Stein Shrinkage and Second-Order Efficiency for semiparametric estimation of the shift

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Abstract. The problem of estimating the shift (or, equivalently, the center of symmetry) of an unknown symmetric and periodic function f observed in Gaussian white noise is considered. Using the blockwise Stein method, a penalized profile likelihood with a data-driven penalization is introduced so that the estimator of the center of symmetry is defined as the maximizer of the penalized profile likelihood. This estimator has the advantage of being independent of the functional class to which the signal f si assumed to belong and, furthermore, is shown to be semiparametrically adaptive and efficient.

Moreover, the second-order term of the risk expansion of the proposed estimator is proved to behave at least as well as the second-order term of the risk of the best possible estimator using monotone smoothing filter. Under mild assumptions, this estimator is shown to be second-order minimax sharp adaptive over the whole scale of Sobolev balls with smoothness $\beta > 1$. Thus, these results extend those of [10], where second-order asymptotic minimaxity is proved for an estimator depending on the functional class containing f and $\beta \geq 2$ is required.

1. Introduction

1.1. **Motivation.** The increasing interest to the semiparametric models in recent years is mainly explained by the compromise they offer between the relative simplicity of parametric inference and the flexibility of nonparametric modeling. In many semiparametric models, though the ignorance of an infinite dimensional nuisance parameter, the finite-dimensional parameter of interest can be estimated as well as if the nuisance parameter were known. In most situations, there are many estimators having this feature. Moreover, most of them depend on the functional class to which the unknown nuisance parameter is assumed to belong.

The aim of the present paper is to propose a second-order efficient and entirely data-dependent estimator in the problem of shift estimation when the observed shifted signal is corrupted by the Gaussian white noise. This problem has been studied in [10], and the goal of the present paper is to complete the inference developed there. In particular, the estimator proposed in [10] and proved to be second-order minimax when the signal belongs to a Sobolev ball, depends on the parameters of the Sobolev ball in question. In the present work, we propose an estimator of the

shift parameter that is independent of the parameters of the Sobolev ball and is second-order minimax simultaneously for a large scale of Sobolev balls.

To further motivate the study of the second-order efficiency in semiparametric inference, let us briefly recall a popular general method of construction of efficient estimators in a semiparametric model: the profile likelihood maximization. The asymptotic properties of this method are studied in [32] and, in a more general fashion, in [24]. Profile likelihood techniques are effectively applied in a number of contexts such as laser vibrometry signals [22], varying coefficient partially linear models [13], and so forth.

If the parameter of interest is partitioned as $(\vartheta; f)$, with ϑ being a low-dimensional parameter of interest and f a higher dimensional nuisance parameter, and $l_n(\vartheta; f)$ is the log-likelihood of the model, then the profile likelihood for ϑ is defined as $pl_n(\vartheta) = \sup_{f \in \mathcal{F}} l_n(\vartheta; f)$ and the Profile Likelihood Estimator (PLE) is $\vartheta_{PLE} = \arg\max_\vartheta pl_n(\vartheta)$. Thus, the nuisance parameter f is eliminated by taking the sup over all possible values of f in some a priori chosen class $\mathcal{F}$.

A natural question arises: what is the best way of choosing the class F and what is the impact of this choice on the accuracy of the PLE? The theory fails to answer this question as long as only the first-order term of the risk is considered. It seems that the most appealing way to study the dependence on $\mathcal{F}$ of the accuracy of the PLE is to consider the second-order term of the quadratic risk. This approach is developed in [16, 17] for partial linear models, in [10] for a nonlinear model with a shift parameter, and in [7] for a model with scaling parameter. In these papers, the second-order asymptotic minimaxity of the proposed estimators is proved and optimal constants are obtained.

Another important point is that the study of the second-order asymptotics of the risk allows one to refine the choice of the tuning parameters, if there are, of the estimating procedure. This argument is used in [19, 21, 23] in order to propose a rate-optimal choice of a tuning parameter.

The results of the above mentioned papers grant an increasing importance to the second-order terms in that they show that, in a semiparametric estimation problem, the second-order term is not dramatically smaller than the first-order term, especially when the nuisance parameter is not very smooth (or not very sparse). Thus, the investigation of second-order efficient estimators is not only a challenging theoretical problem, but is also of practical interest.

1.2. **Methodology.** The main goal of the present paper is to define an estimator of ϑ which adapts automatically to the smoothness of f and is simultaneously second-order efficient over a large variety of Sobolev balls. For linear models, such a procedure has firstly been proposed by Golubev and Härdle in [17]. They use the well known idea of unbiased risk minimization in order to determine the data driven filter. However, their procedure is provably second-order efficient only when the (data driven) filter and the contrast function are based on independent subsamples. This sample-splitting technique is frequently used in theory (see e.g. [2], [26, p. 19]), but it is rather unattractive from the practical point of view.

In this paper, we focus on the shift estimation of a periodic symmetric signal and use the celebrated Stein shrinkage (see [20, 29]) for defining the data-driven version of

the penalized profile likelihood. While there is a huge number of papers concerning the Stein's shrinkage in nonparametric estimation ([4, 5, 6, 8, 9, 11, 28], see also [34] for a comprehensive discussion), this is to our knowledge the first paper where Stein's shrinkage is applied in the context of semiparametric estimation.

The construction we use is closer to the one of [9] in that a penalized version of the Stein estimator with weakly geometrically increasing blocks is considered. It seems that, unlike in nonparametric inference, in semiparametric inference it is crucial to use the penalized shrinkage in order to get the second-order efficiency. An important advantage of using the Stein Shrinkage is that, unlike the procedure proposed in [17], our procedure is not based on the sample-splitting technique and, nevertheless, enjoys the property of second-order asymptotic minimaxity.

Note also that our results hold when the function f has arbitrary Sobolev smoothness $\beta > 1$. However, for $\beta < 2$, the penalized profile likelihood (PPL) is not necessarily concave over the whole parameter space, therefore we use a two-step procedure. We first construct an initial $\sqrt{n}$-consistant estimator $\bar{\vartheta}$ of ϑ, and then maximize the PPL over a shrinking neighborhood of $\bar{\vartheta}$, where we prove that the PPL is concave with probability close to one.

1.3. **Structure of the paper.** Section 2 describes the model and introduces the Penalized Maximum Likelihood Estimator (PMLE) based on a filtering sequence h. In Section 3, the local concavity of the PPL is proved and the PMLE based on a data dependent choice of h is proposed. Oracle inequalities for adaptive PMLE and its second-order efficiency over Sobolev balls are stated and proved in Section 4. Finally, Section 5 contains the definition of a preliminary estimator and the technical details of the proofs.

2. A simple semiparametric model

Consider the "signal in Gaussian white noise model", that is the observations $(x^\varepsilon(t), \, t \in [-1/2, 1/2])$ with

$$dx^\varepsilon(t) = f_\vartheta(t) \, dt + \varepsilon dW(t), \qquad t \in [-1/2, 1/2], \tag{1}$$

are available, where $W(t)$ is a Brownian motion. Assume that the signal has the form $f_\vartheta(t) = f(t - \vartheta)$, where $f : \mathbb{R} \to \mathbb{R}$ is a symmetric periodic function having 1 as smallest period. More precisely, we assume that the function f belongs to the set $\mathcal{F}_0 = \cup_{\rho > 0} \mathcal{F}_0(\rho)$ with

$$\mathcal{F}_0(\rho) = \left\{ f \in L^2_{loc} \, : \, f(x) = f(-x) = f(x+1), \, \forall x \in \mathbb{R}; \, |f_1| \geq \rho \right\},$$

where we denote by L^2_{loc} is the set of all locally squared integrable functions and by $f_1 = \int_{-1/2}^{1/2} f(t) \cos(2\pi t) \, dt$.

The goal is to estimate the parameter $\vartheta \in \Theta \subset \,] - T, T]$ with $T < 1/4$. As explained in [10], the assumption $T < 1/4$ is necessary for the identifiability of the parameter ϑ. In this context, the unknown function f is considered as an infinite dimensional nuisance parameter.

The first-order asymptotic properties of estimators in closely related models have been studied in [15, 18, 31, 30]. Note also that the model we have just defined is

an idealized version of the symmetric location model [33, 23] and the shifted curves model [14].

Let us introduce the "sequence model" equivalent to (1). To do this, for any integer $k \geq 1$, we denote

$$
\begin{aligned}
x_k &= \sqrt{2} \int_{-1/2}^{1/2} \cos(2\pi kt) \, dx^\varepsilon(t), \\
x_k^* &= \sqrt{2} \int_{-1/2}^{1/2} \sin(2\pi kt) \, dx^\varepsilon(t).
\end{aligned}
\tag{2}
$$

Clearly,

$$
\begin{aligned}
x_k &= f_k \cos(2\pi k\vartheta) + \varepsilon \xi_k, \\
x_k^* &= f_k \sin(2\pi k\vartheta) + \varepsilon \xi_k^*,
\end{aligned}
\tag{3}
$$

where $f_k = \sqrt{2} \int_{-1/2}^{1/2} \cos(2\pi kt) f(t) \, dt$ and $(\xi_k, \xi_k^*, k = 1, 2, \ldots)$ are independent standard Gaussian random variables. Note that, the laws of likelihood processes (indexed by $(\vartheta, f) \in \mathbb{R} \times \mathcal{F}_0$) of the models (1) and (3) coincide implying thus the equivalence of these models. It is useful to mention here that the Fisher information in the problem of estimating ϑ with fixed f is

$$
I^\varepsilon(f) = \varepsilon^{-2} \int_{-1/2}^{1/2} f'(x)^2 \, dx = \varepsilon^{-2} \sum_{k \in \mathbb{N}} (2\pi k)^2 f_k^2.
$$

In this paper, we estimate the parameter ϑ by a version of the method of profile likelihood maximization (cf. [35, p. 106]), which is also known as penalized maximum likelihood estimator [10]. We recall briefly its definition. Write $\mathbf{P}_{\vartheta,f}$ (resp. $\mathbf{E}_{\vartheta,f}$) for probability measure (resp. expectation) induced by x^ε on the canonical space $C([-1/2, 1/2])$ equipped with the Wiener measure. As no confusion is possible, we use the same notation in the "sequence model" given by (3). The Radon density $L_\varepsilon(\tau, f, \cdot)$ of $\mathbf{P}_{\tau,f}$ with respect to $\mathbf{P}_{0,0}$ is given by

$$
\begin{aligned}
L_\varepsilon(\tau, f, x^\varepsilon) &= \exp\left(\varepsilon^{-2} \int_{-1/2}^{1/2} f(t - \tau) \, dx^\varepsilon(t) - \frac{\varepsilon^{-2}}{2} \int_{-1/2}^{1/2} f^2(t) \, dt \right) \\
&= \exp\left(\frac{\sqrt{2}}{\varepsilon^2} \sum_{k=1}^{\infty} f_k \int_{-1/2}^{1/2} \cos[2\pi k(t - \tau)] \, dx^\varepsilon(t) - \frac{1}{2\varepsilon^2} \sum_{k=1}^{\infty} f_k^2 \right).
\end{aligned}
$$

Easy algebra yields

$$
\max_{(f_k)_k \in \mathbb{R}^{\mathbb{N}}} L_\varepsilon(\tau, f, x^\varepsilon) = \exp\left\{ \varepsilon^{-2} \sum_{k=1}^{\infty} \left(\int_{-1/2}^{1/2} \cos[2\pi k(t - \tau)] \, dx^\varepsilon(t) \right)^2 \right\}.
\tag{4}
$$

For any $\tau \in \mathbb{R}$, this expression is equal to infinity for almost all paths x^ε. Thus, it is necessary to impose some restrictions on the set over which the maximization is done. This is equivalent to considering a penalized profile likelihood. In this paper, we focus on the penalization corresponding to ellipsoids in ℓ^2. More details on this method can be found in [10], we here content ourselves with giving the final definition.

