mathrm{End}_A(L)$ and $L^\vee = \mathrm{Hom}_A(L, A)$. We have a commutative diagram

$$\begin{tikzcd} L \otimes_E L^\vee \otimes_A L \otimes_E L^\vee \arrow[rr, "l\otimes f\otimes l'\otimes f' \mapsto l\otimes f(-)l'\otimes f'", "\sim"'] \arrow[dr, "\tau_L\otimes\tau_L"'] & & L \otimes_E \mathrm{End}_A(L) \otimes_E L^\vee \arrow[dl, "l\otimes\phi\otimes f' \mapsto f'(\phi(l))"] \\ & A \otimes_A A = A & \end{tikzcd}$$

where the horizontal arrow is an isomorphism since L is projective. The image in A of $L \otimes_E \mathrm{id}_L \otimes_E L^\vee$ is equal to J and the diagram shows it is contained in J^2. □

Lemma 4.4. *Let J be an ideal of A which is projective as a left A-module and such that $J^2 = J$. Let M be an A-module. Then, $\mathrm{Hom}_A(J, M) = 0$ if and only if $JM = 0$.*

Proof. Consider $m \in M$ with $Jm \neq 0$. The morphism of A-modules $J \to M$, $j \mapsto jm$ is not zero. This shows the first implication. The image of a morphism of A-modules $J \to M$ is contained in JM, since $J^2 = J$. This proves the Lemma. □

Lemma 4.5. *Let L be a projective object of $\mathcal{C}$ which is a faithful k-module. The following assertions are equivalent*

(i) *$\tau'_{L,P} : L \otimes \mathrm{Hom}_{\mathcal{C}}(L, P) \to P$ is a split injection of k-modules for all projective objects P of $\mathcal{C}$.*
(ii) *$\tau'_{L,A} : L \otimes \mathrm{Hom}_{\mathcal{C}}(L, A) \to A$ is a split injection of k-modules.*
(iii) *$k \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(L)$ and given P a projective object of $\mathcal{C}$, then there is a subobject P_0 of P such that*
  - *P/P_0 is a projective k-module*
  - *$\mathrm{Hom}_{\mathcal{C}}(L, P/P_0) = 0$ and*
  - *$P_0 \simeq L \otimes U$ for some $U \in k$-proj.*

Proof. The equivalence between (i) and (ii) is clear.

Assume (i). Then, $\tau'_{L,L} : L \otimes \mathrm{End}_A(L) \to L$ is injective. Since it is clearly surjective, it is an isomorphism. Since L is a progenerator for k, we obtain $k \xrightarrow{\sim} \mathrm{End}_A(L)$. Let P be a projective object of $\mathcal{C}$. Let $P_0 = \mathrm{im}\, \tau'_{L,P}$, a direct summand of P as a k-module. The map

$$\mathrm{Hom}_A(L, \tau'_{L,P}) : \mathrm{Hom}_A(L, L \otimes \mathrm{Hom}_A(L, P)) \to \mathrm{Hom}_A(L, P)$$

is clearly surjective, hence $\mathrm{Hom}_A(L, P/P_0) = 0$. This proves (iii).

Assume (iii). Let P be a projective object of $\mathcal{C}$. The canonical map $\mathrm{Hom}_A(L, P_0) \to \mathrm{Hom}_A(L, P)$ is an isomorphism. We have canonical isomorphisms $\mathrm{Hom}_A(L, L\otimes U) \xrightarrow{\sim} \mathrm{End}_A(L)\otimes U \xrightarrow{\sim} U$. So, $\tau'_{L,P}$ is injective with image P_0 and (i) holds. □

Remark 4.6. Note that if k has no non-trivial idempotent, then every non-zero projective k-module is faithful.

Let $M(\mathcal{C})$ be the set of isomorphism classes of projective objects L of $\mathcal{C}$ satisfying the equivalent assertions of Lemma 4.5.

Let $\mathrm{Pic}(k)$ be the group of isomorphism classes of invertible k-modules. Given $F \in \mathrm{Pic}(k)$ and $L \in M(\mathcal{C})$, then $L \otimes F \in M(\mathcal{C})$. This gives an action of $\mathrm{Pic}(k)$ on $M(\mathcal{C})$.

Proposition 4.7. *There is a bijection from $M(\mathcal{C})/\mathrm{Pic}(k)$ to the set of indecomposable split heredity ideals of A given by $L \mapsto \mathrm{im}(\tau_L)$.*

Furthermore, the canonical functor $(A/\operatorname{im}\tau_L)$-mod $\to A$-mod *induces an equivalence between* $(A/\operatorname{im}\tau_L)$-mod *and the full subcategory of* $\mathcal{C}$ *of objects* M *such that* $\operatorname{Hom}_{\mathcal{C}}(L,M)=0$.

Proof. We will prove a more precise statement. We will construct inverse maps α,β between $M(\mathcal{C})$ and the set of isomorphism classes of pairs (J,P), where J is an indecomposable split heredity ideal of A and P is a progenerator for $\operatorname{End}_A(J)$ such that $k\xrightarrow{\sim}\operatorname{End}_{\operatorname{End}_A(J)}(P)$. Here, we say that two pairs (J,P) and (J',P') are isomorphic if $J'=J$ and $P'\simeq P$.

Let $L\in M(\mathcal{C})$, let $J=\operatorname{im}\tau_L$ and let $B=\operatorname{End}_A(J)$. By assumption, A/J is a projective k-module. Also, $J^2=J$ by Lemma 4.3. Note that $\operatorname{Hom}_A(L,A)$ is a faithful projective k-module. Since $L\otimes\operatorname{Hom}_A(L,A)\simeq J$, it follows that J is a projective A-module. Also, $\operatorname{End}_k(\operatorname{Hom}_A(L,A))\xrightarrow{\sim}\operatorname{End}_A(J)$ because $k\xrightarrow{\sim}\operatorname{End}_A(L)$. This gives $\operatorname{Hom}_A(L,A)$ a structure of right B-module. Let $P=\operatorname{Hom}_k(\operatorname{Hom}_A(L,A),k)$. This is a progenerator for B and $k\xrightarrow{\sim}\operatorname{End}_B(P)$. We have obtained a pair $(J,P)=\alpha(L)$ as required.

Consider now a pair (J,P). Let $B=\operatorname{End}_A(J)$. Let $L=J\otimes_B P$, a projective A-module. We have $k\xrightarrow{\sim}\operatorname{End}_B(P)\xrightarrow{\sim}\operatorname{End}_A(L)$. Let $i:J\to A$ be the inclusion map. There is $p\in P$ and $f\in\operatorname{Hom}_B(P,\operatorname{Hom}_A(J,A))$ such that $f(p)=i$. Let $g\in\operatorname{Hom}_A(J\otimes_B P,A)$ be the adjoint map. Given $j\in J$, we have $\tau_{J\otimes_B P}(j\otimes_B p\otimes g)=j$. So, $J\subset\operatorname{im}\tau_L$. Finally, we have $\operatorname{Hom}_A(L,A/J)\xrightarrow{\sim}\operatorname{Hom}_B(P,\operatorname{Hom}_A(J,A/J))$. By Lemma 4.4, we have $\operatorname{Hom}_A(J,A/J)=0$, hence $\operatorname{Hom}_A(L,A/J)=0$. So, $\operatorname{im}(\tau_L)\subset J$, hence $\operatorname{im}(\tau_L)=J$. We have an isomorphism of right B-modules $\operatorname{Hom}_B(P,B)\simeq\operatorname{Hom}_k(P,k)$ by Morita theory. We have $\operatorname{End}_A(J)\xrightarrow{\sim}\operatorname{Hom}_A(J,A)$ since $\operatorname{Hom}_A(J,A/J)=0$. So, we have isomorphism of right B-modules

$$\operatorname{Hom}_A(L,A)\simeq\operatorname{Hom}_B(P,\operatorname{Hom}_A(J,A))\simeq\operatorname{Hom}_B(P,B)\simeq\operatorname{Hom}_k(P,k).$$

We deduce

$$J\simeq J\otimes_B P\otimes\operatorname{Hom}_k(P,k)\xrightarrow{\sim}L\otimes\operatorname{Hom}_A(L,A).$$

Now, $\tau_L:L\otimes\operatorname{Hom}_A(L,A)\to J$ is an isomorphism, since it is a surjection between two isomorphic finitely generated projective k-modules. We have constructed $L=\beta(J,P)\in M(\mathcal{C})$ and we have proved that $\beta\alpha=\mathrm{id}$. Since $\operatorname{Hom}_A(L,A)\otimes_B P\simeq\operatorname{Hom}_k(P,k)\otimes_B P\simeq k$, it follows that $\alpha\beta=\mathrm{id}$.

The last assertion of the Proposition is an immediate consequence of Lemma 4.4. □

Remark 4.8. From the previous Theorem, we see that $(A/\operatorname{im}\tau_L)$-mod is a Serre subcategory of A-mod (*i.e.*, closed under extensions, subobjects and quotients).

Let us now study the relation between projective A-modules and projective (A/J)-modules.

Lemma 4.9. *Let* $L\in M(\mathcal{C})$ *and* $J=\operatorname{im}\tau_L$.

Given $P\in\mathcal{C}$-proj, *then* $\operatorname{im}\tau'_{L,P}=JP$ *and* P/JP *is a projective* A/J*-module.*

Conversely, let $Q\in(A/J)$-proj. *Let* $U\in k$-proj *and* $f:U\to\operatorname{Ext}^1_A(Q,L)$ *be a surjection. Let* $0\to L\otimes U^*\to P\to Q\to 0$ *be the extension corresponding to* f *via the canonical isomorphism* $\operatorname{Hom}_k(U,\operatorname{Ext}^1_A(Q,L))\xrightarrow{\sim}\operatorname{Ext}^1_A(Q,L\otimes U^*)$. *Then,* $P\in\mathcal{C}$-proj.

Proof. The first assertion reduces to the case $P=A$, where it is clear.

Let us now consider the second assertion. It reduces to the case $Q=(A/J)^n$ for some positive integer n. The canonical map $A^n\to(A/J)^n$ factors through $\phi:A^n\to P$. Let $\psi=\phi+\mathrm{can}:A^n\oplus L\otimes U^*\to P$ and $N=\ker\psi$. Then, ψ is surjective and there is an exact sequence of A-modules $0\to N\to J^{\oplus n}\oplus L\otimes U^*\to L\otimes U^*\to 0$. Such a sequence splits, hence

$N \simeq L \otimes V$ for some $V \in k$-proj. By construction, $\mathrm{Ext}^1_A(P, L) = 0$, so $\mathrm{Ext}^1_A(P, N) = 0$. It follows that ψ is a split surjection, hence P is projective. □

The following Lemma shows that $M(\mathcal{C})$ behaves well with respect to base change.

Lemma 4.10. *Let L be an object of $\mathcal{C}$. Let R be a commutative noetherian k-algebra. If $L \in M(\mathcal{C})$, then $RL \in M(R\mathcal{C})$.*

The following assertions are equivalent

(i) $L \in M(\mathcal{C})$.
(ii) $k_{\mathfrak{m}}L \in M(k_{\mathfrak{m}}\mathcal{C})$ *for every maximal ideal* $\mathfrak{m}$ *of* k.
(iii) L *is projective over* k *and* $L(\mathfrak{m}) \in M(\mathcal{C}(\mathfrak{m}))$ *for every maximal ideal* $\mathfrak{m}$ *of* k.

Proof. There is a commutative diagram

$$\begin{tikzcd} RL \otimes_R \mathrm{Hom}_{RA}(RL, RA) \arrow[r, "\tau'_{RL,RA}"] & RA \\ R(L \otimes \mathrm{Hom}_A(L, A)) \arrow[u, "\sim"] \arrow[ur, "R\tau'_{L,A}"'] & \end{tikzcd}$$

This shows the first assertion.

Assume (ii). Since $k_{\mathfrak{m}}L$ is a projective $k_{\mathfrak{m}}A$-module for every $\mathfrak{m}$, it follows that L is projective A-module. We obtain also that $\tau'_{L,A}$ is injective and that its cokernel is projective over k. So, (ii)$\Longrightarrow$(i).

Assume (iii). Then, L is a projective A-module. Also, $\tau'_{L,A}$ is injective. The exact sequence $0 \to L \otimes \mathrm{Hom}_A(L, A) \to L \to \mathrm{coker}\, \tau'_{L,A} \to 0$ remains exact after tensoring by $k/\mathfrak{m}$ for every $\mathfrak{m}$, hence $\mathrm{Tor}^k_1(k/\mathfrak{m}, \mathrm{coker}\, \tau'_{L,A}) = 0$ for all $\mathfrak{m}$, so $\mathrm{coker}\, \tau'_{L,A}$ is projective over k. Hence, (iii)$\Longrightarrow$(i).

Finally, (i)$\Longrightarrow$(ii) and (i)$\Longrightarrow$(iii) are special cases of the first part of the Lemma. □

4.1.3. *Definition.* Let $\mathcal{C}$ be (a category equivalent to) the module category of a finite projective k-algebra A. Let Δ be a finite set of objects of $\mathcal{C}$ together with a poset structure.

Given Γ an ideal of Δ, we denote by $\mathcal{C}[\Gamma]$ the full subcategory of $\mathcal{C}$ of objects M such that $\mathrm{Hom}(D, M) = 0$ for all $D \in \Delta \setminus \Gamma$.

We put $\tilde{\Delta} = \{D \otimes U | D \in \Delta,\ U \in k\text{-proj}\}$. We put the order on $\tilde{\Delta}$ given by $D \otimes U < D' \otimes U'$ if $D < D'$. We also put $\Delta_\otimes = \{D \otimes U | D \in \Delta,\ U \in \mathrm{Pic}(k)\}$.

Definition 4.11. *We say that $(\mathcal{C}, \Delta)$ is a highest weight category over k if the following conditions are satisfied:*

(1) *The objects of Δ are projective over k.*
(2) $\mathrm{End}_{\mathcal{C}}(M) = k$ *for all* $M \in \Delta$.
(3) *Given $D_1, D_2 \in \Delta$, if* $\mathrm{Hom}_{\mathcal{C}}(D_1, D_2) \neq 0$, *then* $D_1 \leq D_2$.
(4) $\mathcal{C}[\emptyset] = 0$.
(5) *Given $D \in \Delta$, there is $P \in \mathcal{C}$-proj and $f : P \to D$ surjective such that* $\ker f \in \mathcal{C}^{\tilde{\Delta}_{>D}}$.

We call Δ the set of *standard objects*.

Let $(\mathcal{C}, \Delta)$ and $(\mathcal{C}', \Delta')$ be two highest weight categories over k. A functor $F : \mathcal{C} \to \mathcal{C}'$ is an *equivalence of highest weight categories* if it is an equivalence of categories and if there is a bijection $\phi : \Delta \xrightarrow{\sim} \Delta'$ and invertible k-modules U_D for $D \in \Delta$ such that $F(D) \simeq \phi(D) \otimes U_D$ for $D \in \Delta$.

When k is a field, this corresponds to the usual concept of a highest category [CPS1]. We leave it to the interested reader to extend the definition to the case where Δ is an infinite set (this will cover representations of reductive groups over $\mathbf{Z}$) and to the non split situation where (2) is relaxed.

Lemma 4.12. *Let $\mathcal{C}$ be the module category of a finite projective k-algebra. Let Δ be a finite set of objects of $\mathcal{C}$ together with a poset structure. Let L be a maximal element of Δ.*

Then, $(\mathcal{C},\Delta)$ is a highest weight category if and only if $L\in M(\mathcal{C})$ and $(\mathcal{C}[\Delta\setminus\{L\}],\Delta\setminus\{L\})$ is a highest weight category.

Proof. Assume $(\mathcal{C},\Delta)$ is a highest weight category. Given $D\in\Delta$, let P_D be a projective object of $\mathcal{C}$ with a surjection $P_D\to D$ whose kernel is in $\mathcal{C}^{\tilde{\Delta}_{>D}}$ (Definition 4.11 (5)). Let $P=\bigoplus_{D\in\Delta}P_D$. Then, P is a progenerator for $\mathcal{C}$ (Definition 4.11 (4)).