We call *filtering sequence* or *filter* any $h = (h_k)_{k \in \mathbb{N}} \in [0,1]^{\mathbb{N}}$ such that only a finite number of h_k are non-zero. Define the functional

$$\Phi_\varepsilon(\tau, h) \quad = \quad \sum_{k=1}^{\infty} h_k \left(\int_{-1/2}^{1/2} \cos[2\pi k(t - \tau)] \, dx^\varepsilon(t) \right)^2. \qquad (5)$$

The PMLE of ϑ is then $\hat{\vartheta}_{\text{PMLE}} = \arg\max_\tau \Phi_\varepsilon(\tau, h)$. The role of the sequence h is thus to filter out the irrelevant terms in the right side of (4), that is to assign a value h_k close to zero to the terms corresponding to a small signal-to-noise ratio $|f_k|/\varepsilon$.

For deterministic filters h, the asymptotic behavior of the estimator $\hat{\vartheta}_{\text{PMLE}}$ is studied in [10]. Under some smoothness assumptions on f, for a broad choice of filters h, $\hat{\vartheta}_{\text{PMLE}}$ is proved to be first-order asymptotic efficient. Moreover, it is shown that the second-order term of its risk expansion is $\varepsilon^2 R^\varepsilon[f, h] / \|f'\|^4$, where

$$R^\varepsilon[f, h] \quad = \quad \sum_{k=1}^{\infty} (2\pi k)^2 \big[(1 - h_k)^2 f_k^2 + \varepsilon^2 h_k^2 \big].$$

This result suggests to use the filter $h_{opt} = \arg\min_h R^\varepsilon[f, h]$ for defining the PMLE of ϑ. However, this minimizer is inapplicable since it depends on f. To get rid of this dependence, the minimax approach recommends the utilization of the filter $h_{\mathcal{F}} = \arg\min_h \sup_{f \in \mathcal{F}} R^\varepsilon[f, h]$. If $\mathcal{F}$ is a ball in a Sobolev space, a solution of this minimization problem is given by the Pinsker filter [27]. Although the latter leads to a second-order minimax estimator of ϑ [10, Thm. 2 and 3], it suffers from the well known drawbacks of the minimax theory: the obtained estimator is pessimistic and requires the precise knowledge of the smoothness of the unknown function.

The aim of the present paper is to propose a data-driven filter $\hat{h}$ so that the resulting PMLE of ϑ circumvents these drawbacks. To be more precise, let us call *oracle* related to the class of filters $\widehat{\mathcal{H}}$ the value $\arg\min_{h \in \widehat{\mathcal{H}}} R^\varepsilon[f, h]$. We cannot use this oracle in our estimating procedure because it depends on the unknown f. Nevertheless, the estimator we propose mimics well the behavior of the oracle related to the classe of monotone filters (cf. Section 4.2) for a fixed function f and is second-order sharp adaptive over a broad scale of Sobolev balls.

3. PMLE based on a data-driven filter

3.1. **Local properties of $\Phi_\varepsilon(\tau, h)$.** Let us introduce some auxiliary notation:

$$y_k(\tau) = x_k(\tau) + ix_k^*(\tau) = \sqrt{2} \int_{-1/2}^{1/2} e^{2i\pi k(t - \tau)} dx^\varepsilon(t),$$

$$z_k(\tau) = \xi_k(\tau) + i\xi_k^*(\tau) = \sqrt{2} \int_{-1/2}^{1/2} e^{2i\pi k(t - \tau)} dW(t).$$

We will write $x_k = x_k(0)$, $x_k^* = x_k^*(0)$ and similarly for ξ_k and ξ_k^*. The symmetry of f implies that $x_k(\vartheta) = f_k + \varepsilon \xi_k(\vartheta)$ and $x_k^*(\vartheta) = \varepsilon \xi_k^*(\vartheta)$. Moreover, for all $\tau \in \mathbb{R}$, the random variables $\{\xi_i(\tau), \xi_i^*(\tau); i \in \mathbb{N}\}$ are i.i.d. standard Gaussian. Using this

notation, the functional Φ_ε can be rewritten as follows

$$\Phi_\varepsilon(\tau, h) = \frac{1}{2} \sum_{k=1}^\infty h_k \left(x_k(\vartheta) \cos[2\pi k(\tau - \vartheta)] + x_k^*(\vartheta) \sin[2\pi k(\tau - \vartheta)] \right)^2.$$

Our aim is to show that under some assumptions on h, the function $\Phi_\varepsilon(\cdot, h)$ has, with a probability close to one, a local maximum in a neighborhood of ϑ. Note that the derivative of the function $\tau \mapsto \Phi_\varepsilon(\tau, h)$ is given by

$$\Phi_\varepsilon'(\tau, h) = -\sum_{k=1}^\infty h_k \pi k (x_k(\vartheta)^2 - x_k^*(\vartheta)^2) \sin[4\pi k(\tau - \vartheta)]$$
$$+ \sum_{k=1}^\infty 2 h_k \pi k x_k(\vartheta) x_k^*(\vartheta) \cos[4\pi k(\tau - \vartheta)]. \tag{6}$$

Proposition 1. *Let* $\|f^{(\beta_*)}\|^2 = \sum_k (2\pi k)^{2\beta_*} f_k^2 < \infty$ *for some* $\beta_* > 1$ *and set* $\hat{\beta}_* = \beta_* \wedge 1.5$, $N_\varepsilon = [(\varepsilon^2 \log \varepsilon^{-5})^{-\frac{1}{2\beta_* + 1}}]$. *Let* $h \in [0,1]^{N_\varepsilon}$ *be a random vector depending on* (x_k, x_k^*) *only via* $x_k^2 + x_k^{*2}$. *For any* $\varepsilon < 1$, *there exists an event* $\mathcal{A}_\varepsilon$ *such that* $\mathbf{P}_{\vartheta, f}(\mathcal{A}_\varepsilon^c) \leq 2\varepsilon^4$ *and on* $\mathcal{A}_\varepsilon \cap \left\{ \sum_{k=1}^\infty (1 - h_k)(2\pi k)^2 f_k^2 \leq \|f'\|^2 / 4 \right\}$, *for all* ε *verifying*

$$N_\varepsilon^{1 - \hat{\beta}_*} \leq \frac{\|f'\|}{2\|f^{(\beta_*)}\| + 6\pi}, \tag{7}$$

the function $\tau \mapsto \Phi_\varepsilon(\tau, h)$ *is strictly concave and admits a unique maximum* $\hat{\vartheta}_\varepsilon$ *in the interval* $[\vartheta - (4\pi N_\varepsilon)^{-1}, \vartheta + (4\pi N_\varepsilon)^{-1}]$, *satisfying*

$$|\hat{\vartheta}_\varepsilon - \vartheta| \leq 8\varepsilon \sqrt{\log \varepsilon^{-5}} \, \|f'\|^{-1}. \tag{8}$$

Proof. Set $\Theta_\varepsilon = [\vartheta - (4\pi N_\varepsilon)^{-1}, \vartheta + (4\pi N_\varepsilon)^{-1}]$. Assume that (7) is fulfilled and $\sum_{k=1}^\infty (1 - h_k)(2\pi k)^2 f_k^2 \leq \|f'\|^2 / 4$. On the one hand, the first inequality of Proposition 4 (see Section 5.2 below) implies that there exists an event $\mathcal{A}_\varepsilon$, such that on this event

$$\Phi_\varepsilon''(\tau, h) \leq -\|f'\|^2 + \sum_{k=1}^\infty (1 - h_k)(2\pi k)^2 f_k^2 + \|f'\|^2 / 4 \leq -\|f'\|^2 / 2,$$

for all $\tau \in \Theta_\varepsilon$ and for ε small enough. Therefore, $\Phi_\varepsilon(\cdot, h)$ is strictly concave. On the other hand, the second inequality of Proposition 4 implies that

$$\frac{\Phi_\varepsilon'(\tau, h)}{\tau - \vartheta} \leq -\frac{\|f'\|^2}{2} + \frac{2\varepsilon(\|f'\| \sqrt{\log \varepsilon^{-5}} + 2\pi N_\varepsilon^{1 - \hat{\beta}_*})}{|\tau - \vartheta|}$$
$$\leq -\frac{\|f'\|^2}{2} + \frac{4\varepsilon \sqrt{\log \varepsilon^{-5}} \, \|f'\|}{|\tau - \vartheta|} \tag{9}$$

for sufficiently small values of ε. Therefore, $\pm\Phi_\varepsilon'(\vartheta \pm (4\pi N_\varepsilon)^{-1}, h) < 0$, which guarantees that the maximum $\hat{\vartheta}_\varepsilon$ of $\Phi_\varepsilon(\cdot, h)$ is attained in the interior of Θ_ε and $\Phi_\varepsilon'(\hat{\vartheta}_\varepsilon, h) = 0$. Applying (9) to $\tau = \hat{\vartheta}_\varepsilon$ we get (8). $\qquad\square$

Remark 1. The choice $N_\varepsilon = [(\varepsilon^2 \log \varepsilon^{-5})^{-\frac{1}{2\beta_* + 1}}]$ has a simple interpretation. If $f_k^2 \leq \varepsilon^2$ for some $k \in \mathbb{N}$, then the kth observation in (3) is not relevant for estimating

the parameter ϑ. Let $\mathcal{K} = \{k \in \mathbb{N} : f_k^2 > \varepsilon^2\}$ and $K = \#\mathcal{K}$. Then

$$\sum_{k \in \mathcal{K}} (2\pi k)^{2\beta_*} f_k^2 \geq \varepsilon^2 \sum_{k \in \mathcal{K}} (2\pi k)^{2\beta_*} \geq \varepsilon^2 \sum_{k=1}^{K} (2\pi k)^{2\beta_*} \geq \frac{\varepsilon^2 (2\pi)^{2\beta_*} K^{2\beta_* + 1}}{2\beta_* + 1}.$$

Thus, the number K of Fourier coefficients f_k larger than ε is at most $O(\varepsilon^{-\frac{2}{2\beta_*+1}})$. Thus, for ε small enough, all observations relevant for estimating the parameter ϑ lie in $\{y_1, \ldots, y_N\}$.

Remark 2. In [10], the estimator $\hat{\vartheta}_{\mathrm{PMLE}}$ is defined as the maximizer of $\Phi_\varepsilon(\cdot, h)$ over the whole interval Θ. Instead, we define it as the local maximizer in the neighborhood of a preliminary estimator. This modification is explained by the fact that the function $\tau \mapsto \Phi_\varepsilon(\tau, h)$ is only locally concave when $\beta_* < 2$. Furthermore, the computation of the local minimum is faster than the computation of the global minimum.

Remark 3. Elaborating on the arguments of Proposition 1, it can be shown that Thm. 1 from [10] remains true for $\beta > 1$, provided that $\vartheta_{\mathrm{PMLE}}$ is defined as the local maximizer of $\Phi_\varepsilon(\cdot, h)$ and condition B2 is replaced by $h_k = 0$ for any $k > N_\varepsilon$.

3.2. **Blockwise constant Stein filter with penalization.** Let J be a positive integer and $\kappa_1, \ldots, \kappa_{J+1} \in \{1, \ldots, N_\varepsilon\}$ be a strictly increasing sequence such that $\kappa_1 = 1$. Set $B_j = \{k \in \mathbb{N} : \kappa_j \leq k < \kappa_{j+1}\}$. Let $\mathcal{H}^*(B)$ be the set of all filters $h \in [0, 1]^{N_\varepsilon}$ that are constant on the blocks $B = \{B_j\}_{j=1}^{J}$:

$$h \in \mathcal{H}^*(B) \qquad \Longleftrightarrow \qquad h_k = h_{k'}, \quad \forall\, k, k' \in B_j.$$

An oracle knowing the function f, would choose the best possible filter $h^* \in \mathcal{H}^*(B)$ by minimizing $R^\varepsilon[f, h]$ over $\mathcal{H}^*(B)$. We call h^* oracle choice of filter or simply oracle. Simple computations show that

$$h_k^* = \frac{\|f'\|_{(j)}^2}{\|f'\|_{(j)}^2 + \varepsilon^2 \sigma_j^2}, \qquad k \in B_j, \quad j = 1, \ldots, J,$$

where $\|f'\|_{(j)}^2 = \sum_{k \in B_j} (2\pi k)^2 f_k^2$ and $\sigma_j^2 = \sum_{k \in B_j} (2\pi k)^2$.