By Definition 4.11 (5), L is projective. We deduce that P_D has a submodule $Q_D\simeq L\otimes U_D$ for some $U_D\in k$-proj with $P_D/Q_D\in\mathcal{C}^{\widetilde{(\Delta\setminus\{L\})}_{>D}}$. So, P has a submodule $Q\simeq L\otimes U$ for some $U\in k$-proj with $P/Q\in\mathcal{C}^{\widetilde{\Delta\setminus\{L\}}}\subset\mathcal{C}[\Delta\setminus\{L\}]$. We deduce that $L\in M(\mathcal{C})$. Also, P_D/Q_D is a projective object of $\mathcal{C}[\Delta\setminus\{L\}]$ (Lemma 4.9) and (5) holds for $\mathcal{C}[\Delta\setminus\{L\}]$. So, $(\mathcal{C}[\Delta\setminus\{L\}],\Delta\setminus\{L\})$ is a highest weight category.

Assume now $L\in M(\mathcal{C})$ and $(\mathcal{C}[\Delta\setminus\{L\}],\Delta\setminus\{L\})$ is a highest weight category.

Let $D\in\Delta\setminus\{L\}$ and Q be a projective object of $\mathcal{C}[\Delta\setminus\{L\}]$ as in Definition 4.11 (5). Let $U\in k$-proj and $p:U\to\mathrm{Ext}^1_{\mathcal{C}}(Q,L)$ be a surjection. Via the canonical isomorphism $\mathrm{Hom}_k(U,\mathrm{Ext}^1_{\mathcal{C}}(Q,L))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathcal{C}}(Q,L\otimes U^*)$, this gives an extension $0\to L\otimes U^*\to P\to Q\to 0$. By Lemma 4.9, P is projective (in $\mathcal{C}$). So, (5) holds for $\mathcal{C}$ and $\mathcal{C}$ is a highest weight category. □

Proposition 4.13. *Let $(\mathcal{C},\Delta)$ be a highest weight category. Then,*

- *Given Γ an ideal of Δ, then $(\mathcal{C}[\Gamma],\Gamma)$ is a highest weight category and $\mathcal{C}[\Gamma]$ is the full subcategory of $\mathcal{C}$ with objects the quotients of objects of $\mathcal{C}^{\tilde{\Gamma}}$. This is a Serre subcategory of $\mathcal{C}$.*
- *Given $D_1,D_2\in\Delta$, if $\mathrm{Ext}^i_{\mathcal{C}}(D_1,D_2)\neq 0$ for some i, then $D_1\leq D_2$. Furthermore, $\mathrm{Ext}^i_{\mathcal{C}}(D_1,D_1)=0$ for $i>0$.*
- *Let $P\in\mathcal{C}$-proj and let $\Delta\xrightarrow{\sim}\{1,\dots,n\}$, $\Delta_i\leftrightarrow i$, be an increasing bijection. Then, there is a filtration $0=P_{n+1}\subset P_n\subset\cdots\subset P_1=P$ with $P_i/P_{i+1}\simeq\Delta_i\otimes U_i$ for some $U_i\in k$-proj.*

Proof. By induction, it is sufficient to prove the first assertion in the case where $|\Delta\setminus\Gamma|=1$. It is then given by Lemma 4.12.

Let us now prove the second assertion. Let Ω be a coideal of Δ. Then, every object of $\mathcal{C}^{\tilde{\Omega}}$ has a projective resolution with terms in $\mathcal{C}^{\tilde{\Omega}}$. This shows the first part of the second assertion. The second part follows from the fact that there is a projective P and $f:P\to D_1$ surjective with kernel in $\mathcal{C}^{\tilde{\Omega}}$, with $\Omega=\Delta_{>D_1}$.

The last assertion follows easily by induction on $|\Delta|$ from Lemma 4.12 and its proof. □

Proposition 4.14. *Let k' be a commutative noetherian k-algebra. Let $(\mathcal{C},\Delta)$ be a highest weight category over k. Then $(k'\mathcal{C},k'\Delta)$ is a highest weight category over k' and $(k'\mathcal{C})[k'\Gamma]\simeq k'(\mathcal{C}[\Gamma])$ for all ideals Γ of Δ.*

Proof. Let A be a finite projective k-algebra with an equivalence $\mathcal{C}\xrightarrow{\sim}A$-mod. Let L be a maximal element of Δ. Then, $L\in M(\mathcal{C})$ (Lemma 4.12) and $k'L\in M(k'\mathcal{C})$ (Lemma 4.10).

Let J be the ideal of A corresponding to L. For $\Gamma = \Delta \setminus \{L\}$, we have $\mathcal{C}[\Gamma] \xrightarrow{\sim} (A/J)$-mod, $(k'\mathcal{C})[k'\Gamma] \xrightarrow{\sim} k'(A/J)$-mod, and we deduce that $(k'\mathcal{C})[k'\Gamma] \simeq k'(\mathcal{C}[\Gamma])$.

The Proposition follows by induction on $|\Delta|$ from Lemmas 4.10 and 4.12. □

Testing that $(\mathcal{C}, \Delta)$ is a highest weight category can be reduced to the case of a base field:

Theorem 4.15. *Let $\mathcal{C}$ be the module category of a finite projective k-algebra. Let Δ be a finite poset of objects of $\mathcal{C} \cap k$-proj.*

Then, $(\mathcal{C}, \Delta)$ is a highest weight category if and only if $(\mathcal{C}(\mathfrak{m}), \Delta(\mathfrak{m}))$ is a highest weight category for all maximal ideals $\mathfrak{m}$ of k.

Proof. The first implication is a special case of Proposition 4.14. The reverse implication follows by induction on $|\Delta|$ from Lemmas 4.10 and 4.12. □

4.1.4. *Quasi-hereditary algebras.* Let us recall now the definition of split quasi-hereditary algebras [CPS2, Definition 3.2].

A structure of *split quasi-hereditary algebra* on a finite projective k-algebra A is the data of a poset Λ and of a set of ideals $\mathcal{I} = \{I_\Omega\}_{\Omega \text{ coideal of } \Lambda}$ of A such that

- given $\Omega \subset \Omega'$ coideals of Λ, then $I_\Omega \subset I_{\Omega'}$
- given $\Omega \subset \Omega'$ coideals of Λ with $|\Omega' \setminus \Omega| = 1$, then $I_{\Omega'}/I_\Omega$ is an indecomposable split heredity ideal of A/I_Ω
- $I_\emptyset = 0$ and $I_\Lambda = A$.

The following Theorem shows that notion of highest weight category corresponds to that of split quasi-hereditary algebras.

Theorem 4.16. *Let A be a finite projective k-algebra and let $\mathcal{C} = A$-mod.*

Assume A, together with Λ and $\mathcal{I}$ is a split quasi-hereditary algebra. Given $\lambda \in \Lambda$, let $\Delta(\lambda) \in M((A/I_{\Lambda_{>\lambda}})\text{-mod})$ correspond to $I_{\Lambda_{\geq\lambda}}/I_{\Lambda_{>\lambda}}$. Then, $(\mathcal{C}, \{\Delta(\lambda)\}_{\lambda\in\Lambda})$ is a highest weight category.

Conversely, assume $(\mathcal{C}, \Delta)$ is a highest weight category. Given Ω a coideal of Δ, let $I_\Omega \subset A$ be the annihilator of all objects of $\mathcal{C}[\Delta \setminus \Omega]$. Then, A together with $\{I_\Omega\}_\Omega$ is a split quasi-hereditary algebra and (A/I_Ω)-mod identifies with $\mathcal{C}[\Delta \setminus \Omega]$.

Proof. We prove the first assertion by induction on $|\Lambda|$. Assume A is a split quasi-hereditary algebra. Let $\lambda \in \Lambda$ be maximal and let $\Gamma = \Lambda \setminus \{\lambda\}$. Let $J = I_\lambda$. By Proposition 4.7, we have $\mathcal{C}[\{\Delta(\lambda')\}_{\lambda'\in\Gamma}] \xrightarrow{\sim} (A/J)$-mod. Since A/J is a split quasi-hereditary algebra, it follows by induction that $(\mathcal{C}[\{\Delta(\lambda')\}_{\lambda'\in\Gamma}], \{\Delta(\lambda')\}_{\lambda'\in\Gamma})$ is a highest weight category. By Lemma 4.12, it follows that $(\mathcal{C}, \{\Delta(\beta)\}_{\beta\in\Lambda})$ is a highest weight category.

We prove the second assertion by induction on $|\Delta|$. Let $(\mathcal{C}, \Delta)$ be a highest weight category. Let $\Omega \subset \Omega'$ be coideals of Δ with $|\Omega' \setminus \Omega| = 1$. If $\Omega = \emptyset$, then $\Omega' = \{L\}$ and $L \in M(\mathcal{C})$, hence $I_{\{L\}}$ is an indecomposable split heredity ideal of A (Proposition 4.7). Assume now $\Omega \neq \emptyset$ and let L be a maximal element of Ω. Then, $\mathcal{C}[\Delta \setminus \{L\}] \simeq (A/I_{\{L\}})$-mod (Proposition 4.7). By induction, $I_{\Omega'}/I_\Omega$ is an indecomposable split heredity ideal of A/I_Ω. So, A is a split quasi-hereditary algebra. □

Remark 4.17. Note that, starting from a split quasi-hereditary algebra, we obtain a well defined poset $\Delta_\otimes$, but Δ is not unique, unless $\mathrm{Pic}(k) = 1$.

Remark 4.18. Note that Theorem 4.15 translates, via Theorem 4.16, to a known criterion for split quasi-heredity [CPS2, Theorem 3.3].

4.1.5. *Tilting objects.*

Proposition 4.19. *Let $(\mathcal{C}, \{\Delta(\lambda)\}_{\lambda\in\Lambda})$ be a highest weight category. Then, there is a set $\{\nabla(\lambda)\}_{\lambda\in\Lambda}$ of objects of $\mathcal{C}$, unique up to isomorphism, with the following properties*

- *$(\mathcal{C}^{\mathrm{opp}}, \{\nabla(\lambda)\}_{\lambda\in\Lambda})$ is a highest weight category.*
- *Given $\lambda, \beta \in \Lambda$, then* $\mathrm{Ext}^i_{\mathcal{C}}(\Delta(\lambda), \nabla(\beta)) \simeq \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \beta \\ 0 & \text{otherwise.} \end{cases}$

Proof. Let A be a finite projective k-algebra with A-mod $\xrightarrow{\sim} \mathcal{C}$, together with its structure $\mathcal{I}$ of split quasi-hereditary algebra (Theorem 4.16). Then, A^{opp} together with $\mathcal{I}$ is a split quasi-hereditary algebra [CPS2, Corollary 3.4]. Let $\mathcal{C}^* = A^{\mathrm{opp}}$-mod and $\{\Delta(\lambda^*)\}_{\lambda\in\Lambda}$ be a corresponding set of standard objects.

We have $\mathrm{Ext}^{>0}_A(\Delta(\lambda), \Delta(\beta^*)^*) = 0$ for all β with $\beta \not> \lambda$, since $\Delta(\lambda), \Delta(\beta^*)^* \in (A/I_{>\lambda})$-mod and $\Delta(\lambda)$ is a projective $(A/I_{>\lambda})$-module. Similarly, we have $\mathrm{Ext}^{>0}_{A^{\mathrm{opp}}}(\Delta(\beta^*), \Delta(\lambda)^*) = 0$ if $\lambda \not> \beta$. Since $\mathrm{Ext}^{>0}_A(\Delta(\lambda), \Delta(\beta^*)^*)^* \simeq \mathrm{Ext}^{>0}_{A^{\mathrm{opp}}}(\Delta(\beta^*), \Delta(\lambda)^*)$, we deduce that this vanishes for all λ, β. In the same way, we obtain $\mathrm{Hom}_A(\Delta(\lambda), \Delta(\beta^*)^*) = 0$ for $\beta \neq \lambda$.

Let $\mathfrak{m}$ be a maximal ideal of k. We know that $\mathrm{Hom}_{A(\mathfrak{m})}(\Delta(\lambda)(\mathfrak{m}), \Delta(\lambda^*)(\mathfrak{m})^*) = k/\mathfrak{m}$ (cf eg [CPS1, proof of Theorem 3.11]). Let $U_\lambda = \mathrm{Hom}_A(\Delta(\lambda), \Delta(\lambda^*)^*)$. This is a projective k-module, since $\Delta(\lambda), \Delta(\lambda^*)^* \in (A/I_{>\lambda})$-mod, $\Delta(\lambda^*)^*$ is a projective k-module, and $\Delta(\lambda)$ is a projective $(A/I_{>\lambda})$-module. It follows that U_λ is invertible. Let $\nabla(\lambda) = U_\lambda^* \otimes \Delta(\lambda^*)^*$. Then, $\mathrm{Hom}_A(\Delta(\lambda), \nabla(\lambda)) \simeq k$.

Let us now show the unicity part. Let $\{\nabla'(\lambda)\}_{\lambda\in\Lambda}$ be a set of objects of $\mathcal{C}$ with the same properties. We show by induction that $\nabla'(\lambda) \simeq \nabla(\lambda)$.

Assume this holds for $\lambda > \alpha$. Then, $\{\nabla'(\lambda)^*\}_{\lambda\not>\alpha}$ and $\{\nabla(\lambda)^*\}_{\lambda\not>\alpha}$ are sets of standard objects for a highest weight category structure on $(A/I_{>\alpha})^{\mathrm{opp}}$-mod. The maximality of α shows that $\nabla'(\alpha)^*$ is a projective $(A/I_{>\alpha})^{\mathrm{opp}}$-module, hence it has a filtration $0 = P_n \subset \cdots \subset P_1 = \nabla'(\alpha)^*$ such that $P_i/P_{i+1} \simeq \nabla(\lambda_i)^* \otimes U_i$ for some $U_i \in k$-proj and $\lambda_i \not> \alpha$, as in Proposition 4.13. By assumption, we have $\mathrm{Hom}(\nabla'(\alpha)^*, \Delta(\beta)^* \otimes U) \simeq \mathrm{Hom}(\Delta(\beta), \nabla'(\alpha)) \otimes U = \delta_{\alpha,\beta} \cdot U$ and $\mathrm{Ext}^1(\nabla'(\alpha)^*, \Delta(\beta)^* \otimes U) = 0$, hence there is a unique term in the filtration and $\nabla'(\alpha) \simeq \nabla(\alpha)$. □

We put $\nabla = \{\nabla(\lambda)\}_{\lambda\in\Lambda}$ and $\tilde{\nabla} = \{L \otimes U | L \in \nabla,\ U \in k\text{-proj}\}$.

From Proposition 4.19 and its proof, we deduce:

Proposition 4.20. *Given $\lambda \in \Lambda$, there is a relatively k-injective module I and an injection $g : \nabla(\lambda) \to I$ with* $\operatorname{coker} g \in \mathcal{C}^{\tilde{\nabla}_{>\lambda}}$.

Lemma 4.21. *Let $M \in \mathcal{C} \cap k$-proj. Then, $M \in \mathcal{C}^{\tilde{\Delta}}$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. Similarly, $M \in \mathcal{C}^{\tilde{\nabla}}$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$.*

Proof. The first implication is clear. We show the converse by induction on $|\Lambda|$. Let $M \in \mathcal{C} \cap k$-proj with $\mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$

Let $\lambda \in \Lambda$ be maximal. Let $M_0 = \operatorname{im} \tau_{\Delta(\lambda),M}$, a subobject of M together with a surjective map $f : \Delta(\lambda) \otimes U \to M_0$, where $U = \mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), M) \in k$-proj. Given $\lambda' \neq \lambda$, we have $\mathrm{Hom}_{\mathcal{C}}(M_0, \nabla(\lambda')) = 0$, hence $\mathrm{Ext}^1_{\mathcal{C}}(M/M_0, \nabla(\lambda')) = 0$. We have $M/M_0 \in \mathcal{C}[\{\Delta(\lambda')\}_{\lambda'\neq\lambda}]$, hence $M/M_0 \in \mathcal{C}^{\tilde{\Delta}}$ by induction. So, $\mathrm{Ext}^i_{\mathcal{C}}(M/M_0, \nabla(\lambda')) = 0$ for all $i > 0$ and $\lambda' \in \Lambda$. Consequently, $\mathrm{Ext}^1_{\mathcal{C}}(M_0, \nabla(\lambda')) = 0$ for all $\lambda' \in \Lambda$.

Let $N = \ker f$. We have $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \nabla(\lambda')) = 0$ for $\lambda' \neq \lambda$, hence $\mathrm{Hom}_{\mathcal{C}}(N, \nabla(\lambda')) = 0$ for $\lambda' \neq \lambda$.

By construction, the canonical map $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\lambda) \otimes U) \to \mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), M_0)$ is surjective. So, $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), N) = 0$. Let P be a projective object of $\mathcal{C}$ with a surjection $P \to N$. There is a subobject P_0 of P with $P_0 \simeq \Delta(\lambda) \otimes U'$ for some $U' \in k$-proj and $P/P_0 \in \mathcal{C}^{\widetilde{\{\Delta(\lambda')\}_{\lambda' \neq \lambda}}}$. We obtain a surjection $P/P_0 \to N$. We have $\mathrm{Hom}_{\mathcal{C}}(P/P_0, \nabla(\lambda)) = 0$, hence $\mathrm{Hom}_{\mathcal{C}}(N, \nabla(\lambda)) = 0$. We deduce that $N = 0$, hence $M \in \mathcal{C}^{\tilde{\Delta}}$.

The second statement follows by duality. □

The following Lemma will be useful in §4.2.3.

Lemma 4.22. *Let A be a split quasi-hereditary k-algebra. Let $M \in (A\text{-mod})^{\tilde{\Delta}}$. If $\mathrm{Ext}^1_A(M, N) = 0$ for all $N \in \Delta$, then M is projective.*

Proof. We have $\mathrm{Ext}^1_A(M, N) = 0$ for all $N \in (A\text{-mod})^{\tilde{\Delta}}$. Let $0 \to N \to P \to M \to 0$ be an exact sequence with P projective. Then, N is $\tilde{\Delta}$-filtered (Lemma 4.21), hence $\mathrm{Ext}^1_A(M, N) = 0$ and the sequence splits. □

Recall that the category of perfect complexes for A is the full subcategory of $D^b(A\text{-mod})$ of objects isomorphic to a bounded complex of finitely generated projective A-modules.

Proposition 4.23. *Every object of $\mathcal{C} \cap k$-proj has finite projective dimension. More precisely, a complex of $\mathcal{C}$ that is perfect as a complex of k-modules is also perfect as a complex of $\mathcal{C}$.*

Proof. This is almost [CPS2, Theorem 3.6], whose proof we follow. We show the Proposition by induction on $|\Lambda|$. Consider $\lambda \in \Lambda$ maximal and let J be the ideal of A corresponding to the projective object $L = \Delta(\lambda)$. Note that we have an isomorphism of (A, A)-bimodules $L \otimes L^\vee \xrightarrow{\sim} J$, where $L^\vee = \mathrm{Hom}_A(L, A)$. The exact sequence of (A, A)-bimodules

$$0 \to J \to A \to A/J \to 0$$

induces an exact sequence of functors $A\text{-mod} \to A\text{-mod}$

$$0 \to L \otimes \mathrm{Hom}_A(L, -) \to \mathrm{Id} \to (A/J) \otimes_A - \to 0.$$

Let C be a complex of A-modules. We have a distinguished triangle

$$L \otimes \mathrm{Hom}_A(L, C) \to C \to A/J \otimes^{\mathbf{L}}_A C \rightsquigarrow .$$

Assume C is perfect, viewed as a complex of k-modules. Then, $\mathrm{Hom}_A(L, C)$ is perfect as a complex of k-modules, hence $L \otimes \mathrm{Hom}_A(L, C)$ is perfect as a complex of A-modules. In particular, $A/J \otimes^{\mathbf{L}}_A C$ is an object of $D^b((A/J)\text{-mod})$ that is perfect as a complex of k-modules. By induction, it is perfect as a complex of (A/J)-modules. Since A/J is perfect as a complex of A-modules, it follows that $A/J \otimes^{\mathbf{L}}_A C$ is a perfect complex of A-modules, hence C as well. □

Remark 4.24. Let $\mathcal{T}$ be the full subcategory of $D^b(\mathcal{C})$ of complexes that are perfect as complexes of k-modules. Fix an increasing bijection $\Lambda \xrightarrow{\sim} \{1, \dots, n\}$. Then, $\mathcal{T}$ has a semi-orthogonal decomposition $\mathcal{T} = \langle \Delta_1 \otimes k\text{-perf}, \Delta_2 \otimes k\text{-perf}, \dots, \Delta_n \otimes k\text{-perf} \rangle$. This gives an isomorphism

$$K_0(k\text{-proj})^{\Delta} \xrightarrow{\sim} K_0(\mathcal{T}) = K_0(\mathcal{C}\text{-proj}), \ \{[L_\lambda]\}_{\lambda \in \Lambda} \mapsto \sum_\lambda [\Delta(\lambda) \otimes L]$$

In the isomorphism above, one can replace $\Delta(\lambda)$ by a projective object $P(\lambda)$ as in Definition 4.11, (5), or by $\nabla(\lambda)$, $I(\lambda)$ or $T(\lambda)$. We recover [Do2, Corollary 1.2.g] (case of integral Schur algebras).

Definition 4.25. *An object $T \in \mathcal{C}$ is* tilting *if $T \in \mathcal{C}^{\tilde{\Delta}} \cap \mathcal{C}^{\tilde{\nabla}}$. We denote by $\mathcal{C}$-*tilt *the full subcategory of $\mathcal{C}$ of tilting objects.*

A tilting complex is a perfect complex C with the following properties

- C generates the category of perfect complexes as a full triangulated subcategory closed under taking direct summands and
- $\mathrm{Hom}_{D^b(A\text{-mod})}(C, C[i]) = 0$ for $i \neq 0$.

Note that a tilting module is *not* a tilting complex in general, for the generating property will be missing in general. Nevertheless, there is a tilting module which is a tilting complex, as explained below.

Proposition 4.26. *Let $M \in \mathcal{C}^{\tilde{\Delta}}$. Then, there is $T \in \mathcal{C}$-*tilt *and an injection $i : M \to T$ with* $\operatorname{coker} i \in \mathcal{C}^{\tilde{\Delta}}$.

*Let $\lambda \in \Lambda$. There is $T(\lambda) \in \mathcal{C}$-*tilt *and*

- *an injection $i : \Delta(\lambda) \to T(\lambda)$ with* $\operatorname{coker} i \in \mathcal{C}^{\tilde{\Delta}_{<\lambda}}$*;*
- *a surjection $p : T(\lambda) \to \nabla(\lambda)$ with* $\ker p \in \mathcal{C}^{\tilde{\nabla}_{<\lambda}}$.

*Let $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$. Then, T is a tilting complex. Let $A^r = \mathrm{End}_{\mathcal{C}}(T)$ and $\mathcal{C}^r = A^r$-*mod*. There is an equivalence $\mathbf{R}\,\mathrm{Hom}_{\mathcal{C}}(T, -) : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{C}^r)$. Let $\Lambda^r = \{\lambda^r\}_{\lambda\in\Lambda}$ be the opposite poset to Λ. Let $\Delta(\lambda^r) = \mathbf{R}\,\mathrm{Hom}_{\mathcal{C}}(T, \nabla(\lambda))$. Then, $(\mathcal{C}^r, \{\Delta(\lambda^r)\}_{\lambda^r\in\Lambda^r})$ is a highest weight category.*

Proof. Let us fix an increasing bijection $\Delta \xrightarrow{\sim} \{1, \dots, n\}$, $\Delta_i \leftrightarrow i$. Let $M \in \mathcal{C}^{\tilde{\Delta}}$. We construct by induction an object T with a filtration $0 = T_{n+1} \subset M = T_n \subset \cdots \subset T_0 = T$ such that $T_{i-1}/T_i \simeq \Delta_i \otimes U_i$ for some $U_i \in k$-proj, for $i \leq n$.

Assume T_i is defined for some i, $2 \leq i \leq n$. Let $U_i \in k$-proj with a surjection of k-modules $U_i \to \mathrm{Ext}^1_{\mathcal{C}}(\Delta_i, T_i)$. Via the canonical isomorphism $\mathrm{Hom}_k(U_i, \mathrm{Ext}^1_{\mathcal{C}}(\Delta_i, T_i)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{C}}(\Delta_i \otimes U_i, T_i)$, we obtain an extension

$$0 \to T_i \to T_{i-1} \to \Delta_i \otimes U_i \to 0.$$

By construction, we have $\mathrm{Ext}^1(\Delta_i, T_{i-1}) = 0$, since $\mathrm{Ext}^1_{\mathcal{C}}(\Delta_i, \Delta_i) = 0$ (Proposition 4.13).

We have $T/M, T \in \mathcal{C}^{\tilde{\Delta}}$. By Proposition 4.13, we have $\mathrm{Ext}^1_{\mathcal{C}}(\Delta_i, T) \simeq \mathrm{Ext}^1_{\mathcal{C}}(\Delta_i, T_{i-1}) = 0$. It follows from Lemma 4.21 that T is tilting.

Assume $M = \Delta(\lambda)$. Then, in the construction above, one can replace the bijection $\Delta \xrightarrow{\sim} \{1, \dots, n\}$, $\Delta_i \leftrightarrow i$ by an increasing bijection $\Delta_{<\lambda} \xrightarrow{\sim} \{1, \dots, m\}$, $\Delta_i \leftrightarrow i$, and obtain the same conclusion. This produces a tilting object $T(\lambda)$. It has a $\tilde{\nabla}$-filtration with top term $\nabla(\lambda)$ giving rise to a map $p : T(\lambda) \to \nabla(\lambda)$ as required.

Every object of $\mathcal{C}^{\tilde{\Delta}}$ has finite homological dimension (Proposition 4.23). In particular, T is a perfect complex. We have $\mathrm{Ext}^i_{\mathcal{C}}(T, T) = 0$ for $i \neq 0$ by Lemma 4.21. Let $\mathcal{D}$ be the smallest full triangulated subcategory of $D^b(\mathcal{C})$ containing T and closed under direct summands. By induction, $\mathcal{D}$ contains Δ, hence it contains the projective objects of $\mathcal{C}$. So, T generates the category of perfect complexes and T is a tilting complex.

As a consequence, we have an equivalence $F = \mathbf{R}\,\mathrm{Hom}_{\mathcal{C}}(T, -)$. Note that $\Delta(\lambda^r)$ has non zero homology only in degree 0 by Lemma 4.21 and it is projective over k. Let $P(\lambda^r) = F(T(\lambda))$, an object with homology concentrated in degree 0 and projective. Also, we obtain a surjection $P(\lambda^r) \to \Delta(\lambda^r)$ with kernel filtered by terms $\Delta(\beta^r) \otimes U$ with $U \in k$-proj and $\beta^r > \lambda^r$. This shows that $(\mathcal{C}^r, \{\Delta^r(\lambda)\})$ is a highest weight category. □

The highest weight category $\mathcal{C}^r$ in the Proposition above is called the *Ringel dual* of $\mathcal{C}$.

Proposition 4.27. *Fix a family $\{T(\lambda)\}_{\lambda\in\Lambda}$ as in Proposition 4.26. Then, every tilting object of $\mathcal{C}$ is a direct summand of a direct sum of $T(\lambda)$'s.*

Furthermore, the category $\mathcal{C}^r$ is independent of the choice of the $T(\lambda)$'s, up to equivalence of highest weight categories.

Proof. The first assertion follows using these equivalences from the fact that every projective object of $\mathcal{C}^r$ is a direct summand of a direct sum of $P(\lambda^r)$'s.

Consider another family $\{T'(\lambda)\}$ and the associated $T', \mathcal{C}'^r$. We consider the composite equivalence

$$F : \mathbf{R}\operatorname{Hom}_{\mathcal{C}}(T', -) \circ \mathbf{R}\operatorname{Hom}_{\mathcal{C}}(T, -)^{-1} : D^b(\mathcal{C}^r) \xrightarrow{\sim} D^b(\mathcal{C}'^r).$$

It sends $\Delta(\lambda^r)$ to $\Delta(\lambda'^r)$, hence it sends projective objects to objects with homology only in degree 0, which are projective by Lemma 4.21. So, F restricts to an equivalence of highest weight categories $\mathcal{C}^r \xrightarrow{\sim} \mathcal{C}'^r$. □

Remark 4.28. Note that we don't construct a canonical $T(\lambda)$ (nor a canonical $P(\lambda)$), our construction depends on the choice of projective k-modules mapping onto certain Ext^1's.

Remark 4.29. The theory of tilting modules has been developed by Donkin for algebraic groups over $\mathbf{Z}$, cf [Do1, Remark 1.7].

4.1.6. *Reduction to fields.*

Proposition 4.30. *Let $(\mathcal{C}, \Delta)$ be a highest weight category over k. Let $M \in \mathcal{C} \cap k$-proj. Then, $M \in \mathcal{C}^{\tilde{\Delta}}$ (resp. $M \in \mathcal{C}$-tilt) if and only if the corresponding property holds for $M(\mathfrak{m})$ in $\mathcal{C}(\mathfrak{m})$, for all maximal ideals $\mathfrak{m}$ of k.*

If k is indecomposable, then the same statement holds for the properties of belonging to $\tilde{\Delta}$ or $\Delta_{\otimes}$.

Proof. Given V a k-module, we put $\bar{V} = (k/\mathfrak{m}) \otimes V$. Let C be a bounded complex of projective objects of $\mathcal{C}$ and $N \in \mathcal{C} \cap k$-proj. Then, we have a canonical isomorphism

$$(k/\mathfrak{m}) \operatorname{Hom}_{D^b(A)}(C, N) \xrightarrow{\sim} \operatorname{Hom}_{D^b(\bar{A})}(\bar{C}, \bar{N})$$

(this only needs to be checked for $C = A[i]$, where is it clear). It follows from Proposition 4.23 that we have a canonical isomorphism

$$(k/\mathfrak{m}) \operatorname{Hom}_A(L, N) \xrightarrow{\sim} \operatorname{Hom}_{\bar{A}}(\bar{L}, \bar{N})$$

for $L, N \in \mathcal{C} \cap k$-proj.

Let $M \in \mathcal{C} \cap k$-proj with $M(\mathfrak{m}) \in \mathcal{C}(\mathfrak{m})^{\tilde{\Delta}(\mathfrak{m})}$ for every maximal ideal $\mathfrak{m}$. We show that $M \in \mathcal{C}^{\tilde{\Delta}}$ by induction on the projective dimension of M (which is finite by Proposition 4.23). Let $0 \to L \to P \to M \to 0$ be an exact sequence with P projective. By Lemma 4.21, $L(\mathfrak{m}) \in \mathcal{C}(\mathfrak{m})^{\tilde{\Delta}(\mathfrak{m})}$ for every $\mathfrak{m}$.

By induction, it follows that $L \in \mathcal{C}^{\tilde{\Delta}}$. Let $N \in \nabla$. We have $\operatorname{Ext}^{>0}_{A(\mathfrak{m})}(L(\mathfrak{m}), N(\mathfrak{m})) = 0$. Let $0 \to C^r \to \cdots \to C^0 \to L \to 0$ be a projective resolution. Let $D = 0 \to \operatorname{Hom}_A(C^0, N) \to \operatorname{Hom}_A(C^1, N) \to \cdots \to \operatorname{Hom}_A(C^r, N) \to 0$. We have $H^i(D(\mathfrak{m})) \xrightarrow{\sim} \operatorname{Ext}^i_{A(\mathfrak{m})}(L(\mathfrak{m}), N(\mathfrak{m})) = 0$ for $i > 0$. It follows that the complex D is homotopy equivalent to $H^0(D)$, as a complex of k-modules, and $H^0(D)$ is projective. So, the canonical map $\operatorname{Hom}_A(L, N)(\mathfrak{m}) \to \operatorname{Hom}_{A(\mathfrak{m})}(L(\mathfrak{m}), N(\mathfrak{m}))$ is an isomorphism.

We have a commutative diagram whose horizontal sequences are exact

$$\begin{tikzcd}
\mathrm{Hom}_A(P,N)(\mathfrak{m}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_A(L,N)(\mathfrak{m}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1_A(M,N)(\mathfrak{m}) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{A(\mathfrak{m})}(P(\mathfrak{m}),N(\mathfrak{m})) \arrow[r] & \mathrm{Hom}_{A(\mathfrak{m})}(L(\mathfrak{m}),N(\mathfrak{m})) \arrow[r] & \mathrm{Ext}^1_{A(\mathfrak{m})}(M(\mathfrak{m}),N(\mathfrak{m})) \arrow[r] & 0
\end{tikzcd}$$

We have $\mathrm{Ext}^1_{A(\mathfrak{m})}(M(\mathfrak{m}),N(\mathfrak{m}))=0$ and it follows that $\mathrm{Ext}^1_A(M,N)(\mathfrak{m})=0$. Since $\mathrm{Ext}^1_A(M,N)$ is a finitely generated k-module, it must thus be 0. Lemma 4.21 shows that $M\in\mathcal{C}^{\tilde\Delta}$.