Since the oracle h^* depends on f, we replace it by a suitable estimator. Let us define $R_j^\varepsilon[f, a] = (1 - a)^2 \|f'\|_{(j)}^2 + \varepsilon^2 a^2 \sigma_j^2$ so that $R^\varepsilon[f, h] = \sum_j R_j^\varepsilon[f, h_{\kappa_j}] + \sum_{k > N_\varepsilon} (2\pi k)^2 f_k^2$. Then $h_{\kappa_j}^*$ is the minimizer of $R_j^\varepsilon[f, a]$ over $a \in \mathbb{R}$. Define $\|y'\|_{(j)}^2 = \sum_{k \in B_j} (2\pi k)^2 |y_k|^2$ and $y_k = x_k + i x_k^*$. For small values of $h_{\kappa_j}^*$, the minimizer of

$$\hat{R}_j^\varepsilon[a] := (1 - a)^2 (\|y'\|_{(j)}^2 - 2\varepsilon^2 \sigma_j^2)_+ + \varepsilon^2 a^2 \sigma_j^2,$$

which is an estimator of $R_j^\varepsilon[f, a]$, can be large with respect to $h_{\kappa_j}^*$. To avoid such a configuration, we penalize large values of a and define the estimator $\hat{h}^{JS}$ of h^* as the minimum over $[0, 1]$ of the function:

$$a \mapsto \hat{R}_j^\varepsilon[a] + 2\varphi_j \varepsilon^2 \sigma_j^2 a,$$

where $\varphi_j > 0$ is a factor of penalization tending to zero as $\varepsilon \to 0$. This leads us to the penalized Stein filter

$$\hat{h}_k^{JS} = \left(1 - \frac{\varepsilon^2 \sigma_j^2 (1 + \varphi_j)}{(\|y'\|_{(j)}^2 - 2\varepsilon^2 \sigma_j^2)_+ + \varepsilon^2 \sigma_j^2}\right)_+, \quad \forall k \in B_j. \tag{10}$$

3.3. **Weakly geometrically increasing blocks.** The aim of this section is to propose a concrete scheme for defining the blockwise constant data-driven filter. We use the weakly geometrically increasing blocks introduced by Cavalier and Tsybakov [8, 9]. These blocks have the advantage of being simple (the construction is driven by only one parameter) and of having good approximation properties with respect to the class of monotone filters [9, Lemma 1].

Let $\nu = \nu_\varepsilon$ be a positive integer that increases as ε decreases. Set $\rho_\varepsilon = \nu_\varepsilon^{-1/3}$ and define

$$\kappa_j = \begin{cases} (1 + \nu_\varepsilon)^{j-1}, & j = 1, 2, \\ \kappa_{j-1} + \lfloor \nu_\varepsilon \rho_\varepsilon (1 + \rho_\varepsilon)^{j-2} \rfloor, & j = 3, 4, \ldots, \end{cases} \tag{11}$$

where $\lfloor x \rfloor$ stands for the largest integer strictly smaller than x. Let J be the smallest integer j such that $\kappa_j \geq N_\varepsilon + 1$. We redefine $\kappa_{J+1} = N_\varepsilon + 1$ and set $B_j = \{\kappa_j, \ldots, \kappa_{j+1} - 1\}$ for all $j = 1, \ldots, J$.

3.4. **Brief description of the procedure.** The outlined scheme can be implemented as follows.

1. Choose a real number $\beta_* > 1$ and set $N_\varepsilon = 5 \vee [(\varepsilon^2 \log \varepsilon^{-5})^{-\frac{1}{2\beta_*+1}}]$, $\nu_\varepsilon = [e^{\sqrt{\log N_\varepsilon}}]$ and $\rho_\varepsilon = \nu_\varepsilon^{-1/3}$.
2. Define the sequence $(\kappa_j)_j$ by (11).
3. Set $\varphi_j = \sqrt{24 \log \varepsilon^{-5}/(\kappa_{j+1} - \kappa_j)}$, $\sigma_j^2 = \sum_{\kappa_j \leq k < \kappa_{j+1}} (2\pi k)^2$ and define the data-dependent filter $\hat{h}^{JS}$ by (10).
4. Compute the preliminary estimator $\bar{\vartheta}_\varepsilon$ (cf. Section 5) and set $\bar{\Theta}_\varepsilon = [\bar{\vartheta}_\varepsilon - \delta_\varepsilon, \bar{\vartheta}_\varepsilon + \delta_\varepsilon]$ with $\delta_\varepsilon = \varepsilon \log(\varepsilon^{-2})$.
5. Define $\hat{\vartheta}_\varepsilon$ as the minimum in $\bar{\Theta}_\varepsilon$ of $\Phi_\varepsilon(\cdot, \hat{h}^{JS})$ (see (5)).

Note that the only "free" parameter in this procedure is β_*. In practice, if no information on the regularity of f is available, it appears plausible to assume that f has Sobolev smoothness $\beta_* = 2$.

4. MAIN RESULTS

4.1. **Comparison with the blockwise constant oracle.** In this section, $\hat{h}^{JS}$ denotes the blockwise constant filter defined by (10), φ_j is the penalization we use on the block B_j and $\varphi_\varepsilon = \max_j \varphi_j$. We emphasize that in this section no condition on the blocks B_j is required. Let T_j be the length of the block B_j and $T_\varepsilon = \inf_j T_j$. The oracle choice of h in the class $\mathcal{H}^*(B)$ of all filters constant on the blocks $B = \{B_j\}_j$ is denoted by h^*. Define

$$\hat{\vartheta}_\varepsilon^{JS} = \arg\max_{\tau \in \bar{\Theta}_\varepsilon} \Phi_\varepsilon(\tau, \hat{h}^{JS}), \tag{12}$$

where $\bar{\Theta}_\varepsilon = [\bar{\vartheta}_\varepsilon - \delta_\varepsilon, \bar{\vartheta}_\varepsilon + \delta_\varepsilon]$ and $\bar{\vartheta}_\varepsilon$ is a rate optimal initial estimator of ϑ (cf. Section 5.1). Introduce the functional class

$$\mathcal{F}(\beta_*, L_*, \rho) = \left\{ f \in \mathcal{F}_0(\rho) : \|f^{(\beta_*)}\| \leq L_* \right\},$$

where $\beta_* > 1$, $\rho > 0$, $L_* > 0$ are some constants.

Theorem 1. *Let $\hat{\vartheta}_\varepsilon^{JS}$ be defined by (12) with blocks B_j verifying $\log \varepsilon^{-1} = o(T_\varepsilon)$ as $\varepsilon \to 0$. If the penalty φ_j is equal to $\sqrt{24 T_j^{-1} \log \varepsilon^{-5}}$, then*

$$\varepsilon^{-2} \|f'\|^2 \mathbf{E}_{\vartheta,f}[(\hat{\vartheta}_\varepsilon^{JS} - \vartheta)^2] \leq 1 + (1 + \alpha_\varepsilon) \frac{R^\varepsilon[f, h^*]}{\|f'\|^2},$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in $f \in \mathcal{F}(\beta_*, L_*, \rho)$.

Remark 4. If the block B_j is large, then more observations (x_k, x_k^*) are used for estimating the value of the oracle $h_{\kappa_j}^*$. Hence, it is natural to expect that α_ε decreases as T_ε increases. A thorough inspection of the proof allows to describe this feature with the help of the order relation $\alpha_\varepsilon^2 \asymp T_\varepsilon^{-1} \log \varepsilon^{-1}$.

Proof. Let us denote $\mathcal{E} = \{\hat{h}_k^{JS} \in [h_k^* - \sqrt{8}\varphi_{j(k)}(1 - h_k^*), h_k^*], \ \forall k = 1, \ldots, N_\varepsilon\}$ and $\tilde{h} = \hat{h}^{JS} \mathbb{1}_\mathcal{E}$, where $j(k)$ is the number of the block containing k. Lemma 5 implies that $\mathbf{P}_{\vartheta,f}(\mathcal{E}^c) \leq 2\varepsilon^4$. For ε small enough, we have $(1 - \tilde{h}_k)\mathbb{1}_\mathcal{E} \leq 2(1 - h_k^*)$ and the inequality $\sum_k (1 - \hat{h}_k)(2\pi k)^2 f_k^2 \leq \|f'\|^2/4$ is fulfilled on $\mathcal{E}$.

By virtue of Proposition 3, for sufficiently small values of ε, the event $\mathcal{A}_0 = \Big\{ |\bar{\vartheta}_\varepsilon - \vartheta| \leq \big(\frac{1}{4\pi N_\varepsilon} - \delta_\varepsilon\big) \wedge \big(\delta_\varepsilon - \frac{8\varepsilon\sqrt{\log \varepsilon^{-5}}}{\|f'\|}\big) \Big\}$ verifies $\mathbf{P}_{\vartheta,f}(\mathcal{A}_0^c) = O(\varepsilon^4)$. It can be checked that on $\mathcal{A}_0$, $\Big[\vartheta \pm \frac{8\varepsilon\sqrt{\log \varepsilon^{-5}}}{\|f'\|}\Big] \subset [\bar{\vartheta}_\varepsilon \pm \delta_\varepsilon] \subset \big[\vartheta \pm \frac{1}{4\pi N_\varepsilon}\big]$, where we have used the notation $[\vartheta \pm \delta] := [\vartheta - \delta, \vartheta + \delta]$.

According to Proposition 1, on the event $\mathcal{A}_1 = \mathcal{A}_0 \cap \mathcal{A}_\varepsilon \cap \mathcal{E}$, the function $\Phi_\varepsilon(\cdot, \hat{h})$ is strictly concave and has a unique maximum in $[\bar{\vartheta}_\varepsilon \pm \delta_\varepsilon]$. Hence the estimator $\hat{\vartheta}_\varepsilon^{JS}$ verifies $\Phi_\varepsilon'(\hat{\vartheta}_\varepsilon^{JS}, \tilde{h}) = 0$ on the event $\mathcal{A}_1$. By Taylor's formula, there exists a point $\tilde{\vartheta} \in [\vartheta, \hat{\vartheta}_\varepsilon^{JS}]$ such that

$$0 = \Phi_\varepsilon'(\hat{\vartheta}_\varepsilon^{JS}, \tilde{h}) = \Phi_\varepsilon'(\vartheta, \tilde{h}) + (\hat{\vartheta}_\varepsilon^{JS} - \vartheta)\Phi_\varepsilon''(\vartheta, \tilde{h}) + \frac{(\hat{\vartheta}_\varepsilon^{JS} - \vartheta)^2}{2}\Phi_\varepsilon'''(\tilde{\vartheta}, \tilde{h}).$$

Therefore, on $\mathcal{A}_1$,

$$\hat{\vartheta}_\varepsilon^{JS} - \vartheta = -\frac{\Phi_\varepsilon'(\vartheta, \tilde{h})}{\Phi_\varepsilon''(\vartheta, \tilde{h}) + \frac{1}{2}(\hat{\vartheta}_\varepsilon^{JS} - \vartheta)\Phi_\varepsilon'''(\tilde{\vartheta}, \tilde{h})} \ .$$

Using (6), one checks that

$$\Phi_\varepsilon''(\vartheta, \tilde{h}) = -\sum_{k=1}^\infty \tilde{h}_k (2\pi k)^2 [(f_k + \varepsilon\xi_k(\vartheta))^2 - \varepsilon^2 \xi_k^*(\vartheta)^2].$$

In the sequel, we write ξ_k, ξ_k^* instead of $\xi_k(\vartheta), \xi_k^*(\vartheta)$. On the one hand, Lemmas 6, 7 (with $x^2 = \log \varepsilon^{-5}$) and 8 combined with (22) imply that, on an event $\mathcal{A}_2$ of probability higher than $1 - 4\varepsilon^4$, we have

$$\Phi_\varepsilon''(\vartheta, \tilde{h}) = -\|f'\|^2 + \sum_{k=1}^\infty (1 - h_k^*)(2\pi k)^2 f_k^2 - 2\varepsilon \sum_{k=1}^{N_\varepsilon} h_k^* (2\pi k)^2 f_k \xi_k + o(R^\varepsilon)$$
$$= -\|f'\|^2 (1 - \|f'\|^{-2} R^\varepsilon[f, h^*] - \zeta - o(R^\varepsilon[f, h^*])),$$

where $\zeta = 2\varepsilon\|f'\|^{-2} \sum_k h_k^*(2\pi k)^2 f_k \xi_k$ is a zero mean Gaussian random variable. By virtue of (16), its variance verifies

$$4\varepsilon^2 \sum_{k=1}^{N_\varepsilon} h_k^{*\,2}(2\pi k)^4 f_k^2 \leq \frac{12\varepsilon^2}{\min_j T_j} \sum_{j=1}^J h_{\kappa_j}^{*\,2}\sigma_j^2\|f'\|_{(j)}^2 \leq \frac{12\|f'\|^2 R^\varepsilon[f,h^*]}{T_\varepsilon}. \qquad (13)$$