The other statements follow easily. □

Remark 4.31. If k is decomposable, then being in $\tilde\Delta$ cannot be tested locally — only being a sum of objects of $\tilde\Delta$ can be tested locally.

4.2. **Covers.**

4.2.1. *Double centralizer.* Let k be a commutative noetherian ring and A a finite dimensional k-algebra. Let $\mathcal{C}=A$-mod.

Let P be a finitely generated projective A-module, $B=\mathrm{End}_A(P)$, $F=\mathrm{Hom}_A(P,-):A\text{-mod}\to B\text{-mod}$, and $G=\mathrm{Hom}_B(FA,-):B\text{-mod}\to A\text{-mod}$. The canonical isomorphism $\mathrm{Hom}_A(P,A)\otimes_A -\xrightarrow{\sim}\mathrm{Hom}_A(P,-)$ makes F a left adjoint of G. We denote by $\varepsilon:FG\to\mathrm{Id}$ (resp. $\eta:\mathrm{Id}\to GF$) the corresponding unit (resp. counit). Note that ε is an isomorphism.

The following Lemma is immediate.

Lemma 4.32. *Let $M\in A$-mod. The following assertions are equivalent*

- *the map $\eta(M):M\to GFM$ is an isomorphism*
- *F induces an isomorphism $\mathrm{Hom}_A(A,M)\xrightarrow{\sim}\mathrm{Hom}_B(FA,FM)$*
- *M is a direct summand of a module in the image of G.*

We will consider gradually stronger conditions on F.

Lemma 4.32 gives :

Proposition 4.33. *The following assertions are equivalent:*

- *the canonical map of algebras $A\to\mathrm{End}_B(FA)$ is an isomorphism*
- *for all $M\in A$-proj, the map $\eta(M):M\to GFM$ is an isomorphism*
- *the restriction of F to A-proj is fully faithful.*

Let us name this "double centralizer" situation.

Definition 4.34. *We say that (A,P) (or $(A\text{-mod},P)$) is a* cover *of B if the restriction of $\mathrm{Hom}_A(P,-)$ to A-proj is fully faithful. We say also that $(\mathcal{C},F)$ is a* cover *of B-mod.*

Remark 4.35. Let $E=P\otimes_B -:B\text{-mod}\to A\text{-mod}$. This is a left adjoint of F. The canonical map $\mathrm{Id}\to FE$ is an isomorphism. By Morita theory, the following conditions are equivalent:

- $F:A\text{-mod}\to B\text{-mod}$ is an equivalence with inverse $G\simeq E$
- $F:A\text{-mod}\to B\text{-mod}$ is fully faithful
- for all $M\in A$-mod, the map $\eta(M):M\to GFM$ is an isomorphism
- the adjunction map $EFA\to A$ is an isomorphism.

The "cover" property can be checked at closed points:

Proposition 4.36. *Assume k is regular. If $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of k, then (A, P) is a cover of B.*

Proof. Since (A, P) is a cover of B if and only if $(k_{\mathfrak{m}}A, k_{\mathfrak{m}}P)$ is a cover of $k_{\mathfrak{m}}B$ for every maximal ideal $\mathfrak{m}$ of k, we can assume k is local. We prove now the Proposition by induction on the Krull dimension of k. Let π be a regular element of the maximal ideal of k. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}_B(FA) \arrow[r, "\pi"] \arrow[d, equal] & \mathrm{End}_B(FA) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_B(FA, (k/\pi)FA) & \\
& \mathrm{End}_B(FA) \arrow[r, "\pi"] & \mathrm{End}_B(FA) \arrow[r] & \mathrm{End}_B(FA) \otimes k/\pi \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & A \arrow[r, "\pi"] \arrow[u] & A \arrow[r] \arrow[u] & (k/\pi)A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and the canonical map $\mathrm{End}_B(FA) \otimes k/\pi \to \mathrm{Hom}_B(FA, (k/\pi)FA)$ is injective.

By induction, $((k/\pi)A, (k/\pi)P)$ is a cover of $(k/\pi)B$, hence the canonical map $(k/\pi)A \to \mathrm{Hom}_B(FA, (k/\pi)FA)$ is an isomorphism. It follows that the canonical map $\mathrm{End}_B(FA) \otimes k/\pi \to \mathrm{Hom}_B(FA, (k/\pi)FA)$ is an isomorphism, hence the canonical map $(k/\pi)A \to \mathrm{End}_B(FA) \otimes k/\pi$ is an isomorphism as well. By Nakayama's Lemma, we deduce that the canonical map $A \to \mathrm{End}_B(FA)$ is an isomorphism. □

4.2.2. *Faithful covers.* We assume now that we are given a highest weight category structure $(\mathcal{C}, \Delta)$ on $\mathcal{C}$. If $\mathcal{C}$ is a cover of B-mod, we say that it is a *highest weight cover.*

Definition 4.37. *Let i be a non-negative integer. We say that (A, P) (or $(A\text{-mod}, P)$) is an* i-faithful cover *of B if $F = \mathrm{Hom}_A(P, -)$ induces isomorphisms $\mathrm{Ext}^j_A(M, N) \xrightarrow{\sim} \mathrm{Ext}^j_B(FM, FN)$ for all $M, N \in \mathcal{C}^{\tilde{\Delta}}$ and $j \leq i$. We say also that $(\mathcal{C}, F)$ is an* i-cover *of B-mod.*

Remark 4.38. For i big enough, this will force F to be an equivalence, assuming k is a field.

Remark 4.39. Note that the 0-faithfulness assumption is not satisfied in Soergel's theory on category $\mathcal{O}$ for a complex semi-simple Lie algebra, cf already the case of $\mathfrak{sl}_2$.

Proposition 4.40. *The following assertions are equivalent:*

(1) *$(\mathcal{C}, F)$ is a 0-faithful cover of B-mod.*
(2) *for all $M \in \mathcal{C}^{\tilde{\Delta}}$, the map $\eta(M) : M \to GFM$ is an isomorphism*
(3) *every object of $\mathcal{C}^{\tilde{\Delta}}$ is in the image of G*
(4) *for all $T \in \mathcal{C}$-tilt, the map $\eta(T) : T \to GFT$ is an isomorphism*
(5) *every object of $\mathcal{C}$-tilt is in the image of G.*

Proof. The equivalence of (1), (2) and (3) and the equivalence of (4) and (5) is given by Lemma 4.32.

Assume (4). Let $M \in \mathcal{C}^{\tilde{\Delta}}$. Then there is an exact sequence

$$0 \to M \to T \to N \to 0$$

where $T \in \mathcal{C}$-tilt $N \in \mathcal{C}^{\tilde{\Delta}}$ (Proposition 4.26). We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\eta(M)"] & T \arrow[r] \arrow[d, "\eta(T)", "\sim"'] & N \arrow[r] \arrow[d, "\eta(N)"] & 0 \\ 0 \arrow[r] & GFM \arrow[r] & GFT \arrow[r] & GFN & \end{tikzcd}$$

It follows that $\eta(M)$ is injective for all $M \in \mathcal{C}^{\tilde{\Delta}}$. In particular, in the diagram above, $\eta(N)$ is injective and it follows that $\eta(M)$ is surjective. So, (4) implies (2) and the converse is trivial. □

Proposition 4.41. *Assume $(\mathcal{C}, F)$ is a 0-faithful cover of B-mod. The following assertions are equivalent:*

(1) *$(\mathcal{C}, F)$ is a 1-faithful cover of B-mod.*
(2) *F restricts to an equivalence of exact categories $\mathcal{C}^{\tilde{\Delta}} \xrightarrow{\sim} (B\text{-mod})^{F\tilde{\Delta}}$ with inverse G*
(3) *for all $M \in \mathcal{C}^{\tilde{\Delta}}$, we have $\mathbf{R}^1 G(FM) = 0$*

Proof. If (2) holds, then $\mathrm{Ext}^1_A(M, N) \xrightarrow{\sim} \mathrm{Ext}^1_B(FM, FN)$ for $M, N \in (A\text{-mod})^{\tilde{\Delta}}$, *i.e.*, (1) holds.

We have $\mathbf{R}^1 G(FM) = \mathrm{Ext}^1_B(FA, FM) = 0$, hence (1)$\Longrightarrow$(3).

Assume (3). Let $X, Y \in \mathcal{C}^{\tilde{\Delta}}$ and let $0 \to FX \to U \to FY \to 0$ be an exact sequence. We have an exact sequence $0 \to GFX \to GU \to GFY \to 0$. Since $X \xrightarrow{\sim} GFX$ and $Y \xrightarrow{\sim} GFY$, we deduce that $GU \in \mathcal{C}^{\tilde{\Delta}}$. Now, $FGU \xrightarrow{\sim} U$, hence $U \in F(\mathcal{C}^{\tilde{\Delta}})$. It follows by induction on the length of a $F\tilde{\Delta}$-filtration that $F(\mathcal{C}^{\tilde{\Delta}}) = (B\text{-mod})^{F\tilde{\Delta}}$. So, (3) implies (2). □

The following very useful result shows that 1-faithful quasi-hereditary covers arise naturally as deformations of 0-faithful covers.

Proposition 4.42. *Assume k is regular and KA is split semi-simple. If $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a 0-faithful cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of k, then (A, P) is a 1-faithful cover of B.*

Proof. As in the proof of Proposition 4.36, we can assume k is local with maximal ideal $\mathfrak{m}$.

Let us first assume k is a discrete valuation ring with uniformizing parameter π. Let $N \in (A\text{-mod})^{\tilde{\Delta}}$. The composition of canonical maps $(k/\pi)N \to (k/\pi)GFN \to G((k/\pi)FN)$ is an isomorphism by assumption and the second map is surjective, hence both maps are isomorphisms. By Nakayama's Lemma, it follows that the canonical map $N \to GFN$ is an isomorphism. Since π is regular for k, FA and FN, the Universal Coefficient Theorem (*i.e.*, the isomorphism $(k/\pi) \otimes^{\mathbf{L}}_k \mathbf{R}G(FN) \xrightarrow{\sim} \mathbf{R}G(FN \otimes^{\mathbf{L}}_k (k/\pi))$ gives an exact sequence

$$0 \to (k/\pi)GFN \to G((k/\pi)FN) \to \mathrm{Tor}^k_1(\mathbf{R}^1 G(FN), k/\pi) \to 0.$$

We deduce that $\mathrm{Tor}^k_1(E, k/\pi) = 0$, where $E = \mathbf{R}^1 G(FN)$, hence E is free over k. Note that the canonical map $N' \to GFN'$ is an isomorphism for every $N' \in KA$-mod, hence KB is Morita-equivalent to KA (cf Remark 4.35). Since KB is semi-simple, E is a torsion k-module and this forces $E = 0$. So, the Proposition holds in the case k has Krull dimension 1.

We prove now the Proposition by induction on the Krull dimension of k. Assume the Krull dimension of k is at least 2. There is $\alpha \in k - \{0\}$ such that $A[\alpha^{-1}]$ is isomorphic to a product of matrix algebras over $k[\alpha^{-1}]$. Then $(k_{\mathfrak{p}}/\mathfrak{p})A$ is split semi-simple, whenever $\mathfrak{p}$ is a prime ideal of k with $\alpha \notin \mathfrak{p}$.

We proceed as in the proof of Proposition 4.36. Let $N \in (A\text{-mod})^{\tilde{\Delta}}$ such that $\mathbf{R}^1 GFN \neq 0$. Let Z be the support of $\mathbf{R}^1 GFN$ in $\mathrm{Spec}\, k$, a non-empty strict closed subvariety. Let $\pi \in \mathfrak{m}$ regular with $Z \cap \mathrm{Spec}(k/\pi) \neq \emptyset$ and $\alpha \notin (\pi)$.

We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & GFN \arrow[r, "\pi"] \arrow[d, equal] & GFN \arrow[r] \arrow[d, equal] & G((k/\pi)FN) & & \\
& GFN \arrow[r, "\pi"] & GFN \arrow[r] & (k/\pi)GFN \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & N \arrow[r, "\pi"] \arrow[u] & N \arrow[r] \arrow[u] & (k/\pi)N \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Since the canonical map $(k/\pi)N \to G((k/\pi)FN)$ is an isomorphism, we deduce that the canonical map $N \to GFN$ is an isomorphism. The Universal Coefficient Theorem gives an exact sequence

$$0 \to (k/\pi)GFN \to G((k/\pi)FN) \to \mathrm{Tor}_1^k(\mathbf{R}^1G(FN), k/\pi) \to 0.$$

It follows that $\mathbf{R}^1G(FN)$ has no π-torsion, which is a contradiction. So, $\mathbf{R}^1GN = 0$. We deduce that (A, P) is a 1-faithful cover of B. □

Remark 4.43. In the proof above, the case of a discrete valuation ring has been treated separately, for if k/π is finite, then there might be no element α as needed. An alternative proof would be to take a faithfully flat extension of k to avoid this problem.

4.2.3. *Unicity of faithful covers.*

Definition 4.44. *We say that two highest weight covers $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ of B are* equivalent *if there is an equivalence of highest weight categories $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ making the following diagram commutative*

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "F"] \arrow[dd, "\sim"'] & \\
& B\text{-mod} \\
\mathcal{C}' \arrow[ur, "F'"'] &
\end{tikzcd}$$

The following result shows that a 1-faithful highest weight cover depends only on $F\Delta_\otimes$:

Proposition 4.45. *Let $(\mathcal{C}, F)$ be a 1-faithful highest weight cover of B.*

Given $M \in F\Delta$, there is a pair $(Y(M), p_M)$ unique up to isomorphism with $Y(M) \in B$-mod and $p_M : Y(M) \to M$ a surjection such that $\ker p_M \in (B\text{-mod})^{F\tilde{\Delta}}$ and $\mathrm{Ext}^1_B(Y(M), F\Delta) = 0$.

Given $N \in \Delta$ with $M = F(N)$ and $q_N : P(N) \to N$ a surjective map with $\ker q_N \in \mathcal{C}^{\tilde{\Delta}}$ and $P(N)$ a projective A-module, then $Y(M) = F(P(N))$ and $p_M = F(q_N)$ satisfy the property above.

Let $Y = \bigoplus_{M \in F\Delta} Y(M)$, $A' = \mathrm{End}_B(Y)$, $\Delta' = \mathrm{Hom}_B(Y, F\Delta)$, and $P' = \mathrm{Hom}_{(A')^{\mathrm{opp}}}(Y, A')$. Then, $(A'\text{-mod}, \Delta')$ is a highest weight category and together with $\mathrm{Hom}_{A'}(P', -)$, this is a 1-faithful highest weight cover of B equivalent to $(\mathcal{C}, F)$.

Proof. The unicity follows from Lemma 4.22, while the construction of $(Y(M), p_M)$ with the required properties is clear.

Note that $\bigoplus_{N\in\Delta} P(N)$ is a progenerator for A, since every object of Δ appears as a quotient. We have a canonical isomorphism $\mathrm{End}_A(\bigoplus_{N\in\Delta} P(N)) \xrightarrow{\sim} A'$, hence an equivalence

$$\mathrm{Hom}_A(\bigoplus_{N\in\Delta} P(N), -) : A\text{-mod} \xrightarrow{\sim} A'\text{-mod}$$

giving rise to the commutative diagram of Definition 4.44. □

We deduce a unicity result.

Corollary 4.46. *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be two 1-faithful highest weight covers of B. Assume $F\Delta_\otimes \simeq F'\Delta'_\otimes$. Then, $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ are equivalent highest weight covers.*

4.2.4. *Deformation.* We assume in §4.2.4 that k is a noetherian domain with field of fractions K.

When $K\mathcal{C}$ is split semi-simple, we can restate the definition of a highest weight category structure on $\mathcal{C}$ as follows (cf [DuPaSc2, Lemma 1.6]):

Proposition 4.47. *Let $\mathcal{C}$ be the module category of a finite projective k-algebra and let Δ be a finite poset of objects of $\mathcal{C} \cap k$-proj. Assume $K\mathcal{C}$ is split semi-simple.*

Then, $(\mathcal{C}, \Delta)$ is a highest weight category if and only if there is a bijection $\mathrm{Irr}(K\mathcal{C}) \xrightarrow{\sim} \Delta$, $E \mapsto \Delta(E)$, such that

- *$K\Delta(E) \simeq E$ for $E \in \mathrm{Irr}(K\mathcal{C})$.*
- *for $E \in \mathrm{Irr}(K\mathcal{C})$, there is a projective module $P(E)$ with a filtration $0 = P_r \subset \cdots \subset P_1 = P(E)$ such that $P_1/P_2 \simeq \Delta(E)$ and $P_j/P_{j+1} \simeq \Delta(F_j) \otimes U_j$ for some $F_j > E$ and $U_j \in k$-proj, for $j \geq 2$.*
- *$\bigoplus_{E\in\mathrm{Irr}(K\mathcal{C})} P(E)$ is a progenerator of $\mathcal{C}$.*

Note that $\Delta_\otimes$ is determined by the order on $\mathrm{Irr}(K\mathcal{C})$: given Q a projective object of $\mathcal{C}$ with $KQ \simeq E \oplus \bigoplus_{F>E} F^{a_F}$ for some integers a_F, then, the image of Q by a surjection $KQ \to E$ is isomorphic to $\Delta(E) \otimes U$ for some $U \in \mathrm{Pic}(k)$.

Let B be a finite projective k-algebra with KB split semi-simple. Let $(\mathcal{C}, F)$ be a 1-faithful highest weight cover of B. Then, $(K\mathcal{C}, KF)$ is a 1-faithful highest weight cover of KB, hence $KF : K\mathcal{C} \to KB$-mod is an equivalence and it induces a bijection $\mathrm{Irr}(K\mathcal{C}) \xrightarrow{\sim} \mathrm{Irr}(KB)$. We will say that $(\mathcal{C}, F)$ is a highest weight cover of B for the order on $\mathrm{Irr}(KB)$ coming from the one on $\mathrm{Irr}(K\mathcal{C})$. Given $I \subset \mathrm{Irr}(KB)$, we denote by $(KB)^I$ the sum of the simple KB-submodules of KB isomorphic to elements of I.

Lemma 4.48. *Let $J \subset I$ be coideals of $\mathrm{Irr}(KB)$ such that no two distinct elements of $I \setminus J$ are comparable. Then,*

$$\big((KB)^I \cap B\big) / \big((KB)^J \cap B\big) \simeq \bigoplus_{E\in I\setminus J} F\Delta(E) \otimes U_E.$$

where $U_E \in k$-proj and $\mathrm{rank}_k U_E = \dim_K E$.

Proof. Recall that $\mathcal{C} = A$-mod, $F = \mathrm{Hom}_A(P, -)$ and $B = \mathrm{End}_A(P)$. Since P is $\tilde{\Delta}$-filtered, there is a filtration $P_0 \subset P_1 \subset P$ with $P_0 \in \mathcal{C}^{\tilde{\Delta}(J)}$, $P_1/P_0 \simeq \bigoplus_{E\in I\setminus J} \Delta(E) \otimes U_E$ for some $U_E \in k$-proj, and $P/P_1 \in \mathcal{C}^{\tilde{\Delta}(\mathrm{Irr}(KB)\setminus I)}$. So, we have a filtration $FP_0 \subset FP_1 \subset FP = B$ and $(KB)^I \cap B = FP_1$ and $(KB)^J \cap B = FP_0$, since FP_0 and FP_1 are direct summands of FP as k-modules. Furthermore, $\dim_K KU_E = \dim_K E$ and we are done. □

We can now show that a 1-faithful highest weight cover is determined by the induced order on $\mathrm{Irr}(KB)$.

Theorem 4.49. *Let B be a finite projective k-algebra such that KB is split semi-simple. Fix two orders, $\leq_1$ and $\leq_2$ on $\mathrm{Irr}(KB)$. Let $(\mathcal{C}_1, F_1)$ and $(\mathcal{C}_2, F_2)$ be 1-faithful highest weight covers of B for the orders $\leq_1$ and $\leq_2$.*

Assume $\leq_1$ is a refinement of $\leq_2$. Then, there is an equivalence $\mathcal{C}_1\text{-mod} \xrightarrow{\sim} \mathcal{C}_2\text{-mod}$ of highest weight covers of B inducing the bijection $\mathrm{Irr}(K\mathcal{C}_1) \xrightarrow{\sim} \mathrm{Irr}(KB) \xrightarrow{\sim} \mathrm{Irr}(K\mathcal{C}_2)$.

Proof. Let $E \in \mathrm{Irr}(KB)$, $I = \mathrm{Irr}(KB)_{\geq_1 E}$ and $J = \mathrm{Irr}(KB)_{>_1 E}$. These are coideals for $\leq_1$ and also for $\leq_2$. Using Lemma 4.48, we obtain $F_1\Delta_1(E) \otimes M_E \simeq F_2\Delta_2(E) \otimes N_E$ where $M_E, N_E \in k\text{-proj}$ and $\mathrm{rank}_k M_E = \mathrm{rank}_k N_E = \dim_K E$. Since $\mathrm{End}_B(F_1\Delta_1(E)) = k$, we deduce that $\mathrm{Hom}_B(F_1\Delta_1(E), F_2\Delta_2(E))$ is an invertible k-module and since

$$F_1\Delta_1(E) \otimes \mathrm{Hom}_B(F_1\Delta_1(E), F_2\Delta_2(E)) \xrightarrow{\sim} F_2\Delta_2(E),$$

we obtain $F_2\Delta_2(E) \xrightarrow{\sim} U_E \otimes F_1\Delta_1(E)$ for some $U_E \in \mathrm{Pic}(k)$. The result follows now from Corollary 4.46. □

Remark 4.50. Let us give a variant of Theorem 4.49. Let $\mathcal{C}_1$ be a 1-faithful highest weight cover of B with associated order $\leq_1$ on $\mathrm{Irr}(KB)$. Let $\leq'$ be an order on $\mathrm{Irr}(KB)$ and $\{S'(E)\}_{E\in\mathrm{Irr}(KB)}$ be a set of B-modules such that given $J' \subset I'$ coideals of $\mathrm{Irr}(KB)$ for $\leq'$ such that no two distinct elements of $I' \setminus J'$ are comparable for $\leq'$, we have

$$\Big((KB)^{I'} \cap B\Big) / \Big((KB)^{J'} \cap B\Big) \simeq \bigoplus_{E\in I'\setminus J'} S'(E) \otimes M_E$$

for some $M_E \in k\text{-proj}$ with $\mathrm{rank}_k M_E = \dim_K E$. Assume $\leq_1$ is a refinement of $\leq'$. Then, given $E \in \mathrm{Irr}(KB)$, we have $S'(E) \xrightarrow{\sim} F_1\Delta_1(E) \otimes U_E$ for some $U_E \in \mathrm{Pic}(k)$.

In particular, if $\mathcal{C}_2$ is a 1-faithful highest weight cover of B with associated order $\leq_2$ and if $\leq_2$ is a refinement of $\leq'$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent highest weight covers.

Remark 4.51. It would interesting to investigate when two 1-faithful highest covers are derived equivalent (cf Conjecture 5.6 for the case of Cherednik algebras). This might be achieved through perverse equivalences (cf [ChRou] and [Rou3, §2.6]).

5. Cherednik's rational algebra

We refer to [Rou2] for a survey of the representation theory of rational Cherednik algebras.

5.1. Category $\mathcal{O}$.

5.1.1. Given $H \in \mathcal{A}$, let $\alpha_H \in V^*$ with $H = \ker \alpha_H$ and let $v_H \in V$ such that $\mathbf{C}v_H$ is a W_H-stable complement to H.

The rational Cherednik algebra $\mathbf{A}$ is the quotient of $\mathbf{C}[\{\mathbf{h}_u\}_{u\in U}] \otimes_{\mathbf{C}} T(V \oplus V^*) \rtimes W$ by the relations

$$[\xi, \eta] = 0 \text{ for } \xi, \eta \in V, \quad [x, y] = 0 \text{ for } x, y \in V^*$$

$$[\xi, x] = \langle \xi, x\rangle + \sum_{H\in\mathcal{A}} \frac{\langle \xi, \alpha_H\rangle\langle v_H, x\rangle}{\langle v_H, \alpha_H\rangle}\gamma_H$$

where

$$\gamma_H = \sum_{w\in W_H-\{1\}} \left(\sum_{j=0}^{e_H-1} \det(w)^{-j}(\mathbf{h}_{H,j} - \mathbf{h}_{H,j-1}) \right) w.$$

Remark 5.1. From the definition in [GGOR, §3.1] one gets to the notations here by putting $h_{H,j} = -k_{H,-j}$ (here, we allow the possibility $h_{H,0} \neq 0$ to make twists by linear characters of W more natural).

From the definitions in [EtGi, p.251], W being a finite Coxeter group, one puts $h_{H,0} = 0$ and $h_{H,1} = c_\alpha$ for H the kernel of the root α.

5.1.2. Let k' be a local commutative noetherian $\mathbf{C}[\{\mathbf{h}_u\}]$-algebra with residue field k.

Let $\mathcal{O}$ be the category of finitely generated $k'\mathbf{A}$-modules that are locally nilpotent for $S(V)$. Given $E \in \mathrm{Irr}(W)$, we put $\Delta(E) = k'\mathbf{A} \otimes_{S(V)\rtimes W} E$ and we denote by $\nabla(E)$ the submodule of $k' \operatorname{Hom}_{S(V^*)\rtimes W}(\mathbf{A}, E)$ of elements that are locally finite for $S(V)$. Let $\Delta = \{\Delta(E)\}_{E\in\mathrm{Irr}(W)}$. We define an order on $\mathrm{Irr}(W)$ by $\chi > \chi'$ if $c_{\chi'} - c_\chi \in \mathbf{Z}_{>0}$.

Theorem 5.2. *$(\mathcal{O}, \Delta)$ is a highest weight category with costandard objects the $\nabla(E)$'s.*

Proof. We know that $\mathcal{O} \simeq R$-mod for some finite projective k'-algebra R [GGOR, Corollary 2.8]. By Theorem 4.15, it suffices to check the highest weight category property for $k\mathcal{O}$: this is given by [GGOR, Theorem 2.19]. □

5.2. **Covers of Hecke algebras.**

5.2.1. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{C}[\{\mathbf{h}_u\}]$ and k' be the completion at $\mathfrak{m}$. We view k' as a $\mathbf{k}$-algebra via $\mathbf{q}_u \mapsto e^{2i\pi\mathbf{h}_u}$.

Let $\hat{\mathfrak{m}}$ be the maximal ideal of k' and $k = k'/\hat{\mathfrak{m}}$. Let $h_\cdot = \{h_u\} \in k^U$ be the image of $\mathbf{h}$. Let $\tilde{\Gamma}$ be the subgroup of k generated by $\mathbf{Z}$ and the h_u's. We have an exact sequence

$$0 \to \mathbf{Z} \to \tilde{\Gamma} \xrightarrow{x\mapsto e^{2i\pi x}} e^{2i\pi\tilde{\Gamma}} \to 0$$

and we are in the setting of §3.2.2, where we choose the coarsest order. In particular, the order on $\mathrm{Irr}(W)$ introduced in §5.1.2 is the same as the one defined in §3.2.3.

There is a functor $\mathrm{KZ} : \mathcal{O} \to k'\mathbf{H}$-mod [GGOR, §5.3] (note that in the definition of the Hecke algebra in [GGOR, §5.2.5], one should read $e^{-2i\pi\mathbf{k}_{H,j}}$ instead of $e^{2i\pi\mathbf{k}_{H,j}}$). By [GGOR, §5.3, 5.4], there is a projective object P_{KZ} of $\mathcal{O}$ and an isomorphism $k'\mathbf{H} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}}(P_{\mathrm{KZ}})$ such that the functor KZ is isomorphic to $\mathrm{Hom}_{\mathcal{O}}(P_{\mathrm{KZ}}, -)$.

Theorem 5.3. *$(k\mathcal{O}, \mathrm{KZ})$ is a highest weight cover of $k\mathbf{H}$.*

Assume $x_{H,j} \neq x_{H,j'}$ for all $H \in \mathcal{A}$ and $j \neq j'$. Then, $(\mathcal{O}, \mathrm{KZ})$ is a 1-faithful highest weight cover of $k'\mathbf{H}$.

Proof. The first statement is [GGOR, Theorem 5.16]. The second statement follows, via Proposition 4.42, from [GGOR, Proposition 5.9]. □

Proposition 5.4. *Assume $\Gamma_{tor} = 1$. Then, $k\mathcal{O}$ and $k\mathbf{H}$ are semi-simple.*

Proof. The semi-simplicities of $k\mathcal{O}$ and of $k\mathbf{H}$ are equivalent (cf Theorem 5.3). The algebra $k\mathbf{H}$ depends only on the h_u's up to shifts by integers. So, in order to prove that $k\mathbf{H}$ is semi-simple, we can assume that the restriction of $t \mapsto e^{2i\pi t}$ to the subgroup Γ_0 of $\mathbf{C}$ generated by the h_u's gives an isomorphism $\Gamma_0 \xrightarrow{\sim} \Gamma$. Then, given $\chi, \chi' \in \mathrm{Irr}(W)$, we have $c_\chi - c_{\chi'} \in \mathbf{Z}$ if and only if

$c_\chi = c_{\chi'}$. In particular, no two distinct elements of $\mathrm{Irr}(W)$ are comparable. So, $\mathcal{O}$ is semi-simple and $k\mathbf{H}$ as well. □

Proof of Theorem 3.5. Without loss of generality, we may assume that k has finite transcendence degree over $\mathbf{Q}$. Then, there is an embedding of k in $\mathbf{C}$ and we can assume $k = \mathbf{C}$. Now, the result follows from Proposition 5.4. □

5.2.2. We denote by $\mathcal{O}(h.)$ the category $k\mathcal{O}$.

From Theorems 4.49 and 5.3, we deduce a translation principle for category $\mathcal{O}$:

Theorem 5.5. *Assume $x_{H,j} \neq x_{H,j'}$ for all $H \in \mathcal{A}$ and $j \neq j'$. Let $\tau \in \mathfrak{t}_{\mathbf{Z}}$ and assume the order on $\mathrm{Irr}(W)$ defined by $h.$ is the same as the one defined by $h. + \tau$. Then, there is an equivalence $\mathcal{O}(h.) \xrightarrow{\sim} \mathcal{O}(h. + \tau)$ of quasi-hereditary covers of $k\mathbf{H}$.*

It would be interesting to describe precisely which τ's satisfy the assumptions of the Theorem.

Conjecture 5.6. *Given any $\tau \in \mathfrak{t}_{\mathbf{Z}}$, then $D^b(\mathcal{O}(h.)) \xrightarrow{\sim} D^b(\mathcal{O}(h. + \tau))$.*

Remark 5.7. Let $\kappa \in \mathbf{Q}_{>0}$ with $\kappa \notin (\frac{1}{e_H}\mathbf{Z}) \setminus \mathbf{Z}$ for all $H \in \mathcal{A}$. Assume $h_{H,j} = 0$ for all $j \neq 0$ and $h_{H,0} = \kappa$, for all H. Let τ be given by $\tau_{H,j} = 0$ for $j \neq 0$ and $\tau_{H,0} = 1$. Then τ satisfies the assumption of the Theorem, *i.e.*, the order defined by $h.$ is the same as the one defined by $\tau + h.$.

We conjecture that, for general W, the shift functor associated to ζ a linear character of W gives an equivalence if $h.$ and $h. + \tau$ define the same order on $\mathrm{Irr}(W)$, where τ is the element corresponding to ζ. Note that shift functors are compatible with the KZ functor, hence when they are equivalences, they are equivalences of highest weight covers of the Hecke algebra as in Theorem 5.5.

When W has type A_{n-1}, Gordon and Stafford proved that the shift functor is an equivalence (parameter $\notin \frac{1}{2} + \mathbf{Z}$) [GoSt1, Proposition 3.16].

Note that equivalences arise also from twists [GGOR, §5.4.1]:

Proposition 5.8. *Let $\zeta \in W^\wedge$. We have an equivalence $\mathcal{O}(h.) \xrightarrow{\sim} \mathcal{O}(\theta_\xi(h.))$ compatible, via* KZ*, with the isomorphism $\theta_\xi : \mathbf{H}(\exp h.) \xrightarrow{\sim} \mathbf{H}(\exp \theta_\xi(h.))$.*

5.2.3. We show now that Hecke algebras do not change, up to isomorphism of $\mathbf{C}$-algebras, by field automorphisms acting on parameters. As a consequence, we show that category $\mathcal{O}$ doesn't change if $h.$ is rescaled by a positive integer, as long as the denominators do not change.

We fix K_0 be a subfield of $\mathbf{C}$ such that the reflection representation V of W is defined over K_0.