Therefore, by Rosenthal's inequality, $\mathbf{E}_{\vartheta,f}[\zeta^{2p}]^{1/p} = o(R^\varepsilon[f,h^*])$ for any $p > 0$. On the other hand, in view of Lemma 9, there is an event $\mathcal{A}_3$ such that $\mathbf{P}_{\vartheta,f}(\mathcal{A}_3^c) = O(\varepsilon^4)$ and

$$(\hat\vartheta_\varepsilon^{JS} - \vartheta)\Phi_\varepsilon'''(\tilde\vartheta, \tilde h) = o(R^\varepsilon)$$

on this event. Using the inequality $(1-x)^{-2} \leq 1 + 2x + 16x^2$ for all $x \in [-1/2, 1/2]$, we get

$$\|f'\|^4(\hat\vartheta_\varepsilon^{JS} - \vartheta)^2 = \frac{\Phi_\varepsilon'(\vartheta, \tilde h)^2}{\left(1 - \|f'\|^{-2}R^\varepsilon[f,h^*] - \zeta - o(R^\varepsilon)\right)^2}$$

$$\leq \Phi_\varepsilon'(\vartheta, \tilde h)^2\left(1 + \frac{2R^\varepsilon[f,h^*]}{\|f'\|^2} + 2\zeta + o(R^\varepsilon)\right) + o(\varepsilon^2 R^\varepsilon),$$

on the event $\mathcal{A}_4 = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$. Using Lemma 3, we infer that

$$\mathbf{E}_{\vartheta,f}[\Phi_\varepsilon'(\vartheta, \tilde h)^2] \leq \varepsilon^2\|f'\|^2 + \varepsilon^2 \sum_{k=1}^\infty (2\pi k)^2[(h_k^{*\,2} - 1)f_k^2 + h_k^{*\,2}\varepsilon^2]$$

$$= \varepsilon^2(\|f'\|^2 - R^\varepsilon[f,h^*]).$$

Combining these relations with Lemmas 10 and 11, we get an event $\mathcal{A}$ such that $\mathbf{P}_{\vartheta,f}(\mathcal{A}^c) = O(\varepsilon^4)$ and

$$\|f'\|^4\mathbf{E}_{\vartheta,f}[(\hat\vartheta_\varepsilon^{JS} - \vartheta)^2 \mathbb{1}_\mathcal{A}] \leq \mathbf{E}_{\vartheta,f}[\Phi_\varepsilon'(\vartheta, \tilde h)^2]\left(1 + \frac{2R^\varepsilon[f,h^*]}{\|f'\|^2}\right) + o(\varepsilon^2 R^\varepsilon[f,h^*])$$

$$\leq \varepsilon^2\|f'\|^2 + \varepsilon^2 R^\varepsilon[f,h^*] + o(\varepsilon^2 R^\varepsilon[f,h^*]).$$

Since $|\hat\vartheta_\varepsilon^{JS} - \vartheta| \leq 1$, we have $\mathbf{E}_{\vartheta,f}[(\hat\vartheta_\varepsilon^{JS} - \vartheta)^2 \mathbb{1}_{\mathcal{A}^c}] \leq \mathbf{P}_{\vartheta,f}(\mathcal{A}^c) = O(\varepsilon^4)$. In view of (23), $\varepsilon^2 = o(R^\varepsilon[f,h^*])$. Therefore $\mathbf{E}_{\vartheta,f}[(\hat\vartheta_\varepsilon^{JS} - \vartheta)^2 \mathbb{1}_{\mathcal{A}^c}] = O(\varepsilon^4) = o(\varepsilon^2 R^\varepsilon[f,h^*])$ and the assertion of the theorem follows. $\qquad\square$

4.2. **Comparison with the monotone oracle.** Now we consider the class $\mathcal{H}_{\mathrm{mon}}$ of filters having decreasing components, that is

$$\mathcal{H}_{\mathrm{mon}} = \left\{h \in [0,1]^{N_\varepsilon} : h_k \geq h_{k+1}, \ 1 \leq k \leq N_\varepsilon - 1\right\}.$$

The class $\mathcal{H}_{\mathrm{mon}}$ is of high interest in statistics because it contains the most common filters such as the projection filter, the Pinsker filter, the Tikhonov or smoothing spline filter and so forth.

Proposition 2. *Set* $\gamma_\varepsilon = \max_{1 \leq j \leq J-1}(\sigma_{j+1}^2/\sigma_j^2)$. *Then*

$$R^\varepsilon[f,h^*] \leq \gamma_\varepsilon \inf_{h \in \mathcal{H}_{\mathrm{mon}}} R^\varepsilon[f,h] + \varepsilon^2\sigma_1^2.$$

A more general version of this result is Lemma 1 in [9]. Since the proof in our setting is simple, we give it below.

Proof. Let h be a filter from $\mathcal{H}_{\mathrm{mon}}$. Define $\bar{h}$ by $\bar{h}_k = h_{\kappa_j}$ if $k \in B_j$ for some j and $\bar{h}_k = 0$ if $k > N_\varepsilon$. Since the components of h are decreasing, we have $1 - \bar{h}_k \le 1 - h_k$ and therefore

$$R^\varepsilon[f, \bar{h}] \le \sum_{k=1}^{N_\varepsilon} (1 - h_k)^2 (2\pi k)^2 f_k^2 + \varepsilon^2 \sum_{j=1}^{J} h_{\kappa_j}^2 \sigma_j^2. \tag{14}$$

Again by monotonicity of h, we have $h_{\kappa_j} \le h_k$ for all $k \in B_{j-1}$. Hence,

$$\sum_{j=2}^{J} h_{\kappa_j}^2 \sigma_j^2 \le \gamma_\varepsilon \sum_{j=2}^{J} h_{\kappa_j}^2 \sigma_{j-1}^2 \le \gamma_\varepsilon \sum_{k=1}^{N_\varepsilon} h_k^2 (2\pi k)^2.$$

Combining this inequality with (14) and bounding h_1 by 1, we get $R^\varepsilon[f, \bar{h}] \le \gamma_\varepsilon R^\varepsilon[f, h] + \sigma_1^2$. Since $\bar{h} \in \mathcal{H}^*$ and h^* minimizes $R^\varepsilon[f, h]$ over all $h \in \mathcal{H}^*$, we have $R^\varepsilon[f, h^*] \le \gamma_\varepsilon R^\varepsilon[f, h] + \varepsilon^2 \sigma_1^2$. This inequality holds for every $h \in \mathcal{H}_{\mathrm{mon}}$, therefore the assertion of the proposition follows. $\square$

Combining this proposition with Theorem 1 we get the following result.

Corollary 1. Assume that the conditions of Theorem 1 are fulfilled, then

$$\varepsilon^{-2} \|f'\|^2 \mathbf{E}_{\vartheta, f}[(\hat{\vartheta}_\varepsilon^{JS} - \vartheta)^2] \le 1 + \gamma_\varepsilon (1 + \alpha_\varepsilon) \frac{\min_{h \in \mathcal{H}_{\mathrm{mon}}} R^\varepsilon[f, h]}{\|f'\|^2},$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in $f \in \mathcal{F}(\beta_*, L_*, \rho)$.

Remark 5. For the blocks defined by (11), we have $T_\varepsilon = \nu_\varepsilon \rho_\varepsilon (1 + \rho_\varepsilon)$, $\sigma_1^2 \le 4\pi^2 \nu_\varepsilon^3$ and $-\nu_\varepsilon \rho_\varepsilon + \nu_\varepsilon (1 + \rho_\varepsilon)^j \le \kappa_{j+1} \le 1 + \nu_\varepsilon (1 + \rho_\varepsilon)^j$. One also checks that $\gamma_\varepsilon = \max_j \sigma_{j+1}^2 / \sigma_j^2$ is asymptotically equivalent to $(1 + \rho_\varepsilon)^3 \sim 1 + 3\rho_\varepsilon$ as $\varepsilon \to 0$. Therefore the factor in the oracle inequality of Corollary 1 is of order $(1 + 3\rho_\varepsilon + \alpha_\varepsilon)$. We have already mentioned that $\alpha_\varepsilon^2 = O(T_\varepsilon^{-1} \log \varepsilon^{-1})$. The trade-off between α_ε and ρ_ε leads us to $\rho_\varepsilon \asymp \nu_\varepsilon^{-1/3}$. This clarifies our choice of ρ_ε slightly differing from the one of [9].

Remark 6. In [8, 28, 34] the weakly geometrically increasing blocks are defined by $T_j = \lfloor \nu (1 + \rho)^{j-1} \rfloor$. This type of blocks does not lead to a sharp oracle inequality in our case, since we need not only $\max(T_{j+1}/T_j) \to 1$, but also $\max(\kappa_{j+1}/\kappa_j) \to 1$ as $\varepsilon \to 0$.

4.3. **Second-order minimax sharp adaptation.** To complete the theoretical analysis, we show below that the estimator $\hat{\vartheta}_\varepsilon^{JS}$ corresponding to the blocks (11) enjoys minimax properties over a large scale of Sobolev balls. Assume that $\bar{f} \in \mathcal{F}(\beta^*, L^*, \rho)$ and define

$$\mathcal{F}_{\delta, \beta, L}(\bar{f}) = \left\{ f = \bar{f} + v \; : \; \|v\| \le \delta, \; \|v^{(\beta)}\| \le L \right\}.$$

Theorem 2. *Assume that ν_ε verifies $\varepsilon^{\frac{2}{2\beta+1}} \nu_\varepsilon \to 0$ as $\varepsilon \to 0$ and the conditions of Theorem 1 are fulfilled. If $\delta = \delta_\varepsilon$ tends to zero as $\varepsilon \to 0$ and $\bar{f} \in \mathcal{F}(\beta^*, L^*, \rho)$ with $\beta^* > \beta \ge \beta_*$, then the estimator $\hat{\vartheta}_\varepsilon^{JS}$ defined in Section 3.4 satisfies*

$$\sup_{\vartheta \in \Theta, f \in \mathcal{F}_{\delta, \beta, L}(\bar{f})} \varepsilon^{-2} \|f'\|^2 \mathbf{E}_{\vartheta, f}[(\hat{\vartheta}_\varepsilon^{JS} - \vartheta)^2] \le 1 + (1 + o(1)) \frac{C(\beta, L) \varepsilon^{\frac{4\beta-4}{2\beta+1}}}{\|\bar{f}'\|^2},$$

when $\varepsilon \to 0$, with $C(\beta, L) = \frac{1}{3}\left(\frac{\beta-1}{2\pi(\beta+2)}\right)^{\frac{2\beta-2}{2\beta+1}}(L(2\beta+1))^{\frac{3}{2\beta+1}}$. Moreover, the following lower bound holds:

$$\inf_{\tilde{\vartheta}_\varepsilon} \sup_{\vartheta \in \Theta, f \in \mathcal{F}_{\delta,\beta,L}(\bar{f})} \varepsilon^{-2}\|f'\|^2 \mathbf{E}_{\vartheta,f}[(\tilde{\vartheta}_\varepsilon - \vartheta)^2] \geq 1 + (1 + o(1))\frac{C(\beta,L)\varepsilon^{\frac{4\beta-4}{2\beta+1}}}{\|\bar{f}'\|^2},$$

where the inf is taken over all possible estimators $\tilde{\vartheta}_\varepsilon$.