Proposition 5.9. *Let $q. \in \mathbf{T}(\mathbf{C})$ with finite order. Then, there exists a K_0-algebra A and an isomorphism of $\mathbf{C}$-algebras $\mathbf{C} \otimes_{K_0} A \simeq \mathbf{H}(q.)$.*

Proof. Let $h. \in \mathbf{Q}^U$ such that $q. = e^{2i\pi h.}$. Consider the category $\mathcal{O}_{K_0}$ for the rational Cherednik algebra defined over K_0, with parameter $h.$. The simple objects of $\mathcal{O}_{K_0}$ remain simple in $\mathbf{C} \otimes_{K_0} \mathcal{O}_{K_0}$, hence there is a projective object P_{KZ,K_0} of $\mathcal{O}_{K_0}$ such that $\mathbf{C} \otimes_{K_0} P_{\mathrm{KZ},K_0} \simeq P_{\mathrm{KZ}}$. Then, $A = \mathrm{End}_{\mathcal{O}_{K_0}}(P_{\mathrm{KZ},K_0})$ satisfies the requirement of the Proposition. □

For Hecke algebras, the next result, which is an immediate consequence of Proposition 5.9, answers positively (over $\mathbf{C}$) a problem raised by Radha Kessar. In type A, the result is due to Chuang and Miyachi [ChMi]. Note that their result covers also fields of positive characteristic.

Theorem 5.10. *Let $q_. \in \mathbf{T}(\mathbf{C})$ with finite order and let σ be an automorphism of $K_0(\{q_u\})/K_0$. Then, we have an isomorphism of $\mathbf{C}$-algebras: $\mathbf{H}(\sigma(q_.)) \simeq \mathbf{H}(q_.)$.*

Remark 5.11. The previous two results can be lifted. We use the notations of the proof of Proposition 5.9.

Let $\mathfrak{m}_0$ be the maximal ideal of $K_0[\{\mathbf{h}_u\}]$ generated by the $\mathbf{h}_u - h_u$ and let k_0 be the completion at $\mathfrak{m}_0$. Let $k_{q_.}$ be the completion of $\mathbf{C}[\{\mathbf{q}_u^{\pm 1}\}]$ at the maximal ideal generated by the $\mathbf{q}_u - q_u$. Let $R = \mathbf{C}[[\{X_u\}]]$ and consider the morphisms of algebras $k_0 \to R$, $\mathbf{h}_u - h_u \mapsto X_u$ and $k_{q_.} \to R$, $\mathbf{q}_u - q_u \mapsto e^{2i\pi X_u}$. As in Proposition 5.9, one shows there is a $K_0[\{\mathbf{h}_u\}]$-algebra A_0 and an isomorphism of R-algebras $R \otimes_{k_0} A_0 \simeq R \otimes_{k_{q_.}} k_{q_.}\mathbf{H}$.

Consider now the setting of Theorem 5.10. We have an isomorphism of R-algebras $R \otimes_{k_{\sigma(q_.)}} k_{\sigma(q_.)}\mathbf{H} \simeq R \otimes_{k_{q_.}} k_{q_.}\mathbf{H}$.

Theorem 5.12. *Let $q_. \in \mathbf{T}(\mathbf{C})$ with finite order and let $r \in \mathbf{Z}_{>0}$ such that there is an automorphism $\sigma \in K_0(\{q_u\})/K_0$ with $\sigma(q_.) = q_.^r$. Assume $x_{H,j} \neq x_{H,j'}$ for all $H \in \mathcal{A}$ and $j \neq j'$.*

Then, there is an equivalence $\mathcal{O}(h_.) \xrightarrow{\sim} \mathcal{O}(rh_.)$, which identifies highest weight covers of $\mathbf{H}(\sigma(q_.)) \simeq \mathbf{H}(q_.)$.

Proof. The order on $\mathrm{Irr}(W)$ induced by $rh_.$ is the same as the order induced by $h_.$. So, via the isomorphism of Remark 5.11, $\mathcal{O}(h_.)$ and $\mathcal{O}(rh_.)$ deform to 1-faithful highest weight covers of the same algebra (Theorem 5.3) and the result follows from Theorem 4.49. □

Let us restate the previous Theorem in the case of Weyl groups and equal parameters, where it takes a simpler form.

Corollary 5.13. *Assume W is a Weyl group, $h_{u,1} = 0$, $h_{u,0} = h$ is constant and $h \in (\frac{1}{d}\mathbf{Z}) \setminus (\frac{1}{2} + \mathbf{Z})$ for some $d \in \mathbf{Z}_{>0}$. Given $r \in \mathbf{Z}_{>0}$ prime to d, there is an equivalence $\mathcal{O}(h) \xrightarrow{\sim} \mathcal{O}(rh)$, which identifies highest weight covers of $\mathbf{H}(q^r) \simeq \mathbf{H}(q)$.*

Finally, let us relate characters. Define

$$\mathbf{eu} = \sum_{b \in \mathcal{B}} b^\vee b + \sum_{H \in \mathcal{A}} \sum_{j=1}^{e_H - 1} \sum_{w \in W_H} (h_{H,j} - h_{H,0}) \det(w)^{-j} w$$

where $\mathcal{B}$ is a basis of V and $\{b^\vee\}_{b \in \mathcal{B}}$ is the dual basis of V^*. Given $M \in \mathcal{O}$ and $a \in \mathbf{C}$, we denote by M_a the generalized a-eigenspace of $\mathbf{eu}$ on M, a finite dimensional vector space. The character of M is an element of $\mathbf{Z}[[t]] \cdot \mathrm{Irr}(W)$ given by $\chi_M(w,t) = \mathrm{Tr}_M(w \cdot t^{\mathbf{eu}}) \in \mathbf{C}[[t]]$ (here, $w \in W$). Given $E \in \mathrm{Irr}(W)$, one has $\chi_{\Delta(E)}(w,t) = \frac{\mathrm{Tr}_E(w) t^{c'_E}}{\det_{V^*}(1 - wt)}$ (cf e.g. [EtCh, §2.1]).

The following result was conjectured by Etingof. It follows immediately from Theorem 5.12.

Proposition 5.14. *With the assumptions of Theorem 5.12, we have*

$$\chi_{L_{rh_.}(E)}(w,t) = \frac{\det_{V^*}(1 - wt^r)}{\det_{V^*}(1 - wt)} \chi_{L_{h_.}(E)}(w, t^r).$$

In particular, $L_{rh_.}(E)$ is finite-dimensional if and only if $L_{h_.}(E)$ is finite-dimensional and when this is the case, we have $\dim L_{rh_.}(E) = r^{\dim V} \dim L_{h_.}(E)$.

5.2.4. We discuss now blocks of "defect 1" and show their structure depends only on their number of simple objects.

Given d a positive integer, recall that a Brauer tree algebra associated to a line with d vertices (and exceptional multiplicity 1) is a $\mathbf{C}$-algebra Morita-equivalent to the principal block of the Hecke algebra of the symmetric group S_d at parameter $(q_0, q_1) = (e^{2i\pi/d}, -1)$ (cf [Ben, §4.18] for a general definition). Consider now

$$\widetilde{\mathrm{Br}}_d = \mathrm{End}_{\mathrm{Br}_d}(\mathrm{Br}_d \oplus \mathbf{C})$$

where $\mathbf{C}$ is the trivial representation of Br_d. This is a quasi-hereditary algebra whose module category is ubiquitous in rational representation theory. It occurs as perverse sheaves on $\mathbf{P}^d$ for the partition $\mathbf{A}^0 \coprod \mathbf{A}^1 \coprod \cdots \coprod \mathbf{A}^d$.

We assume here that the algebra $\mathbf{H}$ is endowed with a symmetrizing form t: here, t in a linear form $\mathbf{H} \to \mathbf{k}$ with $t(ab) = t(ba)$ for all $a, b \in \mathbf{H}$ and the pairing $\mathbf{H} \times \mathbf{H} \to \mathbf{k}$, $(a, b) \mapsto t(ab)$, is perfect. This is well-known to exist for W a finite Coxeter group (take $t(T_w) = \delta_{1w}$) and it is known to exist for the infinite series $G(r, p, n)$ [MalMat].

Let $\mathfrak{n} \subset \hat{\mathfrak{m}}$ be a prime ideal such that $R = k'/\mathfrak{n}$ is a discrete valuation ring. Denote by π a uniformizing parameter for R. Denote by K the field of fractions of R. Its residue field is k.

Let $\mathcal{A}$ be a block of $R\mathcal{O}$. We assume $K\mathcal{A}$ is semi-simple. We denote by $\mathrm{Irr}_{\mathcal{A}}(W)$ the set of $E \in \mathrm{Irr}(W)$ such that $\Delta(E) \in \mathcal{A}$. We denote by B the block of $R\mathbf{H}$ corresponding, via the KZ-functor, to $\mathcal{A}$ [GGOR, Corollary 5.18]. Given $\chi \in \mathrm{Irr}_{\mathcal{A}}(W)$, we denote by $s_\chi \in R$ the Schur element of χ_K: the primitive idempotent of $Z(K\mathbf{H})$ corresponding to χ_K is $s_\chi^{-1} \sum_a \chi_K(a) a^\vee$, where a runs over a basis of $\mathbf{H}$ over $\mathbf{k}$ and $(a^\vee)$ is the dual basis.

The following Theorem gives the structure of blocks with defect one. Theorem 5.15 was known for W of type A_n in case the order of h in $\mathbf{C}/\mathbf{Z}$ is $n+1$ [BerEtGi, Theorem 1.4]. When W is a Coxeter group, the statement about $\mathbf{H}$ in Theorem 5.15 goes back to Geck [Ge1, Theorem 9.6] (in the case of equal parameters, but the proof applies to unequal parameters as well) and we follow part of his proof.

Theorem 5.15. *Let $d = |\mathrm{Irr}_{\mathcal{A}}(W)|$. Assume for every $\chi \in \mathrm{Irr}_{\mathcal{A}}(W)$, we have $\pi^{-1} s_\chi \in R^\times$ ("defect 1"). Then, $>$ is a total order on $\mathrm{Irr}_{\mathcal{A}}(W)$, kB is Morita equivalent to Br_d and $\mathcal{A}$ is equivalent to $\widetilde{\mathrm{Br}}_d$-mod. In particular, if $\chi_1 < \cdots < \chi_d$ are the elements of $\mathrm{Irr}_{\mathcal{A}}(W)$, then, for $n = 1, \ldots, d$, we have*

$$[L(\chi_n)] = \sum_{i=1}^{n} (-1)^{i+n} [\Delta(\chi_i)].$$

Proof. Given $E \in \mathrm{Irr}_{\mathcal{A}}(W)$, we denote by $L(E)$, $\Delta(E)$ and $P(E)$ the corresponding simple, standard and projective objects of $k\mathcal{A}$. Let $\mathrm{Irr}_{\mathcal{A}}(W)^0$ be the set of $E \in \mathrm{Irr}_{\mathcal{A}}(W)$ such that $\mathrm{KZ}(L(E)) \neq 0$.

Brauer's theory of blocks of finite groups of defect 1 carries to $R\mathbf{H}$ (cf [Ge1, Propositions 9.1-9.4] for the case of Weyl groups) and shows that

(i) $[\Delta(E) : L(F)] \in \{0, 1\}$ for $E \in \mathrm{Irr}_{\mathcal{A}}(W)$ and $F \in \mathrm{Irr}_{\mathcal{A}}(W)^0$.
(ii) Given $E \in \mathrm{Irr}_{\mathcal{A}}(W)^0$, there is a unique $F \in \mathrm{Irr}_{\mathcal{A}}(W)$ distinct from E such that $[P(E)] = [\Delta(E)] + [\Delta(F)]$.
(iii) Given $E \in \mathrm{Irr}_{\mathcal{A}}(W)$, then $\mathrm{KZ}(\Delta(E))$ is uniserial.

Let $E \in \mathrm{Irr}_{\mathcal{A}}(W)$. Let $E_1 \neq E_2 \in \mathrm{Irr}_{\mathcal{A}}(W)$ distinct from E and such that $L(E_1)$ and $L(E_2)$ are composition factors of $\Delta(E)$.

Since $[P(E_1)] = [\Delta(E_1)] + [\Delta(E)]$, the reciprocity formula shows that $[\Delta(E_2) : L(E_1)] = 0$. We have $[P(E_2)] = [\Delta(E_2)] + [\Delta(E)]$ by the reciprocity formula, so we have an exact sequence

$$0 \to \Delta(E) \to P(E_2) \to \Delta(E_2) \to 0.$$

Let $\Omega L(E_2)$ be the kernel of a projective cover $P(E_2) \to L(E_2)$. Let M be the kernel of a surjective map $\Delta(E_2) \to L(E_2)$. We have an exact sequence

$$0 \to \Delta(E) \to \Omega L(E_2) \to M \to 0.$$

Since $\mathrm{Hom}(M, L(E_1)) = 0$ and $\mathrm{Hom}(\Delta(E), L(E_1)) = 0$, it follows that $\mathrm{Hom}(\Omega L(E_2), L(E_1)) = 0$, hence $\mathrm{Ext}^1(L(E_2), L(E_1)) = 0$. Similarly, one shows that $\mathrm{Ext}^1(L(E_1), L(E_2)) = 0$.

Let N be the kernel of a surjective map $\Delta(E) \to L(E)$. We have shown that N is semi-simple. Since $\mathrm{KZ}(\Delta(E))$ is uniserial, we deduce that $\mathrm{KZ}(N)$ is simple or 0. So, we have proven

(iv) Given $E \in \mathrm{Irr}_{\mathcal{A}}(W)$, there is at most one $F \in \mathrm{Irr}_{\mathcal{A}}(W)^0$ distinct from E and such that $[\Delta(E) : L(F)] \neq 0$.

The decomposition matrix of B has at most two non-zero entries in each row and in each column. It follows that kB is a Brauer tree algebra associated to a line (cf [Ge1, Theorem 9.6]). In particular, the order $>$ on $\mathrm{Irr}_{\mathcal{A}}(W)$ is a total order. Also, there is a unique $E' \in \mathrm{Irr}(W)$ such that $\mathrm{KZ}(L(E')) = 0$. We have $P(E') = \Delta(E')$ and $\mathrm{KZ}(\Delta(E'))$ is a simple module. Via an appropriate identification of kB-mod with Br_d-mod, it corresponds to the trivial module $\mathbf{C}$. Since $k\mathcal{A} \simeq \mathrm{End}_{k\mathbf{H}}\left(\bigoplus_{E \in \mathrm{Irr}_{\mathcal{A}}(W)} \mathrm{KZ}(P(E))\right)$-mod, it follows that $k\mathcal{A} \simeq \widetilde{\mathrm{Br}}_d$-mod. $\square$

Let us give a concrete application of the previous result. Assume there is $r \in \mathbf{Z}_{>0}$ such that for all u, we have $h_u = \frac{a_u}{r}$ for some $a_u \in \mathbf{Z}$. The Schur element s_χ is the specialization at $\mathbf{q}_u = \mathbf{q}^{a_u}$ of the generic Schur element $\mathbf{s}_\chi$ of χ, where $\mathbf{q} = e^{2i\pi\mathbf{h}}$ and $\pi = \mathbf{h} - \frac{1}{r}$. The assumption "$\pi^{-1}s_\chi \in R^\times$" will be satisfied if and only if the r-th cyclotomic polynomial in $\mathbf{q}$ (over K_0) divides $\mathbf{s}_\chi$ exactly once.

In case $a_u = 1$ for all u and W is a finite Coxeter group, then the principal block satisfies the assumption if and only if $\Phi_r(q)$ divides the Poincaré polynomial of W exactly once. Note that in such a case the other blocks either satisfy the assumption or are simple.

We list now for each finite exceptional irreducible Coxeter group W all simple finite dimensional representations in a block $\mathcal{A}$ of defect 1 and provide their character. We assume $a_u = 1$ for all u. We denote by $\phi_{m,b}$ an irreducible representation of W of dimension m whose first occurrence in $S(V)$ is in degree b. When we use this notation, there is a unique irreducible representation of W with that property. For example, $\phi_{1,0} = \mathbf{C}$ is the trivial representation and $\phi_{\dim V,1} = V$. Computations have been performed in GAP, using the CHEVIE package [CHEVIE]. The blocks are described in [GePfe, Appendix F].

F_4
- $h = 1/12$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/8$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{4,1} + t^2\phi_{1,0}$.

H_3
- $h = 1/10$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/6$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{3,1} + t^2\phi_{1,0}$.

H_4
- $h = 1/30$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/20$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{4,1} + t^2\phi_{1,0}$.
- $h = 1/15$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{4,1} + t^2(\phi_{1,0} + \phi_{9,2}) + t^3\phi_{4,1} + t^4\phi_{1,0}$ and $L(\phi_{4,7}) = t^2\phi_{4,7}$.

- $h = 1/12$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{4,1} + t^2(\phi_{1,0} + \phi_{9,2}) + t^3(\phi_{4,1} + \phi_{16,3}) + t^4(\phi_{1,0} + \phi_{9,2}) + t^5\phi_{4,1} + \phi_{1,0}$.
- $h = 1/10$, $L(\phi_{4,1}) = t^3(\phi_{4,1} + t\phi_{1,0} + t^2\phi_{4,1})$.

E_6
- $h = 1/12$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/9$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{6,1} + t^2\phi_{1,0}$.

E_7
- $h = 1/18$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/14$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{7,1} + t^2\phi_{1,0}$.
- $h = 1/10$, $L(\phi_{7,1}) = t^{9/2}(\phi_{7,1} + t\phi_{1,0} + t^2\phi_{7,1})$.

E_8
- $h = 1/30$, $L(\mathbf{C}) = \phi_{1,0}$.
- $h = 1/24$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{8,1} + t^2\phi_{1,0}$.
- $h = 1/20$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{8,1} + t^2(\phi_{1,0} + \phi_{35,2}) + t^3\phi_{8,1} + t^4\phi_{1,0}$.
- $h = 1/18$, $L(\phi_{8,1}) = t^{5/3}(\phi_{8,1} + t(\phi_{1,0} + \phi_{28,8}) + t^2\phi_{8,1})$.
- $h = 1/15$, $L(\mathbf{C}) = \phi_{1,0} + t\phi_{8,1} + t^2(\phi_{1,0} + \phi_{35,2}) + t^3(\phi_{8,1} + \phi_{112,3}) + t^4(\phi_{1,0} + \phi_{35,2} + \phi_{210,4}) + t^4(\phi_{8,1} + \phi_{112,3}) + t^5(\phi_{1,0} + \phi_{35,2}) + t^6\phi_{8,1} + t^7\phi_{1,0}$ and $L(\phi_{8,1}) = t^2(\phi_{8,1} + t(\phi_{1,0} + \phi_{28,8} + \phi_{35,2}) + t^2(2\phi_{8,1} + \phi_{160,7}) + t^3(\phi_{1,0} + \phi_{28,8} + \phi_{35,2}) + t^4\phi_{8,1})$.
- $h = 1/12$, $L(\phi_{28,8}) = t^5(\phi_{28,8} + t(\phi_{8,1} + \phi_{56,19}) + t^2\phi_{28,8})$.

Remark 5.16. Consider a block $\mathcal{A}$ of defect 1. Then, $\mathcal{A}$ has at most one finite-dimensional simple module. If $\mathcal{A}$ has a finite-dimensional simple module, it is $L(E)$ where c'_E is minimal and we have $|\operatorname{Irr}_{\mathcal{A}}(W)| \geq 1 + \dim V$, since $L(E)$ has a projective resolution over $\mathbf{C}[V]$ of length $|\operatorname{Irr}_{\mathcal{A}}(W)|$.

Remark 5.17. It would be interesting to see if $|\operatorname{Irr}_{\mathcal{A}}(W)| \leq 1+\dim V$ for any block $\mathcal{A}$ satisfying the assumption of Theorem 5.15. Also, in case of equal parameters with order e in $\mathbf{C}/\mathbf{Z}$, is it true that $|\operatorname{Irr}_{\mathcal{A}}(W)| \leq e$?

6. Case $W = B_n(d)$

6.1. **Combinatorics.**

6.1.1. Let W be the complex reflection group of type $B_n(d)$ (*i.e.*, $G(d,1,n)$) for some integers $n, d \geq 1$. This is the subgroup of $\mathrm{GL}_n(\mathbf{C})$ of monomial matrices whose non-zero entries are d-th roots of unity. The subgroup of permutations matrices is the symmetric group $\mathfrak{S}_n$. It is generated by the transpositions $s_1 = (1,2), \ldots, s_{n-1} = (n-1, n)$. Let s_0 be the diagonal matrix with diagonal coefficients $(e^{2i\pi/d}, 1, \ldots, 1)$. Then, W is generated by $s_0, s_1, \ldots, s_{n-1}$. We identify its subgroup of diagonal matrices with the group of functions $\{1, \ldots, n\} \to \mu_d$, where μ_d is the group of d-th roots of unity of $\mathbf{C}$. Let H_i be the reflecting hyperplane of s_i.

A partition of n is a non-increasing sequence (finite or infinite) $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots)$ of non-negative integers with sum n and we write $|\alpha| = n$. We identify two partitions that differ only by zeroes. We denote by ${}^t\alpha$ the transposed partition. We denote by $\mathcal{P}(n)$ the set of partitions of n.

A multipartition of n is a d-tuple of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(d)})$ with $\sum_i |\lambda^{(i)}| = n$. We denote by l_r the largest integer such that $\lambda^{(r)}_{l_r} \neq 0$. We put

$$I_\lambda(r) = \{\sum_{i=1}^{r-1} |\lambda^{(i)}| + 1, \sum_{i=1}^{r-1} |\lambda^{(i)}| + 2, \ldots, \sum_{i=1}^{r} |\lambda^{(i)}|\}.$$

Given $i,j\geq 1$, we put

$$b_{i,j}^{(r)}=\begin{cases}({}^t\lambda^{(r)})_j-i & \text{if } ({}^t\lambda^{(r)})_j>i\\ 0 & \text{otherwise}\end{cases}\quad\text{and } d_{i,j}^{(r)}=\begin{cases}\lambda_i^{(r)}-j & \text{if } \lambda_i^{(r)}>j\\ 0 & \text{otherwise.}\end{cases}$$

We put $\mathfrak{S}_\lambda=\mathfrak{S}_{I_\lambda(1)}\times\cdots\mathfrak{S}_{I_\lambda(d)}$ and $B_\lambda(d)=\mu_d^{\{1,\dots,n\}}\rtimes\mathfrak{S}_\lambda$. We denote by $\mathcal{P}(d,n)$ the set of multipartitions of n.

Given $\alpha\in\mathcal{P}(n)$, we denote by χ_α the corresponding irreducible character of $\mathfrak{S}_n$. Given $\lambda\in\mathcal{P}(d,n)$, we denote by χ_λ the corresponding irreducible character of $B_n(d)$. Let us recall its construction. We denote by $\phi^{(r)}$ the one-dimensional character of $(\mu_d)^{I_\lambda(r)}\rtimes\mathfrak{S}_{I_\lambda(r)}$ whose restriction to $(\mu_d)^{I_\lambda(r)}$ is $\det^{r-1}$ and whose restriction to $\mathfrak{S}_{I_\lambda(r)}$ is trivial. Then,

$$\chi_\lambda=\mathrm{Ind}_{B_\lambda(d)}^{B_n(d)}(\phi^{(1)}\chi_{\lambda^{(1)}}\otimes\cdots\otimes\phi^{(d)}\chi_{\lambda^{(d)}}).$$

Lemma 6.1. *Let $\lambda\in\mathcal{P}(d,n)$. Then, given $0\leq l\leq d-1$, we have*

$$\frac{1}{\chi_\lambda(1)}\langle(\chi_\lambda)_{|\langle s_0\rangle},\det{}^l\rangle=\frac{|\lambda^{(l+1)}|}{n}$$

and

$$\frac{1}{\chi_\lambda(1)}\langle(\chi_\lambda)_{|\langle s_1\rangle},\det\rangle=\frac{1}{2}+\frac{1}{n(n-1)}\sum_r\sum_{i,j}(b_{i,j}^{(r)}-d_{i,j}^{(r)}).$$

Proof. By Frobenius reciprocity and Mackey's formula, we have

$$\mathrm{Res}_{\langle s_0\rangle}\chi_\lambda=\sum_{i=1}^d\frac{|\lambda^{(i)}|(n-1)!}{\prod_{r=1}^d|\lambda^{(r)}|!}\left(\prod_{r=1}^d\chi_{\lambda^{(r)}}(1)\right)\cdot\det{}^{i-1}$$

hence

$$\frac{1}{\chi_\lambda(1)}\mathrm{Res}_{\langle s_0\rangle}\chi_\lambda=\frac{1}{n}\sum_{i=1}^d|\lambda^{(i)}|\cdot\det{}^{i-1}.$$

We have

$$\mathrm{Res}_{\langle s_1\rangle}\chi_\lambda=\sum_{1\leq r\leq d,|\lambda^{(r)}|>1}\frac{|\lambda^{(r)}|(|\lambda^{(r)}|-1)(n-2)!}{\prod_{i=1}^d|\lambda^{(i)}|!}\left(\prod_{1\leq i\leq d,i\neq r}\chi_{\lambda^{(i)}}(1)\cdot\mathrm{Res}_{\mathfrak{S}_2}\chi_{\lambda^{(r)}}\right)+$$
$$+\sum_{r=1}^d\frac{|\lambda^{(r)}|(n-|\lambda^{(r)}|)(n-2)!}{2\prod_{i=1}^d|\lambda^{(i)}|!}\left(\prod_{i=1}^d\chi_{\lambda^{(i)}}(1)\right)(1+\det)$$

hence

$$\frac{1}{\chi_\lambda(1)}\mathrm{Res}_{\langle s_1\rangle}\chi_\lambda=\frac{1}{n(n-1)}\Bigg(\sum_{1\leq r\leq d,|\lambda^{(r)}|>1}\frac{|\lambda^{(r)}|(|\lambda^{(r)}|-1)}{\chi_{\lambda^{(r)}}(1)}\cdot\mathrm{Res}_{\mathfrak{S}_2}\chi_{\lambda^{(r)}}+$$
$$+\sum_{r=1}^d\frac{|\lambda^{(r)}|(n-|\lambda^{(r)}|)}{2}\cdot(1+\det)\Bigg).$$

Now, we have (cf Remark 3.3 and [GePfe, Theorem 10.5.2] for the generic degrees)

$$\frac{1}{\chi_{\lambda^{(r)}}(1)}\langle \mathrm{Res}_{\mathfrak{S}_2}\chi_{\lambda^{(r)}}, \det\rangle = \frac{1}{2} + \frac{1}{|\lambda^{(r)}|(|\lambda^{(r)}|-1)}\sum_{i,j}(b_{i,j}^{(r)} - d_{i,j}^{(r)})$$

and the second result follows. □

6.1.2. Assume $d \neq 1$ and $n \neq 1$. The braid group B_W has generators $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ and relations [BrMaRou, Theorem 2.26]

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1,\ \sigma_0\sigma_1\sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_1\sigma_0 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \geq 1.$$

The canonical morphism $B_W \to W$ is given by $\sigma_i \mapsto s_i$.

Put $\mathbf{x}_i = \mathbf{x}_{H_0,i}$, $\mathbf{x} = \mathbf{x}_{H_1,0}$ and put $\mathbf{x}_{H_1,1} = -1$. Similarly, we will write $h_i = h_{H_0,i}$, $h = h_{H_1,0}$ and assume $h_{H_1,1} = 0$.

The Hecke algebra $\mathbf{H}$ is the quotient of $\mathbf{Z}[\mathbf{q}^{\pm 1}, \mathbf{x}_0^{\pm 1}, \ldots, \mathbf{x}_{d-1}^{\pm 1}][B_W]$ by the ideal generated by $(\sigma_0 - \mathbf{x}_0)(\sigma_0 - \mathbf{x}_1)\cdots(\sigma_0 - \mathbf{x}_{d-1})$ and $(\sigma_i - \mathbf{q})(\sigma_i + 1)$ for $1 \leq i \leq n-1$ (this differs from the algebra $\mathbf{H}$ of §3.1.1 since we have already specialized $\mathbf{x}_{H_1,1}$ to -1).

When $d = 1$, then B_W has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \geq 1.$$

The canonical morphism $B_W \to W$ is given by $\sigma_i \mapsto s_i$.

Put $\mathbf{x} = \mathbf{x}_{H_1,0}$ and assume $\mathbf{x}_{H_1,1} = -1$. Similarly, let $h = h_{H_1,0}$ and assume $h_{H_1,1} = 0$.

The Hecke algebra $\mathbf{H}$ is the quotient of $\mathbf{Z}[\mathbf{q}^{\pm 1}][B_W]$ by the ideal generated by $(\sigma_i - \mathbf{q})(\sigma_i + 1)$ for $1 \leq i \leq n-1$.

When $n = 1$, then B_W is an infinite cyclic group with one generator σ_0. The canonical morphism $B_W \to W$ is given by $\sigma_0 \mapsto s_0$.

Put $\mathbf{x}_i = \mathbf{x}_{H_0,i}$ and $h_i = h_{H_0,i}$.

The Hecke algebra $\mathbf{H}$ is the quotient of $\mathbf{Z}[\mathbf{x}_0^{\pm 1}, \ldots, \mathbf{x}_{d-1}^{\pm 1}][B_W]$ by the ideal generated by $(\sigma_0 - \mathbf{x}_0)(\sigma_0 - \mathbf{x}_1)\cdots(\sigma_0 - \mathbf{x}_{d-1})$.

We denote by T_i the image of σ_i in $\mathbf{H}$. Note that $\mathbf{Q}(\mathbf{q}, \mathbf{x}_0, \ldots, \mathbf{x}_{d-1})\mathbf{H}$ is split semi-simple [ArKo].

From Lemma 6.1, we obtain

Proposition 6.2. *Let $\lambda \in \mathcal{P}(d,n)$. We have*

$$c_{\chi_\lambda} = d\sum_{2\leq r\leq d}|\lambda^{(r)}|(h_{r-1} - h_0) - d\left(\frac{n(n-1)}{2} + \sum_{r,i,j}(b_{i,j}^{(r)} - d_{i,j}^{(r)})\right)h.$$

We put the dominance order $\trianglelefteq$ on $\mathcal{P}(d,n)$: $\lambda \trianglelefteq \mu$ if

$$\sum_{i=1}^{r-1}|\lambda^{(i)}| + \sum_{j=1}^{s}|\lambda_j^{(r)}| \leq \sum_{i=1}^{r-1}|\mu^{(i)}| + \sum_{j=1}^{s}|\mu_j^{(r)}|$$

for all $1 \leq r \leq d$ and $s \geq 0$.

Lemma 6.3. *Let $\lambda, \mu \in \mathcal{P}(d,n)$. Then, $\lambda \lhd \mu$ and there is no $\lambda' \in \mathcal{P}(d,n)$ with $\lambda \lhd \lambda' \lhd \mu$ if and only if one (or more) or the following holds :*

(a) *there is $s < d$ with*

- $\mu^{(r)} = \lambda^{(r)}$ *for* $r \neq s, s+1$
- $\mu^{(s)} = (\lambda_1^{(s)}, \dots, \lambda_{l_s}^{(s)}, 1)$
- $\mu_1^{(s+1)} = \lambda_1^{(s+1)} - 1$ *and* $\mu_j^{(s+1)} = \lambda_j^{(s+1)}$ *for* $j > 1$.

(b) *there are* s *and* i *with*
- $\mu^{(r)} = \lambda^{(r)}$ *for* $r \neq s$,
- $\mu_j^{(s)} = \lambda_j^{(s)}$ *for* $j \neq i, i+1$, $\mu_i^{(s)} = \lambda_i^{(s)} + 1$ *and* $\mu_{i+1}^{(s)} = \lambda_{i+1}^{(s)} - 1$.

(c) *there are* s *and* $i < i'$ *with*
- $\mu^{(r)} = \lambda^{(r)}$ *for* $r \neq s$,
- $\mu_j^{(s)} = \lambda_j^{(s)}$ *for* $j \neq i, i'$ *and* $\mu_i^{(s)} - 1 = \mu_{i'}^{(s)} + 1 = \lambda_i^{(s)} = \lambda_{i'}^{(s)}$.

Proof. Assume $\lambda \lhd \mu$ and there is no $\lambda' \in \mathcal{P}(d,n)$ with $\lambda \lhd \lambda' \lhd \mu$. Take s minimal such that $\lambda^{(s)} \neq \mu^{(s)}$.