Proof. According to Lemma 12, there exists a filter $\lambda^* \in \mathcal{H}_{\mathrm{mon}}$ such that

$$\sup_{f \in \mathcal{F}_{\delta,\beta,L}(\bar{f})} R^\varepsilon[f, \lambda^*]/\|f'\|^2 \leq (1 + o(1))C(\beta,L)\varepsilon^{\frac{4\beta-4}{2\beta+1}} \sup_f \|f'\|^{-2}.$$

Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, we have $\sup_{f \in \mathcal{F}_{\delta,\beta,L}(\bar{f})} \|f'\|^{-2} = (1+o(1))\|\bar{f}'\|^{-2}$. Combining this result with Corollary 1, we get the first inequality. The second inequality is Theorem 2 of [10]. Although the latter is stated for $\beta \geq 2$, the inspection of its proof shows that the same claim is true for any $\beta > 1$. $\qquad\square$

5. Preliminary estimator and technical lemmas

5.1. Preliminary estimator.
Having the observation $(x^\varepsilon(t), |t| \leq 1/2)$, we can compute x_1, x_1^* by (2). Then we have $x_1 = f_1 \cos(2\pi\vartheta) + \varepsilon\xi_1$ and $x_1^* = f_1 \sin(2\pi\vartheta) + \varepsilon\xi_1^*$, where ξ_1, ξ_1^* are independent standard Gaussian random variables. We define

$$\bar{\vartheta}_\varepsilon = \frac{1}{2\pi}\arctan\left(\frac{x_1^*}{x_1}\right),$$

if $x_1 \neq 0$ and $\bar{\vartheta}_\varepsilon = 1/4$ if $x_1 = 0$. One easily checks that $\bar{\vartheta}_\varepsilon$ is the maximum likelihood estimator in the model induced by observations (x_1, x_1^*). The following result describes its asymptotic behavior.

Proposition 3. *If ε is sufficiently small, then*

$$\sup_{|\vartheta| \leq T} \mathbf{P}_{\vartheta,f}\left(|\bar{\vartheta}_\varepsilon - \vartheta| \geq x\right) \leq \exp\left(-2(x/\varepsilon)^2 f_1^2 \cos^2(2\pi T)\right),$$

for all $x \in [0, 1/2]$ and for all $T < 1/4$.

Proof. Let us introduce $X = \sqrt{\xi_1^2 + \xi_1^{*2}}$. One checks that $\sin[2\pi(\bar{\vartheta}_\varepsilon - \vartheta)] = \varepsilon(\xi_1 \sin(2\pi\bar{\vartheta}_\varepsilon) - \xi_1^* \cos(2\pi\bar{\vartheta}_\varepsilon))f_1^{-1}$. The Cauchy-Schwarz inequality implies that $|\sin[2\pi(\bar{\vartheta}_\varepsilon - \vartheta)]| \leq \varepsilon X|f_1^{-1}|$. Since $\vartheta \in [-1/4, 1/4]$ and $\bar{\vartheta}_\varepsilon \in [-T, T]$, we have $\frac{\sin[2\pi(\bar{\vartheta}_\varepsilon - \vartheta)]}{2\pi(\bar{\vartheta}_\varepsilon - \vartheta)} \geq \cos(2\pi T)$. Therefore,

$$\mathbf{P}_{\vartheta,f}\left(|\bar{\vartheta}_\varepsilon - \vartheta| \geq x\right) \leq \mathbf{P}_{\vartheta,f}\left(\frac{|\sin[2\pi(\bar{\vartheta}_\varepsilon - \vartheta)]|}{2\pi\cos(2\pi T)} \geq x\right)$$
$$\leq \mathbf{P}_{\vartheta,f}\left(X \geq 2x\varepsilon^{-1}\pi|f_1|\cos(2\pi T)\right),$$

and the fact that $X^2/2$ follows the exponential law completes the proof. $\qquad\square$

5.2. **Proofs of Lemmas used in Proposition 1.** Let us start with some basic facts that will be often used in the proofs. For any $n, m, p \in \mathbb{N}$, we have

$$n^p(n - m) \geq \sum_{k=m+1}^{n} k^p \geq \frac{n^p(n - m)}{p + 1}. \tag{15}$$

Applying this inequality to $p = 2$, we get

$$\max_{k \in B_j}(2\pi k)^2 \leq 3\sigma_j^2/T_j. \tag{16}$$

Assume now that ξ is a random variable of law $\mathcal{N}(0, 1)$. For any $\sigma^2 \leq 1/4$, we have $(1 - 2\sigma^2)^{-1} \leq 2$ and $(1 - 2\sigma^2)^{-1/2} \leq e^{2\sigma^2}$, therefore

$$\mathbf{E}[e^{(\mu + \sigma\xi)^2}] = \frac{e^{\frac{\mu^2}{(1 - 2\sigma^2)}}}{\sqrt{1 - 2\sigma^2}} \leq e^{\frac{\mu^2}{(1 - 2\sigma^2)} + 2\sigma^2} \leq \exp\left(2\mu^2 + 2\sigma^2\right). \tag{17}$$

Using the more precise inequalities $(1 - 2\sigma^2)^{-1} \leq 1 + 4\sigma^2$ and $\log(1 - 2\sigma^2)^{-1} \leq 2\sigma^2 + 4\sigma^4$, we get $\mathbf{E}[e^{(\mu + \sigma\xi)^2}] \leq \exp(\mu^2 + \sigma^2 + 2\sigma^2(2\mu^2 + \sigma^2))$ or equivalently,

$$\mathbf{E}[e^{2\mu\sigma\xi + \sigma^2(\xi^2 - 1)}] \leq e^{2\sigma^2(2\mu^2 + \sigma^2)}. \tag{18}$$

Throughout this section, we assume that $\|f^{(\beta_*)}\|^2 = \sum_k (2\pi k)^{2\beta_*} f_k^2 < \infty$ for some $\beta_* > 1$, $N_\varepsilon = [(\varepsilon^2 \log \varepsilon^{-5})^{-\frac{1}{2\beta_* + 1}}]$ and $h \in [0, 1]^{N_\varepsilon}$ is a random vector depending on (x_k, x_k^*) only via $x_k^2 + x_k^{*2}$. Without loss of generality, we give the proofs in the case $\vartheta = 0$.

Lemma 1. *Set $\hat{\beta}_* = \beta_* \wedge 1.5$. For all τ such that $4\pi|\tau - \vartheta| \leq N_\varepsilon^{-1}$,*

$$\left|\Phi_\varepsilon''(\tau, h) + \sum_{k=1}^{N_\varepsilon}(2\pi k)^2 h_k f_k^2\right| \leq N_\varepsilon^{2 - 2\hat{\beta}_*}\left(\|f^{(\beta_*)}\| + 2\pi\hat{X}/\sqrt{\log \varepsilon^{-5}}\right)^2,$$

where $\hat{X} = \max_{1 \leq k \leq N_\varepsilon} \sqrt{\xi_k^2 + \xi_k^{*2}}$.

Proof. One easily checks that

$$\Phi_\varepsilon''(\vartheta + \tau, h) = -\sum_{k=1}^{N_\varepsilon}(2\pi k)^2 h_k f_k^2 \cos[4\pi k\tau]$$

$$- 2\varepsilon \sum_{k=1}^{N_\varepsilon}(2\pi k)^2 h_k f_k(\xi_k \cos[4\pi k\tau] + \xi_k^* \sin[4\pi k\tau])$$

$$- \varepsilon^2 \sum_{k=1}^{N_\varepsilon}(2\pi k)^2 h_k\left[(\xi_k^2 - \xi_k^{*2})\cos[4\pi k\tau] + 2\xi_k^*\xi_k \sin[4\pi k\tau]\right].$$

On the one hand, thanks to inequality $|1 - \cos x| \leq |x|$,

$$\left|\sum_{k=1}^{N_\varepsilon}(2\pi k)^2 h_k f_k^2(1 - \cos[4\pi k\tau])\right| \leq 2\tau \sum_{k=1}^{N_\varepsilon}(2\pi k)^3 f_k^2$$

$$\leq 2\tau(2\pi N_\varepsilon)^{3 - 2\hat{\beta}_*}\|f^{(\beta_*)}\|^2$$

$$\leq N_\varepsilon^{2 - 2\hat{\beta}_*}\|f^{(\beta_*)}\|^2.$$

On the other hand, in view of the Cauchy-Schwarz inequality, it holds $\xi_k \cos[4\pi k\tau] + \xi_k^* \sin[4\pi k\tau] \leq \hat{X}$ and $(\xi_k^2 - \xi_k^{*2})\cos[4\pi k\tau] + 2\xi_k^*\xi_k \sin[4\pi k\tau] \leq \hat{X}^2$. Therefore, it holds

$$\left| \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k f_k (\xi_k \cos[4\pi k\tau] + \xi_k^* \sin[4\pi k\tau]) \right| \leq \hat{X} \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 |f_k|$$

$$\leq \hat{X} \|f^{(\beta_*)}\| \sqrt{\sum_{k=1}^{N_\varepsilon} (2\pi k)^{4-2\beta_*}} \leq 2\pi \hat{X} N_\varepsilon^{\frac{5}{2}-\hat{\beta}_*} \|f^{(\beta_*)}\|,$$

and

$$\left| \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k \left[(\xi_k^2 - \xi_k^{*2})\cos[4\pi k\tau] + 2\xi_k^*\xi_k \sin[4\pi k\tau] \right] \right| \leq 4\pi^2 N_\varepsilon^3 \hat{X}^2.$$

Taking into account the identity $\varepsilon^2 N_\varepsilon^3 = N_\varepsilon^{2-2\beta_*}/\log(\varepsilon^{-5})$, for all τ verifying $|\tau| \leq (4\pi N_\varepsilon)^{-1}$, we get

$$\left| \Phi_\varepsilon''(\vartheta + \tau, h) + \sum_{k=1}^{N_\varepsilon} h_k (2\pi k)^2 f_k^2 \right| \leq N_\varepsilon^{2-2\hat{\beta}_*} (\|f^{(\beta_*)}\|^2 + 2\pi \hat{X}/\sqrt{\log \varepsilon^{-5}})^2$$

and the assertion of the lemma follows. $\square$

Lemma 2. *Let $h \in [0,1]^{N_\varepsilon}$ be a random vector depending on (x_k, x_k^*) only via $x_k^2 + x_k^{*2}$. For any $x \in [0, \sqrt{N_\varepsilon/6}]$, it holds*

$$\mathbf{P}_{\vartheta,f}\left(|\Phi_\varepsilon'(\vartheta, h)| > 2x\varepsilon(\|f'\| + 2\pi\varepsilon N_\varepsilon^{3/2}) \right) \leq 2e^{-x^2}$$

Proof. The random variables $X_k = (2\pi k)(f_k + \varepsilon\xi_k)\xi_k^*$, $k = 1, \ldots, N_\varepsilon$ and $h_1, \ldots, h_N$ fulfill the conditions of Lemma 14 with $\varrho_k^* = 1$, $T_k = 1/(2\sqrt{2}\pi k\varepsilon)$, $g_k^2 = (2\pi k)^2(f_k^2 + \varepsilon^2)$, since due to (17),

$$\mathbf{E}[e^{tX_k}] = \mathbf{E}[e^{(2\pi kt)^2(f_k+\varepsilon\xi_k)^2/2}] \leq e^{(2\pi kt)^2(f_k^2+\varepsilon^2)}. \tag{19}$$

By definition, $\Phi_\varepsilon'(\vartheta, h) = \varepsilon \sum_{k=1}^{N_\varepsilon} h_k X_k$, and therefore,

$$\mathbf{P}_{\vartheta,f}\left(|\Phi_\varepsilon'(\vartheta, h)| \geq 2x\varepsilon \Big(\sum_{k=1}^{N_\varepsilon} (2\pi k)^2(f_k^2 + \varepsilon^2) \Big)^{1/2} \right) \leq 2e^{-x^2}$$

for all $x \in [0, (\sum_k (2\pi k)^2(f_k^2+\varepsilon^2))^{1/2}/(2\sqrt{2}\pi N_\varepsilon \varepsilon)]$. To complete the proof, it suffices to remark that

$$\frac{\sum_{k=1}^{N_\varepsilon}(2\pi k)^2(f_k^2 + \varepsilon^2)}{8\pi^2 N_\varepsilon^2 \varepsilon^2} \geq \frac{\sum_{k=1}^{N_\varepsilon} k^2}{2N_\varepsilon^2} \geq \frac{N_\varepsilon}{6}$$

and $\sum_{k=1}^{N_\varepsilon}(2\pi k)^2(f_k^2 + \varepsilon^2) \leq (\|f'\| + \varepsilon(2\pi)N_\varepsilon^{3/2})^2$. $\square$

Proposition 4. *Assume that $\|f^{(\beta_*)}\| < \infty$ for some $\beta_* > 1$ and set $\hat{\beta}_* = \beta_* \wedge 1.5$. There exists an event $\mathcal{A}_\varepsilon$ such that for every $\varepsilon < 1/2$, $\mathbf{P}_{\vartheta,f}(\mathcal{A}_\varepsilon) \geq 1 - 2\varepsilon^4$ and on*

$\mathcal{A}_\varepsilon$ it holds:

$$\Phi''_\varepsilon(\tau, h) \leq -\sum_{k=1}^{N_\varepsilon} h_k (2\pi k)^2 f_k^2 + N_\varepsilon^{2-2\hat{\beta}_*} \left(\|f^{(\beta_*)}\| + 3\pi \right)^2 \tag{20}$$

$$\frac{\Phi'_\varepsilon(\tau, h)}{\tau - \vartheta} \leq -\sum_{k=1}^{N_\varepsilon} h_k (2\pi k)^2 f_k^2 + N_\varepsilon^{2-2\hat{\beta}_*} \left(\|f^{(\beta_*)}\| + 3\pi \right)^2$$
$$+ \frac{2\varepsilon(\|f'\|\sqrt{\log \varepsilon^{-5}} + 2\pi N_\varepsilon^{1-\beta_*})}{|\tau - \vartheta|}. \tag{21}$$

for all $\tau \in [\vartheta - (4\pi N_\varepsilon)^{-1}, \vartheta + (4\pi N_\varepsilon)^{-1}]$.