Assume first that $|\lambda^{(s)}| < |\mu^{(s)}|$. We denote by m_s the largest integer such that $\mu_{m_s}^{(s)} \neq 0$. If $\mu_{m_s}^{(s)} \neq 1$, then $\lambda \lhd \nu \lhd \mu$, where $\nu^{(r)} = \mu^{(r)}$ for $r \neq s$ and $\nu^{(s)} = (\mu_1^{(s)}, \dots, \mu_{m_s-1}^{(s)}, \mu_{m_s}^{(s)} - 1, 1)$, and this is a contradiction. So, $\mu_{m_s}^{(s)} = 1$. Let $\xi \in \mathcal{P}(d,n)$ be given by $\xi^{(r)} = \mu^{(r)}$ for $r \neq s, s+1$, $\xi^{(s)} = (\mu_1^{(s)}, \dots, \mu_{m_s-1}^{(s)})$ and $\xi^{(s+1)} = (\mu_1^{(s+1)} + 1, \mu_2^{(s+1)}, \mu_3^{(s+1)}, \dots)$. Then, $\lambda \unlhd \xi \lhd \mu$. So, $\lambda = \xi$ and we are in the case (a).

Assume now $|\lambda^{(s)}| = |\mu^{(s)}|$. Let $\xi = (\mu^{(1)}, \dots, \mu^{(s)}, \lambda^{(s+1)}, \dots, \lambda^{(d)})$. Then, $\lambda \lhd \xi \unlhd \mu$, hence $\xi = \mu$. Now, it is a classical fact about partitions that (b) or (c) holds (cf e.g. [JamKe, Theorem 1.4.10]).

The other implication is clear. □

Proposition 6.4. *Assume* $h \leq 0$ *and* $h_s - h_{s-1} \geq (1-n)h$ *for* $1 \leq s \leq d-1$.

Let $\lambda, \mu \in \mathcal{P}(d,n)$. *If* $\lambda \unlhd \mu$, *then* $c_{\chi_\lambda} \geq c_{\chi_\mu}$.

Proof. It is enough to prove the Proposition in the case where $\lambda \neq \mu$ and there is no λ' with $\lambda \lhd \lambda' \lhd \mu$. We use the description of Lemma 6.3.

Assume we are in case (a). Then,

$$c_{\chi_\lambda} - c_{\chi_\mu} = d(h_s - h_{s-1}) + dh(l_s + \lambda_1^{(s+1)} - 1).$$

In case (b), we have

$$c_{\chi_\lambda} - c_{\chi_\mu} = -dh(\mu_i^{(s)} - \mu_{i+1}^{(s)})$$

and in case (c), we have

$$c_{\chi_\lambda} - c_{\chi_\mu} = -dh(i' - i + 1).$$

The Proposition follows easily. □

Remark 6.5. One should compare the above order on $\mathcal{P}(d,n)$ depending on h and the h_i's to the order given by Jacon's a-function [Jac3, Definition 4.1] and to the order defined by Yvonne [Yv, §3.3].

6.2. **The "classical" q-Schur algebras.**

6.2.1. We recall here a generalization of Dipper and James' construction (cf [Do3]) of q-Schur algebras for type A_{n-1} (case $d = 1$ below). As a first generalization, q-Schur algebras of type B_n (case $d = 2$ below) have been introduced by Dipper, James, and Mathas [DiJaMa1], and Du and Scott [DuSc2]. The constructions have been then extended by Dipper, James, and Mathas to the complex reflection groups $B_n(d)$ [DiJaMa2].

6.2.2. The subalgebra of $\mathbf{H}$ generated by $T_1, \dots, T_{n-1}$ is the Hecke algebra of $\mathfrak{S}_n$, viewed as a Coxeter group with generating set $(s_1 = (1,2), \dots, s_{n-1} = (n-1,n))$. Given $w = s_{i_1} \cdots s_{i_r} \in \mathfrak{S}_n$, we put $T_w = T_{i_1} \cdots T_{i_r}$. We put $L_i = \mathbf{q}^{1-i} T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1}$.

Let $\lambda \in \mathcal{P}(d,n)$. We put $m_\lambda = \left(\sum_{w \in \mathfrak{S}_\lambda} T_w\right) \left(\prod_{i=2}^d \prod_{j=1}^{a_i} (L_j - \mathbf{x}_i)\right)$, where $a_i = |\lambda^{(1)}| + \cdots + |\lambda^{(i-1)}|$.

We put $M(\lambda) = m_\lambda \mathbf{H}$, a right $\mathbf{H}$-module, and $P = \bigoplus_{\lambda \in \mathcal{P}(d,n)} M(\lambda)$. Let $\mathcal{S} = \mathcal{S}(d,n) = \mathrm{End}_{\mathbf{H}^{\mathrm{opp}}}(P)^{\mathrm{opp}}$ (Dipper, James, and Mathas consider a Morita equivalent algebra, where in the definition of P the sum is taken over all multicompositions of n).

Theorem 6.6. *$(\mathcal{S}, P)$ is a quasi-hereditary cover of $\mathbf{H}$, for the order given by the dominance order on $\mathcal{P}(d,n)$.*

Assume k is a complete discrete valuation ring such that

$$(q+1) \prod_{i \neq j} (x_i - x_j) \in k^\times \text{ and } \prod_{i=1}^n (1 + q + \cdots + q^{i-1}) \prod_{\substack{1 \le i < j \le d \\ -n < r < n}} (q^r x_i - x_j) \neq 0.$$

Then $(k\mathcal{S}, kP)$ is a 1-faithful quasi-hereditary cover of $k\mathbf{H}$.

Proof. The first assertion is known [Mat2, Theorems 4.14 and 5.3]. The non-vanishing assumption is exactly the condition required to ensure that $K\mathbf{H}$ is split semi-simple [Ar1], where K is the field of fractions of k. By [Mat1, Corollary 6.11 and Theorem 6.18], given T a tilting module for $k\mathcal{S}$, there is some $k\mathbf{H}$-module M such that $\mathrm{Hom}_{k\mathbf{H}}(\mathrm{Hom}_{k\mathcal{S}}(kP, k\mathcal{S}), M) \simeq T$. The second part follows now from Propositions 4.40 and 4.42. □

Remark 6.7. In type A, these results are classical. Under the assumption that $(1+q)(1+q+q^2) \neq 0$ and k is a field, then $k\mathcal{S}(1,n)$ is a 1-faithful cover [HeNa, Theorem 3.8.1]. See also [Do4, §10] for a different approach.

We put $S(\lambda) = \mathrm{Hom}_{\mathcal{S}}(P, \Delta(\lambda))$.

6.3. **Comparison.** In §6.3, we take k, k' as in §5.2.1.

6.3.1. The following result identifies category $\mathcal{O}$ under certain assumptions.

Theorem 6.8. *Assume $(q+1)\prod_{i \neq j}(x_i - x_j) \neq 0$. Assume $h \le 0$ and $h_{s+1} - h_s \ge (1-n)h$ for $0 \le s \le d-2$.*

Then, $k\mathcal{O}$ and $k\mathcal{S}$-mod are equivalent highest weight covers of $k\mathbf{H}$: there is an equivalence $k\mathcal{O} \xrightarrow{\sim} k\mathcal{S}$-mod sending the standard object associated to $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ to the standard object associated to χ.

Proof. By Theorems 5.3 and 6.6, $\mathcal{O}$ and $k'\mathcal{S}$-mod are 1-faithful highest weight covers of $k'\mathcal{H}$. The order on irreducible characters in $k'\mathcal{H}$ coming from $k'\mathcal{S}$ is a refinement of the one coming from $\mathcal{O}$, by Proposition 6.4. The Theorem follows now from Theorem 4.49. □

Note that, under the assumptions of the Theorem, $\mathcal{O}$ and $k'\mathcal{S}$-mod are equivalent highest weight covers of $k'\mathbf{H}$ as well.

Remark 6.9. Using Proposition 5.8, we obtain other parameter values for which $k\mathcal{O}$ is equivalent to $k\mathcal{S}$-mod (for example, replacing h by $-h$ in the Theorem). The Theorem should hold without the assumption $(q+1)\prod_{i \neq j}(x_i - x_j) \neq 0$, but the methods developed here cannot handle this general case.

Remark 6.10. This suggests to look for a construction similar to that of §6.2 of q-Schur algebras of type $B_n(d)$ for orders on $\mathcal{P}(d,n)$ coming from other choices of h and h_i's. Recent work of Gordon [Go] provides an order based on the geometry of Hilbert schemes that is probably more relevant that the orders used here.

It might be possible to produce explicit "perverse complexes" and obtain the other q-Schur algebras by perverse tilts (cf Conjecture 5.6).

6.3.2. Let us restate the previous Theorem in the case $W = \mathfrak{S}_n$. In that case, $\mathcal{S}(1,n)$ is the q-Schur algebra of $\mathfrak{S}_n$, Morita equivalent to a quotient of the quantum group $U_q(\mathfrak{gl}_n)$. The following result solves a conjecture of [GGOR, Remark 5.17] (under the assumption $h \notin \frac{1}{2}+\mathbf{Z}$).

Theorem 6.11. *Assume $h \notin \frac{1}{2}+\mathbf{Z}$. Then, there is an equivalence of highest weight categories $k\mathcal{O} \xrightarrow{\sim} k\mathcal{S}(1,n)$-mod sending the standard object associated to $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ to the standard object associated to* $\begin{cases} \chi & \text{if } h \leq 0 \\ \chi \otimes \det & \text{if } h > 0. \end{cases}$

This shows the characters of simple objects of $\mathcal{O}$ are given by canonical basis elements in the Fock space for $\hat{\mathfrak{sl}}_r$, where r is the order of k in $\mathbf{C}/\mathbf{Z}$, according to Varagnolo-Vasserot's proof [VarVas1] of Leclerc-Thibon's conjecture [LeTh] (a generalization of Ariki's result [Ar2] proving Lascoux-Leclerc-Thibon's conjecture [LaLeTh]). Cf §6.5 for a conjectural generalization to the case $d > 1$.

Gordon and Stafford [GoSt2, Proposition 6.11] deduce from this result a description of the maximal dimensional components of the characteristic cycle of the simple objects (a cycle in $\mathrm{Hilb}^n \mathbf{C}^2$). If these characteristic cycles were equidimensional, they would thus be known and one could deduce what are the support varieties in $\mathbf{C}^{2n}/\mathfrak{S}_n$ of the simple objects in $\mathcal{O}$.

Remark 6.12. One can expect to obtain a different proof of Theorem 6.11 via the work of Suzuki [Su], which relates representations of rational Cherednik algebras of type A with representations at negative level of affine Lie algebras of type A.

Note that an analog of Theorem 6.11 has been proven by Varagnolo and Vasserot for trigonometric (or elliptic) Cherednik algebras [VarVas2].

6.4. **Orbit decomposition.** Let $s \in \{0,\dots,d-1\}$ such that $(q^i x_r - q^{i'} x_{r'}) \in k^\times$ for $0 \leq r < s$, $s \leq r' < d$ and $0 \leq i, i' \leq n$. There is a bijection

$$\coprod_{m=0}^{n} \mathcal{P}(s,m) \times \mathcal{P}(d-s,n-m) \xrightarrow[\sim]{\cup} \mathcal{P}(d,n)$$
$$(\alpha^{(1)},\dots,\alpha^{(s)}),(\beta^{(1)},\dots,\beta^{(d-s)}) \mapsto (\alpha^{(1)},\dots,\alpha^{(s)},\beta^{(1)},\dots,\beta^{(d-s)}).$$

We write $\mathbf{H}_{x_0,\dots,x_{d-1}}(n)$ for the algebra $k\mathbf{H}$ (which depends further on q).

In [DiMa, Theorem 1.6], Dipper and Mathas construct an equivalence

$$F : \left(\bigoplus_{m=0}^{n} \mathbf{H}_{x_0,\dots,x_{s-1}}(m) \otimes \mathbf{H}_{x_s,\dots,x_{d-1}}(n-m)\right)\text{-mod} \xrightarrow{\sim} \mathbf{H}_{x_0,\dots,x_{d-1}}(n)\text{-mod}$$

with the property that $F(S(\alpha) \otimes S(\beta)) = S(\alpha \cup \beta)$ [DiMa, Proposition 4.11].

Assume we are in the setting of §5.2.1. We write $\mathcal{O}_{h_0,\dots,h_{d-1}}(n)$ for the category $k\mathcal{O}$.

Theorem 6.13. *Assume* $(q+1)\prod_{i\neq j}(x_i-x_j)\neq 0$. *Let* $s\in\{0,\dots,d-1\}$ *such that* $q^i x_r\neq q^{i'}x_{r'}$ *for* $0\leq r<s$, $s\leq r'<d$ *and* $0\leq i,i'\leq n$. *Then, there is an equivalence of highest weight categories*

$$\left(\bigoplus_{m=0}^{n}\mathcal{O}_{h_0,\dots,h_{s-1}}(m)\otimes\mathcal{O}_{h_s,\dots,h_{d-1}}(n-m)\right)\text{-mod}\xrightarrow{\sim}\mathcal{O}_{h_0,\dots,h_{d-1}}(n)\text{-mod}\,.$$

It sends $\Delta(\alpha)\otimes\Delta(\beta)$ *to* $\Delta(\alpha\cup\beta)$ *and it is compatible with* F.

Proof. Fix m and consider $\alpha\in\mathcal{P}(s,m)$ and $\beta\in\mathcal{P}(d-s,n-m)$. We have

$$\frac{c_{\chi_\alpha}}{s}+\frac{c_{\chi_\beta}}{d-s}=\sum_{2\leq r\leq s}|\alpha^{(r)}|(h_{r-1}-h_0)+\sum_{s+1\leq r\leq d}|\beta^{(r-s)}|(h_{r-1}-h_s)-$$
$$-\left(\frac{m(m-1)+(n-m)(n-m-1)}{2}+\sum_{r,i,j}(b_{i,j}^{(r)}-d_{i,j}^{(r)})\right)h$$

so

$$d(\frac{c_{\chi_\alpha}}{s}+\frac{c_{\chi_\beta}}{d-s})=c_{\chi_{\alpha\cup\beta}}+d(n-m)(h_0-h_s)+m(n-m).$$

We deduce that if $\chi_\alpha\leq\chi_{\alpha'}$ and $\chi_\beta\leq\chi_{\beta'}$, then $\chi_{\alpha\cup\beta}\leq\chi_{\alpha'\cup\beta'}$.

The result follows now from Theorems 5.3 and 4.49. □

Remark 6.14. The Theorem should hold without the assumption $(q+1)\prod_{i\neq j}(x_i-x_j)\neq 0$.

Remark 6.15. Note that this Theorem applies to more general 1-faithful highest weight covers (in particular, to the classical one, where we recover [DiMa, Theorem 1.5], with the additional assumption that $(q+1)\prod_{i\neq j}(x_i-x_j)\in k^\times$).

Remark 6.16. We put an equivalence relation on $\{0,1,\dots,d-1\}$: r and r' are equivalent if there is $a\in\{-n,\dots,n\}$ such that $x_{r'}=q^a x_r$. Then, $\mathcal{O}$ is equivalent to

$$\bigoplus_{\substack{m:(\{0,\dots,d-1\}/\sim)\to\mathbf{Z}_{\geq 0}\\ \sum_I m(I)=n}}\ \bigotimes_{I\in\{0,\dots,d-1\}/\sim}\mathcal{O}_{\{h_i\}_{i\in I}}(m(I)).$$

6.5. **Uglov's higher level Fock spaces.** Let $e>1$ be an integer and let $s_\cdot=(s_0,\dots,s_{d-1})\in\mathbf{Z}^d$. Let $h=\frac{1}{e}$ and $h_j=\frac{s_j}{e}-\frac{j}{d}$. Uglov [Ug] has introduced a q-deformed Fock space of level d associated to the multicharge $s_\cdot$, together with a standard and a canonical basis, both parametrized by d-multipartitions.

Yvonne [Yv] conjectured that, for suitable values of the s_i's, the multiplicities of simple modules in standard modules for classical q-Schur algebras are equal to the corresponding coefficients of the transition matrix between the standard and the canonical basis. He showed that this is compatible with the Jantzen sum formula.

Now, Theorem 6.8 shows that Yvonne's conjecture can be restated for category $\mathcal{O}$ and particular values of the s_i's. We conjecture that, for arbitrary s_i's, the multiplicities of simple modules in standard modules in $\mathcal{O}$ are equal to the corresponding coefficients of the transition matrix between the standard and the canonical basis. We also expect that the q-coefficients measure the level in the filtration induced by the Shapovalov form. It should be possible to prove a sum formula for Cherednik algebras and obtain a result similar to Yvonne's.

Remark 6.17. In order to prove the conjecture (in the case "$(q+1)\prod_{i\neq j}(x_i - x_j) \neq 0$"), it would suffice to construct a (deformation of a) highest weight cover of the Hecke algebra of a geometrical nature, where character formulas can be computed.

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