Proof. According to Lemma 1, we have

$$\Phi''_\varepsilon(\tau, h) \leq -\sum_{k=1}^{N_\varepsilon} h_k (2\pi k)^2 f_k^2 + N_\varepsilon^{2-2\hat{\beta}_*} \left(\|f^{(\beta_*)}\| + 2\pi \hat{X}/\sqrt{\log \varepsilon^{-5}} \right)^2.$$

Since for every k, $(\xi_k^2 + \xi_k^{*2})/2$ follows the exponential law with mean 1, we have

$$\mathbf{P}\left(4\hat{X}^2 > 9 \log \varepsilon^{-5}\right) \leq N_\varepsilon \mathbf{P}(X_1^2/2 > \log \varepsilon^{-5}) \leq N_\varepsilon \varepsilon^5 \leq \varepsilon^4.$$

This inequality completes the proof of (20).

To prove (21), note that for some $\tilde{\tau} \in [\vartheta, \tau]$, we have $\Phi'_\varepsilon(\tau, h) = \Phi'_\varepsilon(\vartheta, h) + (\tau - \vartheta)\Phi''_\varepsilon(\tilde{\tau}, h)$. Lemma 2 and (20) yield (21). $\qquad \square$

5.3. **Lemmas used in Theorem 1.** Let us start with some simple algebra allowing to obtain a rough evaluation of $R^\varepsilon[f, h^*]$, where h^* is the ideal filter an oracle would choose in the class of blockwise constant filters. For this filter h^*, it holds

$$R^\varepsilon[f, h^*] = \sum_{j=1}^{J} \frac{\varepsilon^2 \sigma_j^2 \|f'\|_{(j)}^2}{\|f'\|_{(j)}^2 + \varepsilon^2 \sigma_j^2} + \sum_{k > N_\varepsilon} (2\pi k)^2 f_k^2.$$

Using the explicit form of h_k^*, we get

$$R^\varepsilon[f, h^*] = \sum_{k=1}^{\infty} (1 - h_k^*)(2\pi k)^2 f_k^2 \geq \varepsilon^2 \sum_{k=1}^{N_\varepsilon} h_k^* (2\pi k)^2. \tag{22}$$

Since $\varepsilon^2 N_\varepsilon^3 \to 0$ as $\varepsilon \to 0$, we have $R^\varepsilon[f, h^*] \leq C N_\varepsilon^{2-2\beta_*} \to 0$ as $\varepsilon \to 0$. In view of $\|f'\|_{(1)}^2 \geq 4\pi^2 \rho^2$ and $\varepsilon^2 \sigma_1^2 \leq 4\pi^2 \varepsilon^2 N_\varepsilon^3 \to 0$ as $\varepsilon \to 0$, for ε small enough the inequality $\varepsilon^2 \sigma_1^2 \leq \|f'\|_{(1)}^2$ holds. Therefore,

$$R^\varepsilon[f, h^*] \geq \frac{\varepsilon^2 \sigma_1^2 \|f'\|_{(1)}^2}{\|f'\|_{(1)}^2 + \varepsilon^2 \sigma_1^2} \geq \frac{\varepsilon^2 \sigma_1^2}{2}. \tag{23}$$

Hence, for every function f, the quantity $R^\varepsilon[f, h^*]$ tends to zero as $\varepsilon \to 0$ slower than ε^2 and faster than $\varepsilon^2 N_\varepsilon^3$.

Lemma 3. *It holds* $\mathbf{E}[\Phi'_\varepsilon(\vartheta, \tilde{h})^2] \leq \varepsilon^2 \sum_{k=1}^{N_\varepsilon} h_k^{*2}(2\pi k)^2(f_k^2 + \varepsilon^2).$

Proof. Using (6), one checks that

$$\Phi'_\varepsilon(\vartheta, \tilde{h}) = \varepsilon \sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)(f_k + \varepsilon\xi_k(\vartheta))\xi_k^*(\vartheta). \tag{24}$$

In the sequel, we write ξ_k, ξ_k^* instead of $\xi_k(\vartheta), \xi_k^*(\vartheta)$. For any $k' \neq k$, the random variable $\tilde{h}_k(f_k + \varepsilon\xi_k)\xi_k^*\tilde{h}_{k'}(f_{k'} + \varepsilon\xi_{k'})\xi_{k'}^*$ is symmetric. Therefore it has zero mean and

$$\mathbf{E}[\Phi'_\varepsilon(\vartheta, \tilde{h})^2] = \varepsilon^2 \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 \mathbf{E}\big[\tilde{h}_k^2 (f_k + \varepsilon\xi_k)^2 \xi_k^{*2}\big]$$

$$\leq \varepsilon^2 \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k^{*2} \mathbf{E}\big[(f_k + \varepsilon\xi_k)^2 \xi_k^{*2}\big],$$

and the assertion of the lemma follows. $\qquad\square$

Lemma 4. *Let us denote*

$$\eta_j = \frac{2\varepsilon \sum_{k \in B_j} (2\pi k)^2 f_k \xi_k + \varepsilon^2 \sum_{k \in B_j} (2\pi k)^2 (\xi_k^2 + \xi_k^{*2} - 2)}{\|f'\|_{(j)}^2 + \varepsilon^2 \sigma_j^2}.$$

For any positive x such that $x^2 \leq T_j/10$, it holds

$$\mathbf{P}\left(|\eta_j| > x\sqrt{24(1 - h_{\kappa_j}^*)T_j^{-1}}\right) \leq 2e^{-x^2}.$$

Proof. Set $Y_k = 2\varepsilon(2\pi k)^2 f_k \xi_k + \varepsilon^2 (2\pi k)^2 (\xi_k^2 + \xi_k^{*2} - 2)$, $\sigma = \varepsilon(2\pi k)\sqrt{t}$ and $\mu = (2\pi k)f_k\sqrt{t}$. Using (18) we infer that

$$\mathbf{E}[e^{tY_k}] = \mathbf{E}[e^{2\mu\sigma\xi_k + \sigma^2(\xi_k^2 - 1)}]\,\mathbf{E}[e^{\sigma^2(\xi_k^{*2} - 1)}] \leq e^{4\sigma^2(\mu^2 + \sigma^2)}$$

$$= e^{4\varepsilon^2(2\pi k)^4 t^2(f_k^2 + \varepsilon^2)}$$

as soon as $\sigma = \varepsilon(2\pi k)\sqrt{t} \leq 1/2$, or equivalently $t \leq 1/4\varepsilon^2(2\pi k)^2$. By [25, Thm. 2.7], for any $x > 0$, we get

$$\mathbf{P}\left(\left|\sum_{k \in B_j} Y_j\right| \geq x\left(2\sum_{k \in B_j} 4\varepsilon^2(2\pi k)^4(f_k^2 + \varepsilon^2)\right)^{1/2}\right) \leq 2e^{-x^2(1 \wedge Q_\varepsilon x^{-1})}$$

where

$$Q_\varepsilon = \frac{(8\varepsilon^2 \sum_{k \in B_j} (2\pi k)^4(f_k^2 + \varepsilon^2))^{1/2}}{4(2\varepsilon\pi\kappa_{j+1})^2}.$$

It is clear that

$$Q_\varepsilon \geq \frac{(8\sum_{k \in B_j}(2\pi k)^4)^{1/2}}{4(2\pi\kappa_{j+1})^2} \geq \frac{(8(2\pi)^4\kappa_{j+1}^4 T_j/5)^{1/2}}{4(2\pi\kappa_{j+1})^2} \geq \sqrt{T_j/10}$$

$$\sum_{k \in B_j}(2\pi k)^4(f_k^2 + \varepsilon^2) \leq (2\pi\kappa_{j+1})^2(\|f'\|_{(j)}^2 + \varepsilon^2\sigma_j^2) \leq 3T_j^{-1}\sigma_j^2(\|f'\|_{(j)}^2 + \varepsilon^2\sigma_j^2)$$

$$\leq \frac{3(1 - h_{\kappa_j}^*)(\|f'\|_{(j)}^2 + \varepsilon^2\sigma_j^2)^2}{T_j\varepsilon^2}.$$

These inequalities combined with the identity $\eta_j = \sum_{k \in B_j} Y_j/(\|f'\|_{(j)}^2 + \varepsilon^2\sigma_j^2)$ yield the desired result. $\qquad\square$

Lemma 5. *Assume that $\varphi_j^2 T_j \geq 24 \log \varepsilon^{-5}$ and $\varphi_j < 1$. Then*

$$\mathbf{P}\left(\hat{h}_k^{JS} \in \left[h_k^* - \frac{2(1-h_k^*)\varphi_j}{(1-\varphi_j)}, h_k^*\right]\right) \geq 1 - 2\varepsilon^5.$$

Proof. Note that

$$\hat{h}_k^{JS} = \left(1 - \frac{(1-h_k^*)(1+\varphi_j)}{1+\eta_j}\right)\mathbb{1}_{\{\eta_j > -h_k^* + \varphi_j(1-h_k^*)\}}. \tag{25}$$

One checks that $\hat{h}_k^{JS} > h_k^*$ if and only if $\eta_j > \varphi_j$. Therefore, $\mathbf{P}(\hat{h}_k^{JS} > h_k^*) = \mathbf{P}(\eta_j > \varphi_j) \leq \varepsilon^4$. Similarly,

$$\mathbf{P}\left(\hat{h}_k^{JS} < h_k^* - \frac{2(1-h_k^*)\varphi_j}{(1-\varphi_j)}\right) = \mathbf{P}\left(\varphi_j - (1+\varphi_j)h_k^* \leq \eta_j \leq -\varphi_j\right)$$
$$+ \mathbf{P}(\eta_j < \varphi_j - (1+\varphi_j)h_k^*)\mathbb{1}_{\{h_k^* > \frac{2(1-h_k^*)\varphi_j}{(1-\varphi_j)}\}}$$
$$\leq \mathbf{P}(\eta_j \leq -\varphi_j),$$

since $h_k^* > \frac{2(1-h_k^*)\varphi_j}{(1-\varphi_j)}$ if and only if $\varphi_j - (1+\varphi_j)h_k^* < -\varphi_j$. Therefore, using Lemma 4,

$$\mathbf{P}\left(\hat{h}_k^{JS} \in \left[h_k^* - \frac{2(1-h_k^*)\varphi_j}{(1-\varphi_j)}, h_k^*\right]\right) \geq \mathbf{P}(|\eta_j| \leq \varphi_j) \geq 1 - 2\varepsilon^5$$

and the assertion of the lemma follows. $\qquad\square$

Lemma 6. *For any positive x verifying $x^2 \leq T_\varepsilon/5$ it holds*

$$\mathbf{P}\left(\left|\sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^2(\xi_k^2 - \xi_k^{*2})\right| \geq \frac{12\sqrt{2}x}{\sqrt{T_\varepsilon}} R^\varepsilon[f, h^*]\right) \leq 2Je^{-x^2}.$$

Proof. Using (16), we get $\left(\sum_{k \in B_j}(2\pi k)^4\right)^{1/2} \leq 3\sigma_j^2/\sqrt{T_j}$. This inequality combined with the fact that $\tilde{h}_k \leq h_k^*$, allows us to bound the probability of the event of interest by

$$\mathbf{P}\left(\left|\sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^2(\xi_k^2 - \xi_k^{*2})\right| \geq \sqrt{32}\, x \sum_{j=1}^{J} h_{\kappa_j}^* \left(\sum_{k \in B_j}(2\pi k)^4\right)^{\frac{1}{2}}\right)$$
$$\leq \sum_{j=1}^{J} \mathbf{P}\left(\tilde{h}_{\kappa_j}\left|\sum_{k \in B_j}(2\pi k)^2(\xi_k^2 - \xi_k^{*2})\right| \geq \sqrt{32}\, x h_{\kappa_j}^*\left(\sum_{k \in B_j}(2\pi k)^4\right)^{1/2}\right)$$
$$\leq \sum_{j=1}^{J} \mathbf{P}\left(\left|\sum_{k \in B_j}(2\pi k)^2(\xi_k^2 - \xi_k^{*2})\right| \geq \sqrt{32}\, x\left(\sum_{k \in B_j}(2\pi k)^4\right)^{1/2}\right).$$

The desired result follows now from Lemma 13 and (15). $\qquad\square$

Lemma 7. *For any $x > 0$,*

$$\mathbf{P}\left(\varepsilon\left|\sum_{k=1}^{N_\varepsilon}(\tilde{h}_k - h_k^*)(2\pi k)^2\xi_k f_k\right| \geq 5x\sqrt{\frac{\varphi_\varepsilon}{T_\varepsilon}} R^\varepsilon[f, h^*]\right) \leq 2Je^{-x^2}.$$

Proof. Remark first that

$$(\tilde{h}_{\kappa_j} - h^*_{\kappa_j})^2 \leq h^{*\,2}_{\kappa_j} \wedge 8(1 - h^*_{\kappa_j})^2 \varphi_j^2 \leq \sqrt{8}\varphi_j h^*_{\kappa_j}(1 - h^*_{\kappa_j})$$

for all $j = 1, \ldots, J$. Set $Y_j = \varepsilon \sum_{k \in B_j} (2\pi k)^2 \xi_k f_k$. The random variables $Y_1, \ldots, Y_J$ are independent zero mean Gaussian with variance

$$\mathbf{E}[Y_j^2] = \varepsilon^2 \sum_{k \in B_j} (2\pi k)^4 f_k^2 \leq 3\varepsilon^2 \sigma_j^2 T_j^{-1} \sum_{k \in B_j} (2\pi k)^2 f_k^2.$$

Therefore, $\mathbf{P}(|Y_j| \geq \sqrt{6/T_j}\, x\varepsilon \sigma_j \|f'\|_{(j)}) \leq 2e^{-x^2}$ and consequently,

$$\mathbf{P}\left(\sum_{j=1}^n |(\tilde{h}_{\kappa_j} - h^*_{\kappa_j})Y_j| \geq \sqrt{\frac{6}{T_\varepsilon}}\, x\varepsilon \sum_{j=1}^J \sqrt{4\varphi_j h^*_{\kappa_j}(1 - h^*_{\kappa_j})}\sigma_j \|f'\|_{(j)} \right) \leq 2Je^{-x^2}.$$

To complete the proof, note that

$$\varepsilon \sum_{j=1}^J \sqrt{h^*_{\kappa_j}(1 - h^*_{\kappa_j})}\, \sigma_j \|f'\|_{(j)} \leq \frac{1}{2} \sum_{j=1}^J (1 - h^*_{\kappa_j}) \|f'\|_{(j)}^2 + \frac{1}{2} \sum_{j=1}^J h^*_{\kappa_j}(\varepsilon \sigma_j)^2$$

and the right side is bounded by $R^\varepsilon[f, h^*]$. $\qquad\square$

Lemma 8. *We have* $\sum_{k=1}^{N_\varepsilon} (\tilde{h}_k - h^*_k)(2\pi k)^2 f_k^2 \leq 4\varphi_\varepsilon R^\varepsilon[f, h^*]$.

Proof. The desired inequality is trivially fulfilled on $\mathcal{E}^c$, while on $\mathcal{E}$ we have $0 \leq h^*_k - \tilde{h}_k \leq 4\varphi_\varepsilon(1 - h^*_k)$, $\forall k \in B_j$, and hence $\sum_{k=1}^\infty |\tilde{h}_k - h^*_k|(2\pi k)^2 f_k^2 \leq 4\varphi_\varepsilon \sum_{k=1}^{N_\varepsilon} (1 - h^*_k)(2\pi k)^2 f_k^2 \leq 4\varphi_\varepsilon R^\varepsilon[f, h^*]$. $\qquad\square$

Lemma 9. *Set* $\hat{X} = \max_{1 \leq k \leq N_\varepsilon} \sqrt{\xi_k^2(\vartheta) + \xi_k^{*\,2}(\vartheta)}$. *There exists an event of probability at least* $1 - 2\varepsilon^4$ *such that on this event, for all* $\tau \in \mathbb{R}$, *we have*

$$\frac{|\Phi_\varepsilon'''(\tau, \tilde{h})|}{R^\varepsilon[f, h^*]} \leq \frac{12|\tau - \vartheta| \cdot \|f'\|^2}{\varepsilon^2 T_\varepsilon} + \frac{12\|f'\|\sqrt{\log \varepsilon^{-8}}}{\varepsilon T_\varepsilon}$$
$$+ \frac{16\sqrt{3}\,\pi N_\varepsilon |\tau - \vartheta|\hat{X}\|f'\|}{\varepsilon\sqrt{T_\varepsilon}} + 4\pi N_\varepsilon \hat{X}^2.$$

Proof. Using (6), one checks that

$$\Phi_\varepsilon'''(\tau, h) = 2 \sum_{k=1}^\infty h_k(2\pi k)^3 [(f_k + \varepsilon\xi_k(\vartheta))^2 - \varepsilon^2 \xi_k^*(\vartheta)^2] \sin[4\pi k(\tau - \vartheta)]$$
$$- 4\varepsilon \sum_{k=1}^\infty h_k(2\pi k)^3 (f_k + \varepsilon\xi_k(\vartheta))\xi_k^*(\vartheta) \cos[4\pi k(\tau - \vartheta)].$$

Without loss of generality, we assume in the sequel that $\vartheta = 0$. Then

$$
\begin{aligned}
\left|\Phi_\varepsilon'''(\tau, \tilde{h})\right| &\leq 2 \sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^3 f_k^2 |\sin(4\pi k\tau)| + 4\varepsilon \sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^3 |f_k \xi_k^*| \\
&\quad + 4\varepsilon \sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^3 \left| f_k \left[\xi_k \sin(4\pi k\tau) + \xi_k^*(\cos(4\pi k\tau) - 1) \right] \right| \\
&\quad + 2\varepsilon^2 \sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^3 \left| (\xi_k^2 - \xi_k^{*2}) \sin(4\pi k\tau) - 2\xi_k \xi_k^* \cos(4\pi k\tau) \right|.
\end{aligned}
$$

Using the inequalities $|\sin(4\pi k\tau)| \leq 4\pi k |\tau|$ and

$$
\sum_{k=1}^{N_\varepsilon} \tilde{h}_k (2\pi k)^4 f_k^2 \leq \sum_{j=1}^{J} h_{\kappa_j}^* \|f'\|_{(j)}^2 (2\pi \kappa_{j+1})^2 \leq \frac{3\|f'\|^2}{T_\varepsilon} \sum_{j=1}^{J} h_{\kappa_j}^* \sigma_j^2,
$$

as well as the inequality $\varepsilon^2 \sum_{j=1}^{J} h_{\kappa_j}^* \sigma_j^2 \leq R^\varepsilon[f, h^*]$, we get the desired bound for the first sum. The bound on the second term is obtained using Lemma 14, the well known bound on the Laplace transform of a Gaussian distribution and the inequality

$$
\begin{aligned}
\sum_{k=1}^{N_\varepsilon} h_k^{*2} (2\pi k)^6 f_k^2 &\leq \sum_{j=1}^{J} h_{\kappa_j}^{*2} (2\pi \kappa_{j+1})^4 \|f'\|_{(j)}^2 \leq \frac{9}{T_\varepsilon^2} \sum_{j=1}^{J} h_{\kappa_j}^{*2} \sigma_j^4 \|f'\|_{(j)}^2 \\
&\leq \frac{9\|f'\|^2}{T_\varepsilon^2} \max_j h_{\kappa_j}^{*2} \sigma_j^4 \leq \frac{9\|f'\|^2 R^\varepsilon[f, h^*]^2}{\varepsilon^4 T_\varepsilon^2}.
\end{aligned}
$$

The bounds on the two remaining sums are obtained by combining the inequalities

$$
\begin{aligned}
|\xi_k \sin(4\pi k\tau) + \xi_k^*(\cos(4\pi k\tau) - 1)| &\leq |4\pi k\tau| \cdot |\xi_k \cos(4\pi k\tilde{\tau}) - \xi_k^* \sin(4\pi k\tilde{\tau})| \\
&\leq 4\pi k |\tau| \hat{X},
\end{aligned}
$$

$$
|(\xi_k^2 - \xi_k^{*2}) \sin(4\pi k\tau) - 2\xi_k \xi_k^* \cos(4\pi k\tau)| \leq \hat{X}^2
$$

with arguments similar to those used to bound the first two sums. $\qquad\square$

Lemma 10. *For any $x > 0$, it holds*

$$
\mathbf{P}\left(\left| \Phi_\varepsilon'(\vartheta, \tilde{h}) - \Phi_\varepsilon'(\vartheta, h^*) \right|^2 \geq 12\varepsilon^2 x^2 \varphi_\varepsilon R^\varepsilon[f, h^*] \right) \leq 2e^{-x^2(1 \wedge \sqrt{T_\varepsilon/8x^2})}.
$$

Proof. Let us denote $X_k = (2\pi k)(f_k + \varepsilon \xi_k)\xi_k^*$. According to (24),

$$
\Phi_\varepsilon'(\vartheta, \tilde{h}) - \Phi_\varepsilon'(\vartheta, h^*) = \varepsilon \sum_{j=1}^{k} (\tilde{h}_k - h_k^*) X_k.
$$

According to (19), for all $t \leq 1/(2\sqrt{2}\pi k\varepsilon)$, we have $\mathbf{E}[e^{tX_k}] \leq e^{t^2(2\pi k)^2(f_k^2 + \varepsilon^2)}$. Thus the conditions of Lemma 14 are fulfilled with $\varrho_k = \tilde{h}_k - h_k^*$, $\varrho_k^* = 3\sqrt{h_k^* \varphi_j(1 - h_k^*)}$, $T_k = 1/(\sqrt{8}\pi k\varepsilon)$ and $g_k^2 = (2\pi k)^2(f_k^2 + \varepsilon^2)$. Therefore,

$$
\mathbf{P}\left(\left| \Phi_\varepsilon'(\vartheta, \tilde{h}) - \Phi_\varepsilon'(\vartheta, h^*) \right|^2 \geq 12\varepsilon^2 x^2 \varphi_\varepsilon \sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k^*(1 - h_k^*)(f_k^2 + \varepsilon^2) \right) \leq 2e^{-x^2}
$$

for any $x > 0$ verifying

$$x^2 \leq \frac{3 \sum_{k=1}^{N_\varepsilon} \varphi_{j(k)} (2\pi k)^2 h_k^* (1 - h_k^*)(f_k^2 + \varepsilon^2)}{8\varepsilon^2 \max_j \varphi_j h_{\kappa_j}^* (1 - h_{\kappa_j}^*)(2\pi \kappa_{j+1})^2}.$$

To complete the proof, it suffices to remark that

$$\sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k^* (1 - h_k^*)(f_k^2 + \varepsilon^2) = \sum_{j=1}^{J} h_{\kappa_j}^* (1 - h_{\kappa_j}^*)[\|f'\|_{(j)}^2 + \varepsilon^2 \sigma_j^2] \leq R^\varepsilon[f, h^*]$$

and

$$\frac{3 \sum_{k=1}^{N_\varepsilon} \varphi_{j(k)} (2\pi k)^2 h_k^* (1 - h_k^*)(f_k^2 + \varepsilon^2)}{8\varepsilon^2 \max_j \varphi_j h_{\kappa_j}^* (1 - h_{\kappa_j}^*)(2\pi \kappa_{j+1})^2} \geq \frac{3 \max_j \varphi_j h_{\kappa_j}^* (1 - h_{\kappa_j}^*) \sigma_j^2}{8 \max_j \varphi_j h_{\kappa_j}^* (1 - h_{\kappa_j}^*)(2\pi \kappa_{j+1})^2}.$$

Since $\sigma_j^2 \geq T_j (2\pi \kappa_{j+1})^2 / 3$, the assertion of the lemma follows.   □

Lemma 11. *Let* $\zeta = 2\varepsilon \|f'\|^{-2} \sum_k h_k^* (2\pi k)^2 f_k \xi_k$. *For any event* $\mathcal{A}$ *verifying* $\mathbf{P}(\mathcal{A}^c) = O(\varepsilon^4)$, *we have*

$$\mathbf{E}[\Phi'_\varepsilon(\vartheta, h^*)^2 \zeta \mathbb{1}_\mathcal{A}] = o(\varepsilon^2 R^\varepsilon[f, h^*]).$$

Proof. We have

$$\mathbf{E}[\Phi'_\varepsilon(\vartheta, h^*)^2 \zeta] = \varepsilon^2 \mathbf{E}\left[\sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k^{*2} (f_k + \varepsilon \xi_k(\vartheta))^2 \xi_k^*(\vartheta)^2 \zeta \right]$$

$$= 2\varepsilon^3 \mathbf{E}\left[\sum_{k=1}^{N_\varepsilon} (2\pi k)^2 h_k^{*2} f_k \xi_k \zeta \right] = 4\varepsilon^4 \|f'\|^{-2} \sum_{k=1}^{N_\varepsilon} (2\pi k)^4 h_k^{*3} f_k^2$$

$$\leq 12 T_\varepsilon^{-1} \varepsilon^4 \sum_{j=1}^{J} h_{\kappa_j}^{*\,3} \sigma_j^2 \leq 12 T_\varepsilon^{-1} \varepsilon^2 R^\varepsilon[f, h^*].$$

Using the Rosenthal inequality, one easily checks that $\mathbf{E}[\Phi'_\varepsilon(\vartheta, h^*)^{2p}] = O(\varepsilon^{2p})$ and $\mathbf{E}[\zeta^{2p}] = o(R^\varepsilon[f, h^*]^p)$ for any integer $p > 0$. Therefore, the Cauchy-Schwarz inequality yields,

$$\left| \mathbf{E}[\Phi'_\varepsilon(\vartheta, h^*)^2 \zeta \mathbb{1}_{\mathcal{A}^c}] \right| \leq o\big(\varepsilon^2 \sqrt{R^\varepsilon[f, h^*]}\big) \sqrt{\mathbf{P}(\mathcal{A}^c)} = o\big(\varepsilon^4 \sqrt{R^\varepsilon[f, h^*]}\big)$$

and the assertion of the lemma follows.   □

5.4. **Lemma used in Theorem 2.** We assume that $f \in \mathcal{F}_{\delta,\beta,L}(\bar{f})$ with $\bar{f} \in \mathcal{F}(\beta^*, L^*, \rho)$ and $\beta^* > \beta \geq \beta_*$. For the sake of completeness we give below a suitable version of the Pinsker theorem [27].

Lemma 12. *Set* $\gamma_\varepsilon = 1/\log \varepsilon^{-2}$, $W_\varepsilon = \big(\frac{L}{\varepsilon^2} \frac{(\beta+2)(2\beta+1)}{(2\pi)^{2\beta}(\beta-1)}\big)^{1/(2\beta+1)}$ *and define*

$$\lambda_k^* = \begin{cases} 1, & k \leq \gamma_\varepsilon W_\varepsilon, \\ \left[1 - \left(\frac{k}{W_\varepsilon} \right)^{\beta-1} \right]_+, & k > \gamma_\varepsilon W_\varepsilon. \end{cases}$$

The filter λ_k^* *satisfies*

$$\sup_{f \in \mathcal{F}_{\delta,\beta,L}} R^\varepsilon[f, \lambda^*] \leq (1 + o(1)) C(\beta, L) \varepsilon^{\frac{4\beta-4}{2\beta+1}}.$$

Proof. Set $v = f - \bar{f}$. Using the inequality $(\bar{f}_k + v_k)^2 \leq 2z^{-1}\bar{f}_k^2 + (1 + z)v_k^2$, $\forall z \in [0, 1]$, we obtain

$$R^\varepsilon[f, \lambda^*] = \sum_{k > \gamma_\varepsilon W_\varepsilon} (2\pi k)^2 (1 - \lambda_k^*)^2 (\bar{f}_k + v_k)^2 + \varepsilon^2 \sum_{k=1}^\infty (2\pi k)^2 \lambda_k^{*2}$$

$$\leq 2z^{-1} \sum_{k > \gamma_\varepsilon W_\varepsilon} (2\pi k)^2 (1 - \lambda_k^*)^2 \bar{f}_k^2 + (1 + z)R^\varepsilon[v, \lambda^*].$$

Since $\bar{f} \in \mathcal{F}(\beta^*, L^*, \rho)$, we have

$$\sum_{k > \gamma_\varepsilon W_\varepsilon} (2\pi k)^2 (1 - \lambda_k^*)^2 \bar{f}_k^2 \leq L^*(\gamma_\varepsilon W_\varepsilon)^{2-2\beta^*} = o(W_\varepsilon^{2-2\beta}). \tag{26}$$

On the other hand, setting $\tilde{\lambda}_k^* = (1 - (k/W_\varepsilon)^{\beta-1})_+$,

$$R^\varepsilon[v, \lambda^*] - R^\varepsilon[v, \tilde{\lambda}^*] \leq \varepsilon^2 \sum_{k \leq \gamma_\varepsilon W_\varepsilon} (2\pi k)^2 \leq 4\pi^2 \varepsilon^2 (\gamma_\varepsilon W_\varepsilon)^3 = o(\varepsilon^2 W_\varepsilon^3).$$

Using the relation $W_\varepsilon^{2-2\beta} = O(\varepsilon^2 W_\varepsilon^3)$ and choosing $z = z_\varepsilon$ appropriately, we get

$$\sup_{f \in \mathcal{F}_{\delta,\beta,L}(\bar{f})} R^\varepsilon[f, \lambda^*] \leq (1 + o(1)) \sup_{v \in \mathcal{W}(\beta,L)} R^\varepsilon[v, \tilde{\lambda}^*] + o(\varepsilon^2 W_\varepsilon^3),$$

where $\mathcal{W}(\beta, L)$ is the Sobolev ball $\{v : \sum_{k \geq 1} (2\pi k)^{2\beta} f_k^2 \leq L\}$. It then follows from [1, Thm. 1 and Example 1, p. 265] (with $\alpha = \beta - 1$ and $\delta = 3$) that $\sup_{v \in \mathcal{W}(\beta,L)} R^\varepsilon[v, \tilde{\lambda}^*] = C(\beta, L) \varepsilon^{\frac{4\beta-4}{2\beta+1}} (1 + o(1))$. To conclude, it suffices to remark that $o(\varepsilon^2 W_\varepsilon^3) = o(\varepsilon^{\frac{4\beta-4}{2\beta+1}})$. $\square$

5.5. Auxiliary general results.

Lemma 13. *Assume that* $a_1, a_2, \ldots, a_n \in \mathbb{R}$ *and* $\varsigma = \sum_{k=1}^n a_k(\xi_k^2 - \xi_k^{*2})$, *where* $(\xi_1, \ldots, \xi_n, \xi_1^*, \ldots, \xi_n^*)$ *is a zero mean Gaussian vector with identity covariance matrix. For any* $y \in [0, \|a\|/\max_k a_k]$, *it holds*

$$\mathbf{P}\left(\varsigma^2 \geq 32y^2 \sum_{k=1}^n a_k^2\right) \leq 2e^{-y^2}.$$

Proof. Using the formula of the Laplace transform of a chi-squared distribution, for any $t \in [-(\sqrt{8}a_k)^{-1}, (\sqrt{8}a_k)^{-1}]$, we get

$$\mathbf{E}[e^{a_k t(\xi_k^2 - \xi_k^{*2})}] = \frac{1}{1 - 4a_k^2 t^2} \leq e^{8a_k^2 t^2}.$$

Applying [25, Thm. 2.7] with $g_k = 16a_k^2$ and $x = \sqrt{32}y\|a\|$, we get the desired result. $\square$

Lemma 14. *Let* $X_1, \ldots, X_n$ *be independent symmetric random variables. Let* $\boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_n)$ *be a random vector satisfying* $|\varrho_j| \leq \varrho_j^*$, $\forall j = 1, \ldots, n$ *with some deterministic sequence* $(\varrho_j^*)_{j=1}^n$ *and* $\mathscr{L}(\boldsymbol{\varrho}|X_j = x) = \mathscr{L}(\boldsymbol{\varrho}|X_j = -x)$ *for all* $j \in \{1, \ldots, n\}$. *If*

$$\mathbf{E}[e^{tX_j}] \leq e^{t^2 g_j^2}$$

for some sequence $(g_j)_{j=1,\ldots,n}$ and for $|t| \le T_j$, then

$$\mathbf{P}\left(\left|\sum_{j=1}^{n} \varrho_j X_j\right|^2 \ge 4x^2 \sum_{j=1}^{n} \varrho_j^{*2} g_j^2\right) \le 2e^{-x^2(1 \wedge Q_n x^{-1})}, \ \forall x > 0$$

where $Q_n = (\sum_{j=1}^{n} \varrho_j^{*2} g_j^2)^{1/2} \min_j(T_j/\varrho_j^*)$.

Proof. Set $Y_j = \varrho_j X_j$ and $\bar{Y}_j = \varrho_j^* X_j$. For any $p_1, \ldots, p_n \in \mathbb{N}^n$, the expectation $\mathbf{E}[Y_1^{p_1} \cdot \ldots \cdot Y_n^{p_n}]$ vanishes if at least one p_j is odd. Therefore, $\mathbf{E}[(\sum_j Y_j)^k] = 0$ if k is odd and $\mathbf{E}[(\sum_j Y_j)^k] \le \mathbf{E}[(\sum_j \bar{Y}_j)^k]$ if k is even. Hence

$$\mathbf{E}\left[\exp\left(t\sum_j Y_j\right)\right] = \sum_{k=0}^{\infty} \frac{t^k \mathbf{E}[(\sum_j Y_j)^k]}{k!} \le \sum_{k=0}^{\infty} \frac{t^{2k} \mathbf{E}[(\sum_j \bar{Y}_j)^{2k}]}{(2k)!}$$
$$= \mathbf{E}[e^{t\sum_j \varrho_j^* X_j}] \le e^{t^2 \sum_j \varrho_j^{*2} g_j^2}, \quad \forall |t| \le \min_j(T_j/\varrho_j^*).$$

According to the Markov inequality, for every $t > 0$,

$$\mathbf{P}\left(\left|\sum_{j=1}^{n} \varrho_j X_j\right| \ge y\right) \le 2e^{-ty}\mathbf{E}[e^{t\sum_j Y_j}] \le 2\exp\left(-ty + t^2\sum_{j=1}^{n} \varrho_j^{*2} g_j^2\right).$$

Setting $t = (x \wedge Q_n)/(\sum_j \varrho_j^{*2} g_j^2)^{1/2}$ and $y^2 = 4x^2 \sum_j \varrho_j^{*2} g_j^2$ we get the desired result. □

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